3-2 概率分布部分定理推导

1 泊松定理推导

$$\lim_{n\to\infty} C_n^k p^k (1-p)^{n-k} = \lim_{n\to\infty} \frac{\boxed{n!}}{k!\boxed{(n-k)!}} p^k (1-p)^{n-k}$$

$$\xrightarrow{k\ll n} \lim_{n\to\infty} \frac{1}{k!}\boxed{n^k p^k}(1-p)^{n-k} = \lim_{n\to\infty} \frac{(np)^k}{k!}(\boxed{1-p})^{n-k}$$

$$\xrightarrow{e^{-p}\sim 1+\sum_{i=1}^{n}\frac{(-p)^i}{i!}\sim 1-p} \frac{(np)^k}{k!}(e^{-p})^{n-k} = \frac{(np)^k}{k!}(e^{-np}\boxed{e^{pk}})$$

$$\xrightarrow{kp\ll np} \frac{(np)^k}{k!}(e^{-np}) = \frac{\lambda^k}{k!}e^{-\lambda}$$

2 指数无记忆性推导

0) $P\{X \le x\} = F(x) = 1-e^{\lambda x}, x>0$

1) $P\{X > t\} = \int_t^{+\infty} \lambda e^{-\lambda t} dt = e^{-\lambda t}, t>0$

2) $P\{X > t+s \mid X > s\} = \dfrac{P\{X > t+s\}}{P\{X > s\}}$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P\{X > t\}, t,s>0$$

你还能活多久和你活了多久没有关系

3-3 多维随机变量

3 泊松分布合并

$X_1 \sim P(\lambda_1)$、$X_2 \sim P(\lambda_2)$

$$P\left\{X_1+X_2=m\right\} = \sum_k P\left\{X_1=k\right\}P\left\{X_2=m-k\right\}$$

$$= \sum_k e^{-\lambda_1}\frac{\lambda_1^k}{k!}\cdot e^{-\lambda_2}\frac{\lambda_2^{m-k}}{(m-k)!} = e^{-\lambda_1-\lambda_2}\sum_k \frac{\lambda_1^k}{k!}\frac{\lambda_2^m\lambda_2^{-k}}{(m-k)!}$$

$$= e^{-\lambda_1-\lambda_2}\frac{\lambda_2^m}{m!}\boxed{\sum_{k=1}^{m}\frac{m!}{k!(m-k)!}\frac{\lambda_1^k}{\lambda_2^k}}$$

$$= e^{-\lambda_1-\lambda_2}\frac{\lambda_2^m}{m!}\boxed{\left(1+\frac{\lambda_1}{\lambda_2}\right)^m}\text{(二项式)} = \frac{e^{-\lambda_1-\lambda_2}}{m!}(\lambda_2+\lambda_1)^m$$

3-4 期望方差公式推导

4 二项分布期望方差

二项分布形式 $X \sim B(n,p)$；$P\{X=k\} = C_n^k p^k (1-p)^{n-k}$

(1) **期望公式**： $EX = \sum_{k=1}^{n} k\cdot P\{X=k\}$

$$= \sum_{k=1}^{n} k\cdot C_n^k p^k (1-p)^{n-k} = \sum_{k=1}^{n} k\cdot\frac{n!}{(n-k)!k!}p^k q^{n-k}$$

$$= \boxed{np}\sum_{k=1}^{n}\frac{\boxed{(n-1)}!}{(n-k)!\boxed{(k-1)}!}\boxed{p^{k-1}}q^{n-k} \quad \text{提取 } n\text{、}p$$

$$= np\sum_{k=1}^{n}\boxed{C_{n-1}^{k-1}}p^{k-1}q^{n-k} = np\sum_{\boxed{k=0}}^{\boxed{n-1}} C_{n-1}^{\boxed{k}}p^{\boxed{k}}q^{n-\boxed{(k+1)}}$$

$$= np\sum_{k=0}^{n-1} C_{n-1}^{k}p^{k}q^{n-k-1} \xrightarrow{t=n-1} np\sum_{k=0}^{t} C_t^k p^k q^{t-k}$$

$$\boxed{\left[(a+b)^n = \sum_{i=0}^{n} C_n^i a^i b^{n-i}\right]} = np(p+q)^t = np$$

(2) **方差公式**： $DX = EX^2 - (EX)^2$

方法一

$$= \left[\sum_{k=1}^{n} k^2 C_n^k p^k (1-p)^{n-k}\right] - (np)^2$$

$$= \sum_{k=1}^{n} k\cdot np C_{n-1}^{k-1} p^{k-1}(1-p)^{n-k} - (np)^2 \text{ 同上，提取 } np$$

$$= np(\sum_{k=1}^{n}\boxed{(k-1)C_{n-1}^{k-1}p^{k-1}q^{n-k}} + \sum_{k=1}^{n}\boxed{C_{n-1}^{k-1}p^{k-1}q^{n-k}}) - (np)^2$$

$$= np(\sum_{k=0\,or\,1}^{n-1}\boxed{k\cdot C_{n-1}^{k}p^{k}q^{n-1-k}} + \sum_{k=0}^{n-1}\boxed{C_{n-1}^{k}p^{k}q^{n-1-k}}) - (np)^2$$

$$= np(\sum_{k=1}^{n-1}\boxed{k\cdot P\{X=k\}} + \sum_{k=0}^{n-1}\boxed{C_{n-1}^{k}p^{k}q^{n-1-k}}) - (np)^2$$

$$= np(\boxed{EX_{n-1}} + \boxed{1}) - (np)^2$$

$$= np[\boxed{(n-1)p} + \boxed{1}] - (np)^2 = np(1-p)$$

方法二

设随机变量 $X_i = \begin{cases} 1, & \text{第}i\text{次实验成功} \\ 0, & \text{第}i\text{次实验失败} \end{cases}$，则 $X = \sum_{i=1}^{n} X_i$

则 $X_i \sim B(1,p)$，故 $D(X_i) = p(1-p)$（0-1 分布）

对于独立的 $X_i, X_j (i \ne j)$，有

$$D(X) = \sum_{i=1}^{n} D(X_i) = np(1-p)$$

5 几何分布期望方差（级数）

几何分布： $P\{X=k\} = p(1-p)^{k-1}$

(1) **期望公式** $EX = \sum_{k=1}^{\boxed{+\infty}} k\cdot p(1-p)^{k-1}$

$$\sum_{k=1}^{+\infty} k\cdot p(1-p)^{k-1} = p\sum_{k=1}^{+\infty} kq^{k-1} = p\sum_{\boxed{k=1}}^{+\infty}(q^k)' = p\sum_{\boxed{k=0}}^{+\infty}(q^k)'$$

$$= p\left(\frac{1}{1-q}\right)' = p\frac{-1\cdot\frac{dq}{dp}}{(1-q)^2} = \frac{p}{(1-q)^2} = \frac{1}{p}$$

(2) **方差公式**

$DX = EX^2 - (EX)^2$

$$EX^2 = \sum_{k=1}^{+\infty} k^2\cdot p(1-p)^{k-1} = p\sum_{k=1}^{+\infty} k^2 q^{k-1} = p\sum_{k=1}^{+\infty} k\left(q^k\right)'$$

$$= p\sum_{\boxed{k=0}}^{+\infty}\left((k+1)q^k - q^k\right)' = p\sum_{k=0,1}^{+\infty}\left(q^{k+1}\right)'' - \sum_{k=0,1}^{+\infty}\left(q^k\right)'$$

$$= p\left[\left(\frac{q^2}{1-q}\right)'' - \left(\frac{q}{1-q}\right)'\right] = p\left[\left(\frac{1}{1-q}\right)'' - \left(\frac{1}{1-q}\right)'\right]$$

$$= p\left[\frac{2}{(1-p)^3} - \frac{1}{(1-p)^2}\right] = \frac{2-p}{p^2}$$

$$DX = EX^2 - (EX)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

3-5 大数定理推导

1 切比雪夫不等式

离散型: $P\{|X-EX|\geq\varepsilon\}=\sum\limits_{|x_i-EX|\geq\varepsilon}p_i\ ,(p_i=P\{X=x_i\})$

因为这里的取值就是$|x-EX|\geqslant\varepsilon$，所以:

$$\leq\sum_{|x_i-EX|\geq\varepsilon}\left(\frac{|x_i-E(X)|}{\varepsilon}\right)^2p_i\leq\frac{1}{\varepsilon^2}\sum_{|x_i-EX|\geq\varepsilon}|x_i-E(X)|^2p_i$$

$$\leq\frac{1}{\varepsilon^2}\sum|x_i-E(X)|^2p_i=\frac{1}{\varepsilon^2}DX$$

连续型: $P\{|X-EX|\geq\varepsilon\}=\int_{|x-EX|\geq\varepsilon}f(x)dx$

$$\leq\int_{|x-EX|\geq\varepsilon}\left(\frac{|x-E(X)|}{\varepsilon}\right)^2f(x)dx\ ,\ \frac{|x-EX|}{\varepsilon}\geq1$$

由于积分项都是正的，所以可以拓展积分范围来放大

$$\leq\frac{1}{\varepsilon^2}\int_{-\infty}^{+\infty}|x-E(X)|^2f(x)dx=\frac{D(x)}{\varepsilon^2}$$

2 切比雪夫大数定律证明

$\{X_i\}$是①两两不相关的随机变量序列，所有② X_i 都有方差，且③方差有上限(存在常数 C，使得 $D(X_i)\leq C,(i=1,2,\ldots)$)

$$\text{则}\frac{1}{n}\sum_{i=1}^nX_i\xrightarrow[n\to+\infty]{P}\frac{1}{n}\sum_{i=1}^nEX_i$$

$$\text{或}\lim_{n\to\infty}P\left\{\left|\frac{1}{n}\sum_{i=1}^nX_i-\frac{1}{n}\sum_{i=1}^nEX_i\right|<\varepsilon\right\}=1$$

证明：有切比雪夫不等式 $P\{\boxed{|X-\mu|}\geq\varepsilon\}\leq\frac{\sigma^2}{\varepsilon^2}$，带入得

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^nX_i-\frac{1}{n}\sum_{i=1}^nEX_i\right|\geq\varepsilon\right\}\xrightarrow{X=\sum_{i=1}^nX_i}$$

$$P\left\{\frac{1}{n}|X-EX|\geq\varepsilon\right\}=P\left\{|X-EX|\geq\boxed{n\varepsilon}\right\}\leq\frac{DX}{\boxed{n\varepsilon}^2}$$

$\therefore$ 当 $n\to+\infty$ 时，有 $\frac{DX}{n^2\varepsilon^2}=0$，

$$\text{原式}=1-\lim_{n\to\infty}P\left\{\left|\frac{1}{n}\sum_{i=1}^nX_i-\frac{1}{n}\sum_{i=1}^nEX_i\right|\geq\varepsilon\right\}=0$$

证毕。

3 棣莫弗-拉普拉斯定理证明

看看就行

$$C_n^kp^kq^{n-k}=\frac{n!}{k!(n-k)!}p^kq^{n-k}$$

$$\simeq\frac{n^ne^{-n}\sqrt{2\pi n}}{k^ke^{-k}\cdot(n-k)^{n-k}e^{-(n-k)}\cdot\sqrt{2\pi k}\sqrt{2\pi(n-k)}}p^kq^{n-k}$$

$$=\sqrt{\frac{n}{2\pi k(n-k)}}\frac{n^n}{k^k(n-k)^{n-k}}p^kq^{n-k}$$

$$=\sqrt{\frac{n}{2\pi k(n-k)}}\left(\frac{np}{k}\right)^k\left(\frac{nq}{n-k}\right)^{n-k}\xrightarrow[p+q=1]{\frac{k}{n}\to p}\frac{1}{\sqrt{2\pi npq}}\left(\frac{np}{k}\right)^k\left(\frac{nq}{n-k}\right)^{n-k}$$

$$=\frac{1}{\sqrt{2\pi npq}}\exp\left\{\ln\left(\left(\frac{np}{k}\right)^k\right)+\ln\left(\left(\frac{nq}{n-k}\right)^{n-k}\right)\right\}$$

$$=\frac{1}{\sqrt{2\pi npq}}\exp\left\{-k\ln\left(\frac{k}{np}\right)+(k-n)\ln\left(\frac{n-k}{nq}\right)\right\}$$

$$=\frac{1}{\sqrt{2\pi npq}}\exp\left\{-k\ln\left(\frac{np+x\sqrt{npq}}{np}\right)+(k-n)\ln\left(\frac{n-np-x\sqrt{npq}}{nq}\right)\right\}$$

$$\xrightarrow{p+q=1}\frac{1}{\sqrt{2\pi npq}}\exp\left\{-k\ln\left(1+x\sqrt{\frac{q}{np}}\right)+(k-n)\ln\left(1-x\sqrt{\frac{p}{nq}}\right)\right\}$$

$$\xrightarrow[\text{of }\ln(x+1)]{Taylor}\frac{1}{\sqrt{2\pi npq}}\exp\left\{-k\left(x\sqrt{\frac{q}{np}}-\frac{x^2q}{2np}+\cdots\right)+(k-n)\left(-x\sqrt{\frac{p}{nq}}-\frac{x^2p}{2nq}-\cdots\right)\right\}$$

$$=\frac{1}{\sqrt{2\pi npq}}\exp\left\{\left(-np-x\sqrt{npq}\right)\left(x\sqrt{\frac{q}{np}}-\frac{x^2q}{2np}+\cdots\right)+\left(np+x\sqrt{npq}-n\right)\left(-x\sqrt{\frac{p}{nq}}-\frac{x^2p}{2nq}-\cdots\right)\right\}$$

$$=\frac{1}{\sqrt{2\pi npq}}\exp\left\{\left(-np-x\sqrt{npq}\right)\left(x\sqrt{\frac{q}{np}}-\frac{x^2q}{2np}+\cdots\right)-\left(nq-x\sqrt{npq}\right)\left(-x\sqrt{\frac{p}{nq}}-\frac{x^2p}{2nq}-\cdots\right)\right\}$$

$$=\frac{1}{\sqrt{2\pi npq}}\exp\left\{\left(-x\sqrt{npq}+\frac{1}{2}x^2q-x^2q+\cdots\right)+\left(x\sqrt{npq}+\frac{1}{2}x^2p-x^2p-\cdots\right)\right\}$$

$$=\frac{1}{\sqrt{2\pi npq}}\exp\left\{-\frac{1}{2}x^2q-\frac{1}{2}x^2p-\cdots\right\}=\frac{1}{\sqrt{2\pi npq}}\exp\left\{-\frac{1}{2}x^2(p+q)-\cdots\right\}$$

$$\simeq\frac{1}{\sqrt{2\pi npq}}\exp\left\{-\frac{1}{2}x^2\right\}=\frac{1}{\sqrt{2\pi npq}}e^{\frac{-(k-np)^2}{2npq}}$$

3-5 数理统计基础

4 样本数字特性推导

① $EX=E\overline{X}=\mu$ 不用推导，我有脑子的

② $D\overline{X}=\frac{\sigma^2}{n}$ 推导: $D\overline{X}=D\left(\frac{\sum X_i}{n}\right)=\frac{1}{n^2}\sum_{i=1}^nDX_i=\frac{\sigma^2}{n}$

③ $ES^2=DX=\sigma^2$

推导: $ES^2=E\left(\frac{1}{n-1}\sum_{i=1}^n(X_i-\overline{X})^2\right)$，其中 $\overline{X}=\frac{1}{n}\sum_{i=1}^nX_i$

$$=\frac{1}{n-1}E\left(\sum_{i=1}^n(X_i^2-2X_i\overline{X}+\overline{X}^2)\right)=\frac{1}{n-1}E\left(\sum_{i=1}^nX_i^2-2\overline{X}\sum_{i=1}^nX_i+n\overline{X}^2\right)$$

$$=\frac{1}{n-1}\left(\sum_{i=1}^nEX_i^2-2\sum_{i=1}^nE(X_i\overline{X})+\sum_{i=1}^nE\overline{X}^2\right)$$

恒有① $EX_i=\mu$ ② $DX_i=\sigma^2$ ③ $EX_i^2=DX_i+(EX_i)^2=\sigma^2+\mu^2$

④ $E(X_i\overline{X})=\frac{1}{n}E\sum_{j=1}^nX_iX_j=\frac{1}{n}\left[E(X_i^2)+(n-1)EX_iEX_j\right],(i\neq j)$

$$=\frac{1}{n}\left[\sigma^2+\mu^2+(n-1)\mu^2\right]=\frac{1}{n}(\sigma^2+n\mu^2)$$

⑤ $E\overline{X}^2=\frac{1}{n}\sum_{i=1}^n\left(E\boxed{X_i\overline{X}}\right)=\frac{1}{n}\sum_{i=1}^n\frac{1}{n}(\sigma^2+n\mu^2)=\frac{1}{n}(\sigma^2+n\mu^2)$

$$=\frac{1}{n-1}\left(\sum_{i=1}^n\boxed{\sigma^2+\mu^2}-2\sum_{i=1}^n\boxed{\frac{1}{n}(\sigma^2+n\mu^2)}+\sum_{i=1}^n\boxed{\frac{1}{n}(\sigma^2+n\mu^2)}\right)$$

$$=\frac{1}{n-1}\left(n\boxed{\sigma^2+\mu^2}-2\boxed{\sigma^2+n\mu^2}+\boxed{\sigma^2+n\mu^2}\right)$$

$$=\frac{1}{n-1}(n-1)\sigma^2=\sigma^2$$

$$=\frac{1}{n-1}E\left(\sum_{i=1}^nX_i^2-2(\overline{X}\boxed{n\overline{X}}+n\overline{X}^2\right)$$

$$=\frac{1}{n-1}E\left[\boxed{\sum_{i=1}^nX_i^2}-\boxed{n\overline{X}^2}\right]$$

3-6 数理统计基本概念

1 抽样分布证明

设总体 $X \sim N(\mu,\sigma^2)$，$X_1,X_2,\ldots X_n$ 是来自总体的样本

$\overline{X}$ 为样本均值，S^2 是样本方差

(1) $S^2=\dfrac{1}{n-1}\left(\sum\limits_{i=1}^{n}X_i^2-n\overline{X}^2\right)$ (1.1)

$$S^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i^2-2X_i\overline{X}+\overline{X}^2\right)$$

$$=\frac{1}{n-1}\left[\sum_{i=1}^{n}X_i^2-2\overline{X}\sum_{i=1}^{n}X_i+\sum_{i=1}^{n}\overline{X}^2\right]$$

$$=\frac{1}{n-1}\left[\sum_{i=1}^{n}X_i^2-2n\overline{X}^2+n\overline{X}^2\right]=\frac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2-n\overline{X}^2\right)$$

(2) $\overline{X}\sim N(\mu,\dfrac{\sigma^2}{n})$ (1.2)

$$E\overline{X}=E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=\mu;\quad D\overline{X}=D\left(\sum_{i=1}^{n}\frac{X_i}{n}\right)=\frac{1}{n^2}n\sigma^2=\frac{\sigma^2}{n}$$

$\overline{X}$ 是 $X_1,X_2,\ldots X_n$ 的线性组合，$\overline{X}$ 服从正态分布，即

$\overline{X}\sim N(\mu,\dfrac{\sigma^2}{n})$ 进而 $U=\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$

(3) $\chi^2=\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$ (1.3)

$$S^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2\Rightarrow\boxed{\frac{(n-1)S^2}{\sigma^2}}=\sum_{i=1}^{n}\frac{(X_i-\overline{X})^2}{\sigma^2}$$

$$=\frac{1}{\sigma^2}\sum_{i=1}^{n}\left[(X_i-\mu)-(\overline{X}-\mu)\right]^2=\boxed{\sum_{i=1}^{n}(\frac{X_i-\mu}{\sigma})^2}-\boxed{n(\frac{\overline{X}-\mu}{\sigma})^2}$$

☆左边 $\sum\limits_{i=1}^{n}(\dfrac{X_i-\mu}{\sigma})^2\sim\chi^2(n)$，右边 $\sqrt{n}(\dfrac{\overline{X}-\mu}{\sigma})\sim N(0,1)$

即 $n(\dfrac{\overline{X}-\mu}{\sigma})^2\sim\chi^2(1)$

左边—右边 $=\chi^2(n-1)$，证毕。

(4) $T=\dfrac{\overline{X}-\mu}{S/\sqrt{n}}\sim t(n-1)$ (1.4)

由（2）得 $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$，由（3）得 $\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\Bigg/\sqrt{\frac{(n-1)S^2}{\sigma^2}}\rightarrow\frac{\underline{X}}{\sqrt{\underline{Y}/k}},(\underline{Y}\sim\chi^2(k),\underline{X}\sim N(0,1))$$

$$=\frac{\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2}\Big/(n-1)}}=\frac{\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\dfrac{S}{\sigma}}=\frac{\overline{X}-\mu}{S/\sqrt{n}}\sim t(n-1)$$

证毕

(5) $\chi^2=\dfrac{1}{\sigma^2}\sum\limits_{i=1}^{n}\left(X_i-\mu\right)^2\sim\chi^2(n)$ (1.5)

$\dfrac{(X_i-\mu)}{\sigma}\sim N(0,1)$，证毕

(6) $U=\dfrac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}}\sim N(0,1)$ (1.6)

$\overline{X}\sim N(\mu_1,\dfrac{\sigma_1^2}{n_1})$ 且 $\overline{Y}\sim N(\mu_2,\dfrac{\sigma_2^2}{n_2})$ 得

$\overline{X}-\overline{Y}\sim N(\mu_1-\mu_2,\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2})$，于是

$\dfrac{\left[\left(\overline{X}-\overline{Y}\right)-\left(\mu_1-\mu_2\right)\right]}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}}\sim N(0,1)$ 证毕

(7) $F=\dfrac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}=\dfrac{S_1^2}{S_2^2}\Big/\dfrac{\sigma_1^2}{\sigma_2^2}\sim F(n_1-1,n_2-1)$ (1.7)

由（3）得 $\chi_1^2=\dfrac{(n_1-1)S_1^2}{\sigma_1^2}\sim\chi^2(n_1-1)$，$\chi_2^2=\dfrac{(n_2-1)S_2^2}{\sigma_2^2}\sim\chi^2(n_2-1)$

$F=\dfrac{\underline{\chi^2(n_1)}/\underline{n_1}}{\underline{\chi^2(n_2)}/\underline{n_2}}\rightarrow$

$$\boxed{\frac{\chi_1^2/(n_1-1)}{\chi_2^2/(n_2-1)}}=\frac{\dfrac{S_1^2}{\sigma_1^2}}{\dfrac{S_2^2}{\sigma_2^2}}=\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}\sim\boxed{F(n_1-1,n_2-1)}$$ 证毕

(8) 如果 $\sigma_1^2=\sigma_2^2$

那么 $T=\dfrac{\overline{X}-\overline{Y}-(\mu_1-\mu_2)}{S_\omega\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}\sim t(n_1+n_2-2)$ (1.8)

其中，$S_\omega^2=\dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$

当 $\sigma_1^2=\sigma_2^2=\sigma^2$ 时，由（6）得

$$\frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}}\sim N(0,1)$$

由（3）得 $\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$ 继而

$$\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{\sigma^2}\sim\chi^2(n_1+n_2-2)$$

$$\frac{\dfrac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}}}{\sqrt{\dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{\sigma^2}\Big/(n_1+n_2-2)}}=\frac{\dfrac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}}}{\sqrt{\dfrac{S_\omega^2}{\sigma^2}}}$$

$$\frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}\sqrt{\dfrac{S_\omega^2}{\sigma^2}}}=\frac{\overline{X}-\overline{Y}-(\mu_1-\mu_2)}{S_\omega\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}\sim t(n_1+n_2-2)$$

证毕

3-7 正态总体的置信区间证明

1 正态总体的置信区间(置信水平为$\boxed{1-\alpha}$)

$X \sim N(\mu,\sigma^2)$，设样本 $X_1, X_2, \ldots X_n$ 来自 X

补充分位点定义：

正态 N： $P\{X > z_\alpha\} = \alpha$

咖方 χ^2： $P\{\chi^2 > \chi_\alpha^2(n)\} = \alpha$

T： $P\{T > t_\alpha(n)\} == \alpha$、$P\{|T| > t_{\boxed{\alpha/2}}(n)\} = \alpha$

学生 F： $P\{F > F_\alpha(n_1,n_2)\} = \alpha$

(1) 求 $\boxed{\mu}$，σ^2 $\boxed{\text{已知√}}$，求取 μ 的置信区间

由公式(1.2)可得，

$\overline{X} \sim N(\mu,\frac{\sigma^2}{n})$；于是 $P\left\{\left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right| < z_{\alpha/2}\right\} = 1-\alpha$；

展开得到：$P\left\{-z_{\alpha/2} < \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right\} = 1-\alpha$

$$\Rightarrow P\left\{-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu - \overline{X} < +z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\} = 1-\alpha$$

即 μ 的置信区间为 $\left(\overline{X}-\frac{\sigma}{\sqrt{n}}z_{\alpha/2}, \overline{X}+\frac{\sigma}{\sqrt{n}}z_{\alpha/2}\right)$

(2) 求 $\boxed{\mu}$，σ^2 $\boxed{\text{未知?}}$，求取 μ 的置信区间

将 σ^2 换为无偏估计 S^2 由公式(1.4)可得

$T = \frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$；于是

$$P\left\{\left|\frac{\overline{X}-\mu}{S/\sqrt{n}}\right| < t_{\alpha/2}(n-1)\right\} = 1-\alpha;$$

展开得到

$$P\left\{\overline{X}-\frac{S}{\sqrt{n}}t_{\alpha/2}(n-1) < \mu < \overline{X}+\frac{S}{\sqrt{n}}t_{\alpha/2}(n-1)\right\} = 1-\alpha$$

即 μ 的置信区间为

$$\left(\overline{X}-\frac{S}{\sqrt{n}}t_{\alpha/2}(n-1), \overline{X}+\frac{S}{\sqrt{n}}t_{\alpha/2}(n-1)\right)$$

(3) 求 $\boxed{\sigma^2}$，μ $\boxed{\text{未知?}}$，求取 σ^2 的置信区间

σ^2 的无偏估计是 S^2，由公式(1.3)可得

$\chi^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$；于是（$\chi^2$ 不对称）

$$P\left\{\chi^2_{\boxed{1-\alpha/2}}(n-1) < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\boxed{\alpha/2}}(n-1)\right\} = 1-\alpha$$

所以可得置信区间 $\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}(n-1)}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}(n-1)}\right)$

其他，略