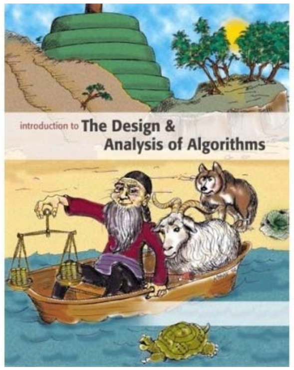
Introduction to

Algorithm Design and Analysis

[13] Undirected Graph



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In the Last Class…

- **Directed Acyclic Graph**
  - Topological Order
  - Critical Path Analysis

- **Strongly Connected Component**
  - Strong Component and Condensation
  - Finding SCC based on DFS

DFS on Undirected Graph

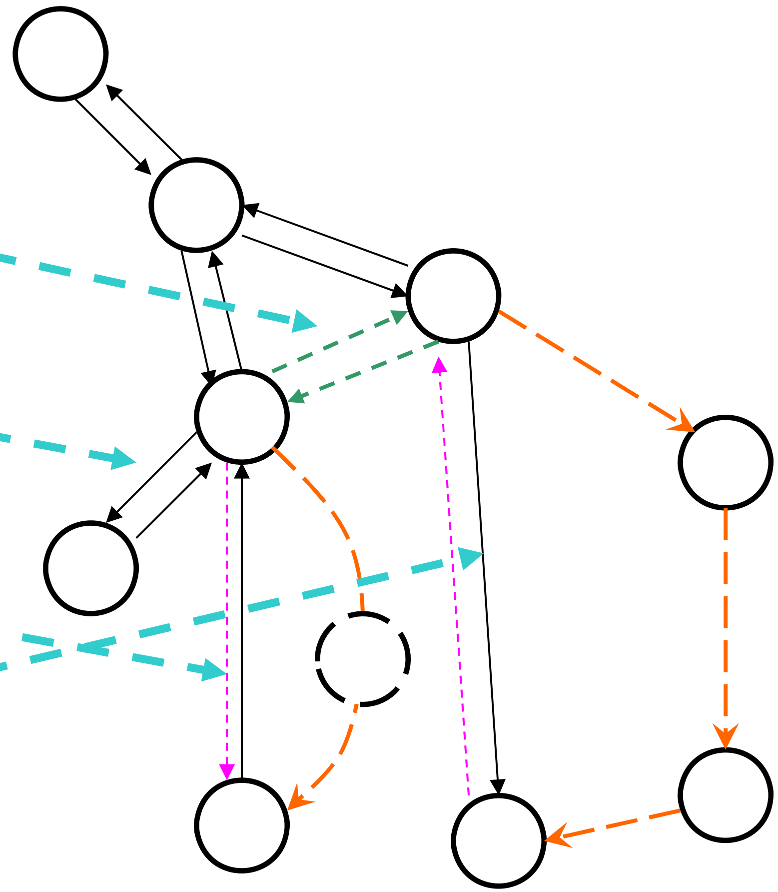
- **Undirected Graph**
  - Symmetric Digraph
  - Undirected Graph DFS Skeleton
- **Biconnected Components**
  - *Articulation Points* and 2-point connectedness
  - *Bridge* and 2-edge connectedness
- **Other undirected graph problems**
  - Orientation of an undirected graph
  - Warm up: Minimum Spanning Tree based on DFS

What is Different for "Undirected"

- **Characteristics of undirected graph traversal**
  - One edge may be traversed for two times in opposite directions.
- **For an undirected graph, DFS provides an orientation for each of its edges**
  - Oriented in the direction in which they are first encountered.

Edges in DFS

- **Cross edge**
  - Not existing
- **Back edge**
  - Back to the direct parent: second encounter
  - Otherwise: **first encounter**
- **Forward edge**
  - Always second encounter, ***and first time as back edge***



Modifications to the DFS Skeleton

- **All the second encounter are bypassed.**
- **So, the *only substantial modification* is for the possible back edges leading to an ancestor, but not direct parent.**
- **We need know the *parent*, that is, the direct ancestor, for the vertex to be processed.**

DFS Skeleton for Undirected Graph

- int dfsSweep(IntList[] *adjVertices*,int n, …)
- int ans;
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- if (color[v]==white)
- int vAns=dfs(*adjVertices*, color, v, -1, …);
- <Process vAns>
- // Continue loop
- return ans;

Recording the parent

DFS Skeleton for Undirected Graph

- int dfs(IntList[] *adjVertices*, int[] color, int v, int p, …)
- int w; IntList remAdj; int ans;
- color[v]=gray;
- <Preorder processing of vertex v>
- remAdj=*adjVertices*[v];
- while (remAdj≠nil)
- w=first(remAdj);
- if (color[w]==white)
- <Exploratory processing for tree edge vw>
- int wAns=dfs(*adjVertices*, color, w, v …);
- < Backtrack processing for tree edge vw , using wAns>
- else if (color[w]==gray && w≠p)
- <Checking for nontree edge vw>
- remAdj=rest(remAdj);
- <Postorder processing of vertex v, including final computation of ans>
- color[v]=black;
- return ans;

Complexity of Undirected DFS

- **$\Theta(m+n)$**
  - If each inserted statement for specialized application runs in constant time
  - The same with directed graph DFS
- **Extra space $\Theta(n)$**
  - For array *color*, or activation frames of recursion.

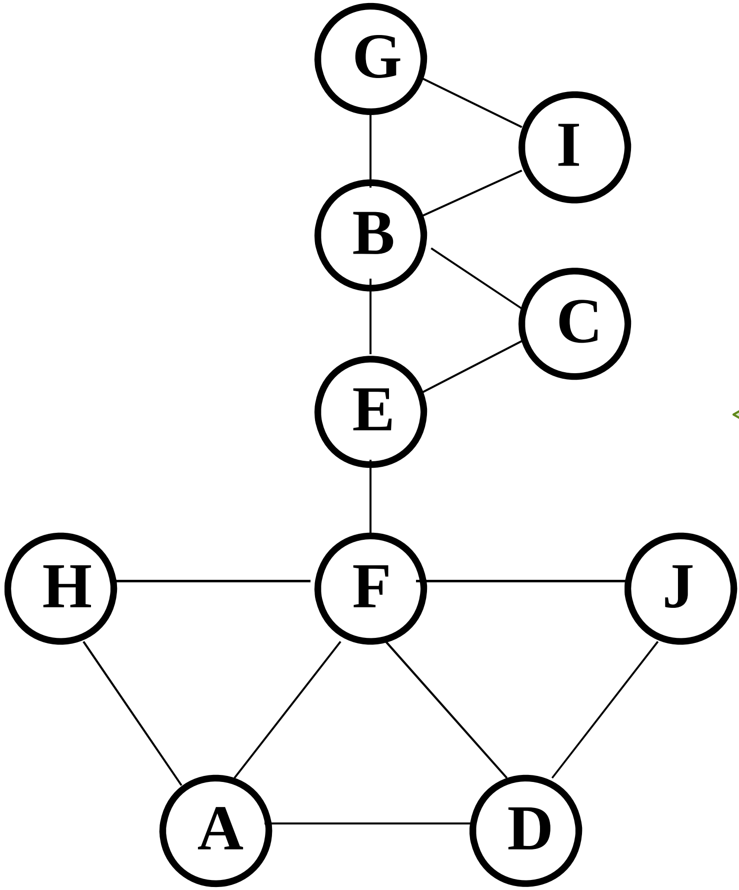
Biconnected Graph

- **Being connected**
  - Tree: acyclic, least (cost) connected
  - Node/edge connected: fault tolerant connectedness
- **Articulation point (2-node connected)**
  - v is an articulation point if deleting v leads to disconnection
- **Bridge (2-edge connected)**
  - uv is a bridge if deleting uv leads to disconnection

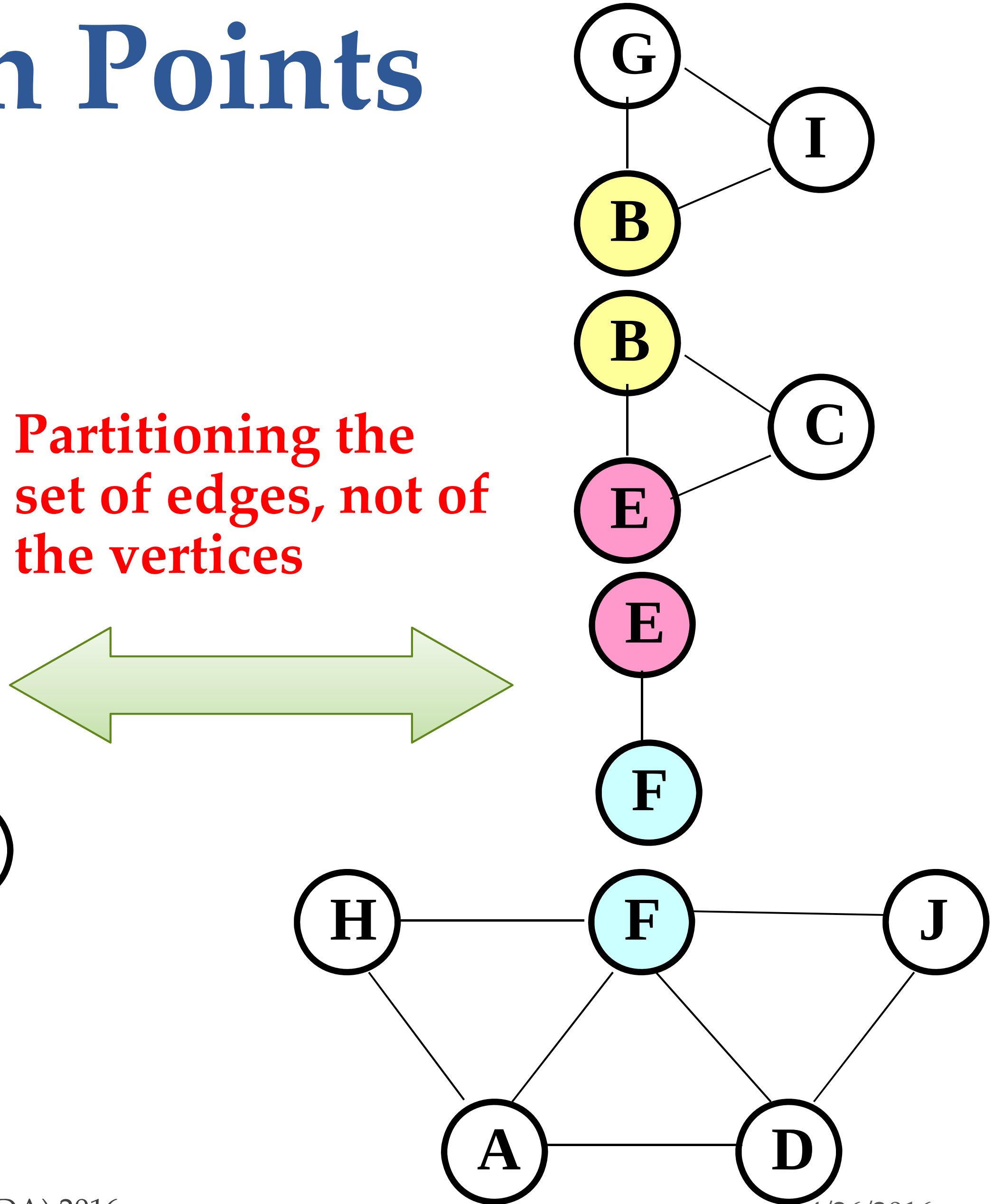
Biconnected Graph

- **Articulation point**
  - v is an articulation point if there **exist** nodes w and x, such that v is in **every** path from w to x (w and x are vertices different from v)
- **Bridge**
  - uv is a bridge if node u and v are connected only by edge uv
- **Bicomponent**
  - Maximal biconnected subgraph

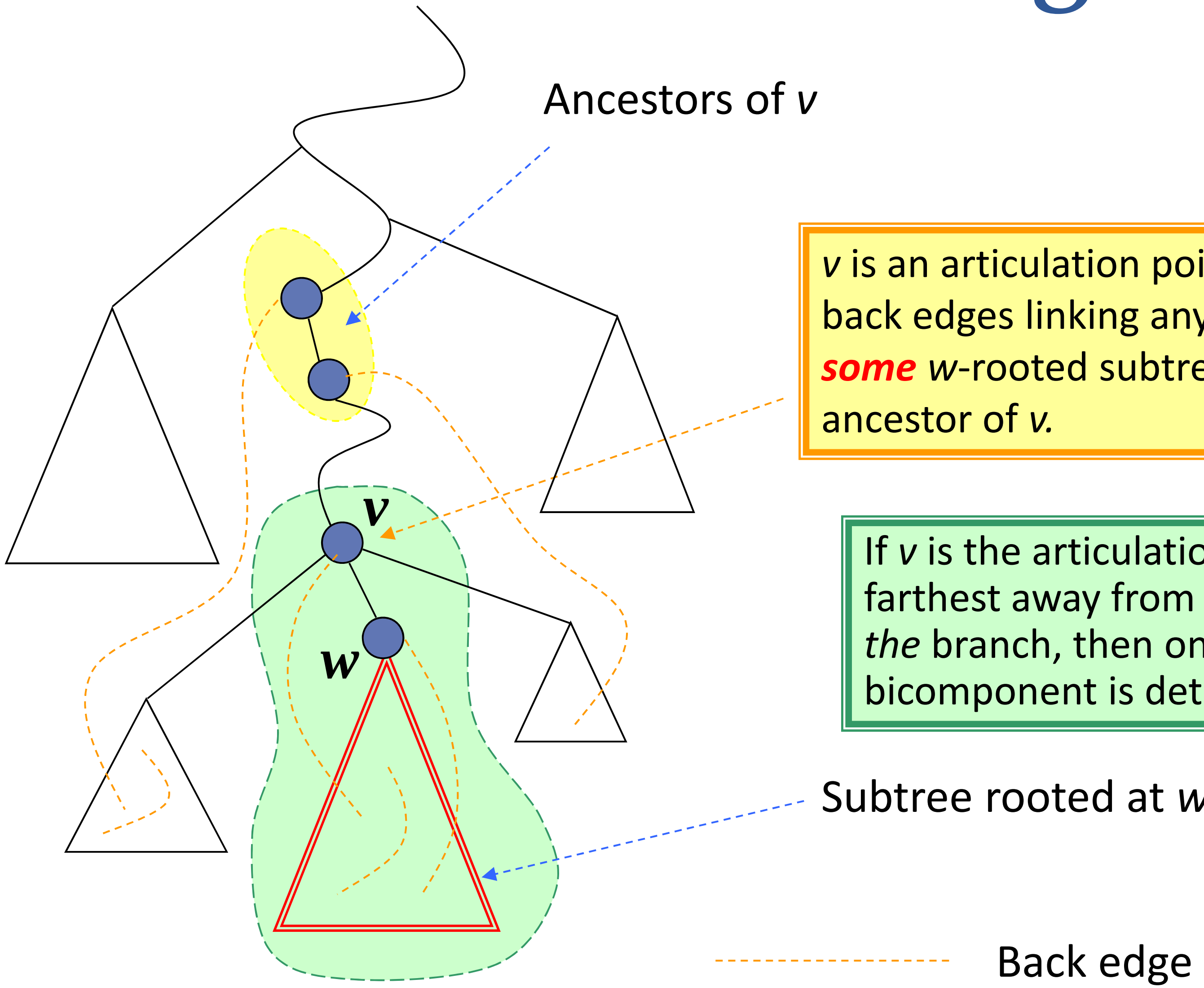
Articulation Points



Partitioning the set of edges, not of the vertices



Articulation Point Algorithm



v is an articulation point iff no back edges linking any vertex in ***some*** *w*-rooted subtree and any ancestor of *v.*

If *v* is the articulation point farthest away from the root on *the* branch, then one bicomponent is detected.

Updating the value of *back*

- v first discovered

  back=discoverTime(v)

- Trying to explore, but a back edge vw from v encountered

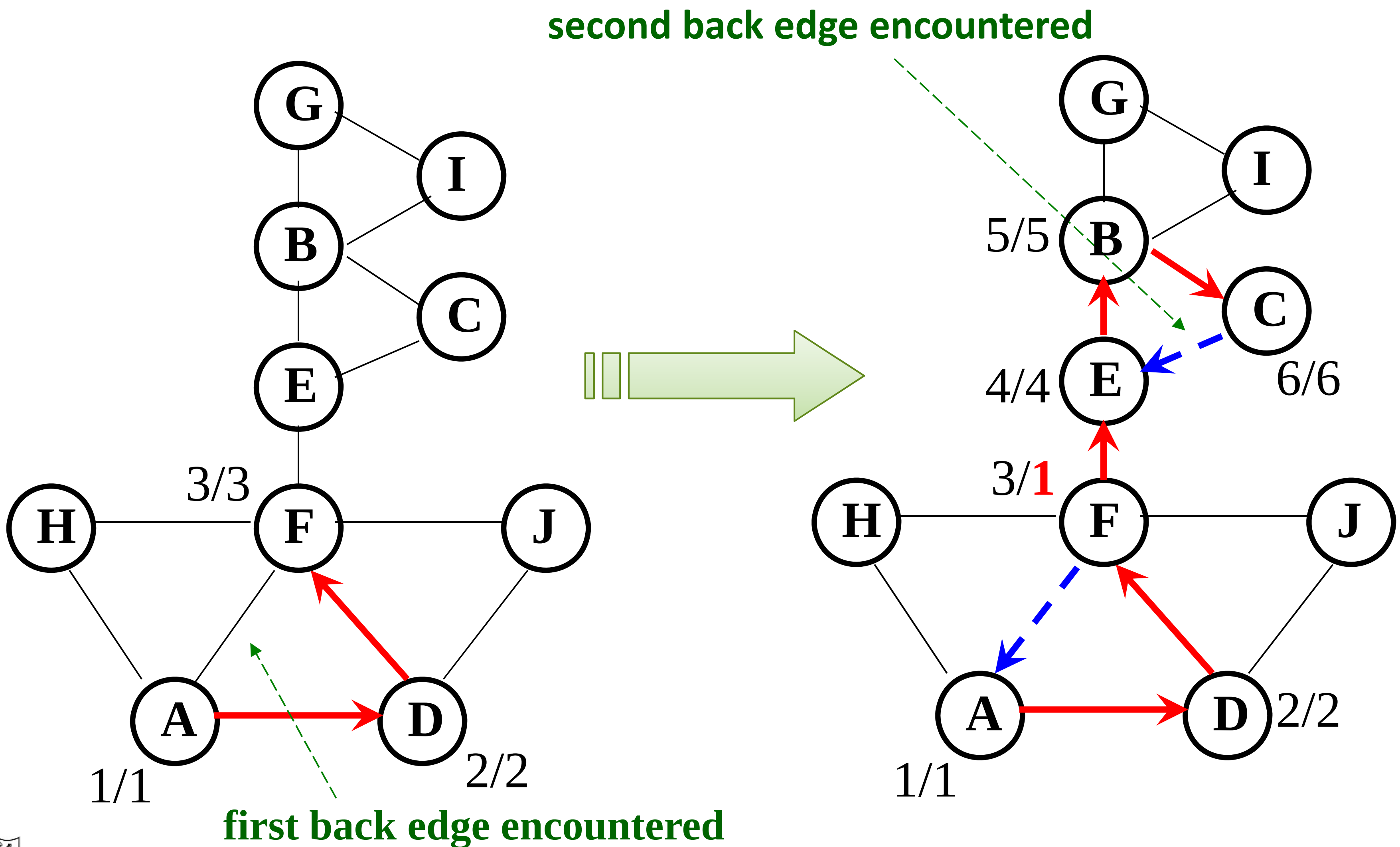
  ***back*=min(*back*, *discoverTime*(*w*))**

- *Backtracking from w to v*

  ***back*=min(*back*, *wback*)**

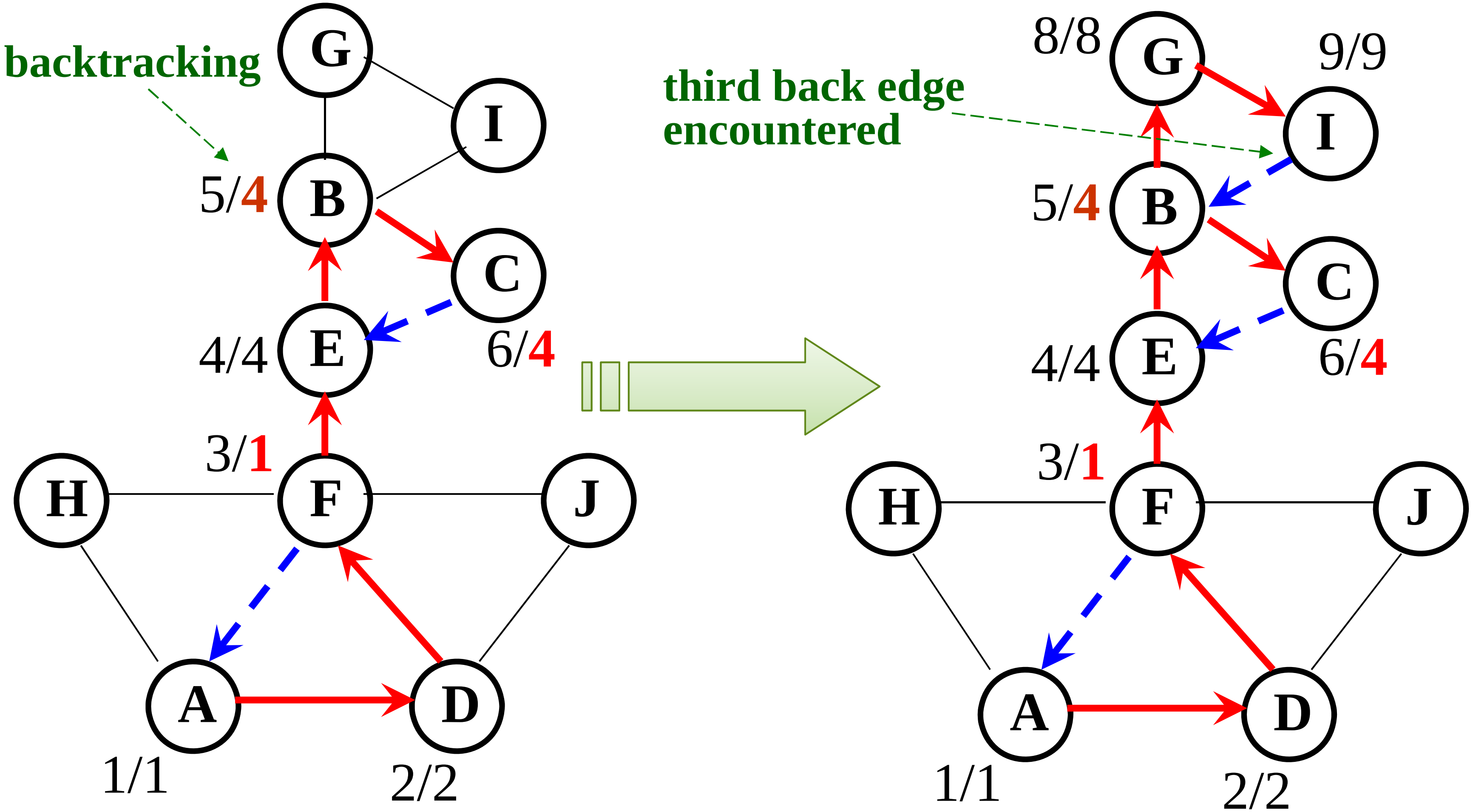
The back value of *v* is the smallest discover time a back edge "sees" from **any** subtree of *v.*

Example



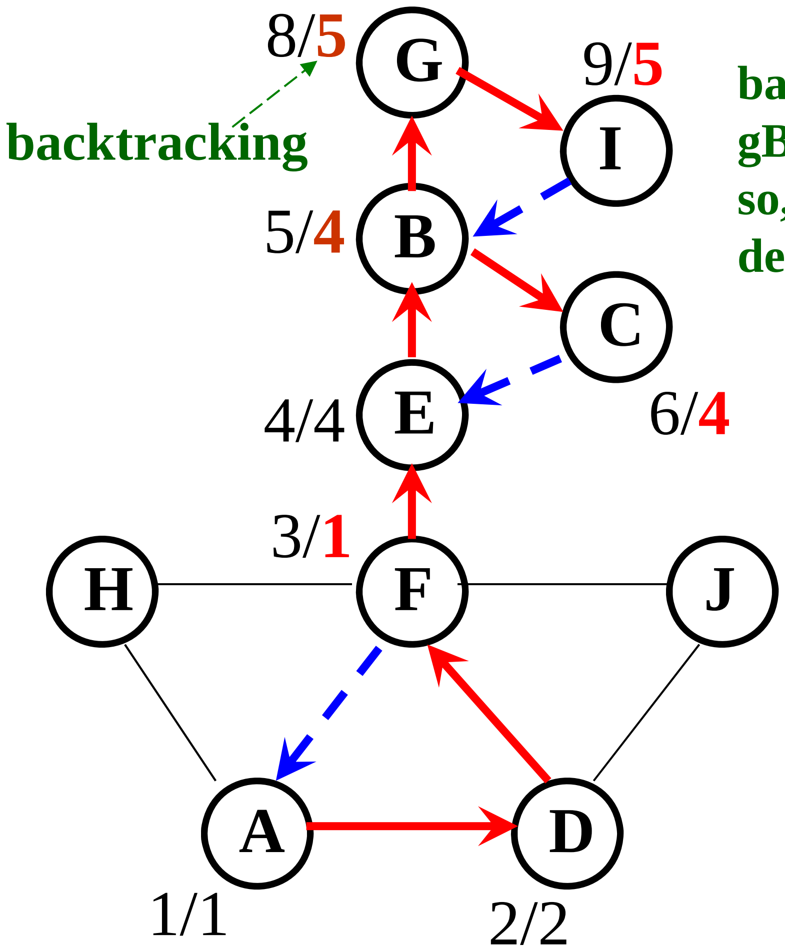
second back edge encountered
G
I
B
C
E
3/3
H
F
J
A
D
1/1
2/2
first back edge encountered
G
I
5/5
B
C
4/4
E
6/6
3/1
H
F
J
A
D
2/2
1/1

Example

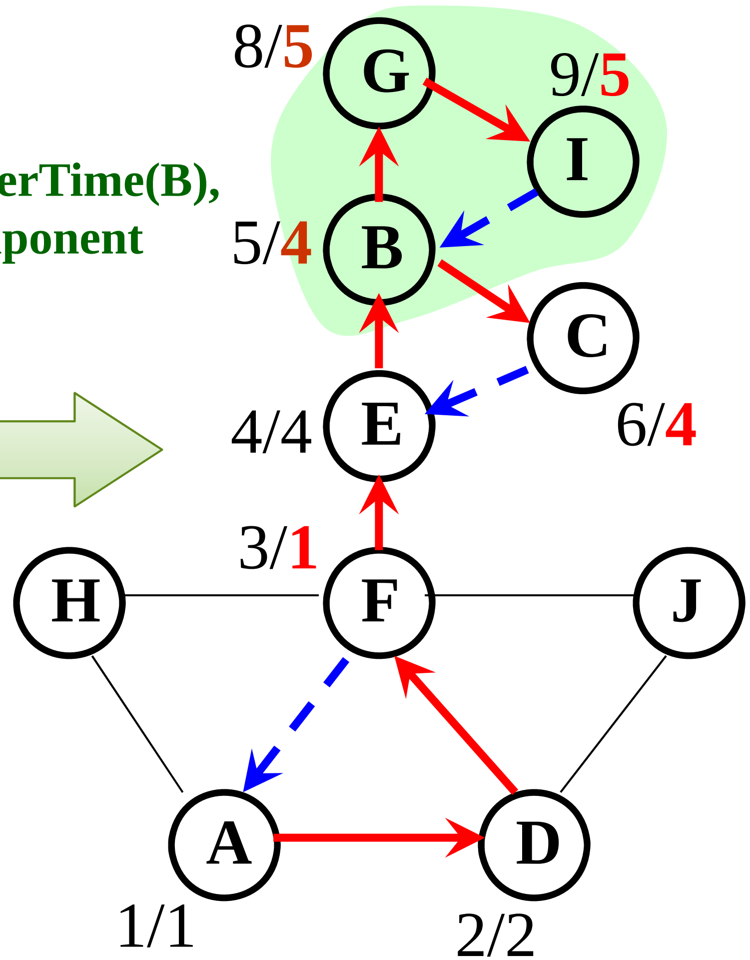


backtracking
G
I
5/4
B
C
4/4
E
6/4
3/1
H
F
J
A
D
1/1
2/2
8/8
G
9/9
third back edge
encountered
I
5/4
B
C
4/4
E
6/4
3/1
H
F
J
A
D
1/1
2/2

Example



backtracking:
gBack=discoverTime(B),
so, first bicomponent
detected.



Keeping the Track of Backing

- **Tracking data**
  - For each vertex *v*, a local variable *back* is used to store the required information, as the value of *discoverTime* of some vertex.
- **Testing for bicomponent**
  - At backtracking from *w* to *v*, the condition implying a bicomponent is:
    - $wBack \geq discoverTime(v)$

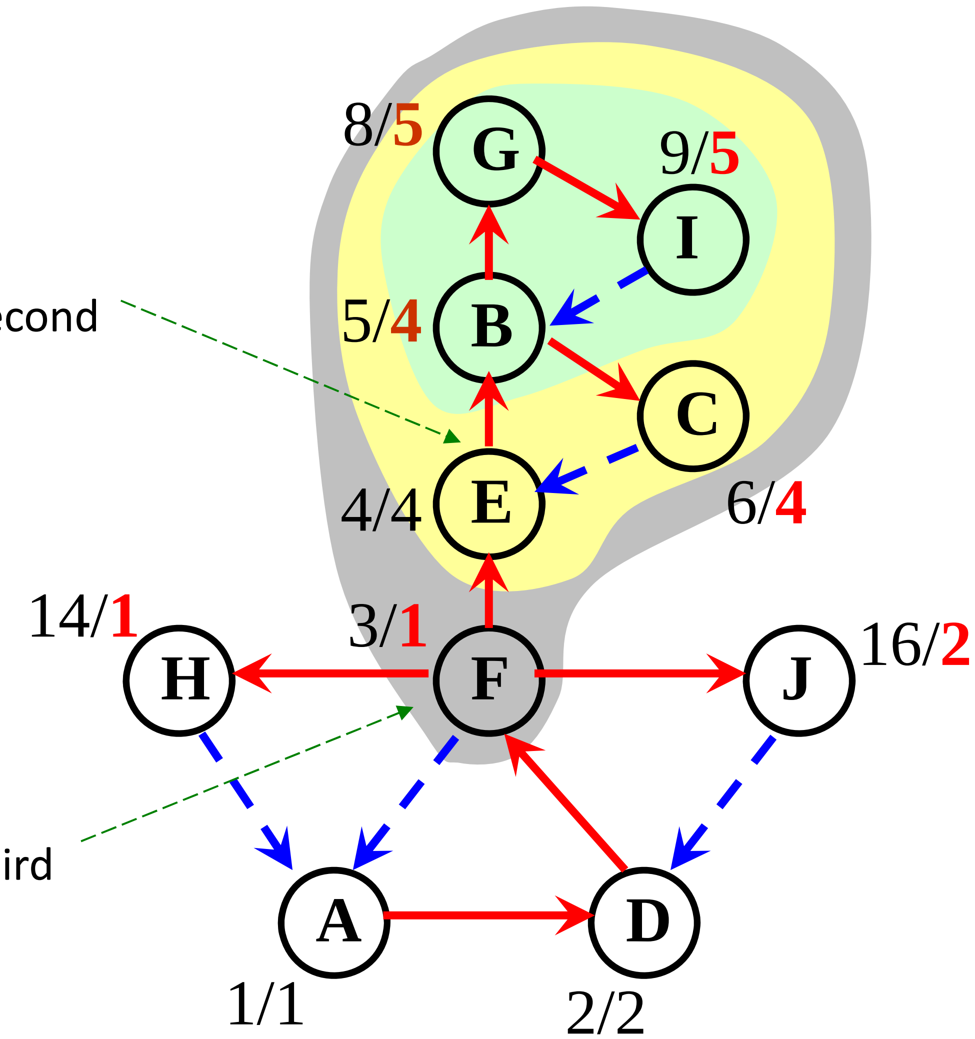
(where *wBack* is the returned back value for *w*)

when back is no less than the discover time of *v*, there is at least one subtree of *v* connected to other part of the graph only by *v*.

Example

Backtracking from B to E:
bBack=discoverTime(E), so, the second bicomponent is detect

Backtracking from E to F:
eBack>discoverTime(F), so, the third bicomponent is detect



Articulation Point Algorithm

- int bicompDFS(v)
- color[v]=gray; time++; discoverTime[v]=time;
- back=discoverTime[v];
- while (there is an untraversed edge vw)
- <push vw into edgeStack>
- if (vw is a tree edge)
- wBack=bicompDFS(w);
- if (wBack≥discoverTime[v])
- Output a new bicomponent
- by popping edgeStack down through vw ;
- back=min(back, wBack);
- else if (vw is a back edge)
- back=min(discoverTime[w], back);
- time++; finishTime[v]=time; color[v]=black;
- return back;

Outline of core procedure

Correctness

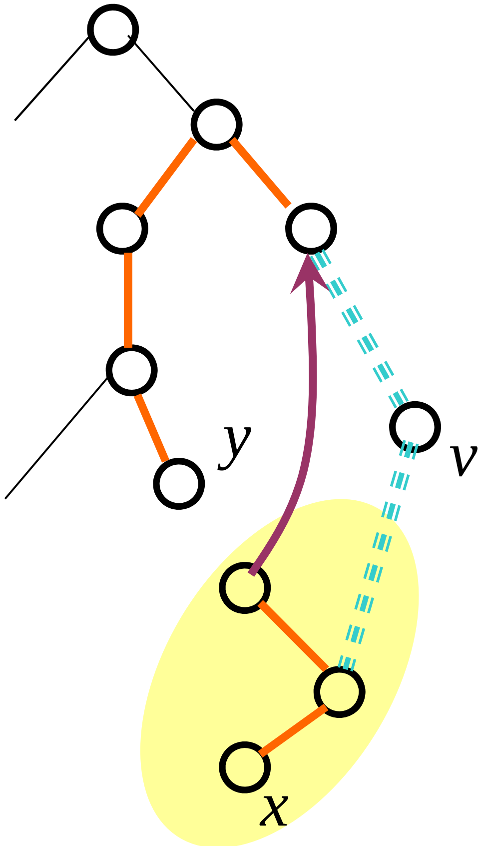
- **We have seen that:**
  - If v is the articulation point farthest away from the root on the branch, then one bicomponent is detected.
- **So, we need only prove that:**
  - In a DFS tree, a vertex(not root) v is an articulation point **if and only if** (1) v is not a leaf; (2) **some** subtree of v has **no back edge** incident with a proper ancestor of v.

Characteristics of Articulation Point

- **In a DFS tree, a vertex(not root) v is an articulation point if and only if (1)v is not a leaf; (2) some subtree of v has no back edge incident with a proper ancestor of v.**
- **$\Leftarrow$ Trivial**
- $\Rightarrow$
  - By definition, v is on **every** path between some x,y(different from v).
  - At least one of x,y is a proper descendent of v(otherwise, $x \leftrightarrow \text{root} \leftrightarrow y$ not containing v).
  - By contradiction, suppose that **every** subtree of v has a back edge to a proper ancestor of v, we can find a xy-path not containing v for all possible cases(only 2 cases)

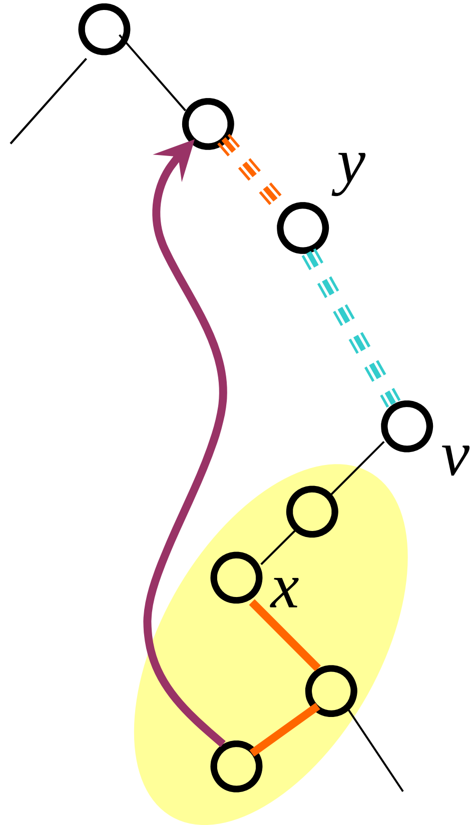
Case 1

suppose that **every** subtree of *v* has a back edge to a proper ancestor of *v*, and, exactly one of *x*, *y* is a descendant of *v*.



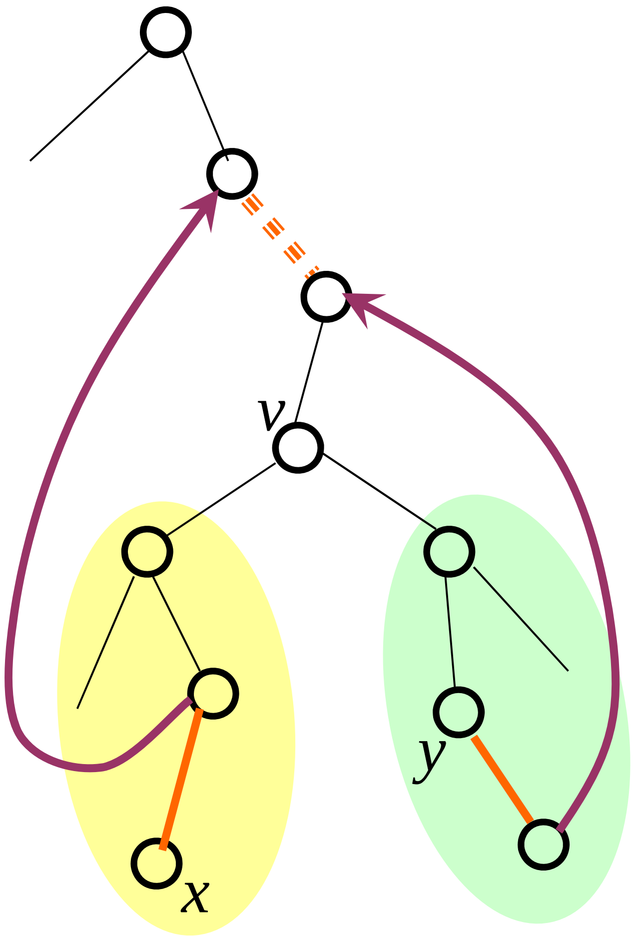
Case 1.1: another is not an ancestor of *v*

Case 1.2: another is an ancestor of *v*

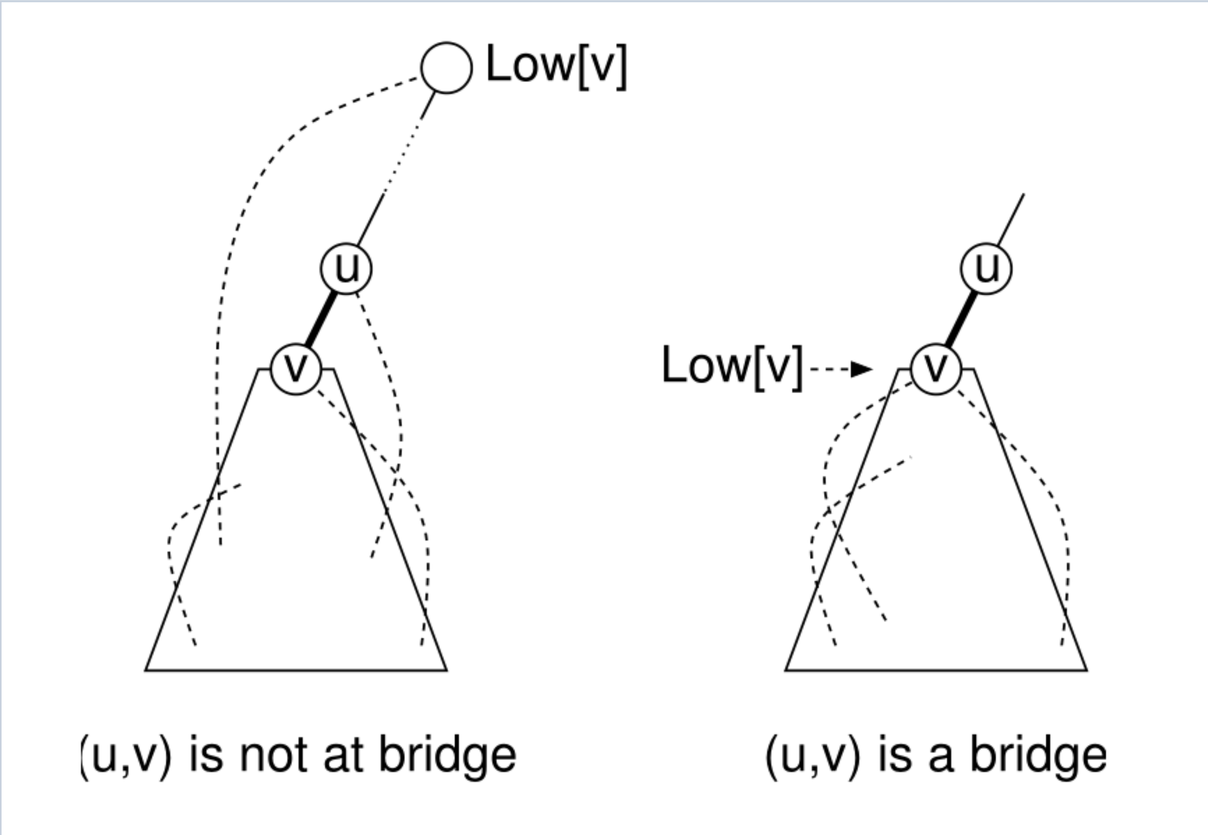


Case 2

suppose that **every** subtree of *v* has a back edge to a proper ancestor of *v*, and, both *x*, *y* are descendants of *v*.



Finding the Bridge



Finding the Bridge

- **Edge uv is a bridge iff**
    - (assuming that u is the parent and v is the child)

a) Edge uv is a tree edge in DFS

b) There is no subtree rooted at v to a proper ancestor of v (including u)

Why!

Edge Finding Algorithm

Bridge Finding by DFS

```
findBridges(u) {
  color[u] = gray;
  Low[u] = d[u] = ++time;          // set discovery time and init Low
  for each (v in Adj(u)) {
    if (color[v] == white) {       // (u,v) is a tree edge
      pred[v] = u;                 // v's parent is u
      findBridges(v);
      Low[u] = min(Low[u], Low[v]) // update Low[u]
      if (Low[v] > d[u]) Report that (u,v) is a bridge
    }
    else if (v != pred[u])        // (u,v) is a back edge
      Low[u] = min(Low[u], d[v])  // update Low[u]
  }
}
```

Orientation for Undirected Graphs

- **Orientation**
  - o Give each edge a direction
  - o Satisfying pre-specified constraints
    - E.g., the "in-degree of each vertex is at least 1"
- **Possible or not?**
  - o If possible, how to?
- **As for "in-degree ≥ 1"**
  - o Orientation possible iff. the graph has at least a circle
    - Find the end point of some back edge
    - A second DFS from this end point

Warm Up for MST

MST: Minimum Spanning Tree

- **Get MST in O(m+n) time**
  - Given that edges weights are only 1 and 2
- **Graph traversal is sufficient**
  - DFS over “weight 1 edges” only
  - DFS over “weight 2 edges” only

Thank you!

Q & A

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