1 性质

1.唯一性 $\lim\limits_{n\to\infty} x_n = A \Rightarrow A$唯一

2.有界性 $\lim\limits_{n\to\infty} x_n = A \Rightarrow |x_n| \leqslant M$

3.保号性$x_n \geqslant 0, \lim\limits_{n\to\infty} x_n = A$, 则$A \geqslant 0$

2 重要公式

① $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1 \Rightarrow \lim\limits_{\varphi(x)\to 0}\dfrac{\sin\varphi(x)}{\varphi(x)} = 1$

② $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e \Rightarrow \lim\limits_{\varphi(x)\to 0}(1+\varphi(x))^{\frac{1}{\varphi(x)}} = e$

③ $\lim\limits_{n\to 0^+}\sqrt[n]{n} = 1$

等价无穷小:

当 $x\to 0$ 时,

$1-\cos x \sim \dfrac{1}{2}x^2$ $\qquad \ln(1+x) \sim x$

$e^x - 1 \sim x$ $\qquad \tan x \sim x + \dfrac{x^3}{3}$

$\sin x \sim x$

$\arcsin x \sim x$ $\qquad \arctan x \sim x$

$a^x - 1 \sim \boxed{x\ln a}$

$(1+x)^a - 1 \sim \boxed{ax}$

3 泰勒展开

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}}{n!}(x-x_0)^n$$

佩亚诺余项表达式

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}}{n!}(x-x_0)^n + R_n(x) \text{ 其中}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, (\xi \in (x_0, x))$$

几个重要泰勒展开式

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + o(x^n)$$

$$\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + (-1)^{n-1}\frac{1}{n}x^n + o(x^n)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + o(x^n)$$

$$\alpha^x = \sum_{i=0}^{n}\frac{\ln^n\alpha}{n!}x^n + o(x^n)$$

$$= 1 + x\ln\alpha + \frac{\ln^2\alpha}{2}x^2 + \cdots + \frac{\ln^n\alpha}{n!}x^n + o(x^n)$$

级数形式 (记忆规律——减则无括号)

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$$

$$\frac{1}{a+x} = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{a}\right)^n$$

$$\frac{1}{a-x} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{x}{a}\right)^n$$

$$e^x = \sum_{\boxed{n=0}}^{\infty}\frac{x^n}{n!}$$

4 求导公式

$(C)'=0$

$(x^{\mu})'=\mu x^{\mu-1}$

$(\sin x)'=\cos x$

$(\cos x)'=-\sin x$

$(\tan x)'=\sec^2 x$

$(\cot x)'=-\csc^2 x$

$(\sec x)'=\sec x\cdot\tan x$

$(\csc x)'=-\csc x\cdot\cot x$

$(a^x)'=a^x\ln a\ (a>0,a\neq 1)$

$(\log_a x)'=\dfrac{1}{x\cdot\ln a}\ (a>0,a\neq 1)$

$(\arcsin x)'=\dfrac{1}{\sqrt{1-x^2}}$

$(\arccos x)'=-\dfrac{1}{\sqrt{1-x^2}}$

$(\arctan x)'=\dfrac{1}{1+x^2}$

$(\operatorname{arccot} x)'=-\dfrac{1}{1+x^2}$

5 积分公式

$\int x^{\mu}\,dx=\dfrac{x^{\mu+1}}{\mu+1}+C\ (\mu\neq -1)$

$\int \dfrac{1}{x}dx=\ln|x|+C$

$\int \dfrac{dx}{1+x^2}=\arctan x+C$

• $\int \dfrac{1}{a^2+x^2}dx=\dfrac{1}{a}\arctan\dfrac{x}{a}+C$

$\int \dfrac{1}{1-x^2}dx=\dfrac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right|+C$

☆ $\int \dfrac{1}{a^2-x^2}dx=\boxed{\dfrac{1}{2a}}\ln\left|\dfrac{a+x}{a-x}\right|+C$

$\int \dfrac{1}{\sqrt{1-x^2}}dx=\arcsin x+C_1=-\arccos x+C_2$

• $\int \dfrac{1}{\sqrt{a^2-x^2}}dx=\arcsin\dfrac{x}{a}+C$

• $\int \dfrac{1}{\sqrt{x^2\pm a^2}}dx=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

$-\int \dfrac{1}{\sin^2 x}d\cos x$

$=-\int \dfrac{1}{1-\cos^2 x}d\cos x$

$\int \sin x\,dx=-\cos x+C$

$\int \dfrac{1}{\sin x}dx=-\dfrac{1}{2}\ln\left|\dfrac{1+\cos x}{1-\cos x}\right|+C=\dfrac{1}{2}\ln\left|\dfrac{1-\cos x}{1+\cos x}\right|+C$

$\int \dfrac{1}{\sin^2 x}dx=-\dfrac{1}{\tan x}+C$

$\int \cos x\,dx=\sin x+C$

$\int \dfrac{1}{\cos x}dx=\dfrac{1}{2}\ln\left|\dfrac{1+\sin x}{1-\sin x}\right|+C=\ln\left|\sec x+\tan x\right|+C$

$\int \dfrac{1}{\cos^2 x}dx=\tan x+C$

$\int \tan x\,dx=-\ln|\cos x|+C$

$\int \dfrac{1}{\tan x}dx=\ln|\sin x|+C$

$\int e^x\,dx=e^x+C$

$\int a^x\,dx=\dfrac{a^x}{\ln a}+C$

6 渐近线

若 $\lim\limits_{x\to\infty}f(x)=b$，则称 $y=b$ 为曲线 $f(x)$ 的水平渐近线

若 $\lim\limits_{x\to x_0}f(x)=\infty$，则称 $x=x_0$ 为曲线 $f(x)$ 的垂直渐近线

若 $\lim\limits_{x\to\infty}[f(x)-(ax+b)]=0$,其中 $\begin{cases}a=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}\\ b=\lim\limits_{x\to\infty}[f(x)-ax]\end{cases}$,则称 $y=ax+b$ 为斜渐近线

7 常用不等式

$\sin x<x<\tan x,\quad x\in(0,\dfrac{\pi}{2})$

$\dfrac{x}{1+x}<\ln(1+x)<x,\quad x\in(0,+\infty)$