Scanning approximation of logit

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Here we define the logistic function with three parameters, L being the maximum value, μ the inflection point, and k the exponents. We thus have:

$$logit(x) = l(x) = \frac{L}{1 + e^{-k(x-\mu)}}$$

We then approximate the logistic curve with dynamics generated by n box models. We will have:

$$f(x) = \sum_{i=1}^{n} \beta_i f_i(x)$$

where:

$$f_i(x) = \sum_j C_{ij} e^{-\sigma_j x}$$

C_{ij} are constants and σ_j being the rate of movement between boxes. Though in the following derivation the exact form of the approximation is irrelevant.

With the above definitions, I will show that it suffice to scan through only one dimension to gain enough information about the goodness of fit of the approximation to the logistic curve with any parameter given that the approximation is made by a class of function $\mathcal{F}$ that is closed under scalar multiplication ($f \in \mathcal{F} \rightarrow b * f \in \mathcal{F}, \forall b \in \mathbb{R}$) and scaling on the x direction ($f(x) \in \mathcal{F} \rightarrow f(b * x) \in \mathcal{F}, \forall b \in \mathbb{R}$). First I will show that we do not need to scan through different value of L.

Suppose $f(x) \in \mathcal{F}$ minimizes:

$$\int_0^t \left(\frac{1}{1 + e^{-k(x-\mu)}} - f(x)\right)^2 dx$$

then $f^*(x) = Lf(x)$ have error:

$$\int_0^t \left(\frac{L}{1 + e^{-k(x-\mu)}} - f^*(x)\right)^2 dx = L^2 \int_0^t \left(\frac{1}{1 + e^{-k(x-\mu)}} - f(x)\right)^2 dx$$

suppose $\exists f'(x) \neq f^*(x)$ s.t.

$$\int_0^t \left(\frac{L}{1 + e^{-k(x-\mu)}} - f'(x)\right)^2 dx < \int_0^t \left(\frac{L}{1 + e^{-k(x-\mu)}} - f^*(x)\right)^2 dx = L^2 \int_0^t \left(\frac{1}{1 + e^{-k(x-\mu)}} - f(x)\right)^2 dx$$

then

$$\int_0^t (\frac{L}{1+e^{-k(x-\mu)}} - f'(x))^2 dx = L^2 \int_0^t (\frac{1}{1+e^{-k(x-\mu)}} - \frac{1}{L} f'(x))^2 dx < L^2 \int_0^t (\frac{1}{1+e^{-k(x-\mu)}} - f(x))^2 dx$$

we will thus have $\frac{1}{L} f'(x)$ is a better approximate then $f(x)$ which contradicts the first statement. Thus, logistic curve with $L = \lambda$ will have their best approximate taking the form $\lambda f(x)$ where $f(x)$ is the best approximation of the logistic curve with the same value of k and μ but $L = 1$, and the goodness of fit will be different by a factor of λ^2.

Secondly I will show that if $\mu k = C$, the best approximate are easily convertible and the integrated squared error is only off by a factor. In our particular case, this is done by scaling the rate constant between boxes.

for any $g(x)$, with a substitution of $x = \frac{s}{u}$:

$$\int_0^t (\frac{1}{1+e^{-k(x-\mu)}} - g(x))^2 dx = \frac{1}{u} \int_0^{ut} (\frac{1}{1+e^{-k(\frac{s}{u}-\mu)}} - g(\frac{s}{u}))^2 ds$$

$$= \frac{1}{u} \int_0^{ut} (\frac{1}{1+e^{-\frac{k}{u}(s-u\mu)}} - g(\frac{s}{u}))^2 ds = \frac{1}{u} \int_0^{ut} (\frac{1}{1+e^{-k'(s-\mu')}} - g(\frac{s}{u}))^2 ds$$

where $k' = k/u$ and $\mu' = u\mu$ and hence the value of ku is constant.

now suppose $f(x)$ minimizes

$$\int_0^t (\frac{1}{1+e^{-k(x-\mu)}} - f(x))^2 dx$$

for some k, μ, then $f^*(x) = f(x/u)$ will also minimize:

$$\int_0^{ut} (\frac{1}{1+e^{-k'(x-\mu')}} - f^*(x))^2 dx$$

again, suppose $\exists f'(x) \neq f^*(x)$ s.t.

$$\int_0^{ut} (\frac{1}{1+e^{-k'(x-\mu')}} - f'(x))^2 dx < \int_0^{ut} (\frac{1}{1+e^{-k'(x-\mu')}} - f^*(x))^2 dx = \frac{1}{u} \int_0^t (\frac{1}{1+e^{-k(x-\mu)}} - f(x))^2 dx$$

we will have $f'^*(x) = f'(ux)$ s.t.

$$\frac{1}{u} \int_0^t (\frac{1}{1+e^{-k(x-\mu)}} - f'^*(x))^2 dx < \frac{1}{u} \int_0^t (\frac{1}{1+e^{-k(x-\mu)}} - f(x))^2 dx$$

which contradicts the first assumption.

Interestingly, the value of the logistic curve at time = 0 is $L/(1 + e^{k\mu})$ and hence we only need to know the approximation and squared error to logistic curves with asymptope of 1 and scanning through different starting value to know the best approximation and error to all logistic curves.