AMATH 515: RECOVERING A CORRUPTED IMAGE WITH PROXIMAL GRADIENT DESCENT

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1. Introduction

In this report, we will implement and test algorithms to recover a corrupted image. In particular, we will be applying a mask to the painting "Son of Man" by René Magritte and recovering a sparse matrix representation with the Discrete Cosine Transform (DCT) applied onto the image. A downsampled version and full size version of the painting was used, where CVXPY was applied to the downsampled version and a proximal gradient descent algorithm (PGD) was applied to both the downsampled image and the full image. Accuracy was quantified by calculating the mean square error on the masked out pixels, and we saw both CVXPY and PGD performed similarly on the downsampled image. This report further explored the effects of altering the parameter λ and its effects on image recovery, where we can see how the accuracy decreases when λ becomes too large or small.

2. Theory

For $\phi(x) = \|x\|_1$, we get the soft thresholding operator when we define

$$\text{prox}_{\lambda\phi}(x) = \arg\min_z \lambda\|z\|_1 + \frac{1}{2}\|z - x\|^2 = \begin{cases} x_i + \lambda & x_i < 0 \\ x_i - \lambda & x_i > 0 \\ 0 & x_i = 0 \end{cases} \tag{1}$$

which was derived from taking the derivative with respect to z and finding the critical points.

Let $X, Y, M \in \mathbb{R}^{N_y \times N_x}$ where X is the DCT of an image which we want to find, Y is the true/uncorrupted image, and M is a masking matrix with entries $M_{i,j} \in \{0, 1\}$ where its zero entries correspond to the missing pixels in the corrupted image. Our objective function is

$$f(X) = \frac{1}{2}\|M \odot (\text{iDCT}(X)_{ij} - Y_{ij})\|_F^2 = \frac{1}{2}\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}(M_{ij} \cdot (\text{iDCT}(X)_{ij} - Y_{ij}))^2 \tag{2}$$

Taking its gradient and using A, A^{-1} and the flattened forms of each matrix with DCT and iDCT respectively (noting that A is also unitary with $A^{-1} = A^T$), we get

$$\nabla f(X) = \frac{1}{2} \cdot 2(A^{-1})^T(\text{vec}(M) \odot (A^{-1}\text{vec}(X) - \text{vec}(Y))) \tag{3}$$

$$= A(\text{vec}(M) \odot (A^{-1}\text{vec}(X) - \text{vec}(Y))) \tag{4}$$

We can also note that we are taking multiplying A and A^{-1} against vectors this gradient, which we can then replace to get

$$\nabla f(X) = \text{DCT}(M \odot (\text{iDCT}(X) - Y)) \tag{5}$$

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We can define

$$\|\nabla f(X_1) - \nabla f(X_2)\| = \|A(\text{vec}(M) \odot (A^{-1}\text{vec}(X_1) - \text{vec}(Y))) - A(\text{vec}(M) \odot (A^{-1}\text{vec}(X_2) - \text{vec}(Y)))\| \tag{6}$$

$$= \|A(M \odot (A^{-1}\text{vec}(X_1 - X_2)))\| \leq \|AM\|\|A^{-1}\|\|(X_1 - X_2)\| \tag{7}$$

We then see $L = \|AM\|\|A^{-1}\| \leq \|A\|\|M\|\|A^{-1}\| = 1$ because A is unitary and the entries of M are only either 0 or 1. This can be used to define a step side of $1/L$ similar to the step size used in gradient descent.

3. Methods

The goal of this report is to recover a corrupted image by imposing the sparsity of its DCT with an ℓ_1 regularized regression problem given by

$$\min_x \quad \frac{1}{2}\left\|M \odot (\text{iDCT}(X) - Y)\right\|_2^2 \;+\; \lambda\left\|\text{vec}(X)\right\|_1 \tag{8}$$

$$= \min_x \quad \frac{1}{2}\left\|\text{vec}(M) \odot (A^{-1}\text{vec}(X) - \text{vec}(Y))\right\|_2^2 \;+\; \lambda\left\|\text{vec}(X)\right\|_1 \tag{9}$$

for a regularization parameter $\lambda > 0$. This is applied to the painting "Son of Man" by René Magritte after a masking matrix M is generated an applied to the full image and a downscaled version. A proximal gradient descent (PGD) method was applied to the full and downscaled image while CVXPY was applied to the downscaled version only. Each step of PGD was iterated as

$$x_{k+1} = \text{prox}_{\lambda\phi}(x_k - h\nabla f)) \tag{10}$$

using Equation 1 until the max iterations was reached or $\|x_{k+1} - x_k\| \leq$ tol for some tolerance.

Measuring how well the true image was recovered was quantified by the mean squared error of the estimates compared to the true image evaluated on the pixels covered by the mask, divided by the variance on the pixels. After creating the algorithms, they were also tested on various values of λ_i to see how that affected the image recovery. The parameters were defined in Table 1.

Parameter	Value
λ	0.01
x_0	$0.5 \cdot [1, \ldots, 1]$
h	0.1
max_iter	5000
tol	$1e-4$
λ_i	[0.001, 0.01, 0.1, 0.5, 1]

Table 1. Initial Parameters for setting up the algorithm.

4. Results

For the downscaled images, CVXPY and the PGD provided a reconstruction with an MSE of 0.048016 and 0.0480589, respectively (Figure 1).

For the full size image, a MSE of 0.009316 was calculated, with the recovered image as shown (Figure 2).

Given different values of lambda, the image was recovered with varying degrees to accuracy, where the lowest MSE was from $\lambda = 0.01$, while increasing or decreasing λ lead to an increased MSE (Table 2). We can see in the plots that as λ increased, the man's silhouette becomes increasing indistinguishable, where even as the pixels in the centered remained darker compared to the red of the image, it no longer retained the shape of the man, especially for $\lambda = 1$ (Figure 3).

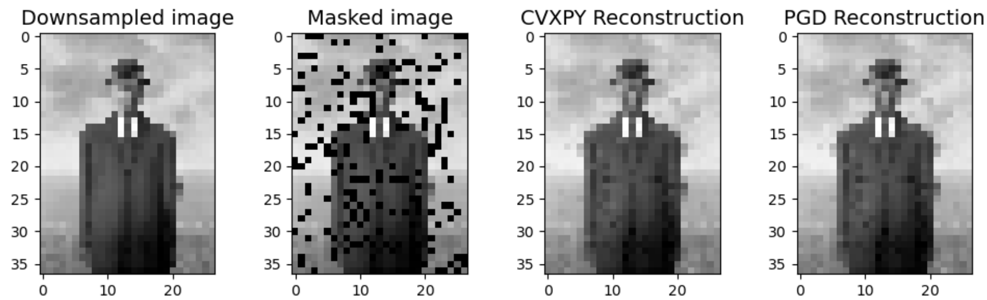

FIGURE 1. From left to right, this is plot of the downsampled image, the downsampled image with a masked applied, the reconstruction from CVXPY and the reconstruction from PGD.

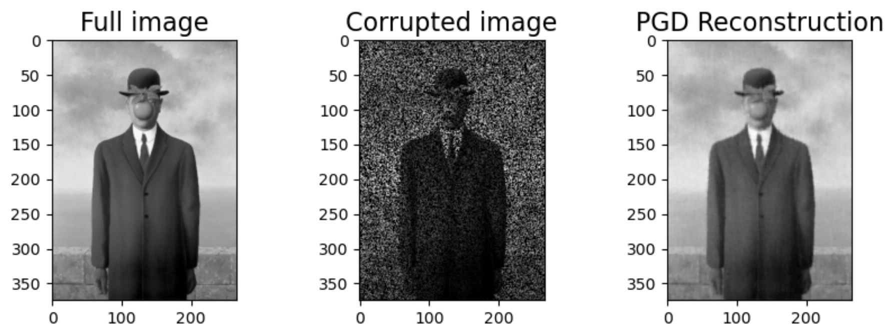

FIGURE 2. From left to right, this is plot of the full image, the full image with a masked applied, the reconstruction from PGD.

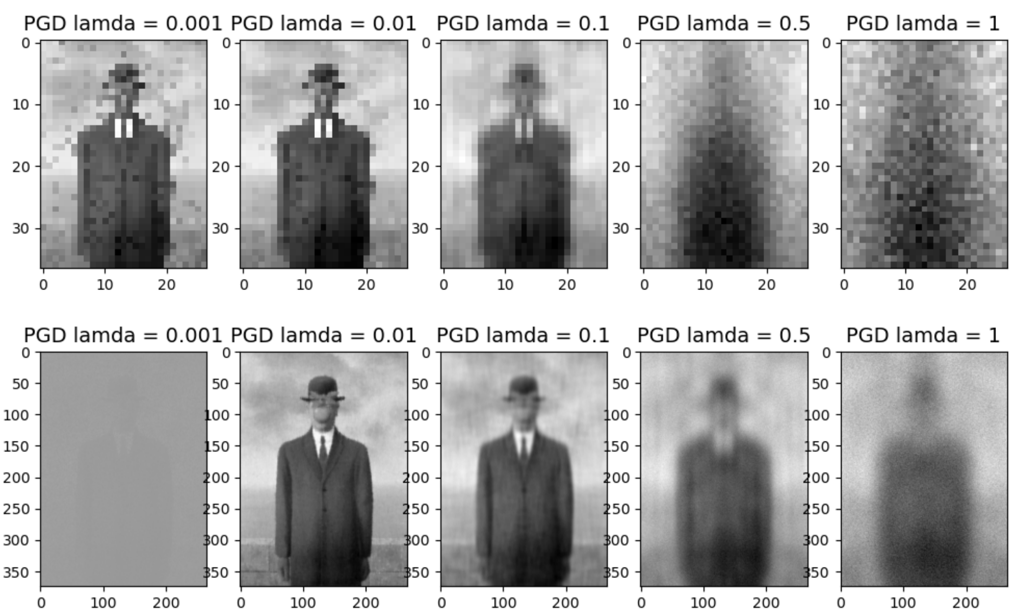

FIGURE 3. From left to right, this is the reconstruction of corrupted image using PGD with $\lambda = [0.001, 0.01, 0.1, 0.5, 1]$ on the downsampled image (top) and full image (bottom).

λ	Downsampled MSE	Full MSE
0.001	0.105137	1.120262
0.01	0.048059	0.009316
0.1	0.049639	0.021894
0.5	0.114897	0.076462
1	0.208182	0.14615

TABLE 2. Calculated MSE for the different values of lambda from running PGD on the downsampled image and full size image.

5. Discussion and Conclusion

From our plots, we see CVXPY and PGD performed similarly on recovering the downsampled image. Running PGD on the full image led to an even smaller MSE value, where the original image was mostly recovered, though blurrier with less contrast for the colors of the image.

There was also an ideal λ with a value of 0.01 which led to the lowest MSE and clearest recovery of the original image. When λ increased, the image became blurrier, mostly retaining the darker pixels in the center of the image. An interesting note was that when λ decreased for the full image, the image appeared to almost be a uniform color, with a very faint outline of the man in the center.

For future directions, it will be interesting to explore other ideas posed, such as how altering the mask density or shape can affect the results.

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