Analogies of the Abraham-Minkowski Controversy

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Abstract

Arguments that lead to the Abraham-Minkowski controversy may be applied to waves of non-electromagnetic nature, such as sound in an ideal gas column or in a solid, for which the controversy may be solved using through mechanical reasoning. The results lead to a solution of the electromagnetic case.

1 Introduction

1.1 Statement

The mechanical properties of the electromagnetic field may be obtained from the corresponding conservation equations with sources due to the interaction of fields with matter. Thus, given the Lorentz force per unit volume

$$\boldsymbol{f} = \rho \boldsymbol{E} + \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B}, \tag{1}$$

a straightforward elimination of the total charge ρ and current $\boldsymbol{j}$ densities using Maxwell's equations yields

$$\partial_t u + \nabla \cdot \boldsymbol{S} = -\mathcal{P} \tag{2}$$

and

$$\partial_t \boldsymbol{g} + \nabla \cdot (-\mathbf{T}) = -\boldsymbol{f}, \tag{3}$$

where we interpret

$$u = \frac{1}{8\pi}(E^2 + B^2) \tag{4}$$

as the energy density,

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B}, \tag{5}$$

as the energy flux,

$$\boldsymbol{g} = \frac{1}{4\pi c}\boldsymbol{E} \times \boldsymbol{B}, \tag{6}$$

as the momentum density, and

$$T_{ij} = \frac{1}{4\pi}\left(E_i E_j + B_i B_j - \frac{1}{2}(E^2 + B^2)\delta_{ij}\right) \tag{7}$$

as the electromagnetic stress tensor, given by the additive inverse of the momentum flux density. On the right hand sides we interpret the power density

$$\mathcal{P} = -\boldsymbol{E} \cdot \boldsymbol{j} \tag{8}$$

as the sink of the electromagnetic energy, i.e., the energy transferred from field to the moving charges per unit volume per unit time. Similarly, the force per unit volume $\boldsymbol{f}$ is interpreted, given Newton's second law, as the momentum per unit volume that is taken away from the field by the charges per unit time. In these equations, ρ and $\boldsymbol{j}$ represent the total charge and current densities, which, in the presence of materials, would include external as well as conduction, polarization and magnetization contributions.

Within non-dispersive and non-dissipative media one can write the same equations (2) and (3) interpreting ρ and $\boldsymbol{j}$ in the source terms (8) and (1) as the density of *external* charges and currents only, provided provided we modify our expressions for the energy density,

$$u = \frac{1}{8\pi}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}), \tag{9}$$

energy flux,

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{H}, \tag{10}$$

momentum density,

$$\boldsymbol{g} \to \boldsymbol{g}_M = \frac{1}{4\pi c}\boldsymbol{D} \times \boldsymbol{B}, \tag{11}$$

and stress tensor,

$$T_{ij} = \frac{1}{4\pi}\left(E_i D_j + H_i B_j - \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})\delta_{ij}\right), \tag{12}$$

where

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{E} = \boldsymbol{\epsilon}\boldsymbol{E} \tag{13}$$

and

$$\boldsymbol{H} = \boldsymbol{B} - 4\pi M = \boldsymbol{\mu}^{-1}\boldsymbol{B}. \tag{14}$$

Here, $\boldsymbol{P}$ and $\boldsymbol{M}$ are the (generalized) polarization and magnetization related to the internal contributions to the charge and current densities through

$$\rho^{\text{int}} = -\nabla \cdot \boldsymbol{P} \tag{15}$$

and

$$\boldsymbol{j}^{\text{int}} = \partial_t \boldsymbol{P} + c\nabla \times \boldsymbol{M}. \tag{16}$$

We have assumed that the temporal Fourier transform of the fields only have frequencies within a given transparency window.

In the former context, Eqs. (2) and (3) account explicitly for the exchange of energy and momentum between the fields and all the charges in the system, while in the latter context, it accounts for energy and momentum exchanged with the external charges only, so we expect that Eqs. (4)-(7) account for the energy and momentum carried by the electromagnetic field alone, while Eqs. (9)-(12) account for energy and momentum carried by both the field and the material. We remark that Eq. 3, together with 11 and 12 is only valid within an *homogeneous* system. In an inhomogeneous system, a system without translational invariance, momentum is not conserved, though space itself is homogeneous and total momentum is conserved, Eqs. 3, 6 and 7 hold, when $\boldsymbol{f}$ is interpreted as the force on the total charges and currents.

Eq. (11) for the momentum density is known as the Minkowski expression. Theoretical reasons have been advanced to question the interpretation of (11) as the momentum density of an electromagnetic field in matter, and it has been proposed that it ought to be replaced by the Abraham expression,

$$\boldsymbol{g} \to \boldsymbol{g}_A = \frac{1}{4\pi c}\boldsymbol{E} \times \boldsymbol{H}. \tag{17}$$

One of the virtues of the Abraham expression is that the stress-energy 4-tensor

$$(\Theta^{\mu\nu}) = \left(\begin{array}{c|c} u & \boldsymbol{S}/c \\ \hline c\boldsymbol{g} & -\mathbf{T} \end{array}\right) \tag{18}$$

becomes symmetric, with the implication that the relativistic angular momentum becomes a conserved quantity.

There have been many attempts to experimentally test which of the expressions (11) or (17) is the correct one by measuring the force that acts on different solid or liquid systems, with contradictory results so far. These experiments rely on interpretations of the momentum flux in terms of the momentum density, notwithstanding the fact that the momentum flux is given by the electromagnetic stress tensor whose expression (12) seems not to be the subject of ongoing controversies.

1.2 Minkowski

There are alternative derivations of expressions (11) and (17) that don't rely directly on Maxwell equations, but are dependent on arguments about the conservation of mechanical quantities alone. For example, consider a plane monochromatic electromagnetic wave propagating through vacuum with wavevector $\boldsymbol{q}$ and frequency $\omega = qc$ that illuminates at an angle the flat surface of a semiinfinite isotropic medium (see Fig. 1.2). From a quantum point of view the wave is made up of free propagating photons each of which has an energy $\mathcal{E}_i = \hbar\omega$ and a momentum $\boldsymbol{p}_i = \hbar\boldsymbol{q}$, which holds uncontested in empty space. Each photon may be reflected with a probability given by the optical reflectance R of the surface. With some other probability it may instead be converted at the surface into an elementary oscilatory excitation of the medium with some wavevector $\boldsymbol{k}$. The temporal invariance of the system implies that in the latter case, the energy of the quantized transmitted excitation ought to equal to the energy of the incoming photon, $\mathcal{E}_t = \mathcal{E}_i = \hbar\omega$. As a flat interface is translationally invariant along its surface, the projection of the momentum parallel to the surface is also a conserved quantity, i.e., the parallel momentum of the transmitted excitation is $\boldsymbol{p}_{t\|} = \boldsymbol{p}_{i\|} = \hbar\boldsymbol{q}_\|$.

On the other hand, the transmitted wave has to obey some condition at the surface that couples it to the incoming wave. Thus the phase relation of the transmitted wave with respect to the incoming wave has to be kept for all times and at all points of the surface, implying that the frequency

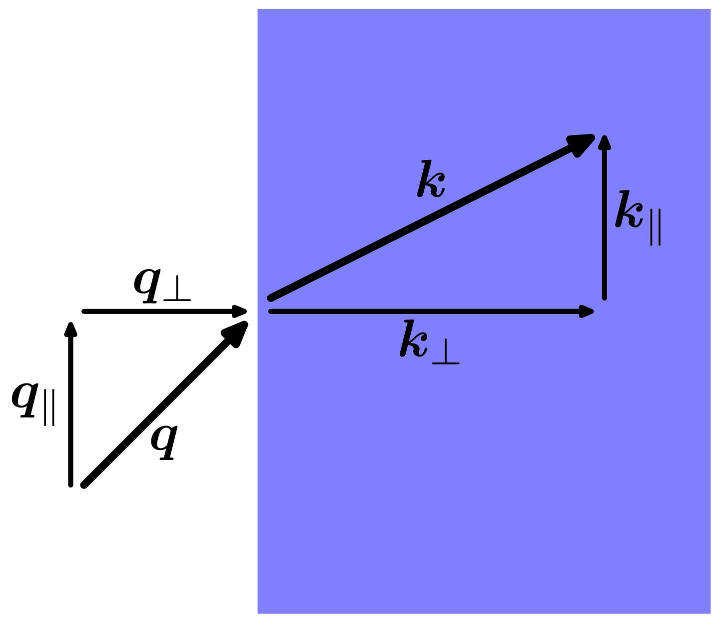

Figure 1: A photon with wavevector $\boldsymbol{q}$ is converted at a flat surface into an excitation with wavevector $\boldsymbol{k}$. Shown are the parallel and perpendicular projections of the wavevectors.

and the surface projection of the wave vector of the transmitted excitation has to equal those corresponding to the incoming wave, i.e. $\boldsymbol{k}_{\parallel} = \boldsymbol{q}_{\parallel}$, which is a generalization of Snell's law. As the medium is isotropic, far from the interface there is no privileged direction for the transmitted wave except for its propagation direction defined by $\boldsymbol{k}$. Thus, we expect the transmitted momentum to be parallel to the wavevector, i.e., $\boldsymbol{p}_t = \zeta \boldsymbol{k}$ for some constant ζ. Projecting this equation onto the surface we finally obtain $\zeta = \hbar$ and $\boldsymbol{p}_t = \hbar k$, a result that is fully consistent with de Broglie's view. Notice that, as $\omega = k v_p$, $\mathcal{E}_t = v_p p_t$, where v_p is the phase velocity of the excitation. Notice that multiplying by the average number of transmitted quanta, this relation may be scaled up to a relation between the total energy $\mathcal{E}$ and the total momentum $\boldsymbol{p}$ of the transmitted wave packet, namely

$$\mathcal{E} = p v_p. \tag{19}$$

This equation is consistent to Minkowski's expression (11), as for a plane wave in an isotropic transparent dispersionless material $\langle g_M \rangle = n\epsilon |E|^2/8\pi c = n\langle u \rangle / c$, with $\langle g_M \rangle$ and $\langle u \rangle$ the averaged momentum and energy densities.

1.3 Abraham

Consider now a block of length L made up of some material into which a wave packet is shot from some heavy device of mass M that is free to move (Fig. 1.3). Assume that L is much larger than the length of the wavepacket with in turn is much larger than the wavelength. If the wave packet carries a momentum $\boldsymbol{p}$, then the launcher recoils with velocity $\boldsymbol{V} = -\boldsymbol{p}/M$. At a time $t = L/v_g$, where v_g is the group velocity of the packet, the wave is absorbed by a similar device situated on the right hand side of the block. By that time the launcher has moved a distance $d = Vt = pL/Mv_g$ towards the left. However, energy $\mathcal{E}$ has been transported by the wave from the launcher to the absorber, and thus at the end the launcher is lighter and the absorber heavier by the relativistic mass $\Delta M = \mathcal{E}/c^2$ which has been transported by the wave a distance L across the block. As there has been no external force acting in the system composed of block, launcher and absorber, the center of mass should remain fixed during the process, so that $Md = \Delta M L$ and thus

$$\mathcal{E} = p\frac{c^2}{v_g} = p\frac{c^2}{v_p}, \tag{20}$$

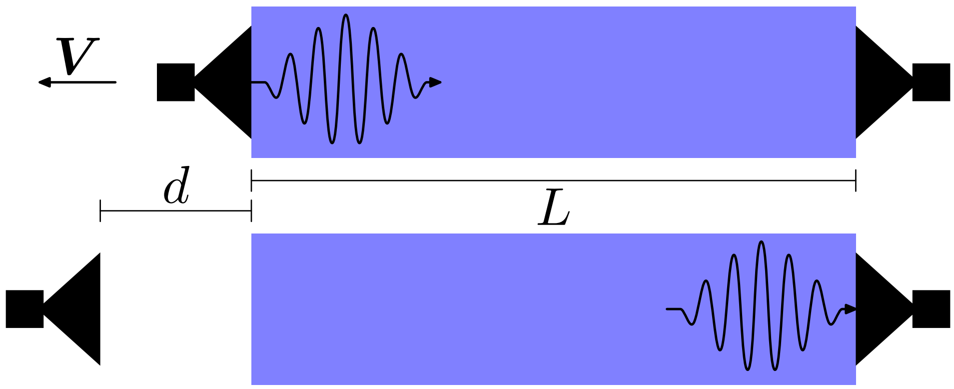

Figure 2: A wave packet is shot from a device at the left side of some material (blue) towards its right side. The launcher recoils towards the left with velocity $\boldsymbol{V}$ (upper panel). At some later time the wave packet has moved a distance L and is absorbed by a device on the right hand side of the material, while the launcher has moved a distance d towards the left (lower panel).

where the last equality is valid for non-dispersive waves, for which $v_g = v_p$. Eq. (20) is consistent with the Abraham expression (17) as $\langle g_A \rangle = n|E|^2/4\pi\mu c = \langle u \rangle/nc$.

The arguments above yielding Eqs. (19) and (20) are independent of the nature of the wave. It could be electromagnetic waves as in the original Abraham-Minkowski controversy, but they could be other kind of non-dispersive waves such as sound waves. Thus the solution of the controversy for one kind of waves may be applied to waves of different kinds.

2 Detailed calculation

One possible explanation of the discrepance between Eq. (20) and (19) is that besides the transport of energy and the consequent relativistic transport of mass, within a material there may be further mechanisms of mass transport. The origin of this mass transport would be the forces acting on the matter, thus transfering mechanical momentum. To account for these, I revisit the argument for Abraham's expression presented in Sec. 1.3. Consider the upper panel of Fig. 3, which shows an electromagnetic pulse being emitted into a material. I assume that pulse is quasimonochromatic, with a pulsewidth that is much larger than the wavelength, but much shorter than the length L of

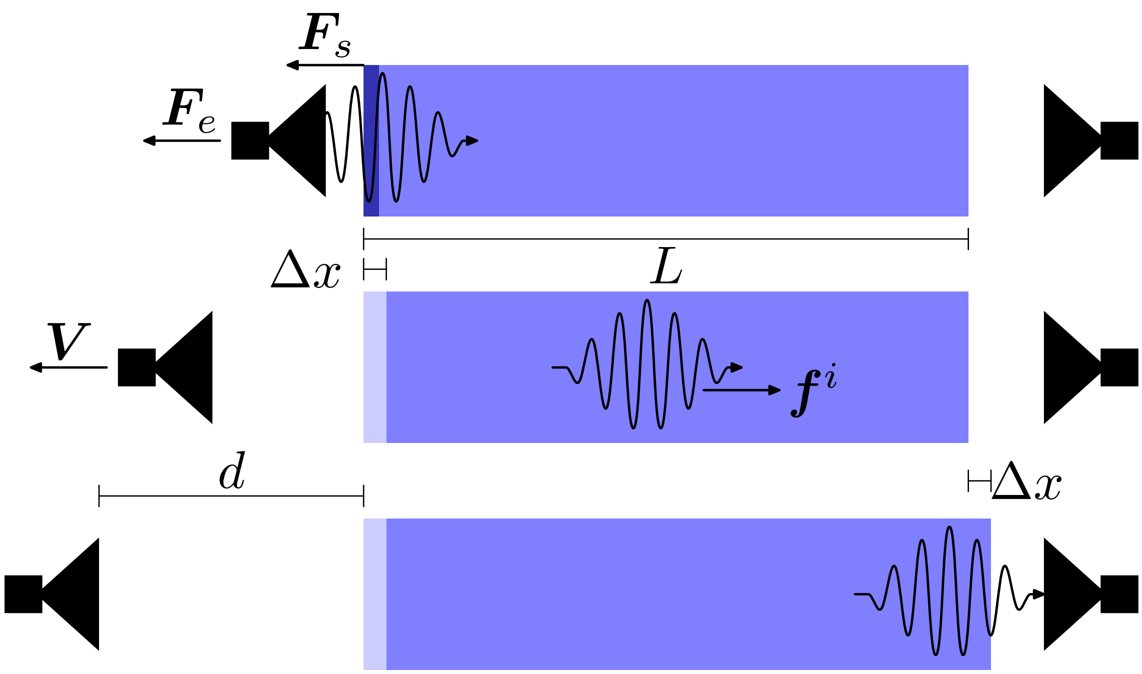

Figure 3: An electromagnetic wave packet is shot into a material (upper panel). The emitter is subject to a recoil force $\boldsymbol{F}_e$ during emission. The electromagnetic interaction with the induced surface currents (dark blue region) produces an additional force $\boldsymbol{F}_s$ which is mechanically transmitted to the emitter. As the package propagates through the material, it acts on the induced bulk currents through a force density $\boldsymbol{f}_i$ (middle panel). When the packet arrives at the detector (bottom panel) the emitter has moved back a distance d.

the material. Using the vacuum stress tensor we obtain the force per unit area acting on the emitter

$$\frac{F_e}{A} = -\frac{1}{8\pi}(E_v^2 + B_v^2), \tag{21}$$

where A is the area and the subindex v indicates that the fields E_v and B_v are to be evaluated in vacuum, in the region between emitter and material. I assume the wave can be described as a plane wave traveling in the direction normal to the surface, so that $E_v = E_m$ and $B_v = H_v = H_m$, where the subindex m indicates that the corresponding fields are evaluated at the material. Thus,

$$\frac{F_e}{A} = -\frac{1}{8\pi}(E_m^2 + H_m^2). \tag{22}$$

Within the material we have a quasimonochromatic wave traveling towards the right, for which $B_m = nE_m$ with $n = \sqrt{\epsilon\mu}$ the index of refraction, and ϵ and μ the permittivity and permeability, so that

$$\frac{F_e}{A} = -\frac{1}{8\pi}\left(1 + \frac{\epsilon}{\mu}\right) E_m^2. \tag{23}$$

If the material has a magnetic response, $\mu \neq 1$, there is and induced surface current $\boldsymbol{K}_s = \hat{x} \times \boldsymbol{M}_m$, where $\boldsymbol{M}_m = (\boldsymbol{B}_m - \boldsymbol{H}_m)/4\pi$ is the magnetization. Its interaction with the $\boldsymbol{B}_m$ and $\boldsymbol{B}_v$ induces a force, which may be calculated either through the Lorentz force or through the stress tensor (Eq. (7)). The latter is better as it avoids the ambiguity due to the discontinuity of $\boldsymbol{B}$ at the surface, and yields

$$\begin{aligned} \frac{F_s}{A} &= \frac{1}{8\pi}((E_v^2 + B_v^2) - (E_m^2 + B_m^2)) = \frac{1}{8\pi}(H_m^2 - B_m^2) \\ &= \frac{1}{8\pi}\left(\frac{1}{\mu} - \mu\right)\epsilon E_m^2 \end{aligned} \tag{24}$$

I assume that the surface is mechanically coupled to the emitter, so that the force is transferred from the surface to the emitter, which thus is subject to a total force

$$\frac{F_t}{A} = \frac{F_e + F_s}{A} = -\frac{1}{2}\left(\frac{1}{\epsilon} + \epsilon\mu\right) u_m \tag{25}$$

where I factored out the energy density

$$u_m = \frac{\boldsymbol{E}_m \cdot \boldsymbol{D}_m + \boldsymbol{H}_m \cdot \boldsymbol{B}_m}{8\pi} = \frac{1}{4\pi}\epsilon E_m^2 \tag{26}$$

During the process force the emitter acquires a momentum $\boldsymbol{P} = M\boldsymbol{V}$ given by

$$P_e = \int_{t_i}^{t_f} dt F_t \tag{27}$$

where t_i corresponds to a time before emission starts and t_f corresponds to a time after emission stops. As the pulse moves within the material without deformation, the dependence of all relevant quantities depend on position and time is through the difference $t - x/v$, where $v = c/n$ is the velocity of the wave within the medium. Thus I can replace the time integral by a position integral

$$\int_{t_i}^{t_f} dt \ldots = \frac{n}{c} \int_0^L dx \ldots \tag{28}$$

where I chose for the spatial integration a fixed time such that the whole pulse is within the material (as in the middle panel of Fig. 3). Thus,

$$P_e = -\frac{n}{2c}\left(\frac{1}{\epsilon} + \mu\right) A \int_0^L dx u_m = -\frac{n}{2c}\left(\frac{1}{\epsilon} + \mu\right) U, \tag{29}$$

with U the total energy of the packet.

Now, within the medium there is a force that acts on the induced charges and currents, given by the force density

$$\boldsymbol{f}_i = \rho_i \boldsymbol{E}_m + \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B}_m, \tag{30}$$

where

$$\rho_i = -\nabla \cdot \boldsymbol{P}, \tag{31}$$

and

$$\boldsymbol{j}_i = \frac{\partial}{\partial t}\boldsymbol{P} + c\nabla \times \boldsymbol{M}, \tag{32}$$

For the transmitted wave $\rho_i = 0$ and

$$\begin{aligned} \boldsymbol{j}_i = \frac{\epsilon - 1}{4\pi}\frac{\partial}{\partial t}\boldsymbol{E}_m + c\frac{\mu - 1}{4\pi}\nabla \times \boldsymbol{H}_m = \frac{\epsilon - 1}{4\pi}\frac{\partial}{\partial t}\boldsymbol{E}_m + \frac{\mu - 1}{4\pi}\frac{\partial}{\partial t}\boldsymbol{D}_m \\ = \frac{\epsilon\mu - 1}{4\pi}\frac{\partial}{\partial t}\boldsymbol{E}_m. \end{aligned} \tag{33}$$

Thus,

$$f_i = \frac{n}{c}\frac{\epsilon\mu - 1}{8\pi}\frac{\partial}{\partial t}E_m^2 = \frac{n}{2c}\left(\mu - \frac{1}{\epsilon}\right)\frac{\partial}{\partial t}u_m. \tag{34}$$

The mechanical momentum of the material has a density

$$\boldsymbol{g}_m = \rho_m \boldsymbol{v}_m, \tag{35}$$

where ρ_m is the mass density and $\boldsymbol{v}_m$ is the local velocity of the material. Thus, we may interpret the mechanical momentum density as the mass flow within the material. Assuming only electromagnetic forces act on the material, we may identify the force density given by Eq. (34) with the rate of change of the mechanical momentum,

$$\frac{\partial}{\partial t}\boldsymbol{g}_m = \boldsymbol{f}_i. \tag{36}$$

Thus, we ignore the convective momentum flux, as it would be of higher than second order in the electromagnetic field, and we ignore elastic forces within the material, which might produce acoustic waves which would carry energy and momentum and would complicate the analysis of the electromagnetic contribution that we want to analyze.

Integrating in time Eq. (36) after substituting Eq. (34) we obtain

$$g_m = \frac{n}{2c}\left(\mu - \frac{1}{\epsilon}\right) u_m. \tag{37}$$

Thus, the mass ΔM that crosses any cross section of the material due to its actual motion is given by

$$\begin{aligned}\Delta M &= A\int dt\, g_m = \frac{n}{2c}\left(\mu - \frac{1}{\epsilon}\right) A\int dt\, u_m \\ &= \frac{n^2}{2c^2}\left(\mu - \frac{1}{\epsilon}\right) A\int dx\, u_m = \frac{1}{2c^2}\left(\epsilon\mu^2 - \mu\right) U\end{aligned} \tag{38}$$

Finally, when the pulse is finally absorbed, after the time $\Delta t = L/v = nL/c$ taken to traverse all of the material, the emitter would have traveled a (negative) distance

$$d = \frac{P_e}{M}\Delta t = -\frac{n^2 L}{2c^2}\left(\frac{1}{\epsilon} + \mu\right) U = -\frac{L}{2Mc^2}\left(\mu + \epsilon\mu^2\right) U \tag{39}$$

On the other hand, a mass ΔM (Eq. (38)) would have moved effectively from the left hand side of the material to its right hand side a distance L. Of course the mass would not have actually moved the distance L, but all of the atoms of the material would have moved some small distance $\Delta x = \Delta M/(A\rho_m)$. Finally, as energy is transfered from emitter to receiver, the emitter becomes lighter and the receiver heavier by a mass ΔM_r or relativistic origin. I assume that the mass of the emitter M is very large, so I may write the conservation of the center of mass as the condition

$$\begin{aligned} 0 &= Md + \Delta ML + \Delta M_r L \\ &= -\frac{L}{2c^2}\left(\mu + \epsilon\mu^2\right)U + \frac{L}{2c^2}\left(\epsilon\mu^2 - \mu\right)U + \Delta M_r L \\ &= -\frac{L}{c^2}\mu U + \Delta M_r L, \end{aligned} \tag{40}$$

from which we obtain the relativistic mass,

$$\Delta M_r = \mu\frac{U}{c^2}, \tag{41}$$

which is clearly **wrong** unless $\mu = 1$.

3 Conclusions

I don't understand nothing. I shouldn't teach EM. Where is the mistake? How can it be corrected?

The mistaken result of Sec. 2 does not allows me going further in the understanding of the AM controversy. I have lots of material written about the momentum flow in other kinds of waves, but I haven't transcribed them yet.

Acknowledgments

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