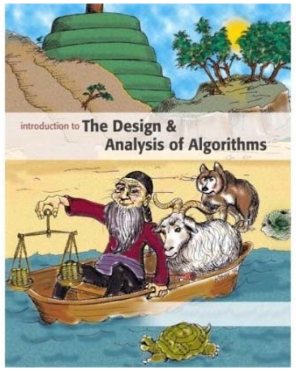
Introduction to

# *Algorithm Design and Analysis*

## [9] Hashing



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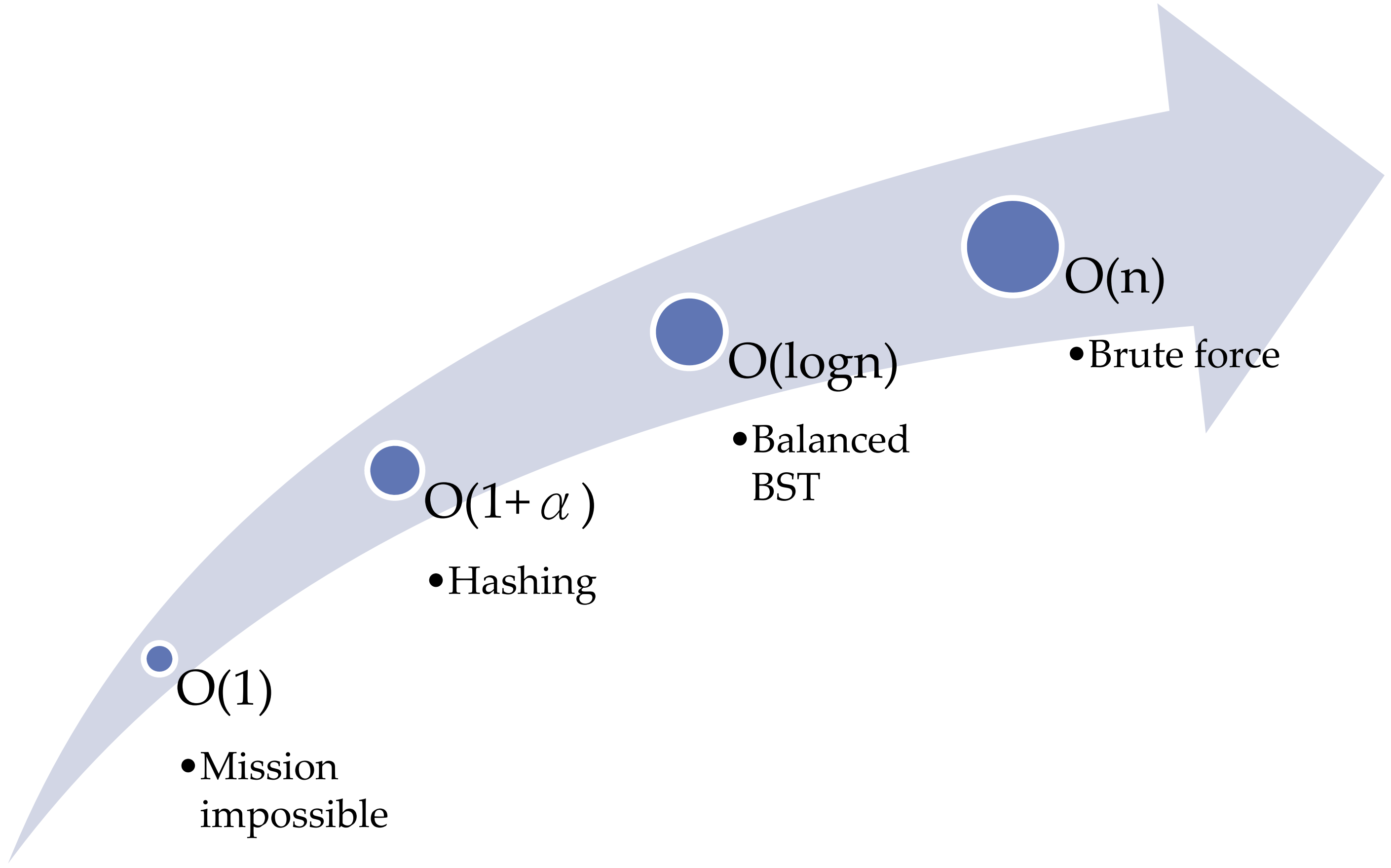
## In the Last Class…

- **The searching problem**
  - “Architecture” of data
- ***logn* search**
  - Binary search
    - In a  more general sense
  - Red-black tree: balanced BST
    - Definition
      - Black height constraint for balance
      - Color constraint for low maintenance cost
    - Operation
      - Insertion, deletion

# Hashing

- **The searching problem**
  - The ambition of hashing
- **Hashing**
  - Brute force table: Direct addressing
  - Basic idea of hashing
- **Collision Handling for Hashing**
  - Closed Address Hashing
  - Open Address Hashing
- **Amortized Analysis**
  - Array doubling

# Cost for Searching

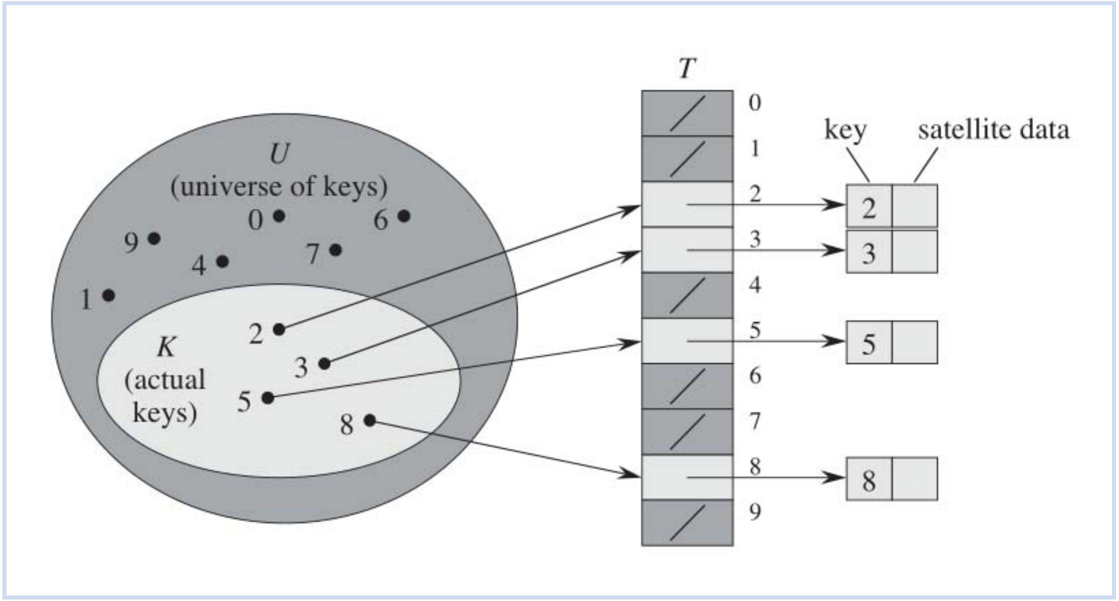


# Cost for Searching

- **Brute force**
  - O(n)
- **Balanced BST**
  - O(logn)
- **Hashing – almost constant time**
  - O(1+$\alpha$)
- **"Mission impossible"**
  - O(1)

# Searching - a Brute Force Approach

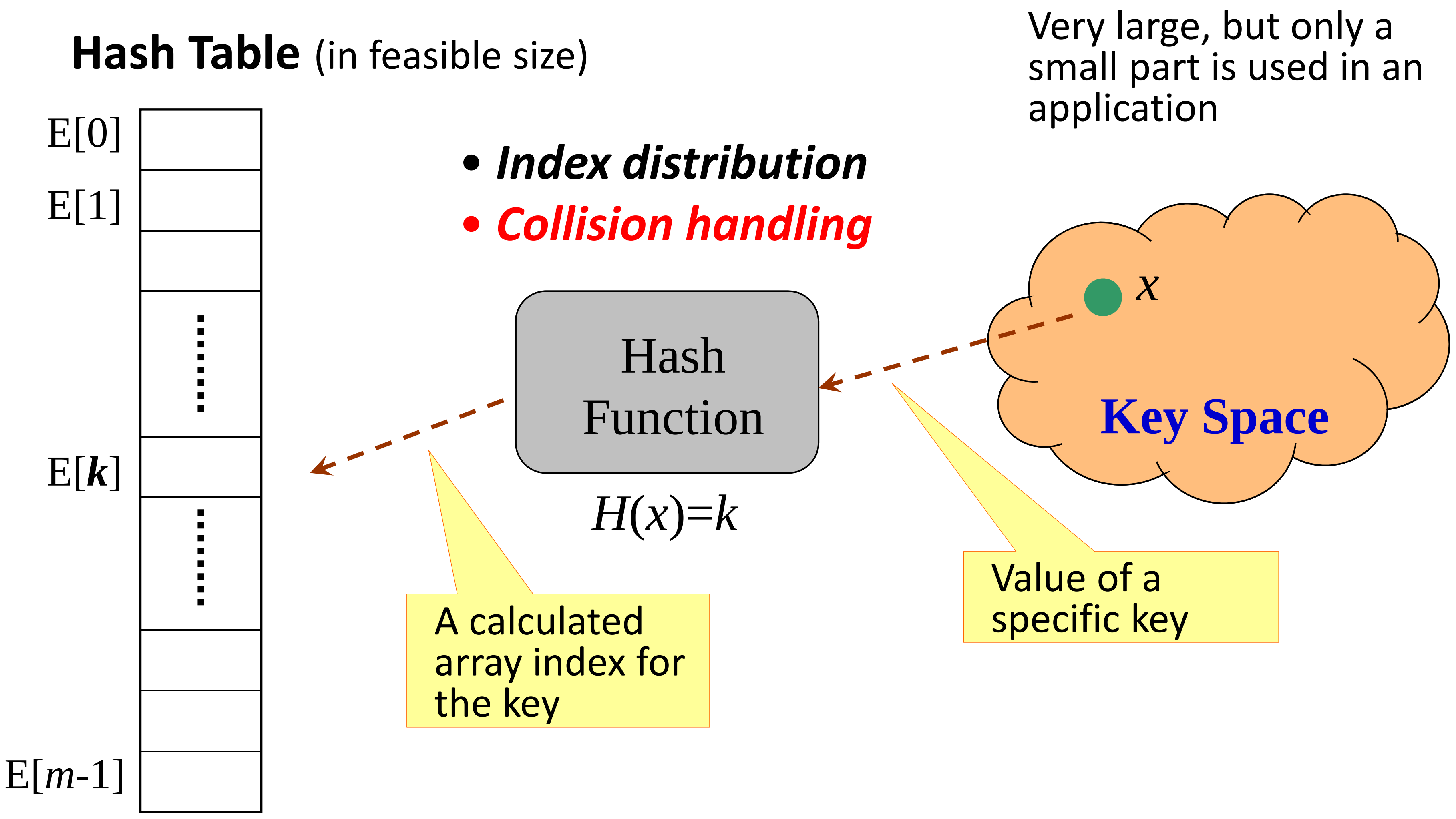
- **Direct-address table**
  - Take into account the *whole universe* of keys



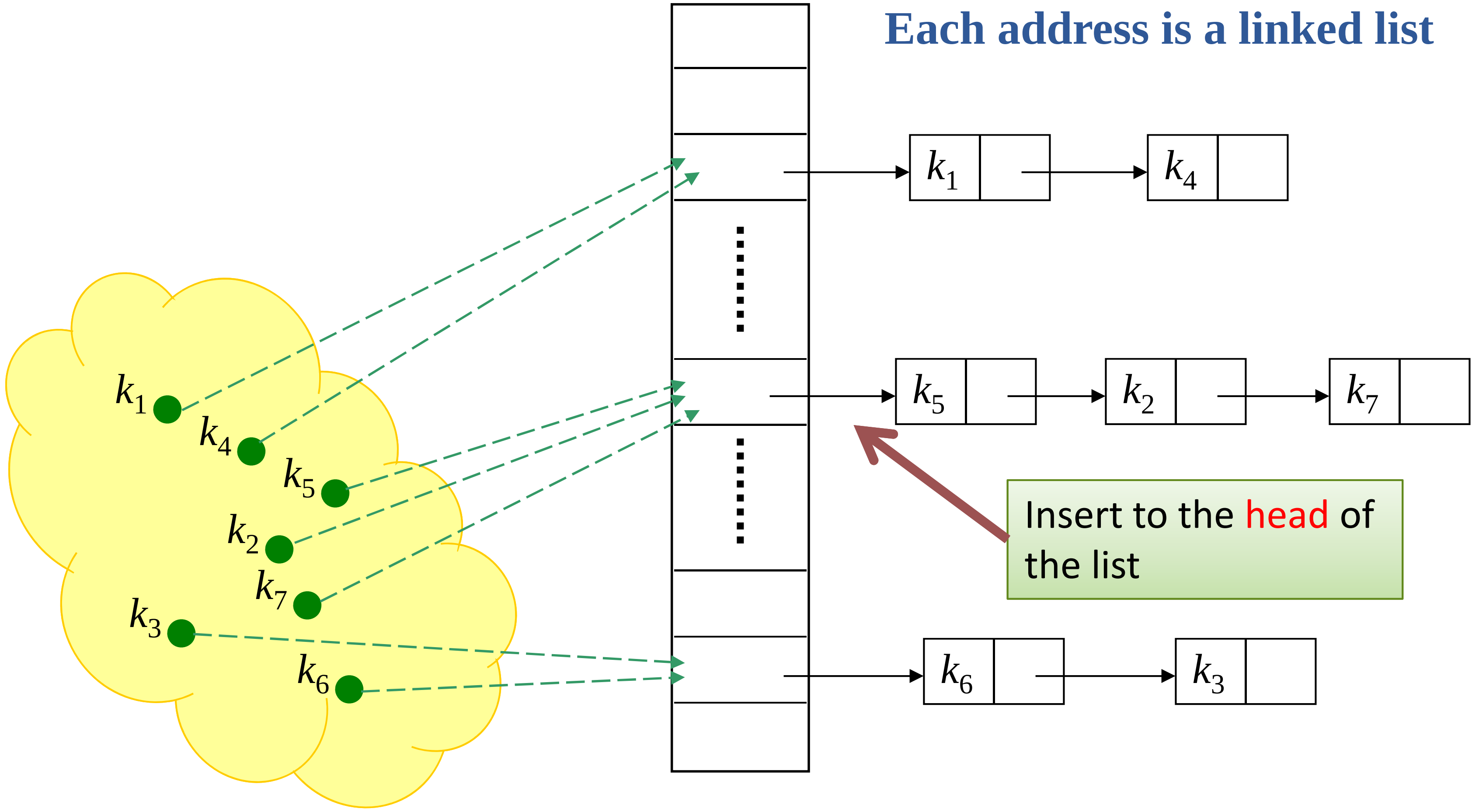
**Direct-address Table**

```
DIRECT-ADDRESS-SEARCH (T, k)
   return T[k]
DIRECT-ADDRESS-INSERT (T, x)
   T[key[x]] := x
DIRECT-ADDRESS-DELETE (T, x)
   T[key[x]] := NIL
```

# Hashing: the Idea


**Hash Table** (in feasible size)
Very large, but only a small part is used in an application
E[0]
E[1]
E[$k$]
E[$m$-1]
• *Index distribution*
• *Collision handling*
Hash Function
$H(x)=k$
$x$
Key Space
A calculated array index for the key
Value of a specific key

# Collision Handling: Closed Address



# Closed Address - Analysis

- **Assumption - simple uniform hashing**
  - For $j$=0,1,2,...,$m$-1, the average length of the list at $E[j]$ is $n/m$.
- **The average cost for an unsuccessful search**
  - Any key that is not in the table is equally likely to hash to any of the $m$ address.
  - Total cost $\Theta(1+ n/m)$
    - The average cost to determine that the key is not in the list $E[h(k)]$ is the cost to search to the end of the list, which is $n/m$.

## Closed Address - Analysis

- **For successful search** **(assuming that $x_i$ is the $i$th element inserted into the table, $i$=1,2,...,$n$)**
  - o For each $i$, the probability of that $x_i$ is searched is $1/n$.
  - o For a specific $x_i$, the number of elements examined in a successful search is $t+1$, where $t$ is the number of elements inserted into the same list as $x_i$, after $x_i$ has been inserted

$$\frac{1}{n}\sum_{i=1}^{n}(1+t)$$

- **How to compute t?**
  - o Consider the *construction* process of the hash table

# Closed Address - Analysis

- **For successful search:** **(assuming that $x_i$ is the $i^{th}$ element inserted into the table, $i$=1,2,…,$n$)**
  - For each $i$, the probability of that $x_i$ is searched is $1/n$.
  - For a specific $x_i$, the number of elements examined in a successful search is $t$+1, where $t$ is the number of elements inserted into the same list as $x_i$, after $x_i$ has been inserted. And for any $j$, the probability of that $x_j$ is inserted into the same list of $x_i$ is $1/m$. So, the cost is:

Cost for computing hashing

$$1 + \frac{1}{n}\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n}\frac{1}{m}\right)$$

Expected number of elements in front of the searched one in the same linked list.

## Closed Address: Analysis

- **The average cost of a successful search:**
  - Define α=n/m as ***load factor***,

$$\text{The average cost of a successful search is :}$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right)=1+\frac{1}{nm}\sum_{i=1}^{n}(n-i)=1+\frac{1}{nm}\sum_{i=1}^{n-1}i$$

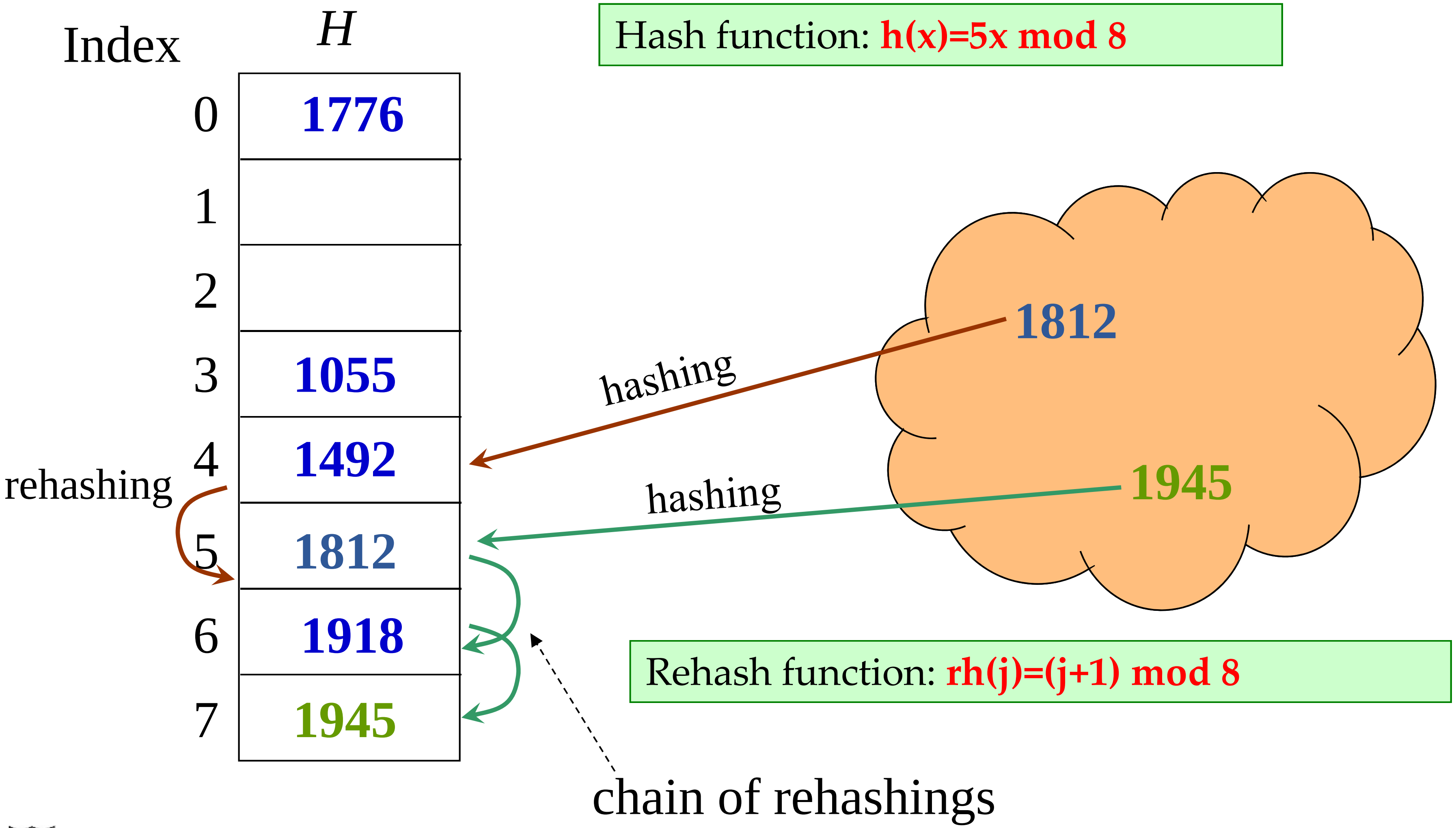
$$=1+\frac{n-1}{2m}=1+\frac{\alpha}{2}-\frac{\alpha}{2n}=\Theta(1+\alpha)$$

Number of elements in front of the searched one in the same linked list

# Collision Handling: Open Address

- **All elements are stored in the hash table**
  - No linked list is used
  - The load factor $\alpha$ cannot be larger than 1
- **Collision is settled by "rehashing"**
  - A function is used to get a new hashing address for each collided address
    - The hash table slots are *probed* successively, until a valid location is found.
- **The probe sequence can be seen as a permutation of (0,1,2,..., $m$-1)**

# Linear Probing: An Example



Hash function: **h(x)=5x mod 8**

Rehash function: **rh(j)=(j+1) mod 8**

# Commonly Used Probing

**Linear probing:**

Given an ordinary hash function $h'$, which is called an auxiliary hash function, the hash function is: **(clustering may occur)**

$$h(k,i) = (h'(k)+i) \bmod m \quad (i=0,1,\ldots,m-1)$$

**Quadratic Probing:**

Given auxiliary function $h'$ and nonzero auxiliary constant $c_1$ and $c_2$, the hash function is: **(secondary clustering may occur)**

$$h(k,i) = (h'(k)+c_1 i+ c_2 i^2) \bmod m \quad (i=0,1,\ldots,m-1)$$

**Double hashing:**

Given auxiliary functions $h_1$ and $h_2$, the hash function is:

$$h(k,i) = (h_1(k)+ ih_2(k)) \bmod m \quad (i=0,1,\ldots,m-1)$$

## Equally Likely Permutations

- **Assumption**
    - Each key is equally likely to have any of the $m!$ permutations of $(1,2...,m)$ as its probe sequence
- **Note**
    - Both linear and quadratic probing have only $m$ distinct probe sequence, as determined by the first probe

# Analysis for Open Address Hashing

- **The average number of probes in an unsuccessful search is at most 1/(1-α) (α=*n*/*m*<1)**
  - o Assuming uniform hashing

The probability of the first probed position being occupied is $\frac{n}{m}$, and that of the $j^{th}(j > 1)$ position occupied is $\frac{n-j+1}{m-j+1}$. So the probability of the number of probes no less than $i$ will be:

$$\frac{n}{m} \cdot \frac{n-1}{m-1} \cdot \frac{n-2}{m-2} \cdot \cdots \cdot \frac{n-i+2}{m-i+2} \leq (\frac{n}{m})^{i-1} = \alpha^{i-1}$$

The the average number of probe is: $\sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^{i} = \frac{1}{1-\alpha}$

See [CLRS] p.1199, C.25

# Analysis for Open Address Hashing

- **The average cost of probes in an successful search is at most** $\frac{1}{\alpha}\ln\frac{1}{1-\alpha}$ **(α=*n*/*m*<1)**
  - o Assuming uniform hashing

To search for the $(i+1)^{th}$ inserted element in the table, the cost is the same as that for inserting it when there are just $i$ elements in the table. At that time, $\alpha = \frac{i}{m}$. So the cost is $\frac{1}{1-\frac{i}{m}} = \frac{m}{m-i}$.
So the average cost for a successful search is:

$$\frac{1}{n}\sum_{i=0}^{n-1}\frac{m}{m-i} = \frac{m}{n}\sum_{i=0}^{n-1}\frac{1}{m-i} = \frac{1}{\alpha}\sum_{i=m-n+1}^{m}\frac{1}{i}$$
$$\leq \frac{1}{\alpha}\int_{m-n}^{m}\frac{dx}{x} = \frac{1}{\alpha}\ln\frac{m}{m-n} = \frac{1}{\alpha}\ln\frac{1}{1-\alpha}$$

For your reference:

Half full: 1.387;

90% full: 2.559

# Hash Function

- **A good hash function satisfies the assumption of simple uniform hashing**
  - Heuristic hashing functions
    - The division method: $h(k)=k \bmod m$
    - The multiplication method: $h(k)=\lfloor m(kA \bmod 1)\rfloor$ $(0<A<1)$
  - No single function can avoid the worst case $\Theta(n)$.
    - So "universal hashing" is proposed.
  - Rich resource about hashing function
    - Gonnet and Baeza-Yates: Handbook of Algorithms and Data Structures, Addison-Wesley, 1991.

## Array Doubling

- **Cost for search in a hash table is $\Theta(1+\alpha)$**
  - If we can keep $\alpha$ constant, the cost will be $\Theta(1)$
- **What if the hash table is more and more loaded?**
  - Space allocation techniques such as array doubling may be needed
- **The problem of "unusually expensive" individual operation**

# Looking at the Memory Allocation

```
hashingInsert(HASHTABLE $H$, ITEM $x$)
    integer $size$=0, $num$=0;
    if $size$=0 then allocate a block of size 1; $size$=1;
    if $num$=$size$ then
        allocate a block of size 2$size$;
        move all item into new table;
        $size$=2$size$;
    insert $x$ into the table;
    $num$=$num$+1;
return
```

Insertion with expansion: cost *size*

Elementary insertion: cost 1

# Worst-case Analysis

- **For *n* execution of insertion operations**
  - A bad analysis: the worst case for one insertion is the case when expansion is required, up to *n*
  - So, the worst case cost is in $O(n^2)$.

*Of course NOT!*

- **Note the expansion is required during the *i*th operation only if $i=2^k$, and the cost of the *i*th operation**

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is exactly the power of 2} \\ 1 & otherwise \end{cases}$$

So the total cost is: $\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \log n \rfloor} 2^j < n + 2n = 3n$

## Amortized Analysis – Why?

- **Unusually expensive operations**
  - E.g., Insert-with-array-doubling
- **Relation between expensive and usual operations**
  - Each piece of the doubling cost corresponds to some previous insert

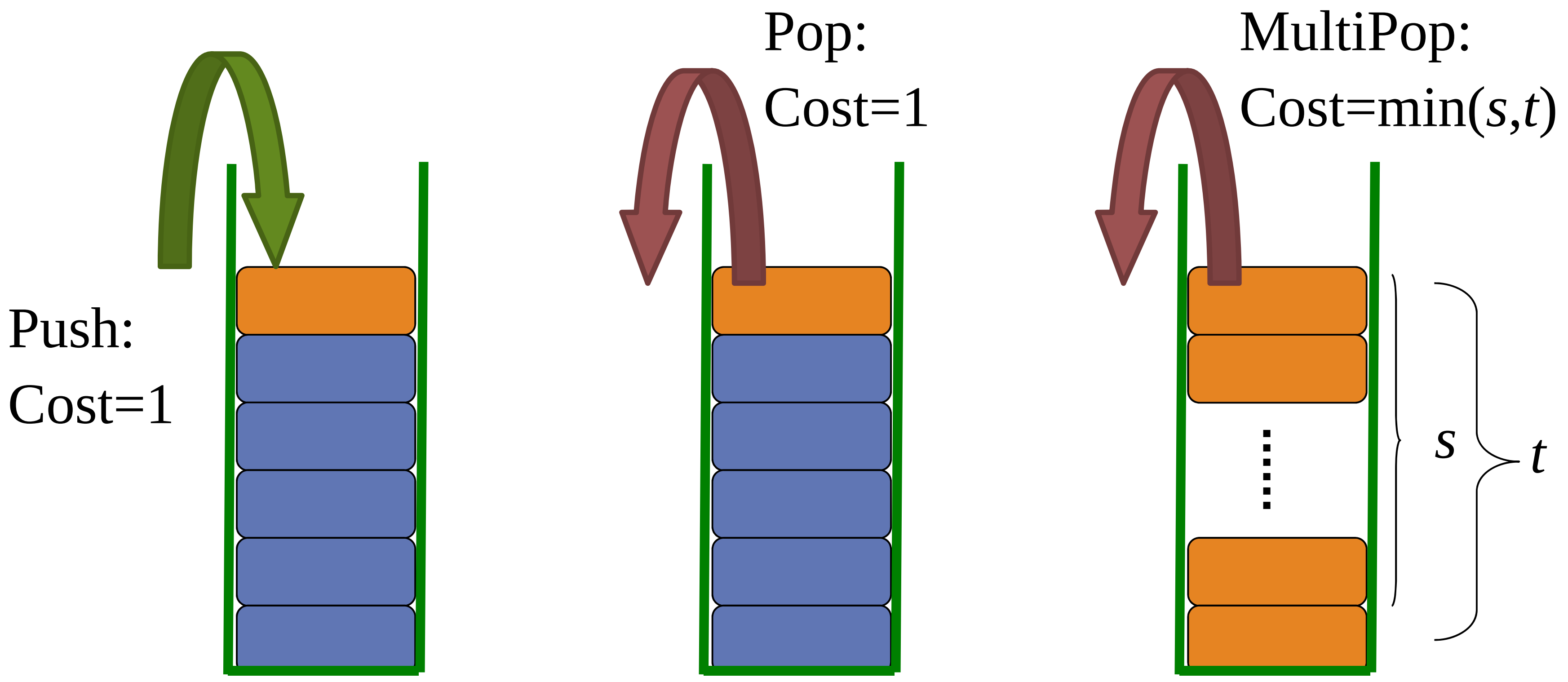
## Amortized Analysis

- **Amortized equation:**

  *amortized cost* = *actual cost* + *accounting cost*

- **Design goals for accounting cost**
  - In any legal sequence of operations, the sum of the accounting costs is nonnegative
  - The amortized cost of each operation is fairly regular, in spite of the wide fluctuate possible for the actual cost of individual operations

# Amortized Analysis: MultiPop Stack



Amortized cost: push:2; pop, multipop: 0

## Amortized Analysis: Binary Counter

| | | |
|---|---|---|
| 0 | 0 0 0 0 0 0 0 0 | 0 |
| 1 | 0 0 0 0 0 0 0 1 | 1 |
| 2 | 0 0 0 0 0 0 1 0 | 3 |
| 3 | 0 0 0 0 0 0 1 1 | 4 |
| 4 | 0 0 0 0 0 1 0 0 | 7 |
| 5 | 0 0 0 0 0 1 0 1 | 8 |
| 6 | 0 0 0 0 0 1 1 0 | 10 |
| 7 | 0 0 0 0 0 1 1 1 | 11 |
| 8 | 0 0 0 0 1 0 0 0 | 15 |
| 9 | 0 0 0 0 1 0 0 1 | 16 |
| 10 | 0 0 0 0 1 0 1 0 | 18 |
| 11 | 0 0 0 0 1 0 1 1 | 19 |
| 12 | 0 0 0 0 1 1 0 0 | 22 |
| 13 | 0 0 0 0 1 1 0 1 | 23 |
| 14 | 0 0 0 0 1 1 1 0 | 25 |
| 15 | 0 0 0 0 1 1 1 1 | 26 |
| 16 | 0 0 0 1 0 0 0 0 | 31 |

Cost measure: bit flip

amortized cost:

set 1: 2

set 0: 0

# Accounting Scheme for Stack Push

- **Push operation with array doubling**
  - No resize triggered: 1
  - Resize($n \to 2n$) triggered: $nt+1$ (t is a constant)
- **Accounting scheme (specifying accounting cost)**
  - No resize triggered: $2t$
  - Resize($n \to 2n$) triggered: $-nt+2t$
- **So, the amortized cost of each individual push operation is $1+2t \in \Theta(1)$**

*Thank you!*

*Q & A*

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