Ho Chi Minh City → Da Nang 27/01/2025

Now we have found expressions for $\frac{\partial}{\partial \theta_j}[C(\underline{\theta})]$ for each $\theta_j \in \underline{\theta}$.

To do next time: ① write iterative equation for gradient descent
② try the same exercise for a 2-hidden-layer network

① Iterative eqn for gradient descent (0-hidden-layer network).

For each $\theta_j \in \underline{\theta} = \{b^{(1)}\} \cup \{w^{(1)}_{i,j} : i \in [0,783] \cap \mathbb{Z},\ j \in [0,9] \cap \mathbb{Z}\}$,

- Initialise $\theta_j^{[0]} :=$ random (between -10 & 10 idk?)
- Step $\theta^{[n+1]} := \theta^{[n]} - \nu \frac{\partial}{\partial \theta_j}[C(\underline{\theta})]$ where $\nu > 0$ is "learning rate".

Can code logic for Step to repeat until we have hit some max number of iterations OR difference between this step vs previous is negligible.

② 2-hidden-layer network

Let L_i denote the number of neurons in layer i (minus 1, due to 0 indexing). We want this:
- $L_0 = 783$ (since 784 pixels in MNIST images)
- $L_1 = 127$
- $L_2 = 63$
- $L_3 = 9$ (10 digits in final layer)

Otherwise, structure is identical to the 0-hidden-layer network I've been working on. So, in particular:
- Each non-zero layer has a bias parameter $b^{(i)}$ added to the weighted sums evaluated in that layer (i.e. $b^{(1)}, b^{(2)}, b^{(3)}$)
- Each possible pairwise connection between layer $i-1$ and the next layer has a weight parameter associated (i.e. for layer i, $W^{(i)} = \{w^{(i)}_{j,k} : i \in [0, L_i] \cap \mathbb{Z},\ j \in [0, L_{i+1}] \cap \mathbb{Z}\}$. $i \in \{1,2,3\}$)
- ReLU activation used: $f(x): \mathbb{R} \to \mathbb{R}$ where $f(x) := \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$
- Cross-entropy loss used for training:

$$C(\underline{\theta}) = \frac{-1}{K} \sum_{k=0}^{K-1} \sum_{j=0}^{9} t_{j,k} \log(\hat{p}_{j,k})$$ as previously defined!

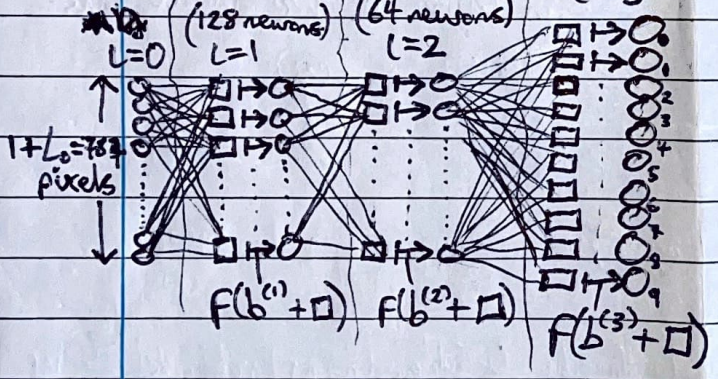
Before differentiating $C(\underline{\theta})$ by all these new parameters:

$$\underline{\theta} = \{b^{(1)}, b^{(2)}, b^{(3)}\} \cup W^{(1)} \cup W^{(2)} \cup W^{(3)}$$

Let us write clearly how neuron values in $C(\underline{\theta})$ are defined...

- $p^{(0)}_{n,k} :=$ n^{th} pixel value in training image k. (valid for $n \in \{0, 1, \ldots, 783\}$)
- $p^{(1)}_{n,k} = f\left(b^{(1)} + \sum_{i=0}^{783} w^{(1)}_{i,n}\, p^{(0)}_{i,k}\right)$ for $n \in \{0, 1, \ldots, L_1 = 127\}$
- $p^{(2)}_{n,k} = f\left(b^{(2)} + \sum_{i=0}^{127} w^{(2)}_{i,n}\, p^{(1)}_{i,k}\right)$ for $n \in \{0, 1, \ldots, L_2 = 63\}$
- $p^{(3)}_{n,k} = f\left(b^{(3)} + \sum_{i=0}^{63} w^{(3)}_{i,n}\, p^{(2)}_{i,k}\right)$ for $n \in \{0, 1, \ldots, L_3 = 9\}$
- Then, $\hat{p}_{n,k} = \dfrac{\exp\{p^{(3)}_{n,k}\}}{\sum_{z=0}^{9} \exp\{p^{(3)}_{z,k}\}}$.

So, in general we have $p^{(l)}_{n,k} = f\left(b^{(l)} + \sum_{i=0}^{L_{l-1}} w^{(l)}_{i,n}\, p^{(l-1)}_{i,k}\right) \quad \forall l \in \{0,1,2,3\}$



then the digit from 0–9 with the highest neuron-value in this final layer is our network's prediction.

(diagram, where □ represents each weighted sum of neurons)

Now we have everything we need, on the next sheet I'll find $\frac{\partial}{\partial \theta_j}[C(\underline{\theta})]$ for all $\theta_j \in \underline{\theta}$...
(start with biases, then do weights. Work from layer 3 to layer 1).
I will use this post in each case to kick things off:

$$\frac{\partial}{\partial \theta_j}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{9} t_{m,k}\,\frac{1}{\hat{p}_{m,k}}\,\frac{\partial}{\partial \theta_j}\left[\hat{p}_{m,k}\right].$$

$$\frac{\partial}{\partial b^{(3)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{j=0}^{q} t_{j,k}\frac{\partial}{\partial b^{(3)}}\left[\log(\hat{p}_{j,k})\right]$$

$b^{(3)}$

$$\frac{\partial}{\partial b^{(3)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{q}\left(\frac{t_{m,k}}{\hat{p}_{m,k}}\frac{\partial}{\partial b^{(3)}}\left[C(\underline{\theta})\right]\frac{\partial}{\partial b^{(3)}}\left[\hat{p}_{m,k}\right]\right)$$

Now, $\frac{\partial}{\partial b^{(3)}}\left[\exp\{p^{(3)}_{m,k}\}\right] = \exp\{p^{(3)}_{m,k}\}\frac{\partial}{\partial b^{(3)}}\left[p^{(3)}_{m,k}\right]$ chain rule & $\frac{\partial}{\partial x}(e^x)=e^x$

using our old friend defined earlier $\sum_{j,k}^{(l)} :=$ "sum of l^{th} bias + weighted sum going into j^{th} neuron in layer l"

and the fact $f'(x)=\begin{cases}1 & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}$

$$= \exp\{p^{(3)}_{m,k}\}\frac{\partial}{\partial x}\left[f(x)\right]\frac{\partial}{\partial b^{(3)}}\left[b^{(3)}+\sum_{i=0}^{63} w^{(3)}_{i,m}p^{(2)}_{i,k}\right]$$

$$= \exp\{p^{(3)}_{m,k}\}\mathbb{I}\{\textstyle\sum^{(3)}_{m,k}>0\}.$$

So, using the quotient rule in essentially the same way we did for the $b^{(1)}$ partial derivative in my 0-hidden-layer network:

$$\hat{p}_{m,k} = \frac{\exp\{p^{(3)}_{m,k}\}}{\sum_{z=0}^{q}\exp\{p^{(3)}_{z,k}\}} = \frac{u}{v}$$

$$u' = \exp\{p^{(3)}_{m,k}\}\mathbb{I}\{\textstyle\sum^{(3)}_{m,k}>0\}.$$

$$v' = \sum_{z=0}^{q}\exp\{p^{(3)}_{z,k}\}\mathbb{I}\{\textstyle\sum^{(3)}_{z,k}>0\}.$$

(derived earlier so copied here!)

$$\Rightarrow \frac{\partial}{\partial b^{(3)}}\left[\hat{p}_{m,k}\right] = \hat{p}_{m,k}\left(\mathbb{I}\{\textstyle\sum^{(3)}_{m,k}>0\} - \sum_{z=0}^{q}\hat{p}_{z,k}\mathbb{I}\{\textstyle\sum^{(3)}_{z,k}>0\}\right)$$

$$\Rightarrow \frac{\partial}{\partial b^{(3)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{q} t_{m,k}\left(\mathbb{I}\{\textstyle\sum^{(3)}_{m,k}>0\} - \sum_{z=0}^{q}\hat{p}_{z,k}\mathbb{I}\{\textstyle\sum^{(3)}_{z,k}>0\}\right)$$

$b^{(2)}$

$$\frac{\partial}{\partial b^{(2)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{q}\left(\frac{t_{m,k}}{\hat{p}_{m,k}}\frac{\partial}{\partial b^{(2)}}\left[\hat{p}_{m,k}\right]\right).$$

Now, $\frac{\partial}{\partial b^{(2)}}\left[\exp\{p^{(3)}_{m,k}\}\right] = \exp\{p^{(3)}_{m,k}\}\frac{\partial}{\partial b^{(2)}}\left[p^{(3)}_{m,k}\right]$

$$= \exp\{p^{(3)}_{m,k}\}\mathbb{I}\{\textstyle\sum^{(3)}_{m,k}>0\}\frac{\partial}{\partial b^{(2)}}\left[b^{(3)}+\sum_{i=0}^{63} w^{(3)}_{i,m}p^{(2)}_{i,k}\right]$$

the $\mathbb{I}\{\sum^{(3)}_{m,k}>0\}$ term comes from differentiating ReLU.

Now, $\frac{\partial}{\partial b^{(2)}}\left[\rho_{i,k}^{(2)}\right] = \frac{\partial}{\partial b^{(2)}}\left[b^{(2)} + \sum_{j=0}^{127} w_{j,i}^{(2)} \rho_{j,k}^{(1)}\right] \cdot \frac{\partial}{\partial x}\left[f(x)\right]$ $\rightarrow x = \sum_{i,k}^{(2)}$

$$= 1 \cdot \mathbb{I}_{\{\sum_{i,k}^{(2)} > 0\}}$$

So, $\frac{\partial}{\partial b^{(2)}}\left[\exp\{\rho_{m,k}^{(3)}\}\right] = \exp\{\rho_{m,k}^{(3)}\}\mathbb{I}_{\{\sum_{m,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,m}^{(3)} \mathbb{I}_{\{\sum_{i,k}^{(2)} > 0\}}\right)$

So, use quotient rule on the following:

$$\hat{\rho}_{m,k} = \frac{u}{v} = \frac{\exp\{\rho_{m,k}^{(3)}\}}{\sum_{z=0}^{9} \exp\{\rho_{z,k}^{(3)}\}}$$

u' is given here ↑

$$v' = \sum_{z=0}^{9} \exp\{\rho_{z,k}^{(3)}\}\mathbb{I}_{\{\sum_{z,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,z}^{(3)} \mathbb{I}_{\{\sum_{i,k}^{(2)} > 0\}}\right)$$

So, $\frac{\partial}{\partial b^{(2)}}\left[\hat{\rho}_{m,k}\right] = \frac{vu' - uv'}{v^2}$

$$= \frac{1}{\left(\sum_{z=0}^{9} \exp\{\rho_{z,k}^{(3)}\}\right)^2}\left(\begin{array}{l}\left(\sum_{z=0}^{9} \exp\{\rho_{z,k}^{(3)}\}\right)\exp\{\rho_{m,k}^{(3)}\}\mathbb{I}_{\{\sum_{m,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,m}^{(3)} \mathbb{I}_{\{\sum_{i,k}^{(2)} > 0\}}\right) \\ -\left(\exp\{\rho_{m,k}^{(3)}\}\right)\sum_{z=0}^{9} \exp\{\rho_{z,k}^{(3)}\}\mathbb{I}_{\{\sum_{z,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,z}^{(3)} \mathbb{I}_{\{\sum_{i,k}^{(2)} > 0\}}\right)\end{array}\right)$$

$$= \hat{\rho}_{m,k}\left(\sum_{i=0}^{63} w_{i,m}^{(3)} \mathbb{I}_{\{\sum_{m,k}^{(3)} > 0, \sum_{i,k}^{(2)} > 0\}} - \sum_{z=0}^{9} \hat{\rho}_{z,k} \sum_{i=0}^{63} w_{i,z}^{(3)} \mathbb{I}_{\{\sum_{z,k}^{(3)} > 0, \sum_{i,k}^{(2)} > 0\}}\right)$$

$$\Rightarrow \frac{\partial}{\partial b^{(2)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{9} t_{m,k}\left(\sum_{i=0}^{63} w_{i,m}^{(3)} \mathbb{I}_{\{\sum_{m,k}^{(3)} > 0, \sum_{i,k}^{(2)} > 0\}} - \sum_{z=0}^{9} \hat{\rho}_{z,k} \sum_{i=0}^{63} w_{i,z}^{(3)} \mathbb{I}_{\{\sum_{z,k}^{(3)} > 0, \sum_{i,k}^{(2)} > 0\}}\right)$$

$$\boxed{b^{(1)}} \quad \frac{\partial}{\partial b^{(1)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{9}\left(\frac{t_{m,k}}{\hat{\rho}_{m,k}} \frac{\partial}{\partial b^{(1)}}\left[\hat{\rho}_{m,k}\right]\right)$$

Now, $\frac{\partial}{\partial b^{(1)}}\left[\exp\{\rho_{m,k}^{(3)}\}\right] = \exp\{\rho_{m,k}^{(3)}\}\frac{\partial}{\partial b^{(1)}}\left[\rho_{m,k}^{(3)}\right]$

$$= \exp\{\rho_{m,k}^{(3)}\}\mathbb{I}_{\{\sum_{m,k}^{(3)} > 0\}}\frac{\partial}{\partial b^{(1)}}\left[b^{(3)} + \sum_{i=0}^{63} w_{i,m}^{(3)} \rho_{i,k}^{(2)}\right]$$

$$= \exp\{\rho_{m,k}^{(3)}\}\mathbb{I}_{\{\sum_{m,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,m}^{(3)} \frac{\partial}{\partial b^{(1)}}\left[\rho_{i,k}^{(2)}\right]\right).$$

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$b^{(1)}$ cont...

Keep unpacking the derivative from prev. page...

$$\frac{\partial}{\partial b^{(1)}}\left[\rho_{i,k}^{(2)}\right] = \frac{\partial}{\partial x}\left[f(x)\right]\frac{\partial}{\partial b^{(1)}}\left[b^{(2)} + \sum_{j=0}^{127} w_{j,i}^{(2)}\rho_{j,k}^{(1)}\right] \qquad x = z_{i,k}^{(2)}$$

$$= \mathbb{I}_{\{z_{i,k}^{(2)} > 0\}}\left(\sum_{j=0}^{127} w_{j,i}^{(2)}\frac{\partial}{\partial b^{(1)}}\left[\rho_{j,k}^{(1)}\right]\right) \qquad y = z_{j,k}^{(1)}$$

$$= \mathbb{I}_{\{z_{i,k}^{(2)} > 0\}}\left(\sum_{j=0}^{127} w_{j,i}^{(2)}\frac{\partial}{\partial y}\left[f(y)\right]\frac{\partial}{\partial b^{(1)}}\left[b^{(1)} + \sum_{n=0}^{783} w_{n,j}^{(1)}\rho_{n,k}^{(0)}\right]\right)$$

$$= \mathbb{I}_{\{z_{i,k}^{(2)} > 0\}}\left(\sum_{j=0}^{127} w_{j,i}^{(2)}\mathbb{I}_{\{z_{j,k}^{(1)} > 0\}}\right) \qquad (=1)$$

So, $$\frac{\partial}{\partial b^{(1)}}\left[\exp\{\rho_{m,k}^{(3)}\}\right] = \exp\{\rho_{m,k}^{(3)}\}\mathbb{I}_{\{z_{m,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,m}^{(3)}\mathbb{I}_{\{z_{i,k}^{(2)} > 0\}}\left(\sum_{j=0}^{127} w_{j,i}^{(2)}\mathbb{I}_{\{z_{j,k}^{(1)} > 0\}}\right)\right)$$

So use quotient rule on following:

$$\hat{\rho}_{m,k} = \frac{u}{v} = \frac{\exp\{\rho_{m,k}^{(3)}\}}{\sum_{z=0}^{9}\exp\{\rho_{z,k}^{(3)}\}}$$

u' is given here

~~$u' = \exp\{\rho_{m,k}^{(3)}\}$~~

$$v' = \sum_{z=0}^{9}\exp\{\rho_{z,k}^{(3)}\}\mathbb{I}_{\{z_{z,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,z}^{(3)}\mathbb{I}_{\{z_{i,k}^{(2)} > 0\}}\left(\sum_{j=0}^{127} w_{j,i}^{(2)}\mathbb{I}_{\{z_{j,k}^{(1)} > 0\}}\right)\right)$$

So, $$\frac{\partial}{\partial b^{(1)}}\left[\hat{\rho}_{m,k}\right] = \frac{vu' - uv'}{v^2}$$

$$= \frac{1}{\left(\sum_{z=0}^{9}\exp\{\rho_{z,k}^{(3)}\}\right)^2}\left(\left(\sum_{z=0}^{9}\exp\{\rho_{z,k}^{(3)}\}\right)\exp\{\rho_{m,k}^{(3)}\}\mathbb{I}_{\{z_{m,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,m}^{(3)}\mathbb{I}_{\{z_{i,k}^{(2)} > 0\}}\left(\sum_{j=0}^{127} w_{j,i}^{(2)}\mathbb{I}_{\{z_{j,k}^{(1)} > 0\}}\right)\right)\right.$$
$$\left. - \exp\{\rho_{m,k}^{(3)}\}\sum_{z=0}^{9}\exp\{\rho_{z,k}^{(3)}\}\mathbb{I}_{\{z_{z,k}^{(3)} > 0\}}\left(\sum_{i=0}^{63} w_{i,z}^{(3)}\mathbb{I}_{\{z_{i,k}^{(2)} > 0\}}\left(\sum_{j=0}^{127} w_{j,i}^{(2)}\mathbb{I}_{\{z_{j,k}^{(1)} > 0\}}\right)\right)\right)$$

$$= \hat{\rho}_{m,k}\left(\sum_{i=0}^{63}\sum_{j=0}^{127}\mathbb{I}_{\{z_{m,k}^{(3)} > 0, z_{i,k}^{(2)} > 0, z_{j,k}^{(1)} > 0\}} - \sum_{z=0}^{9}\sum_{i=0}^{63}\sum_{j=0}^{127}\hat{\rho}_{z,k}\mathbb{I}_{\{z_{z,k}^{(3)} > 0, z_{i,k}^{(2)} > 0, z_{j,k}^{(1)} > 0\}}\right)$$

Define shorthand $\mathbb{I}_{j,k}^{(1)} := \mathbb{I}_{\{z_{j,k}^{(1)} > 0\}}$

$$= \hat{\rho}_{m,k}\left(\sum_{i=0}^{63}\sum_{j=0}^{127}\mathbb{I}_{m,k}^{(3)}w_{i,m}^{(3)}\mathbb{I}_{i,k}^{(2)}w_{j,i}^{(2)}\mathbb{I}_{j,k}^{(1)} - \sum_{z=0}^{9}\sum_{i=0}^{63}\sum_{j=0}^{127}\hat{\rho}_{z,k}\mathbb{I}_{z,k}^{(3)}w_{i,z}^{(3)}\mathbb{I}_{i,k}^{(2)}w_{j,i}^{(2)}\mathbb{I}_{j,k}^{(1)}\right)$$

$$= \hat{\rho}_{m,k}\left(\sum_{i=0}^{63}\sum_{j=0}^{127}\mathbb{I}_{m,k}^{(3)}w_{i,m}^{(3)}\mathbb{I}_{i,k}^{(2)}w_{j,i}^{(2)}\mathbb{I}_{j,k}^{(1)} - \sum_{z=0}^{9}\sum_{i=0}^{63}\sum_{j=0}^{127}\hat{\rho}_{z,k}\mathbb{I}_{z,k}^{(3)}w_{i,z}^{(3)}\mathbb{I}_{i,k}^{(2)}w_{j,i}^{(2)}\mathbb{I}_{j,k}^{(1)}\right).$$

So, $\frac{\partial}{\partial b^{(1)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{q} t_{m,k}\left(\sum_{i=0}^{63}\sum_{j=0}^{127} I_{m,k}^{(3)} w_{i,m}^{(3)} I_{i,k}^{(2)} w_{j,i}^{(2)} I_{j,k}^{(1)} - \sum_{z=0}^{q}\sum_{i=0}^{63}\sum_{j=0}^{127} \hat{p}_{z,k} I_{z,k}^{(3)} w_{i,z}^{(3)} I_{i,k}^{(2)} w_{j,i}^{(2)} I_{j,k}^{(1)}\right)$

$W_{\alpha,\beta}^{(3)}$

$$\frac{\partial}{\partial w_{\alpha,\beta}^{(3)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{q} \frac{t_{m,k}}{\hat{p}_{m,k}} \frac{\partial}{\partial w_{\alpha,\beta}^{(3)}}\left[\hat{p}_{m,k}\right]. \quad \text{Now,}$$

$$\frac{\partial}{\partial w_{\alpha,\beta}^{(3)}}\left[\exp\{p_{m,k}^{(3)}\}\right] = \exp\{p_{m,k}^{(3)}\}\frac{\partial}{\partial w_{\alpha,\beta}^{(3)}}\left[p_{m,k}^{(3)}\right]$$

$$= \exp\{p_{m,k}^{(3)}\} I_{m,k}^{(3)} \frac{\partial}{\partial w_{\alpha,\beta}^{(3)}}\left[b^{(3)} + \sum_{i=0}^{63} w_{i,m}^{(3)} p_{i,k}^{(2)}\right]$$

$$= \begin{cases} 0 & \text{if } m \neq \beta \\ \exp\{p_{\beta,k}^{(3)}\} I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)} & \text{otherwise} \end{cases}$$

So use quotient rule on the following:

$$\hat{p}_{m,k} = \frac{u}{v} = \frac{\exp\{p_{m,k}^{(3)}\}}{\sum_{z=0}^{q}\exp\{p_{z,k}^{(3)}\}}$$

u' is here:

$$u' = \exp\{p_{\beta,k}^{(3)}\} I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)} \mathbb{I}_{\{m=\beta\}}$$

$$v' = \exp\{p_{\beta,k}^{(3)}\} I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)}$$

So, $\frac{\partial}{\partial w_{\alpha,\beta}^{(3)}}\left[\hat{p}_{m,k}\right] = \frac{vu' - uv'}{v^2}$

$$= \frac{1}{\left(\sum_{z=0}^{q}\exp\{p_{z,k}^{(3)}\}\right)^2}\left(\left(\sum_{z=0}^{q}\exp\{p_{z,k}^{(3)}\}\right)\exp\{p_{\beta,k}^{(3)}\} I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)} \mathbb{I}_{\{m=\beta\}} - \exp\{p_{m,k}^{(3)}\}\left(\exp\{p_{\beta,k}^{(3)}\} I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)}\right)\right)$$

$$= \hat{p}_{m,k}\left(\mathbb{I}_{\{m=\beta\}} I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)} - \hat{p}_{\beta,k} I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)}\right)$$

can take: $\left(\mathbb{I}_{\{m=\beta\}} - \hat{p}_{\beta,k}\right) \cdot I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)}$

Therefore we arrive at:

$$\frac{\partial}{\partial w_{\alpha,\beta}^{(3)}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{q} t_{m,k}\left(\mathbb{I}_{\{m=\beta\}} - \hat{p}_{\beta,k}\right) I_{\beta,k}^{(3)} p_{\alpha,k}^{(2)}$$

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$W^{(2)}_{\alpha,\beta}$

Start with $\frac{\partial}{\partial W^{(2)}_{\alpha,\beta}}\left[\exp\{\rho^{(3)}_{m,k}\}\right] = \ldots$

$$\ldots = \exp\{\rho^{(3)}_{m,k}\} I^{(3)}_{m,k} \frac{\partial}{\partial W^{(2)}_{\alpha,\beta}}\left[b^{(3)} + \sum_{i=0}^{63} W^{(3)}_{i,m}\rho^{(2)}_{i,k}\right]$$

$$= \sum_{i=0}^{63} W^{(3)}_{i,m} \frac{\partial}{\partial W^{(2)}_{\alpha,\beta}}\left[\rho^{(2)}_{i,k}\right]$$

Now, $\frac{\partial}{\partial W^{(2)}_{\alpha,\beta}}\left[\rho^{(2)}_{i,k}\right] = I^{(2)}_{i,k} \frac{\partial}{\partial W^{(2)}_{\alpha,\beta}}\left[b^{(2)} + \sum_{j=0}^{127} W^{(2)}_{j,i}\rho^{(1)}_{j,k}\right]$

$$= I^{(2)}_{i,k} \mathbb{I}\{i=\beta\}\rho^{(1)}_{\alpha,k}$$

$$\left(= I^{(2)}_{\beta,k}\rho^{(1)}_{\alpha,k}\mathbb{I}\{i=\beta\}\right)$$

$$\Rightarrow \frac{\partial}{\partial W^{(2)}_{\alpha,\beta}}\left[\exp\{\rho^{(3)}_{m,k}\}\right] = \exp\{\rho^{(3)}_{m,k}\} I^{(3)}_{m,k} \sum_{i=0}^{63} W^{(3)}_{i,m} I^{(2)}_{\beta,k}\rho^{(1)}_{\alpha,k}\mathbb{I}\{i=\beta\}$$

$$= \exp\{\rho^{(3)}_{m,k}\} I^{(3)}_{m,k} W^{(3)}_{\beta,m} I^{(2)}_{\beta,k}\rho^{(1)}_{\alpha,k}$$

Interestingly, there is no index indicator here unlike the $W^{(3)}_{\alpha,\beta}$ derivative. I think this is because the final layer weights do not contribute to ALL 9 OF THE FINAL NEURONS, whereas each 2nd (and 1st) layer weight does contribute to EVERY final neuron!

QUOTIENT RULE

$$\hat{\rho}_{m,k} = \frac{\exp\{\rho^{(3)}_{m,k}\}}{\sum_{z=0}^{9}\exp\{\rho^{(3)}_{z,k}\}} = \frac{u}{v}$$

u' is above.

$$v' = \sum_{z=0}^{9}\exp\{\rho^{(3)}_{z,k}\} I^{(3)}_{z,k} W^{(3)}_{\beta,z} I^{(2)}_{\beta,k}\rho^{(1)}_{\alpha,k}.$$

$$\frac{\partial}{\partial W^{(2)}_{\alpha,\beta}}\left[\hat{\rho}_{m,k}\right] = \frac{vu' - uv'}{v^2} =$$

$$= \frac{1}{\left(\sum_{z=0}^{9}\exp\{\rho^{(3)}_{z,k}\}\right)^2}\left(\left(\sum_{z=0}^{9}\exp\{\rho^{(3)}_{z,k}\}\right)\exp\{\rho^{(3)}_{m,k}\} I^{(3)}_{m,k} W^{(3)}_{\beta,m} I^{(2)}_{\beta,k}\rho^{(1)}_{\alpha,k}\right.$$

$$\left. - \exp\{\rho^{(3)}_{m,k}\}\sum_{z=0}^{9}\exp\{\rho^{(3)}_{z,k}\} I^{(3)}_{z,k} W^{(3)}_{\beta,z} I^{(2)}_{\beta,k}\rho^{(1)}_{\alpha,k}\right) = \ldots$$

$$= \hat{p}_{m,k}\left(I^{(3)}_{m,k} W^{(3)}_{\beta,m} I^{(2)}_{\beta,k} \rho^{(1)}_{\alpha,k} - \sum_{z=0}^{9} \hat{p}_{z,k} W^{(3)}_{\beta,z} I^{(3)}_{z,k} I^{(2)}_{\beta,k} \rho^{(1)}_{\alpha,k}\right)$$

$$\Rightarrow \frac{\partial}{\partial W^{(2)}_{\alpha,\beta}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{9} t_{m,k} I^{(2)}_{\beta,k} \rho^{(1)}_{\alpha,k}\left(I^{(3)}_{m,k} W^{(3)}_{\beta,m} - \sum_{z=0}^{9} \hat{p}_{z,k} I^{(3)}_{z,k} W^{(3)}_{\beta,z}\right)$$

$W^{(1)}_{\alpha,\beta}$

Start with $\frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[\exp\{\rho^{(3)}_{m,k}\}\right] = \ldots$

$$= \exp\{\rho^{(3)}_{m,k}\} I^{(3)}_{m,k} \sum_{i=0}^{63} W^{(3)}_{i,m} \frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[\rho^{(2)}_{i,k}\right]$$

Now $\frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[\rho^{(2)}_{i,k}\right] = I^{(2)}_{i,k} \frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[b^{(2)} + \sum_{j=0}^{127} W^{(2)}_{j,i} \rho^{(1)}_{j,k}\right]$

$$= I^{(2)}_{i,k} \sum_{j=0}^{127} W^{(2)}_{j,i} \frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[\rho^{(1)}_{j,k}\right]$$

$$= I^{(2)}_{i,k} \sum_{j=0}^{127} W^{(2)}_{j,i} I^{(1)}_{j,k} \frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[b^{(1)} + \sum_{n=0}^{783} W^{(1)}_{n,j} \rho^{(0)}_{n,k}\right]$$

$$= I^{(2)}_{i,k} \sum_{j=0}^{127} W^{(2)}_{j,i} I^{(1)}_{0,k} \mathbb{I}\{j=\beta\} \rho^{(0)}_{\alpha,k}$$

$$= I^{(2)}_{i,k} W^{(2)}_{\beta,i} I^{(1)}_{\beta,k} \rho^{(0)}_{\alpha,k} \qquad \text{(since summing over } j \text{ zeroes all except } j=\beta\text{)}$$

So, subbing back in:

$$\frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[\exp\{\rho^{(3)}_{m,k}\}\right] = \exp\{\rho^{(3)}_{m,k}\} I^{(3)}_{m,k} \sum_{i=0}^{63} W^{(3)}_{i,m} I^{(2)}_{i,k} W^{(2)}_{\beta,i} I^{(1)}_{\beta,k} \rho^{(0)}_{\alpha,k}$$

(QUOTIENT RULE) $\hat{p}_{m,k} = \frac{u}{v} = \frac{\exp\{\rho^{(3)}_{m,k}\}}{\sum_{z=0}^{9}\exp\{\rho^{(3)}_{z,k}\}}$. $u' \cdot v' = \sum_{z=0}^{9} \exp\{\rho^{(3)}_{z,k}\} I^{(3)}_{z,k} \sum_{i=0}^{63} W^{(3)}_{i,z} I^{(2)}_{i,k} W^{(2)}_{\beta,i} I^{(1)}_{\beta,k} \rho^{(0)}_{\alpha,k}$.

$$\Rightarrow \frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[\hat{p}_{m,k}\right] = \frac{vu' - uv'}{v^2} = \hat{p}_{m,k}\left(I^{(3)}_{m,k} \sum_{i=0}^{63} W^{(3)}_{i,m} I^{(2)}_{i,k} W^{(2)}_{\beta,i} I^{(1)}_{\beta,k} \rho^{(0)}_{\alpha,k} - \sum_{z=0}^{9} \hat{p}_{z,k} I^{(3)}_{z,k} \sum_{i=0}^{63} W^{(3)}_{i,z} I^{(2)}_{i,k} W^{(2)}_{\beta,i} I^{(1)}_{\beta,k} \rho^{(0)}_{\alpha,k}\right)$$

$$\Rightarrow \frac{\partial}{\partial W^{(1)}_{\alpha,\beta}}\left[C(\underline{\theta})\right] = \frac{-1}{K}\sum_{k=0}^{K-1}\sum_{m=0}^{9} t_{m,k} I^{(1)}_{\beta,k} \rho^{(0)}_{\alpha,k}\left(I^{(3)}_{m,k} \sum_{i=0}^{63} W^{(3)}_{i,m} I^{(2)}_{i,k} W^{(2)}_{\beta,i} - \sum_{z=0}^{9} \hat{p}_{z,k} I^{(3)}_{z,k} \sum_{i=0}^{63} W^{(3)}_{i,z} I^{(2)}_{i,k} W^{(2)}_{\beta,i}\right).$$