

# THE LIBRARIAN

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**Author / Name / Page *n***

Cover: whatever the cover is.

Write for *The Librarian*.

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# Miscellany

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**On orthographic diversity** The effects of orthography are somewhat marginal; nevertheless, orthographic diversity has certain benefits.

**Three observations on the Skripal affair** Analysis of the incentives of various actors is helpful in ruling out some possibilities, but there are still several left.

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SIR,

I note that you have ceased to receive letters. This is something of a travesty, compounded by this modern innovation of ‘notices’. I therefore write this letter in the hope of inducing other readers to write further letters, preferably differing in content from mine; were this not to occur, that would, at least, be equally humorous, though possibly something of a waste of space.

I beg to subscribe myself,  
SENEX

SIR,

I read *The Librarian* not to hear of fawning reviews of *Cabaret* but to hear incisive (and occasionally decisive) analysis and review. I should be much obliged were the present theatre critic replaced by someone whose writing is sounder, and less beholden to social necessity. May I suggest the Mathematic Editor?

Yours faithfully,  
LECTOR INSCRIBENS

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**Note:** articles in *The Librarian* do not necessarily reflect the views of anyone else, of which authors, those connected with them, *The Librarian*, the editors thereof, the Library Committee, the members, the Chair and the Assistant Chair thereof, the library, the librarians, and the school are either subsets or members.

*The Librarian* is the publication of the Library Committee of Westminster School. The existence of a Library Committee dates back to at least December 1879, when the Editor of *The Elizabethan* replied to a letter on the ‘disgraceful’ state of the books in the library, that ‘[s]ome years ago a regular library committee was in existence’. The present state of the library is far removed from its state in the late 1870s; the employment of four librarians, the Library Committee and the general interest

**Adventures in Recreational Mathematics VII** We are introduced to various large numbers in the seventh instalment of the series.

SIR,

I desire to ‘employ the medium’ of your ‘valuable journal’ (or whatever it was that your last editorial said) to draw attention to the seeming disappearance of Deus misereatur at Latin prayers. Even those mistaken in preference should be concerned by this dramatic change. If we are to alter our rituals thus, even Ad te levavi’s adherents may soon find themselves disappointed.

I am, &c.,  
DIES IRÆ

SIR,

For some reason, I seem to have the impression that the last pages of your article on Fukuyama seem to have disappeared. Will the previous practice of filling pages to reach a multiple of four with sudokus be reinstated? On another note, I was rather distressed to note the inconsistency in styling between the display of metadata in the review of Benedict Randall Shaw’s ‘algorithmic music’ and rest of *The Librarian*.

I remain, &c.,  
SENEX II

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of the rest of the school have all combined to ensure that there is little danger of any repetition thereof. The Library Committee broadly exists to support the work of the librarians; some examples of this support include the conveying of pupil views to the librarians, direct support (e.g., in desk duty, and charitable activities), and the publication of *The Librarian*.

Some find that they are unwilling to ‘go all the way to the library’. Consequently, *The Librarian* offers a subscription service. Readers may email the editor<sup>3</sup>, with a specified destination, which must either be an email or a physical location. This is, of course, free, as is *The Librarian* in general. Issues are occasionally uploaded to <https://librarian.cf>, and it is likely to be increasingly frequently updated. The athletically blessed are encouraged to make the journey to the library.

To do good is to be opportunistic; apathy is authenticity; ‘social justice’ is a joke; thus spoke many exiting Abbey on Friday morning.

There are several potential criticisms of the ‘homeless drive’. Some of these may even be legitimate, or correct. Perhaps the assistance of the homeless in Britain is less effective than the assistance of those whose need is even greater, unimaginable though this may be. Yet whatever one is to do on the afternoon of a weekend is unlikely to be nearly as altruistically productive as a homeless drive, except, perhaps, were one to play on-line poker competently and donate the proceeds to the Against Malaria Foundation, which is unlikely.

Equally, one might complain that the ‘homeless drive’ is ‘virtue signalling’. There may even be a case against ‘virtue signalling’ which provides some harm capable of outweighing the positive effect that signalling has—an increase in charitable giving which, at worst, will retain the *status quo*. This is somewhat unlikely; it is unclear to whom this virtue signalling was directed, given that, predictably, the reaction primarily consisted of accusations that the ‘drive’ was primarily intended to provide some material for the curricula vitae of the organisers. It is hardly the case that the organisers are traipsing off to Haiti or Kenya to acquire diarrhoea, and, in the unlikely event that they remain healthy, render unemployed a local for decuple the cost.

Indeed, the main criticism was not that the ‘homeless drive’ was not so much that it was less effective than the giving of malaria nets, or that it was the product of some sort of damaging ‘virtue signalling’ or ‘saviour syndrome’, but that there was no other motivation than entry to university. As pupils, in large part, we are so blinkered that it is inconceivable to us that one could wish to do good for anything other than some sort of ulterior motive, or that there could exist some good whose achievement even for some ulterior motive would not be a desideratum.

Yet, at other times, one finds the same reaction to a different, though related, sort of lecture; those, for example, on social justice. Examples include an assembly on anti-semitism for the present Upper Shell some months ago. The reaction of pupils presents the school with a dilemma. To do nothing is almost to be complicit, especially when one is aware of a certain issue—the prevalence of anti-semitism, for example. Yet the course of action open to the school appears to be actively unhelpful.

Every time the school endorses an idea whose opposite is not viewed as sufficiently beyond the pale to be absurd and render such endorsement useless, opposition to this idea is viewed as countercultural. Conversation about the idea becomes not, therefore, so much about the truth of the idea, as about *whence* the idea comes—a socially unpopular outgroup? the school? a particularly unlikeable teacher? those who are bothered?

The inability of the school to promulgate an idea because it is true, or intuitive, or a behaviour because it is desirable, or kind, is a product of apathy; an inability to and inhibition against thought of any kind. Why think about the effect of a homophobic remark when one can simply dismiss those

who do think as ‘tryhards’, the affected as ‘triggered’, even if they advocate not censorship but this thought, and the idea of that very thought as too much effort—compliant with the authority of the schoolmaster, the Dean, or whoever else? This is mirrored on, and perhaps promoted by, the internet—in the imagination of Breitbart, et al., the left are alternately too focused on argument and serious, and too ‘triggered’ to provide coherent cases for their beliefs.

It is not clear that there is a solution to this dilemma on the part of the school. Yet those pupils who care about social capital, but also are disturbed by this trend, are faced by a choice: change little by an individual conversion and be damned, or do nothing, losing little change in the process, and stay within the mainstream. Most will choose the latter.

In many cases, this assessment is, fortunately, untrue. The recent establishment of ‘The Left’—a leftist society, protests *in re Pink*, and attempts to create new publications—see, e.g., *The Thread*, and *The Westminster Martlet*—could be seen to adequately rebut the claims in this editorial, and to some extent do. However, there are a few limitations to this claim.

First, these activities primarily occur in the upper years. This is not necessarily a particularly important limitation. It is to be expected that, as one grows older, the desire to appear trendy by apathy will decrease, and that some sort of desire for change, or to appear different, will manifest.

Second, these publications and societies do not do anything. *The Librarian*, for example, is about as influential as a strongly-worded letter of protest or a stare at a train passenger whose music is too loud; where there exists something whose prevention is of import, change is unlikely, and potentates continue unmolested.

Third, these publications and societies do not involve the vast majority of pupils. Even the most popular speakers—politicians, for example—attract, at most, a hundred or so; most of the absence of the rest is almost certainly not due to a determination that the politician will be insufficiently dialectically incisive, but, rather, apathy.

This activity, therefore, is insufficient to rebut the initial claim, though it is certainly far from harmful. It will not reduce the prevalence of anti-semitic and homophobic remarks. It will not induce any of us to attempt to achieve ‘social justice’—that is, in easily visualised form, the housing of the homeless, the clothing of the insufficiently clad, the feeding of the hungry, the improvement of the provision of education to those who leave school functionally illiterate and innumerate, &c., nor, more relevantly to the original claim, it seems, will it induce those who disapprove of any effort at rectification to cease their disapproval.

Editorials often will note that ‘there are no easy solutions.’ In this case, there are no apparent solutions. It is just as naïve to hope that our inactivity should halt as it is to hope that disapproval of activity should cease. Perhaps the Bennites were right after all.

## On orthographic diversity

Joshua Loo

*The Librarian* is home to what is certainly unconventional orthography. The use of the diæresis, acute accent, ligatures and ‘-xion’ spellings is often commented upon by readers; many complain that, variously, this is an affectation, it is incorrect, it does not reflect modern usage, and so on.

### A test

Any discussion on orthography must recognise that orthography has a relatively limited effect. It does not change the content of what is said; to the extent that one as a reader changes one’s mind, this is a choice. There are, however, several reasons why the orthographic conservatism of this publication could potentially be beneficial. They do not seem particularly convincing, because orthography is so unimportant; however, given that most of the criticisms of the use of, for example, the diæresis are that it is ‘an affectation’, or ‘antiquated’, they nevertheless are important in relative terms. Three seem particularly pertinent.

The assumption here is that conservative orthography does not inhibit comprehension; were this to occur, that as a harm would outweigh the benefits claimed. It is relatively clear, however, that the diæresis, or acute accent, or ligature, does not impede comprehension. Though it may induce curiosity, there is, at worst, an initial fixed cost. It is relatively trivial for a reader to use the internet to find that ‘coördinate’ has a diæresis to indicate that ‘oo’ has two separate syllables, and so on. Where this is too frequent, there is of course a non-negligible cost.

First, the use of older orthography in particular ensures that readers are familiarised with their use, so that when they read older texts, they are not surprised.

Second, some unconventional orthography is helpful. The diæresis, for example, helps those who are not native speakers of English to understand when words are disyllabic. Of course, this demand must be balanced against the need to avoid confusing native speakers too much. Though the placing of a diæresis on, for example, ‘react’, would be logic, as the ‘ea’ sound is often monosyllabic, because the spelling ‘reäct’ is so uncommon as to not appear in dictionaries or corpora.

Third, the presence of diverse orthographies in general, that is, local prescription, helps to decrease the prevalence of global prescription. The promotion of local prescription necessarily promotes an understanding and acceptance that, at the very least, there exist multiple acceptable orthographies. By contrast, global prescription either promotes the tyranny of the majority or of a specific privileged minority. Competition between orthographies is far superior to such tyranny.

Indeed, the understanding that such tyranny is often problematic is common even amongst those who most describe

themselves as ‘descriptivist’, for they are the most likely to promote, for example, gender-neutral language. Gender-neutral language is, and certainly was, out of step both with the language of the establishment and the masses. With local prescription comes acceptance of difference, and so the *possibility* of change. Equally, of course, it means that one must accept certain remnants from an age gone by—perhaps, even after gender-neutral language becomes the norm, some will insist on the use of the generic ‘he’. Yet this will become fringe usage, and so harm very few; the alternative is a stagnation, where proposed linguistic changes either succeed rapidly or fail. Most of the time, they will fail. Far better is an acceptance of competition between linguistic conventions.

Readers who accept, for example, that some prefer the diæresis, or that some would prefer not to use one on ‘däis’, and so on, are far less likely to promote linguistic standards which, for all the improvements thus far, still are problematic, hence the improvement.

Fourth, orthographic diversity acts as a filter. Some potential readers will be too close-minded to continue to read after the first sighting of a spelling or linguistic decision that they dislike. The benefit here is twofold: first, the reader does not waste their time considering concepts which would be lost upon them, and second, the publication need not provide or alter itself for them, or publish their letters.

Fifth, choice of orthography influences æsthetic flavour; as æsthetic taste differs, it is only natural that there should be diversity in orthography as well.

Is there a cost? There almost certainly exists one—orthographic diversity may cause confusion or issues when one learns a language. Yet it is hardly the case that the principal cause of Britain’s present educational woes with the English language are the product of orthographic diversity, though it may be a minor contributor. The English language has coped without significant ill effect with the gulf between American and Commonwealth orthography; it also manages to proceed despite other differences. Hence, in a comparative between internal and external diversity, and merely external diversity, it makes little difference whether arguments over ligature usage are between Britons and Americans, or Britons and other Britons; the difference still exists.

More importantly, it is not clear why schools need the rest of the world to use uniform orthography. Perhaps children should initially read books that employ standardised orthography, but, thereafter, it is not clear quite what is inhibited. Indeed, to the extent that there are non-obvious differences in usage, it may be better that people are made aware by visible difference that there is occasionally a difference in usage in other, less visible forms. One particularly pertinent example is the use of ‘would’ in countries without a long history of English usage. In India, for example, it is common to hear that the government ‘would

implement a scheme'; it is not that, were something the case, the government would, but rather that the government will.

As contact between speakers of languages from different regions increases, due to the rise of media which are not constrained by geographical distance, such as the internet, understanding of such difference will become more and more essential. Hence, even on the metric upon which it seems most likely that orthographic diversity fails, it seems, instead, that such diversity is beneficial.

# Three observations on the Skripal affair

Joshua Loo

Three observations seem particularly pertinent in the wake of the poisoning of the Skripal affair.

First, it is fairly trivial to narrow down possible perpetrators to those who would have a motive to commit such a crime, and those who were capable of developing the sort of chemical weapon in question. The suggestion of the spokeswoman of the Russian Foreign Ministry, that Sweden, the UK, Czechia and Slovaks were the most likely to have had the capacity to produce such weapons<sup>1</sup> may indeed be true.

What is missing is an explanation as to quite why the Swedes, Czechs, Slovaks, or British should wish to poison a fairly irrelevant spy. In particular, it is fairly trivial to rule out the first three. The Swedes, Czechs, and Slovaks have no interest in the spy; they do not need to demonstrate to anyone that they have chemical weapons capability, and gain far more from the perception that they are responsible international actors than any perception of international power. Equally, the United Kingdom still has one of the strongest armies in Europe<sup>2</sup>; they would not wish to alienate the United Kingdom.

Her Majesty's Government equally has little incentive to poison spies under its protection. Most obviously, it is not a very good idea to poison spies whom one has used in the past, as this is likely to discourage future potential informants, who may think that the government is untrustworthy, or even malevolent. Further, its weakness has been

shown in its response. Britain has no military leverage over Russia; to the extent that it could punish members of the Russian élite by a reduction of their influence in the United Kingdom, advocacy thereof has been responded to with cries of 'politicisation', 'shame', &c..<sup>3,4</sup> Given the government is so unwilling to remove Russians, even those who have significant connections with the Putin régime, from London, it is not clear that it had some secret desire to end up in a conflict with the Russian government.

Somewhere, there is almost certainly someone blaming Israël. This paragraph will be useless, as people who blame Israël for such affairs are unlikely to either read *The Librarian* or be convinced by anything they read, but it is still worth repeating: there is no reason why Israel should wish to poison Skripal, given that he featured in the relationship between two powers whom Israël has no interest in separating further.

What of the Russians? It is fairly clear that the Russian government might have a motive: Skripal, after all, was described a 'traitor' by Russian state television.<sup>5</sup> There are several clear benefits. First, the attack deters future potential informants. Second, it publicly discredits the British state's intelligence apparatus. Third, it shows, by contrast, their strength.

Equally, however, there are other 'Russian' actors who also would have benefited from the attack. There are many independently operating parts of the Russian state, who could plausibly have wished to

<sup>1</sup>RussiahintsUK2018.

<sup>2</sup>GuzelyteNationalDefenceData2016.

<sup>3</sup>CapurroFrontBenchIf2018.

<sup>4</sup>DavidMilibandsays2018.

<sup>5</sup>MonitoringRussianstateTV2018.

demonstrate their loyalty to Putin in the attack. Had it been successful, and killed its target, Putin would have been more likely to reward a particular unit, for example. There are many different law enforcement agencies in Russia,<sup>6</sup> each of which would likely wish to increase their funding, prominence, and power; as the attack can be disowned by the relevant unit, it may have been in the interest of a particular part of the Russian state to attack. Similarly, the Russian state has been associated with criminal organisations, who equally may have wished to please Putin.<sup>7</sup>

It is important to note that these claims are not definitive; there is no evidence to support them. However, establishing that it is *possible* that someone else could simultaneously have had a motive and the ability to attack other than Putin is important.

Another possibility is that a Russian group desired to attack Putin. There was a reasonable risk after the attack of further restrictions on many of Putin's friends and associates in London.

Boris Johnson says that the response of the Russian government does not suggest that it truly believes itself innocent.<sup>8</sup> This is true; most of its statements have been rather combative. However, even were it not to have ordered the attack, it would still benefit, and indeed is benefiting, from the publicity surrounding the case. It would optically be something of a concession, especially for a state which

cares so much about projecting its strength, to grovel before Her Majesty's Government, explaining that it had not ordered the attack; such a strategy would require an explanation of how chemical weapons were lost, and international scrutiny of a most unwelcome kind.

Second, comparisons to Iraq are somewhat helpful, but are not perfect. Craig Murray, whose blog is an excellent read, even for those who disagree with its contents, suggests that there is to some extent a rush to blame Russia.<sup>9, 10</sup> It is certainly politically expedient to do so, and so this is a plausible claim. Murray may be lying—it would be unwise to trust him absolutely—but, at the very least, he provides another narrative which is plausible, and so should be considered.

A comparison to Iraq might suggest that there is some sort of conspiracy within Her Majesty's Government to start a war with Russia. This is fairly obviously a bad idea. A significant proportion of its instigators would die in a nuclear strike from Russia. There is no effective defence against this. Russia's defence against nuclear attack is a system which uses *nuclear bombs* to cause missiles not to hit their targets. Since the West is less willing to take such risks, we instead have systems which work approximately half the time, when the test projectiles have been fired *in idealised conditions*.<sup>11</sup> Britain is obviously behind the United States in such matters, so there is no reason for it to start a war. Fur-

<sup>6</sup>RussiaEuropeMember.

<sup>7</sup>WhitmorePutinMafiaStatecraft2015.

<sup>8</sup>Russiaspyresponse.

<sup>9</sup>CraigMurrayCraigMurray.

<sup>10</sup>Particularly entertaining is his posting of a Twitter thread between a chemist and him, in which a chemist points out that there exist mass spectrometers. The chemist, unfortunately, appears to have failed to have read Murray's article; it seemed that the initial response was to some non-existent claim that it was impossible to identify chemicals.

<sup>11</sup>Couldactuallyshoot2017.



ther, as demonstrated above, it was not particularly politically expedient to highlight British weakness at this point in time. Those who could be behind such a conspiracy clearly operate in knowledge of these concerns. There is no motive which could possibly outweigh the risk of nuclear war arising from an attempt to cause war. Even arms manufacturers would prefer to lose business than to initially sell lots of arms before dying in such a war.

Further, if one trusts Murray, one should note that it seems that the Foreign and Commonwealth Office is not being very successful in attempting to prevent leaks. Consequently, were there such a conspiracy, Murray would have noticed this, and published this on his blog. There are a sufficient number of people who read his blog, especially after the revelations, to grant him effective immunity from prosecution; certainly, there would be a great embarrassment for the government were he to be prosecuted for revealing an attempt to deceive parliament in an attempt to start a war.

Third, discussion has been predictably suboptimal. Consider Corbyn's article in *The Guardian*<sup>12</sup>. There are several issues with the view espoused in the article. First, it is not clear how the responses Corbyn outlined—that 'Russian money be excluded from our political system', 'Magnitsky-style sanctions', and 'measures to tackle the oligarchs[sic] and their loot' can be reconciled with a desire to 'reduce tensions and conflict wherever possible.' Clearly, it is possible to do nothing in this situation; Corbyn takes the intellectually lazy option of using the phrase 'wherever

possible' to omit from his article that there is necessarily a trade-off between strong action and the avoidance of conflict. The article, rather than providing a mechanism by which the optimal trade-off can be determined, ignores it.

Second, Corbyn ignores the possibility that we may not find the ultimate perpetrator with a satisfying degree of certainty. Many crimes remain unsolved, leaving a list of suspects; the risk of this always exists, and this attack is no exception. If we do not find the culprit with any certainty, what then? Should we proceed with something less than certainty? Corbyn is right to call for rigour—the perpetrators ... must be identified', but equally important is what should happen were one unable to definitively determine the identity of the perpetrator.

Third, Corbyn resorts to rhetoric appealing primarily to his own base *in rebus* Libya and Afghanistan. Certainly, the liberation of Libya was suboptimally executed; it is even possible that Libya is worse off than it would have been had Gadaffi stayed. None of these conclusions are, however, *prima facie* true, at least for a significant number of people. Corbyn's call for calm is an important; the inclusion of not particularly rigorous references to these examples reduces the appeal of this argument.

Criticism of Corbyn has, however, focused on his 'politicisation' of the attack. Quite why a political leader should not 'politicise' an issue, when politics clearly is clearly an important determinant of the outcome of such cases, is unclear. Critics need to explain

<sup>12</sup>CorbynSalisburyattackwas2018.

why the normal democratic process of criticism and replacement is inadequate in this scenario. 'Politicisation', that is, the treating of this affair in a political manner, could expel thousands of Russian oligarchs in London, prevent the use of the London Stock Exchange by large numbers of Russian companies, prevent the use of the financial system to launder and transfer money, prevent the use of financial consultancy services in London, and significantly disrupt the lives of large numbers of Putin's cronies; 'politicisation' could also alter our response, whether by inducing

Her Majesty's Government to supply samples to the Russians, or the Organisation for the Prohibition of Chemical Weapons, altering the number of diplomats expelled, or a change in policy on chemical weapons.

As Her Majesty's Government has relatively few options, what seems most likely is that we shall soon forget about the affair; occasionally, articles on progress in the investigation will be found, but little further action will be taken. More interesting and difficult to predict will be the effect on Corbyn's reputation.

## Adventures in Recreational Mathematics VII : numbers that are *not* astronomical in size

### Isky Mathews

This time we will examine some of mathematics' largest numbers; I feel this is a conceptual area that many find interesting, and even amusing, to think about—some of the numbers I will mention here will be so large that to refer to them as *astronomical* would not just be inaccurate, since there is no object that could found in these quantities within the observable universe, but, frankly, *insulting* to their magnitude.

To demonstrate the previous point, we shall begin by considering the number of baryons in the observable universe. Baryons are particles made up of 3 quarks and interact with the strong nuclear force, e.g. protons or neutrons, and we can calculate how many there are using 4 numbers, 3 of which were obtained using data from the Planck Satellite :

- $\rho_{crit}$ , the critical density of the universe ( $= 8.64 \times 10^{-33} \text{kgm}^{-3}$ )
- $\Omega_b$ , the fraction of the universe's energy in baryons ( $= 0.0485$ )
- $L$ , the radius of the observable universe, which is roughly spherical ( $= 4.39 \times 10^{26} \text{cm}$ )
- $m_p$ , the mass of one proton ( $= 1.67 \times 10^{-27} \text{kg}$ )

Now, since  $\rho_{crit}$  is essentially the energy density of the universe,  $\rho_{crit} \times \Omega_b$  is the mass stored in baryons per  $\text{cm}^3$  of the observable universe on average, making  $\rho_{crit} \times \Omega_b \times \frac{4}{3}\pi L^3$  roughly the combined mass of all baryons in the universe. Finally, since a neutron's mass is essentially equivalent to that of a proton, we divide the above expression by  $m_p$  to get

$$\frac{\rho_{crit} \times \Omega_b \times \frac{4}{3}\pi L^3}{m_p} = 8.89 \times 10^{79}$$

which is really quite a big number, in comparison to the numbers of things you encounter for everyday life! However, it was small enough to be expressed, to a fair level of precision and concisely, using a notation we are so familiar with that I barely need to name it: that of the *exponential*. For many, if asked to write down quickly the biggest number they could think of at the time, exponentials or stacked exponentials (those of the form  $a^{b^{c^{d^{\dots}}}}$ ) would be their first thought, since it's so simple—for example, just  $10^{10^2}$  is bigger than the number of baryons in the universe. In fact, our first famous number can be expressed as  $10^{100}$ , a *googol*, and the next as  $10^{10^{100}}$ , a *googolplex*. We shall return to exponentials and the process of stacking them later, for it has great potential to make large numbers.

### Primitive Recursive and Non-Primitive Recursive functions

For now, we take ourselves back to near the beginning of the 20th century, where individuals such as **Gödel**, **Turing** and **Church** were discussing the nature of functions. They realised

that the process of calculating the outputs to most functions could be seen as an iterative process that, most importantly, had a predictable number of steps; for example, to calculate  $2 + 2$ , one could see it as applying  $f(n) = n + 1$  to the input 2 twice. Such functions were called *primitive recursive*, because they *could* be written down or represented recursively, i.e. where they were seen as a series of repeated applications of some function, but could also be written down in a single closed form—all polynomials, exponentials and many more that we are familiar with are primitive recursive. The computer scientist Robert Ackermann is most famous for describing an eponymous function, denoted  $A(m, n)$ , that it was still possible to evaluate but was not primitive recursive defined by these conditions :

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

Let us call a *closed-form* representation of a function a form which uses a finite number of operations and without self reference. Then, an amazing fact is that the Ackermann function's above self-referential or *recursive* definition cannot be written out into a closed form, unlike addition or multiplication—this is what it means for it to not be a primitive-recursive function and it grows extremely quickly—try evaluating it for different inputs! Clearly things like  $A(0, 3) = 4$  and  $A(1, 2) = 4$  are quite small, but then  $A(4, 3)$  is an incredible 19729 digit number :

$$A(4, 3) = 2^{2^{65536}} - 3$$

In fact, it's often difficult to find examples to demonstrate how large the numbers are that the Ackermann function outputs, because nearly all of them are so big that they either can't be written down in any concise manner or, worse, they couldn't be computed within the lifetime of the universe given all the computing available today. Furthermore, Ackermann and his peers were later able to show that functions of this kind<sup>13</sup> *dominate* all primitive recursive functions, i.e. for any primitive recursive function  $f(x)$  and a non-primitive-recursive function  $g(x)$ , there is some input  $n$  so that for all  $m > n$ ,  $g(m) > f(m)$ .

In order to understand and express just *how* quickly such functions grow, we have to use a lovely typographical system developed some years ago by the famous **Donald Knuth**<sup>14</sup>

<sup>13</sup>As in, those that can be evaluated in a finite amount of time but that are not primitive recursive.

<sup>14</sup>A computer scientist and mathematician, perhaps most famous for his remarkably complicated series of volumes *The Art of Computer Programming* (often referred to as the computer scientist's bible!) but also for the typesetting system *T<sub>E</sub>X* which this very publication uses to format its articles!

known as *up-arrow notation*, which is based on the idea of the *hyperoperation hierarchy*. The first operator in the hierarchy is that of the *successor*, an unary operator (meaning that it takes 1 argument) which takes in  $n$  and outputs  $n + 1$ , often written  $n ++$ .<sup>15</sup> Addition can be seen as repeated successorship in that  $a + b$  can be seen as denoting  $a ++$ ,  $b$  times. Multiplication can then be seen as repeated addition in that  $a \times b$  represents  $a + a$ ,  $b$  times. This continues as we go higher up the hierarchy, with each  $n$ th operation  $a *_n b$  representing performing the  $(n - 1)$ th operation to  $a$  by itself,  $b$  times. Knuth created the hyperoperation-notation  $a \uparrow b$  which *starts* at exponentiation (as in  $a \uparrow b = a^b$ ) and by writing more arrows, one goes up the hierarchy, so  $2 \uparrow\uparrow 4 = 2^{2^{2^2}}$ —the name we give for this operation above exponentiation is "tetration" and  $a \uparrow\uparrow\uparrow b$  is called "*a pentated by b*" etc. These operations make writing really large numbers simple and if we *index* the arrows, that is say that  $\uparrow^n$  denotes  $n$  arrows, then we can write down numbers that could never have any practical use—for example, the famous **Graham's number**.

## Graham's Number

This number comes out of a question in a somewhat ill-defined area of mathematics known as Ramsey Theory, which purports to comprehend the conditions under which complex structures are forced to appear; **Ronald Graham** and **Bruce Lee Rothschild**, both legends in this field, came up with the question in 1970. The question requires understanding what a *graph* is in pure mathematics; Benedict Randall Shaw has written a helpful article explaining graph theory in a previous issue of *The Librarian*,<sup>16</sup> but a summary is that any set of points and lines drawn connecting them is a graph. More formally, a graph is a set of points along with a set of pairings defining connections between those points—thus neither the precise coordinate/relative position of points nor the shape of the lines connecting them matters, only the connections.<sup>17</sup>

Given  $n$  points, the graph obtained by adding all possible connections between them is called the *complete graph on  $n$  vertices*, denoted  $K_n$  (e.g.  $K_3$  is like a triangle and  $K_4$  is like a square with its diagonals drawn in). Now, Rothschild and Graham were considering complete graphs of  $n$ -dimensional cubes, which have  $2^n$  vertices each, and properties of the *colourings* of their edges, i.e. the ways in which you can assign different colours to those edges. In particular, they asked what was the smallest value of  $n$  such that every 2-colour colouring, using, for example, red and blue, of the edges of the complete graph on an  $n$ -dimensional cube is *forced* to contain a subset  $S$  of its points such that all the edges between the points in  $S$  are the same colour and such that all points in  $S$  are *coplanar*.<sup>18</sup> They were able to prove that there is such an  $n$  and they knew from checking on paper that  $n > 5$  and so they sought to also put

<sup>15</sup>This is, interestingly, why C++ is called what it is—it was supposed to be the *successor* to C

<sup>16</sup>**RandallShawIntroductionGraphTheory17.**

<sup>17</sup>To be precise, we consider two graphs that have the same number of vertices and the same connections between those vertices but are drawn differently to be distinct graphs or objects but we say they are *isomorphic*, i.e. share all the same graph-theoretic properties.

<sup>18</sup>i.e. are points on a common plane.

an upper-bound on it (Graham's number).<sup>19</sup> It is constructed as follows:

- Let  $G_1 = 3 \uparrow^4 3$  (an amazingly large number, so big that the number of 3s in its power-tower representation couldn't be written in base 10 even if each digit could be assigned to each planck-volume in the observable universe!)
- For each  $n$ , let  $G_{n+1} = 3 \uparrow^{G_n} 3$
- Then Graham's number is  $G_{64}$ .

Can you see how large it is? We urge you to ponder over this for a minute, since there are no metaphors or real world analogies for this...

It is clear from this that uparrow notation becomes inadequate for integers as large as Graham's Number, since there is no way of expressing it concisely if we need to write out all the arrows. Thus, when you have gotten over  $G_{64}$ , we must move on to a better framework that will allow us to see just how large it is "in the grand scheme of things".

## The Fast-Growing Hierarchy or the Grandest of Schemes of Things

The fast-growing hierarchy is a series of functions, built recursively, that grow faster and faster as we go up. We start with the simple function  $f_0(x) := x + 1$  and we say<sup>20</sup> that  $f_1(x) := f_0^x(x)$ , or in other words  $x + x$ . Similarly,  $f_2(x) := f_1^x(x) = x \times x$  and in general for any integer  $n > 0$ ,  $f_n(x) = f_{n-1}^x(x)$ .

So far, there is no difference between this and hyperoperations but now, we can use *ordinals* to give us unbounded growth-rates... There was a previous article<sup>21</sup> introducing readers to the wonderful universe of ordinals but, to simplify their technical definition, they are a clever set-theoretic version of numbers, discovered by **Georg Cantor**, which essentially allows us to have a natural extension of the integers to varying sizes of infinity. The number  $\omega$  is the ordinal "larger" than all the integers but then we still have a well-defined concept of  $\omega + 1$  or  $+2$  or  $+n$  and much, much more. We call  $\omega$  the first *limit ordinal*, meaning that it has no specific predecessor, but rather can be reached as a limit of a strictly increasing sequence, and we call  $2, 3, 4, n, \dots$  and  $\omega, \omega + 1, \omega + n$  etc. *successor ordinals* because they *do* have a well-defined predecessor (i.e. they are the successor of some known ordinal). Thus we have the definition that if  $\alpha$  is a successor ordinal, then  $f_\alpha(x) = f_{\alpha-1}^x(x)$ , and if  $\alpha$  is a limit ordinal and  $S_\alpha$  is a strictly-increasing sequence of ordinals whose limit is  $\alpha$  (as in,  $\alpha$  is the smallest upper-bound for all the terms in  $S_\alpha$ ), with  $S_{\alpha[n]}$  denoting the  $n$ th term of  $S_\alpha$  for some ordinal  $n$ , then  $f_\alpha(x) = f_{S_{\alpha[n]}}(x)$ .

<sup>19</sup>It may be of interest that subsequently we have created a better bound,  $2 \uparrow\uparrow\uparrow 6$ .

<sup>20</sup>Here,  $g^n(x)$ , for some integer  $n$  and some function  $g(x)$ , denotes performing  $g$  to the input  $x$ ,  $n$  times.

<sup>21</sup>**MathewsAdventuresRecreationalMathematics17.**

To give an example,<sup>22</sup>  $f_\omega(x) = f_x(x)$ , since the sequence of integers  $1, 2, 3, \dots, x, \dots$  has the limit  $\omega$  but since  $\omega+1$  is a successor ordinal,  $f_{\omega+1} = f_\omega^x(x)$ . We can observe from these definitions immediately that  $f_\omega(x)$  can't be primitive-recursive, since it grows faster than any  $f_n$  for integer  $n$ , and thus that it is, in a sense, *beyond uparrows*, since it can't be represented in the form  $m \uparrow^k x$ , where  $m, k$  are fixed integers. In fact, it is possible to show that  $f_\omega(x)$  grows at almost exactly the same rate as the Ackermann function that we've seen previously and that  $f_{\omega+1}(64) > G_{64}$ .<sup>23</sup> Now, you can choose your favourite transfinite ordinal and create a function that grows faster than you can imagine, for example  $f_{\omega \times 2}, f_{\omega^2}, f_{\omega^\omega}$  or, if  $\epsilon_0 = \omega^{\omega^{\omega^{\dots}}}$ , then you can even have  $f_{\epsilon_0}$  and larger!

The next issue of The Librarian will contain *ARM: Numbers that are not astronomical in size Part II*, covering *TREE*(3), Kruskal's Tree Theorem and will contain a number that is the largest ever written by any human ever, one of my own design that blows the previous record "out of the water". If you have any questions until then, email **Isky Mathews** or **Benedict Randall Shaw**.

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<sup>22</sup>Some may notice that this definition only applies for integer  $x$  (since there is no 3.2th function in our list, for example)—that's because of the caveat that the fast-growing hierarchy only contains functions defined for ordinal inputs.

<sup>23</sup>They aren't actually comparable in size, since  $f_{\omega+1}(64) > f_\omega^{64}(6) > G_{64}$ .