三点确定椭圆

圆心为零点，三点投影到长短轴上，设长短轴矢量(a, b) (-b, a) ($a^2+b^2=1$)

$$\begin{cases}(X_1,Y_1)=[(x_1,y_1)\cdot(a,b),(x_1,y_1)\cdot(-b,a)]=(ax_1+by_1,-bx_1+ay_1)\\(X_2,Y_2)=[(x_2,y_2)\cdot(a,b),(x_2,y_2)\cdot(-b,a)]=(ax_2+by_2,-bx_2+ay_2)\\(X_3,Y_3)=[(x_3,y_3)\cdot(a,b),(x_3,y_3)\cdot(-b,a)]=(ax_3+by_3,-bx_3+ay_3)\end{cases}$$

椭圆方程组

$$\begin{cases}X_1^2/M^2+Y_1^2/N^2=1\\X_2^2/M^2+Y_2^2/N^2=1\\X_3^2/M^2+Y_3^2/N^2=1\end{cases}$$

M^2和N^2不方便计算，设$m=1/M^2, n=1/N^2$ (m > 0, n > 0) 方程组展开

$$\begin{cases}m(x_1^2a^2+y_1^2b^2+2abx_1y_1)+n(x_1^2b^2+y_1^2a^2-2abx_1y_1)=1 & (1)\\m(x_2^2a^2+y_2^2b^2+2abx_2y_2)+n(x_2^2b^2+y_2^2a^2-2abx_2y_2)=1 & (2)\\m(x_3^2a^2+y_3^2b^2+2abx_3y_3)+n(x_3^2b^2+y_3^2a^2-2abx_3y_3)=1 & (3)\end{cases}$$

先消除ab: (1)式乘以x_2y_2 (2)式乘以x_1y_1 ($x_1y_1\neq0, x_2y_2\neq0$)
相乘之后(2)式减(1)式 设$k=a^2$ 则 $b^2=1-k(k>0)$

$$k(m-n)\underline{[x_1y_1(x_2^2-y_2^2)-x_2y_2(x_1^2-y_1^2)]}+m\underline{(x_1y_1y_2^2-x_2y_2y_1^2)}+n\underline{(x_1y_1x_2^2-x_2y_2x_1^2)}=\underline{x_1y_1-x_2y_2}$$

将里面常量提取出来

$$\begin{cases}k_1 & =x_1y_1(x_2^2-y_2^2)-x_2y_2(x_1^2-y_1^2)\\m_1 & =(x1_1y_1y_2^2-x_2y_2y_1^2)\\n_1 & =(x_1y_1x_2^2-x_2y_2x_1^2)\\c_1 & =x_1y_1-x_2y_2\end{cases}$$

令$m'=m-n$,则$m=m'+n \quad m'可以为负$

$$\begin{cases}km'k_1+(m'+n)m_1+nn_1=c_1\\km'k_1+m'm_1+n(m_1+n_1)=c_1\end{cases}$$

令$n_1'=m_1+n_1$

$$km'k_1+m'm_1+nn_1'=c_1 \quad (a)$$

(3)式得出结果

$$km'k_2+m'm_2+nn_2'=c_2 \quad (b)$$

由(a)和(b)得出, 分别乘k_1k_2:

$$\begin{aligned}n&=[\underline{(c_2k_1-c_1k_2)}-m'\underline{(m_2k_1-m_1k_2)}]/\underline{(n_2'k_1-n_1'k_2)}\\&=s+tm'=\begin{cases}s=(c_2k_1-c_1k_2)/(n_2'k_1-n_1'k_2)\\t=(m_1k_2-m_2k_1)/(n_2'k_1-n_1'k_2)\end{cases}\end{aligned}$$

将上式代入第(a)式得到

> $$\begin{aligned} k &= [\underline{c_1 - sn'_1} - \underline{(tn'_1 + m_1)m'}]/m'k_1 \\ &= o + p/m' = \begin{cases} o = -(tn'_1 + m_1)/k_1 \\ p = \quad (c_1 - sn'_1)/k_1 \end{cases} \end{aligned}$$

k、m、n 都表示成m'的方程式

> $$\begin{cases} n & = s + tm' \\ k & = o + p/m' \quad (c) \\ m & = n + m' \end{cases}$$

代入原方程(1)

> $$\begin{aligned} ab &= [1 - (n + m')y_1^2 - nx_1^2 - km'(x_1^2 - y_1^2)]/2x_1y_1m' \\ &= [1 - n(x_1^2 + y_1^2) - m'y_1^2 - km'(x_1^2 - y_1^2)]/2x_1y_1m' \\ &= [1 - (s + tm')(x_1^2 + y_1^2) - m'y_1^2 - (o + p/m')(x_1^2 - y_1^2)]/2x_1y_1m' \\ &= r/m' + q = \begin{cases} r = [1 - s(x_1^2 + y_1^2) - p(x_1^2 - y_1^2)]/2x_1y_1 \\ q = -[t(x_1^2 + y_1^2) + y_1^2 + o(x_1^2 - y_1^2)]/2x_1y_1 \end{cases} \end{aligned}$$

上式两边平方

> $$k(1 - k) = (ab)^2 = (r/m' + q)^2$$

得出关于m'的一元二次方程

> $$Am'^2 + Bm' + C = 0 \begin{cases} A = o - o^2 - q^2 \\ B = p - 2op - 2rq \\ C = -p^2 - r^2 \end{cases}$$

判断$B^2 - 4AC > 0$

> $$m' = (-B \pm \sqrt{B^2 - 4AC})/2A$$

结合(c)方程组，求出m>0、n>0、k>0。最终椭圆参数如下：

> $$\begin{cases} M & = \sqrt{1/m} \\ N & = \sqrt{1/n} \\ a & = \sqrt{k} \end{cases}$$

以上化简是在第一步乘数不为零的情况下进行的，也是最麻烦情况下的推导，另外需要考虑$x_1y_1 = 0 \quad x_2y_2 = 0 \quad x_3y_3 = 0$

- 其中两者为0时，第三者满足则有解
- 其中一个为0时(直接化简得到k_1, m_1, n_1, c_1), 另外两方程合并得出k_2, m_2, n_2, c_2

一般情况代码

```
float R2D(float angle) {
    return angle / 180 * 3.1415926f;
}

int main()
{
    int M = 20;
    int N = 10;
    float x_1(M*cos(R2D(30.0f))), y_1(N*sin(R2D(30.f)));
    float x_2(M*cos(R2D(60.0f))), y_2(N*sin(R2D(60.f)));
    float x_3(M*cos(R2D(120.0f))), y_3(N*sin(R2D(120.f)));

    float x_1_2 = pow(x_1, 2), y_1_2 = pow(y_1, 2);
    float x_2_2 = pow(x_2, 2), y_2_2 = pow(y_2, 2);
    float x_3_2 = pow(x_3, 2), y_3_2 = pow(y_3, 2);

    float k_1 = x_1*y_1*(x_2_2 - y_2_2) - x_2*y_2*(x_1_2 - y_1_2);
    float m_1 = x_1*y_1*y_2_2 - x_2*y_2*y_1_2;
    float n_1 = x_1*y_1*x_2_2 - x_2*y_2*x_1_2;
    float c_1 = x_1*y_1 - x_2*y_2;

    n_1 += m_1;

    float k_2 = x_1*y_1*(x_3_2 - y_3_2) - x_3*y_3*(x_1_2 - y_1_2);
    float m_2 = x_1*y_1*y_3_2 - x_3*y_3*y_1_2;
    float n_2 = x_1*y_1*x_3_2 - x_3*y_3*x_1_2;
    float c_2 = x_1*y_1 - x_3*y_3;

    n_2 += m_2;

    float s = (c_2*k_1 - c_1*k_2) / (n_2*k_1 - n_1*k_2);
    float t = (m_1*k_2 - m_2*k_1) / (n_2*k_1 - n_1*k_2);

    float o = -(t*n_1 + m_1) / k_1;
    float p = (c_1 - s*n_1) / k_1 ;

    float r = (1 - s*(x_1_2 + y_1_2) - p*(x_1_2 - y_1_2)) / (2 * x_1 * y_1);
    float q = -(t*(x_1_2+y_1_2) + y_1_2 + o*(x_1_2-y_1_2)) / (2 * x_1 * y_1);

    float A = o - o*o - q*q;
    float B = p - 2 * o*p - 2 * r*q;
    float C = -p*p - r*r;

    float m_ = (-B - sqrt(B*B-4*A*C)) / (2 * A);

    float n = s + t * m_;
    float k = o + p / m_;
    float m = n + m_;

    float M_1 = sqrt(1 / m);
    float N_1 = sqrt(1 / n);
    float A_1 = sqrt(k);

    return 0;
}
```

点到椭圆切线

定义：点(x,y) 椭圆上切点$(acos\theta, bsin\theta)$ 切点斜率$(-asin\theta, bcos\theta)$
由于点和切点边线应该与斜率一致，则：

$$\begin{cases} (x,y)-(acos\theta, bsin\theta) & = k(-asin\theta, bcos\theta) \\ (x-acos\theta, y-bsin\theta) & = k(-asin\theta, bcos\theta) \\ (x-acos\theta)/(-asin\theta) & = (y-bsin\theta)/bcos\theta = k \\ ab - xbcos\theta - aysin\theta & = 0 \end{cases}$$

令$t = cos\theta$

$$ab - xbt = \pm\sqrt{1-t^2}ay$$

两边同时平方合并

$$[(xb)^2 + (ay)^2]t^2 - 2abxbt + (ab)^2 - (ay)^2 = 0$$

点到椭圆垂足

线代解法

定义: 点(x,y) 椭圆上垂足$(acos\theta, bsin\theta)$ 垂足切线向量$(-asin\theta, bcos\theta)$
由于点到垂足向量点乘斜率应为0

$$
\begin{aligned}
&\Rightarrow ((x,y)-(acos\theta, bsin\theta))\cdot(-asin\theta, bcos\theta)=0 \\
&\Rightarrow xasin\theta - ybcos\theta-(a^2-b^2)sin\theta cos\theta=0
\end{aligned}
$$

令$k=sin\theta$

$$
\begin{aligned}
&\Rightarrow xak-yb\sqrt{1-k^2}-(a^2-b^2)k\sqrt{1-k^2}=0 \\
&\Rightarrow xak=(yb+(a^2-b^2)k)\sqrt{1-k^2} \\
&\Rightarrow mk=(n+pk)\sqrt{1-k^2}=\begin{cases} m=xa \\ n=yb \\ p=a^2-b^2 \end{cases}
\end{aligned}
$$

两边同时平方

$$
\begin{aligned}
&\Rightarrow (mk)^2=(1-k^2)[n^2+2npk+(pk)^2] \\
&\Rightarrow p^2\underline{k^4}+2np\underline{k^3}+(m^2+n^2-p^2)\underline{k^2}-2np\underline{k}-n^2=0
\end{aligned}
$$

这是一个一元四次方程，直接用求根公式求解

拉格朗日解法

定义：点(a,b) 椭圆上一点(x,y)

$$
\begin{cases}
L(x,y)=(x-a)^2+(y-b)^2+\lambda(mx^2+ny^2-1) & \\
L_x=x-a+\lambda mx=0\Rightarrow x=a/(1+\lambda m) & (1) \\
L_y=y-b+\lambda ny=0\Rightarrow y=b/(1+\lambda n) & (2) \\
mx^2+ny^2=1 & (3)
\end{cases}
$$

将(1)和(2)代入(3)式

$$
m(\frac{a}{1+\lambda m})^2+n(\frac{b}{1+\lambda n})^2=1\Rightarrow ma^2(1+\lambda n)^2+nb^2(1+\lambda m)^2=(1+\lambda m)^2(1+\lambda n)^2
$$

$$
m^2n^2\lambda^4+2(mn^2+nm^2)\lambda^3+(4mn+n^2+m^2-ma^2n^2-nb^2m^2)\lambda^2+2(m+n-mnb^2-mna^2)\lambda+(1-ma^2-nb^2)=0
$$

解出λ并代入(1)式和(2)式

三圆的共切圆

已知三圆参数$(x_1,y_1,r_1),(x_2,y_2,r_2),(x_3,y_3,r_3)$和切圆参数$(x,y,r)$

$$
\begin{cases}
(x-x_1)^2+(y-y_1)^2=(r\pm r_1)^2 & (1) \\
(x-x_2)^2+(y-y_2)^2=(r\pm r_2)^2 & (2) \\
(x-x_3)^2+(y-y_3)^2=(r\pm r_3)^2 & (3)
\end{cases}
$$

将(2)-(1)、(3)-(1)式得出方程组(上式以-号处理)

$$
\begin{cases}
\underline{2(x_1-x_2)}x+\underline{2(y_1-y_2)}y+\underline{2(r_2-r_1)}r+\underline{[(x_2^2+y_2^2-r_2^2)-(x_1^2+y_1^2-r_1^2)]}=0 \\
\underline{2(x_1-x_3)}x+\underline{2(y_1-y_3)}y+\underline{2(r_3-r_1)}r+\underline{[(x_3^2+y_3^2-r_3^2)-(x_1^2+y_1^2-r_1^2)]}=0
\end{cases}
$$

按照$ax+by+cr+d=0$简化上式得到系数如下

$$\Rightarrow (1) = \begin{cases} a_1 = 2(x_1 - x_2) \\ b_1 = 2(y_1 - y_2) \\ c_1 = 2(r_2 - r_1) \\ d_1 = (x_2^2 + y_2^2 - r_2^2) - (x_1^2 + y_1^2 - r_1^2) \end{cases} \quad (2) = \begin{cases} a_2 = 2(x_1 - x_3) \\ b_2 = 2(y_1 - y_3) \\ c_2 = 2(r_3 - r_1) \\ d_2 = (x_3^2 + y_3^2 - r_3^2) - (x_1^2 + y_1^2 - r_1^2) \end{cases}$$

$$\Rightarrow \begin{cases} a_1 x + b_1 y + c_1 r + d_1 = 0 \\ a_2 x + b_2 y + c_2 r + d_2 = 0 \end{cases} \Rightarrow \begin{cases} x = mr + n \\ y = mr + n \end{cases}$$

将x, y代入方程组(1)得出关于r的方程

最小公倍数

已知两个数a, b， 其最小公倍数为am=bn(m,n互质) 以下是正面求解的推导证明过程：
设$b = ka + c(k = b/a \quad c = b \div a)$

$$\begin{aligned} &\underline{a}m = \underline{b}n && (1) \quad a < b \\ \Rightarrow\ &am = (ka + c)n && \\ \Rightarrow\ &\underline{c}\,n = \underline{a}(m - kn) && (2) \quad c < a \end{aligned}$$

推导：观察(1)式和(2)式

- 求解a,b的最小公倍数转化成求解c,a的最小公倍数
- c,a的最小公倍数除以c等于n再乘以b即为a,b的最小公倍数

证明：(2)式n与(1)式n是否相等?
$\because m与n互质，则n与m - kn也互质 \therefore (2)式n与(1)式n相等$

```
int LCM(a, b) {
    int  c = b % a;
    if (c == 0)
       return b;

    return LCM(c, a) / c * b;
}
```

牛顿迭代法解切线和垂足

切线

同向量叉乘为0

$$\begin{cases} F(x) = f'(x) \times (f(x) - P) = 0 & (1) \\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} & (2) \end{cases}$$

$$\Rightarrow \frac{F(x)}{F'(x)} = \frac{f'(x) \times (f(x) - P)}{(f'(x) \times (f(x) - P))'} = \frac{f'(x) \times (f(x) - P)}{f''(x) \times (f(x) - P) + f'(x) \times f'(x)}$$
$$\Rightarrow \frac{V \times V_1}{V \times V_2 + V_1 \times V_1} = \frac{V \times V_1}{V \times V_2}(V = f(x) - P, V_1 = f'(x), V_2 = f''(x))$$

垂足

$$\begin{cases} F(x) = f'(x) \cdot (f(x) - P) = 0 & (1) \\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} & (2) \end{cases}$$

> $$\Rightarrow \frac{F(x)}{F'(x)} = \frac{f'(x)\cdot(f(x)-P)}{(f'(x)\cdot(f(x)-P))'} = \frac{f'(x)\cdot(f(x)-P)}{f''(x)\cdot(f(x)-P)+f'(x)\cdot f'(x)}$$
> $$\Rightarrow \frac{V\cdot V_1}{V\cdot V_2+V_1\cdot V_1}(V=f(x)-P, V_1=f'(x), V_2=f''(x))$$