Problem

In 1975, Steve Selvin wrote a letter to American Statistician describing a problem loosely based on the game show, “Let’s Make a Deal,” which he dubbed, “The Monte Hall Problem.” The problem is:

> A contestant is given 3 doors to choose from. Behind 1 door is a new car and behind the other 2 doors are a goat. The contestant would prefer to win the car and randomly chooses a door. The host, Monte Hall, then opens one of the remaining 2 doors and reveals a goat. Should the contestant keep her initial guess or should she switch her guess to the other closed door?

A writer asked this question to Parade Magazine in 1990, asking the columnist, Marilyn vos Savant, which was the correct choice. Vos Savant told the reader the correct choice was to switch their choice. Over 10,000 readers wrote letters, including nearly 1000 PhDs in fields such as Mathematics, Statistics, and Physics saying vos Savant’s answer was incorrect.

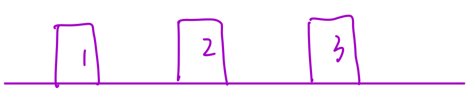
Who is right? Why? Explain your reasoning and show it empirically via simulation.

Answer

Marilyn vos Savant was correct. The probability of winning when keeping the initial choice is $\frac{1}{3}$ and switching is therefore $\frac{2}{3}$.

Assume the initial choice was Door #1. Before the first goat is revealed, the contestant has a 33% chance of being correct. The car will be behind either Doors #2 or #3 66% of the time.

Monte opens one of the other doors. This does not change the probability of the car being behind door #1, but it shows us, with perfect certainty that the car is not behind the door opened. The 66% chance of the car being behind #2 and #3 is still 66%. It cannot be behind the opened door, therefore the door that remains out of #2 and #3 has the full 66% chance of being the car.



第i个门后是car：事件 A_i：$P(A_i) = 1/3$ $\quad i = 1, 2, 3$ $\quad$ independent $\Rightarrow P(A_iB_j) = 1/9$

玩家选择第j门：事件 B_j：$P(B_j) = 1/3$ $\quad j = 1, 2, 3$

庄家从剩下的门中选第k门：事件 K

假设玩家选1门 (B_1)，庄家从2, 3中选 $\Downarrow$ $K = 2, 3$：

因为庄家选的门后面无车

$P(C_3 | A_1B_1) = 1/2 = \frac{P(C_3A_1B_1)}{P(A_1B_1)} \Rightarrow P(C_3A_1B_1) = 1/2 \times 1/9 = \frac{1}{18}$

$P(C_3 | A_2B_1) = 1 = \frac{P(C_3A_2B_1)}{P(A_2B_1)} \Rightarrow P(C_3A_2B_1) = 1/9$

$P(C_3 | A_3B_1) = 0$

所以 $P(B_1C_3)=\sum_{i=1}^{3} P(A_i)\cdot P(B_1|A_i)\cdot P(C_3|A_iB_1) = \frac{1}{3}\times\frac{1}{3}\times(1/2+1+0)=1/6$

$\Rightarrow$ 坚持1门拿到车的概率: $P(A_1|B_1C_3)=\frac{P(A_1B_1C_3)}{P(B_1C_3)}=\frac{\frac{1}{18}}{1/6}=1/3$ $\Rightarrow$ ∴ change!!

换2门拿到车的概率: $P(A_2|B_1C_3)=\frac{P(A_2B_1C_3)}{P(B_1C_3)}=\frac{1/9}{1/6}=\frac{2}{3}$

<table>
<tr><td>Door 1</td><td>Door 2</td><td>Door 3</td></tr>
<tr><td>Contestant pick - closed</td><td>closed</td><td>closed</td></tr>
<tr><td>33% car</td><td>33% car</td><td>33% car</td></tr>
<tr><td>33%</td><td colspan="2">66% car</td></tr>
<tr><td>Closed</td><td>Open - Goat</td><td>Closed</td></tr>
<tr><td>33% Car</td><td>0% car</td><td>66% Car</td></tr>
</table>

Simulations show this is correct. Playing the game 1000 times and repeating that 1000 times show:

Stat	**Value**
Mean	66.74%
Standard Deviation	1.4%