Vibrations Laboratory 2
Free-Free Beam Experiment

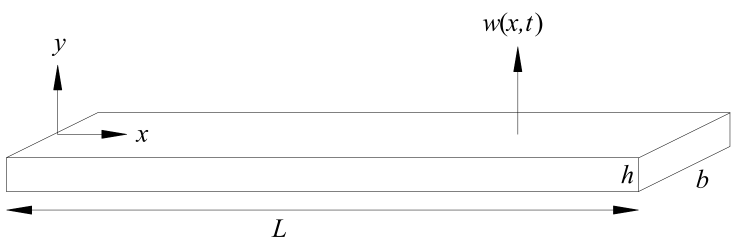
Objectives

Study the vibration characteristics of a free-free beam. The displacement-force amplitude characteristic for the beam will be measured using random and harmonic excitation. Additionally, measured natural frequencies and mode shapes will be compared to theory

Theoretical Background

Freely-vibrating beam

Consider the free-free beam having the typical properties shown in Figure 1. The beam is termed “free-free” since no forces or moments are applied at its ends. The purpose of this analysis is to determine the mode shapes and natural frequencies for the beam.



$A = bh$

$I = \dfrac{bh^3}{12}$

$E = 200\text{GPa}$ (steel)

$\rho = 7800\ \text{kg/m}^3$ (steel)

Figure 1. Schematic and typical properties of a free-free beam.

The analysis begins with the unforced equation of motion for the beam,

$$\rho A\ddot{w} + \gamma\dot{w} + EIw'''' = 0 \tag{1}$$

where the dots correspond to a time partial differentiation, and the primes correspond to a spatial partial differentiation (i.e., $\dot{w} = \frac{\partial w}{\partial t}$ and $w' = \frac{\partial w}{\partial x}$). The damping term $\gamma\dot{w}$ models the internal energy loss mechanisms of the beam. Equation 1 is a partial differential equation which describes the vertical deflection w(x,t) of the beam as a function of space x and time t. Equation 1 can be rewritten as

$$\ddot{w} + \frac{\gamma L}{m}\dot{w} + c^2 w'''' = 0, \quad m = \rho AL\,,\ c = \sqrt{\frac{EI}{\rho A}} \tag{2}$$

A "separation of variables" approach is employed to solve Equation 2. The approach assumes that the solution is composed of the product of two functions, one which is a function of space only and one which is a function of time only,

$$w(x,t) = X(x)T(t) \tag{3}$$

Substitution of Equation 3 into Equation 2 gives, after some manipulation, the result

$$\frac{\ddot{T}+\frac{\gamma L}{m}\dot{T}}{T}=-c^{2}\,\frac{X''''}{X} \tag{4}$$

where the dots correspond to ordinary time differentiation, and the primes correspond to ordinary spatial differentiation. Since the left side of Equation 4 is a function of time only, and the right side is a function of space only, and since both sides are equal for all values of x and t, it must be true that both sides are equal to a constant. This constant will be arbitrarily denoted -ω 2. The spatial and temporal portions of Equation 4 then become, respectively,

$$X'''' - \beta^{4} X = 0\,,\quad \beta^{4} = \frac{\omega^{2}}{c^{2}} \tag{5a}$$

$$\ddot{T}+\frac{\gamma L}{m}\dot{T}+\omega^{2}T=0 \tag{5b}$$

Let us first consider the spatial differential equation, Equation 5a. In the case where $\beta \neq 0$, Equation 5a has the general solution

$$X(x)=a_1\sin(\beta\,x)+a_2\cos(\beta\,x)+a_3\sinh(\beta\,x)+a_4\cosh(\beta\,x) \tag{6}$$

The unknown constants in Equation 6 are eliminated through the application of boundary conditions. These include zero moment at both ends of the beam,

$$EIw''(0,t)=EIw''(L,t)=0 \tag{7}$$

and zero shear at both ends of the beam,

$$EIw'''(0,t)=EIw'''(L,t)=0 \tag{8}$$

Consider the first boundary condition in Equation 7, zero moment at x = 0. Substitution of Equation 3 into this boundary condition gives the simplified boundary condition

$$EIX(0)''T(t)=0 \quad\Rightarrow\quad X(0)''=0 \tag{9}$$

Taking the second spatial derivative of Equation 6, evaluating the expression at x = 0, and setting the result equal to 0 (as required by the right-hand expression in Equation 9) gives the result

$$a_2 = a_4 \tag{10}$$

The following hyperbolic trig identities were used in arriving at this result:

$$\sinh(x)' = \cosh(x)\,, \quad \cosh(x)' = \sinh(x)\,, \quad \sinh(0) = 0\,, \quad \cosh(0) = 1 \tag{11}$$

In a similar fashion, the boundary condition of zero shear at x = 0 gives the result

$$a_3 = a_1 \tag{12}$$

The boundary condition of zero moment at x = L gives the result

$$a_1\left[-\sin(\beta L) + \sinh(\beta L)\right] + a_2\left[-\cos(\beta L) + \cosh(\beta L)\right] = 0 \tag{13}$$

where Equations 10 and 12 have been used to eliminate a3 and a4 from this expression. Finally, the boundary condition of zero shear at x = L gives the result

$$a_1\left[-\cos(\beta L) + \cosh(\beta L)\right] + a_2\left[\sin(\beta L) + \sinh(\beta L)\right] = 0 \tag{14}$$

where Equations 10 and 12 have again been used to eliminate a3 and a4 from the expression. Equations 13 and 14 can be cast into matrix form,

$$[B][a] = [0] \tag{15}$$

where $$[B] = \begin{bmatrix} -\sin(\beta L) + \sinh(\beta L) & -\cos(\beta L) + \cosh(\beta L) \\ -\cos(\beta L) + \cosh(\beta L) & \sin(\beta L) + \sinh(\beta L) \end{bmatrix}, \quad [a] = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad [0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If [B] is an invertible (non-singular) matrix, then it follows that [a] = [B]-1[0] = [0] and therefore a1 = a2 = a3 = a4 = 0. This gives the "trivial" solution X(x) = 0 to Equation 5a. To obtain the non-trivial solution, we require the matrix [B] to be non-invertible (singular), i.e. det[B] = 0. This requirement gives the characteristic equation

$$\cos(\beta L)\cosh(\beta L) = 1 \tag{16}$$

The characteristic equation has infinitely many roots, with the nth root being denoted as (βnL). The roots of the characteristic equation are given in Table 1. It will be shown shortly that the roots of the characteristic equation give the natural frequencies for the beam.

Root	Value
(β_0L)	0 (rigid body)
(β_1L)	4.73004074
(β_2L)	7.85320462
(β_3L)	10.9956078
(β_4L)	14.1371655
(β_5L)	17.2787597
(β_nL)	(2n+1)π/2 , n>5

Table 1. Roots of the characteristic equation for a free-free beam.

Since [B] is required to be singular, then its two rows are linearly dependent, i.e., they differ only by a constant. Hence, only one row is used. The first row in Equation 15 gives the result

$$a_1 = -a_2 \frac{\cosh(\beta_n L) - \cos(\beta_n L)}{\sinh(\beta_n L) - \sin(\beta_n L)} \tag{17}$$

(It would be equally valid to use the second row, as well). Note that the (βL) terms have been replaced by the roots ($\beta_n L$), since these discrete values insure the singularity of the matrix [B]. Substituting Equations 10, 12, and 17 into Equation 6 gives the result

$$X_n(x) = -a_2 \left\{ \cosh\left[(\beta_n L)\frac{x}{L}\right] + \cos\left[(\beta_n L)\frac{x}{L}\right] - \sigma_n \left(\sinh\left[(\beta_n L)\frac{x}{L}\right] + \sin\left[(\beta_n L)\frac{x}{L}\right] \right) \right\} \tag{18}$$

where $\quad n = 1, 2, 3, \ldots \quad$ and $\quad \sigma_n = \dfrac{\cosh(\beta_n L) - \cos(\beta_n L)}{\sinh(\beta_n L) - \sin(\beta_n L)}$

Equation 18 describes the nth flexible "mode shape" for a free-free beam. Each of the n mode shapes satisfies the spatial differential equation, Equation 5a, and associated boundary conditions for the free-free beam. A free-free beam will execute only motions which are some combination of these mode shapes. The first 3 mode shapes for a free-free beam are shown in Figure 1. The mode shapes in the figure have arbitrarily been normalized to have a maximum value of unity. The points where the mode shapes have a value of zero are called "nodes", and these correspond to points on the beam which remain stationary as the beam vibrates.

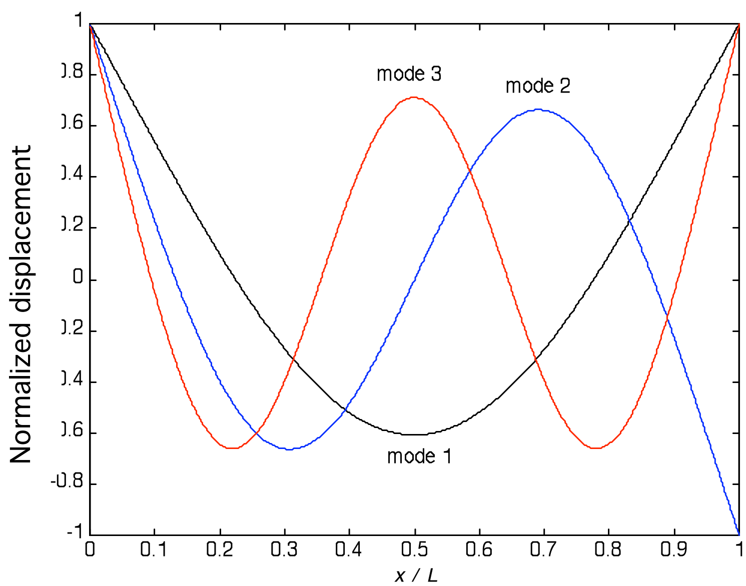


Figure 1. First 3 mode shapes for a free-free beam.

Let us now shift our attention to the temporal differential equation, Equation 5b:

$$\ddot{T} + \omega_n{}^2 T = 0, \beta_n{}^4 = \frac{\omega_n{}^2}{c^2} \tag{19}$$

Notice that the damping term has been excluded. All physical systems have damping, but the exclusion of damping here has been made for the purpose of simplification. Also notice that the arbitrary constant ω has been replaced by the constant ωn. Recall that since only certain discrete values of β are allowed, then it follows that only certain discrete values of ω are allowed. Equation 19 has the solution

$$T_n(t) = A_n \cos(\omega_n t + \phi_n) \tag{20}$$

where the unknown constants A_n and ϕ_n are obtained through the application of initial conditions (i.e., the initial displacement and initial velocity of the beam). The substitution of Equations 18 and 20 into Equation 3 gives an expression for the unforced, undamped motion of a free-free beam:

$$\begin{gathered} w(x,t) = \sum_{n=1}^{\infty} A_n \hat{X}_n(x) \cos(\omega_n t + \phi_n), \\ \hat{X}_n(x) = \cosh\left[(\beta_n L)\frac{x}{L}\right] + \cos\left[(\beta_n L)\frac{x}{L}\right] - \sigma_n\left(\sinh\left[(\beta_n L)\frac{x}{L}\right] + \sin\left[(\beta_n L)\frac{x}{L}\right]\right) \end{gathered} \tag{21}$$

where the rigid body modes (translation and rotation) have been neglected. Notice from the summation index in Equation 21 that the response of the beam depends on all modes of vibration. Additionally, it can be seen that the nth mode shape of vibration occurs at the natural frequency ω_n. By Equation 19, the n nth natural frequency of vibration, associated with the nth mode shape, is given by

$$\omega_n = \beta_n^2 c = \frac{(\beta_n L)^2}{L^2}\sqrt{\frac{EI}{\rho A}} \tag{22}$$

Harmonically-driven beam

The theory associated with a point-driven free-free beam is somewhat more involved than the theory for a freely vibrating beam and will not be presented here in detail. Briefly, the equation motion for a beam which is driven by a point harmonic force input is

$$\rho A \ddot{w} + \gamma \dot{w} + EIw'''' = F\delta(x-a)\cos(\Omega t) \tag{23}$$

where F is the magnitude of the force, δ is the Dirac delta function, a is the location where the force is applied, and Ω is the frequency of the harmonic excitation. It can be shown that Equation 23 has the solution

$$\begin{gathered} w(x,t) = \frac{F}{m}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2} D_n \hat{X}_n(a)\hat{X}_n(x)\cos(\Omega t - \phi_n), \text{ where} \\ D_n = \frac{1}{\sqrt{(1-r_n^2)^2 + (2\xi_n r_n)^2}},\ \phi_n = \arctan\left(\frac{2\xi_n r_n}{1-r_n^2}\right),\ r_n = \frac{\Omega}{\omega_n} \end{gathered} \tag{24}$$

In Equation 24, the rigid body motions (translation and rotation) have been neglected; and, ξn, the modal damping ratio, is an empirical parameter which models energy losses in the beam. As with the freely vibrating beam, the summation sign indicates that the response of the beam depends on all modes of vibration. As expected, the response of the beam becomes large when the force input is large, and when the beam is relatively light. Notice also that when the beam is excited at the nth natural frequency, the parameter D_n becomes very large (for light damping), and the response is dominated by the nth mode shape of motion. On the other hand, if the point force input is located at a node for a particular mode shape, the effect of that mode on the response of the beam will be eliminated.

A quantity of interest (to be measured in this laboratory) for a harmonically forced beam is the displacement-force amplitude characteristic. It is defined as the ratio of the displacement output amplitude at some point on a beam to the amplitude of the force input. The displacement-force amplitude characteristic is equivalent to the magnitude of the displacement-force transfer function, $|T|$. It follows from Equation 24 that the displacement-force amplitude characteristic for a free-free beam, with the harmonic point force excitation located at x = a, and with the measurement point located at x = b, is

$$|T| = \left|\frac{w(b,t)}{F}\right| = \left|\frac{1}{m}\sum_{n=1}^{\infty}\frac{1}{\omega_n^{\,2}} D_n \hat{X}_n(a)\hat{X}_n(b)\cos(\Omega t - \phi_n)\right| \qquad (25)$$

Note that $|T|$ becomes large when $\Omega = \omega n$ and that the contribution from a particular mode becomes small when either the input force or measurement point is at a node location.

Experimental Setup

The student will first measure the displacement-force amplitude characteristic for a free-free beam using a random point excitation. As described in the Theoretical Background section, the displacement-force amplitude characteristic is constructed by computing the ratio of the displacement output amplitude at some point on a beam to the amplitude of the force input, over a range of drive frequencies. An efficient way of measuring the amplitude characteristic is through random excitation. Random excitation simultaneously delivers energy at all frequencies to the beam. The random response of the beam will in turn contain energy at all frequencies. By taking the ratio of the spectral energy of the displacement response to that of the force excitation at each frequency, one will obtain the displacement-force amplitude characteristic. This step is basically achieved by forming the ratio of the FFT's of the displacement response and force excitation.

Figure 1 depicts the experimental setup used to measure the displacement-force amplitude characteristic via random excitation. As shown, an o-ring suspended beam is driven with a shaker. Since the stiffness of the o-rings is small compared to the stiffness of the beam, the beam can be treated as a free-free beam. A computer is used to generate the random input signal. The data acquisition (DAQ) card relays the random signal to a power amplifier, which in turn drives the shaker. The force transducer located between the beam and the shaker at x = a measures the force input to the beam, and an LDV located at x = b is used to measure the displacement response w of the beam.

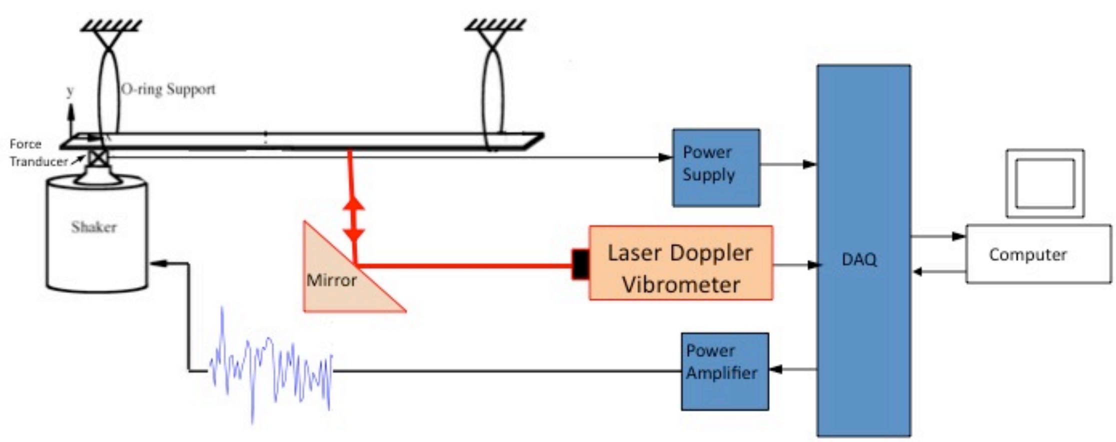


Figure 1. Experimental configuration for measuring the force-displacement amplitude characteristic of a free-free beam via random excitation.

The LDV actually measures velocity $\dot{w}$, but since $\dot{w} = -i\omega w$ for harmonic motion at frequency ω, the output from the LDV is proportional to displacement. The signals from the LDV and force transducer are relayed back to the computer via the power supply boxes and the DAQ card. A MATLAB code is then used to compute the displacement-force amplitude characteristic from the LDV and force transducer signals.

Shown in figure 2 is a sample displacement-force amplitude characteristic for a free-free beam. The characteristic corresponds to a beam which has the dimensions b = 1 inch, h = 0.25 inches, and L = 27 inches. Additionally, the beam is driven at the left-hand end (a = 0), and the displacement is measured at the right-hand end (b = L). The peaks in the characteristic correspond to the natural frequencies of the beam. The student will compare these experimental values to theoretical values calculated using Equation 22.

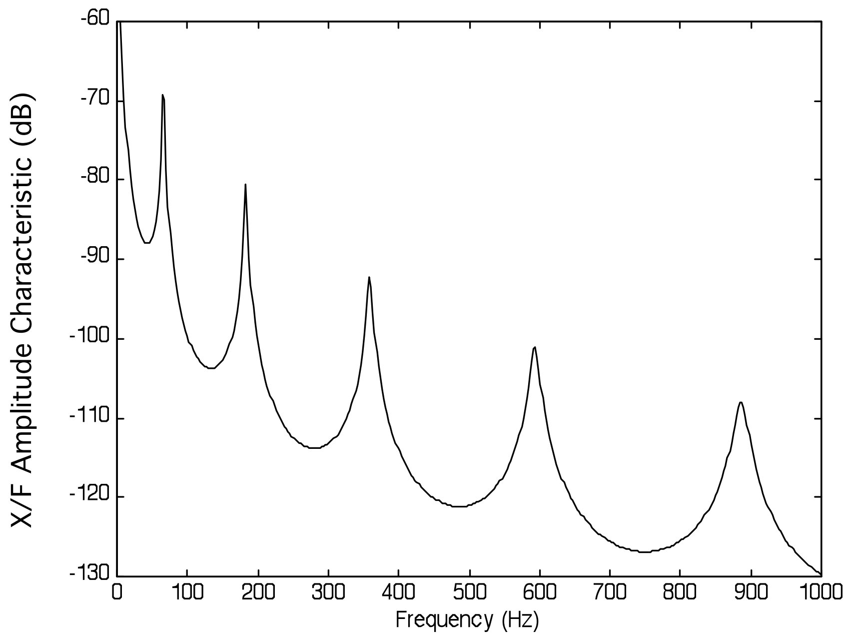


Figure 2. Sample displacement-force amplitude characteristic for a free-free beam.

The student will verify the randomly-acquired displacement-force amplitude characteristic using harmonic excitation. Figure 3 depicts the experimental setup used to measure the amplitude characteristic via harmonic excitation. As shown, a function generator drives the shaker with a harmonic voltage input. As in the random test, the force transducer at x = a measures the force input to the beam, and an LDV located at x = b measures the displacement response of the beam. The force input and displacement response will both be sinusoidal waveforms having the same frequency. The signals from the accelerometer and force transducer are relayed to an oscilloscope via the power supply boxes.

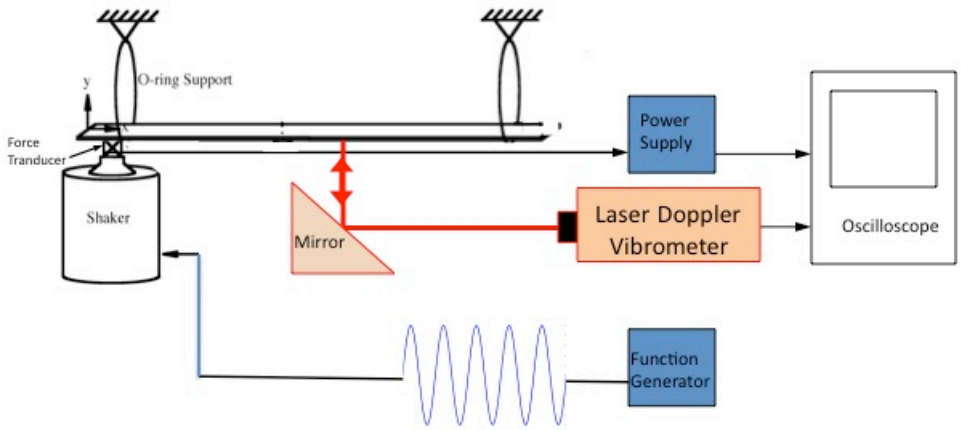


Figure 3. Experimental configuration for measuring the displacement-force amplitude characteristic of a free-free beam via harmonic excitation.

Experimental readings from the accelerometer and force transducer at several different drive frequencies are used to construct the displacement-force amplitude characteristic. Displacement amplitude is calculated from the measured velocity waveform using the equation

$$|\mathrm{X}| = \frac{1}{\omega} \cdot LDV\mathrm{output(v)} \cdot \frac{0.025\mathrm{m/s}}{\mathrm{v}} \tag{1}$$

The sensitivity value 0.025*m/s/v* is the sensitivity of the LDV used in this experiment. Similarly, the force amplitude is calculated from the measured force waveform using the equation

$$|\mathrm{F}| = \text{force transducer output(mv)} \cdot \frac{\mathrm{N}}{11\,\mathrm{mv}} \tag{2}$$

The sensitivity value $g_f = 11$ mv/N is the sensitivity of the force transducer used in this experiment. The value of the amplitude characteristic given by

$$20\log_{10}|T| = 20\log_{10}\left|\frac{X}{F}\right| = 20\log_{10}\left(\frac{\frac{1}{(2\pi \cdot f\,Hz)} \cdot (V_1)_{pp} \cdot \frac{0.025\mathrm{m/s}}{\mathrm{v}}}{V_2\ \mathrm{mv_{pp}} \cdot \frac{N}{11\mathrm{mv}}}\right) \tag{3}$$

The final task for the student is to characterize an arbitrary mode shape, say the nth mode shape. This is achieved by exciting the beam near its nth natural frequency and measuring the beam displacement at several locations along the beam using an accelerometer. Recall from the Theoretical Background section that the response of the beam will be dominated by the nth mode shape of motion when the beam is excited at its nth natural