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Mathematical Expectation

4.1 Mean of a Random Variable

- Let X be a random variable with probability distribution $f(x)$. The mean, or expected value, of X is

$$\mu = E(X) = \sum_x x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

if X is continuous.

- Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x) f(x)$$

if X is discrete,

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

if X is continuous.

- Let X and Y be random variables with joint probability distribution $f(x,y)$. The mean, or expected value of the random variable $g(X,Y)$ is

$$\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y)$$

if X and Y are discrete and

$$\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

if X and Y are continuous.

- If $g(X,Y) = X$, we have

$$E(X) = \begin{cases} \sum_x \sum_y x f(x,y) = \sum_x x g(x) & \text{(discrete)} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} x g(x)\, dx & \text{(continuous)} \end{cases}$$

similarly also If $h(X,Y) = Y$.

4.2 Variance and Covariance of Random Variables.

- Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x) \quad \text{if } X \text{ is discrete,}$$

$$\sigma^2 = E[(X-\mu^2)] = \int_{-\infty}^{\infty} (x-\mu^2) f(x)\, dx \quad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ, is called the standard deviation of X.

Theorem 4.2 The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2$$

Theorem 4.3 Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_{g(X)}]^2\} = \sum_x [g(x) - \mu_{g(X)}]^2 f(x)$$

if X is discrete, and

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_{g(X)}]^2\} = \int_{-\infty}^{\infty} [g(x) - \mu_{g(X)}]^2 f(x)\, dx.$$

if X is continuous.

- Let X and Y be random variables with joint probability distribution $f(x,y)$. The covariance of X and Y is

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y)$$

if X and Y are discrete, and

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\, dx\, dy$$

if X and Y are continuous.

Theorem 4.4 The covariance of two random variables X and Y with means μ_X and μ_Y, respectively, is given by

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

- Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y, respectively, the correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

The correlation coefficient satisfies the inequality $-1 \le \rho_{XY} \le 1$.

4.3 Means and Variances of Linear Combinations of Random Variables.

Theorem 4.5 If a and b are constants, then

$$E(aX+b) = aE(X) + b.$$

- Setting $a=0$, we see that $E(b) = b$
- setting $b=0$, we see that $E(aX) = aE(X)$

Theorem 4.6 The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]$$

Theorem 4.7.

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

- Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$, we see that

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)]$$

- setting $g(X,Y) = X$ and $h(X,Y) = Y$, we see that

$$E[X \pm Y] = E[X] \pm E[Y].$$

Theorem 4.8 Let X and Y be two independent random variables. Then,

$$E(XY) = E(X)E(Y).$$

- Let X and Y be two independent random variables. Then $\sigma_{XY} = 0$.

Theorem 4.9 If X and Y are random variables with joint probability distribution f(x,y) and a, b, c, are constants, then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\,\sigma_{XY}$$

- setting $b=0$, we see that

$$\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2$$

- setting $a=1$ and $b=0$, we see that

$$\sigma^2_{X+c} = \sigma^2_X = \sigma^2$$

- setting $b=0$ and $c=0$, we see that

$$\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2$$

- If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

- If X and Y are independent random variables, then

$$\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

- If $X_1, X_2, \dots, X_n$ are independent random variables, then

$$\sigma^2_{a_1X_1+a_2X_2+\dots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \dots + a_n^2\sigma^2_{X_n}$$

4.4 Chebyshev's Theorem

Theorem 4.10 (Chebyshev's Theorem).

The probability that any random variable X will assume a value within k standard deviations of the mean is at least $1-1/k^2$. that is.

$$P(\mu-k\sigma < X < \mu+k\sigma) \geq 1-\frac{1}{k^2}$$