Ellipse-Ellipse Intersection Detection

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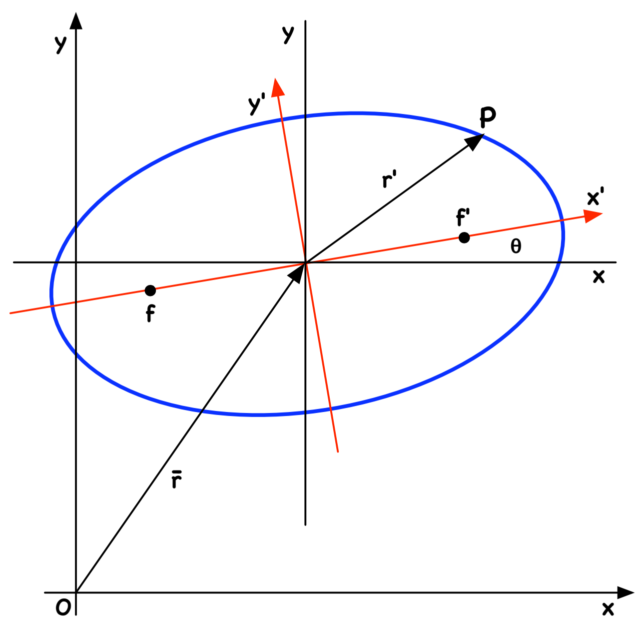
December 13, 2008

Abstract

The goal of this essay is to describe a method to detect intersections between two arbitrary ellipses on the same plane.

1 Describing an arbitrary ellipse

Assume that we have a global orthogonal coordinate system Oxy for the plane containing our ellipses. An arbitrary ellipse may have its center located somewhere other than the center of the global coordinate system and it may have its major axes rotated with respect to the global coordinate system's axes, as illustrated below.



The figure defines several quantities:

- $\bar{r}$, the position vector of the center of the ellipse, with respect to the center of the coordinate system;
- θ, the angle by which the ellipse's own coordinate system is rotated with respect to the global coordinate system's axes;
- f and f', the two foci of the ellipse; and
- $\vec{r}\,'$, the position vector of an arbitrary point P on the ellipse, with respect to the ellipse's center.

In terms of the ellipse's own coordinate system, a point $P = (x', y')$ on the ellipse is described by the equation

$$\frac{(x')^2}{a^2} + \frac{(y')^2}{b^2} = 1$$

where a and b are, respectively, the lengths of the semi-major and semi-minor axes.

This is equivalent to the parameterized equations

$$\begin{aligned} x' &= a\cos\alpha \\ y' &= b\sin\alpha \end{aligned}$$

where $0 \leq \alpha < 2\pi$. With respect to a coordinate system centered at the center of the ellipse, but with axes parallel to those of the global coordinate system, we have:

$$\begin{aligned} P_x &= x'\cos\theta - y'\sin\theta \\ P_y &= y'\cos\theta + x'\sin\theta \end{aligned}$$

Finally, with respect to the global coordinate system, P's coordinates are a simple translation away:

$$\begin{aligned} x &= \bar{x} + a\cos\alpha\cos\theta - b\sin\alpha\sin\theta \\ y &= \bar{y} + b\sin\alpha\cos\theta + a\cos\alpha\sin\theta \end{aligned}$$

where $\bar{x}$ and $\bar{y}$ are the coordinates of the ellipse's center, with respect to the global coordinate system. These two equations give us, then, a parametric representation of an arbitrary ellipse, with respect to our global coordinate system.

1.1 Foci and bounding rectangle

A well-known property of an ellipse is that the sum of the distances from any point on the ellipse to each of its foci is a constant, equal to twice the length of the ellipse's semi-major axis:

$$|P - f| + |P - f'| = 2a\,.$$

What are the coordinates of the two foci of an arbitrary ellipse, with respect to the global coordinate system?

With respect to the ellipse's own coordinate system, they are $x' = \pm\,\epsilon\, a$ and $y' = 0$, where ϵ is the ellipse's ***excentricity***, defined by

$$\epsilon \equiv \sqrt{1 - \frac{b^2}{a^2}}\,.$$

Therefore, with respect to the global coordinate system, the foci have coordinates:

$$\begin{aligned} x &= \bar{x} \pm \epsilon a \cos\theta \\ y &= \bar{y} \pm \epsilon a \sin\theta \end{aligned}$$

What about the ellipse's bounding rectangle? The extrema of the x and y coordinates of a point on the ellipse are easily determined from the solutions to the equations $dx/d\alpha = 0$ and $dy/d\alpha = 0$:

$$\begin{aligned} dx/d\alpha = 0 \;\Rightarrow\; a \sin\alpha \cos\theta + b \cos\alpha \sin\theta = 0 \;\Rightarrow\; \tan\alpha &= -(b/a) \tan\theta \\ dy/d\alpha = 0 \;\Rightarrow\; b \cos\alpha \cos\theta - a \sin\alpha \sin\theta = 0 \;\Rightarrow\; \tan\alpha &= (b/a) / \tan\theta \end{aligned}$$

Thus,

$$\begin{aligned} dx/d\alpha = 0 \;\Rightarrow\; \sin\alpha &= \frac{\mp (b/a) \tan\theta}{\sqrt{1 + (b/a)^2 \tan^2\theta}} \quad \text{and} \quad \cos\alpha = \frac{\pm 1}{\sqrt{1 + (b/a)^2 \tan^2\theta}} \\ dy/d\alpha = 0 \;\Rightarrow\; \sin\alpha &= \frac{\pm b/a}{\sqrt{(b/a)^2 + \tan^2\theta}} \quad \text{and} \quad \cos\alpha = \frac{\pm \tan\theta}{\sqrt{(b/a)^2 + \tan^2\theta}} \end{aligned}$$

From these, it's easy to show that the extreme values of x and y, which are the values that bound the ellipse in a rectangle, are given by:

$$\begin{aligned} x &= \bar{x} \pm \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta} \\ y &= \bar{y} \pm \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta} \end{aligned}$$

2 Detecting intersections between two ellipses

Imagine now that we have two ellipses on the xy plane, characterized by the two sets of parameters $\{\bar{x}_1, \bar{y}_1, a_1, b_1, \theta_1\}$ and $\{\bar{x}_2, \bar{y}_2, a_2, b_2, \theta_2\}$. In order to determine whether or not they intersect one another and, if so, where they do, we'll make use of the fact that a point P is inside of, or on, ellipse 2 if and only if

$$|P - f_2| + |P - f_2'| \leq 2a_2 \,.$$

The idea, then, is to test for each point P on ellipse 1 whether or not it's inside or on ellipse 2. Before carrying out that test, however, let's rewrite the condition above in a way that does not involve square-roots. By rearranging and twice-squaring, we get

$$8\, a_2^2\, |P - f_2|^2 + 8\, a_2^2\, |P - f_2'|^2 + 2\, |P - f_2|^2\, |P - f_2'|^2 \leq 16\, a_2^4 + |P - f_2'|^4 + |P - f_2|^4 \,.$$

In terms of the quantities $\Delta \equiv |P - f_2|^2$ and $\Delta' \equiv |P - f_2'|^2$, the above condition can be written as

$$8\, a_2^2\, (\Delta + \Delta') \leq 16\, a_2^4 + (\Delta - \Delta')^2 \,.$$

Now, what are Δ and Δ'? From their definitions, and for P on ellipse 1, we find:

$$\begin{aligned}
\Delta &= (\bar{x}_1 - \bar{x}_2 \;+\; a_1 \cos\alpha_1 \cos\theta_1 \;-\; b_1 \sin\alpha_1 \sin\theta_1 \;+\; \epsilon_2\, a_2 \cos\theta_2)^2 \\
&\quad + (\bar{y}_1 - \bar{y}_2 \;+\; b_1 \sin\alpha_1 \cos\theta_1 \;+\; a_1 \cos\alpha_1 \sin\theta_1 \;+\; \epsilon_2\, a_2 \sin\theta_2)^2 \\
\Delta' &= (\bar{x}_1 - \bar{x}_2 \;+\; a_1 \cos\alpha_1 \cos\theta_1 \;-\; b_1 \sin\alpha_1 \sin\theta_1 \;-\; \epsilon_2\, a_2 \cos\theta_2)^2 \\
&\quad + (\bar{y}_1 - \bar{y}_2 \;+\; b_1 \sin\alpha_1 \cos\theta_1 \;+\; a_1 \cos\alpha_1 \sin\theta_1 \;-\; \epsilon_2\, a_2 \sin\theta_2)^2
\end{aligned}$$

In terms of the quantities

$$\begin{aligned}
X &\equiv \bar{x}_1 - \bar{x}_2 \;+\; a_1 \cos\alpha_1 \cos\theta_1 \;-\; b_1 \sin\alpha_1 \sin\theta_1 \\
Y &\equiv \bar{y}_1 - \bar{y}_2 \;+\; b_1 \sin\alpha_1 \cos\theta_1 \;+\; a_1 \cos\alpha_1 \sin\theta_1
\end{aligned}$$

we have

$$\begin{aligned}
\Delta &= (X + \epsilon_2\, a_2 \cos\theta_2)^2 + (Y + \epsilon_2\, a_2 \sin\theta_2)^2 \\
\Delta' &= (X - \epsilon_2\, a_2 \cos\theta_2)^2 + (Y - \epsilon_2\, a_2 \sin\theta_2)^2 \,.
\end{aligned}$$

These can be simplified to

$$\begin{aligned}
\Delta &= X^2 + Y^2 + \epsilon_2^2\, a_2^2 + 2\epsilon_2\, a_2\, (X \cos\theta_2 + Y \sin\theta_2) \\
\Delta' &= X^2 + Y^2 + \epsilon_2^2\, a_2^2 - 2\epsilon_2\, a_2\, (X \cos\theta_2 + Y \sin\theta_2)
\end{aligned}$$

from which we obtain

$$\begin{aligned}
8\, a_2^2\, (\Delta + \Delta') &= 16\, a_2^2 \left(X^2 + Y^2 + \epsilon_2^2\, a_2^2\right) \\
(\Delta - \Delta')^2 &= 16\, \epsilon_2^2\, a_2^2 \left(X \cos\theta_2 + Y \sin\theta_2\right)^2 .
\end{aligned}$$

Thus, the condition

$$8\, a_2^2\, (\Delta + \Delta') \leq 16\, a_2^4 + (\Delta - \Delta')^2$$

amounts to

$$X^2 + Y^2 - \epsilon_2^2 \left(X \cos\theta_2 + Y \sin\theta_2\right)^2 - b_2^2 \leq 0 \,.$$

To recap, when the condition above is satisfied, a point P on ellipse 1 (corresponding to some value of the parameter α_1) will also be inside ellipse 2 (if the condition is strictly negative) or on ellipse 2 (if the condition equals zero).

Note that every quantity appearing in the condition above is known and is fixed, except for α_1. Thus, numerically solving the equality for α_1 gives us the point or points where the two ellipses intersect, if they intersect at all.

It could be the case, however, that ellipse 1 is entirely inside ellipse 2, in which case there's no intersection. That situation is easily detected because the condition will be strictly negative for any value of α_1. If the condition is strictly positive for any value of α_1, then no intersection exists and ellipse 1 is not inside ellipse 2. By exchanging the ellipses and testing again, we can determine if ellipse 2 is entirely inside of ellipse 1, or not.

In any case, if a non-degenerate intersection exists (that is, one where the two ellipses do not coincide), there will be exactly one or two solutions for α_1, from which the intersection points can be determined. Once the intersection points have been determined, it's also relatively easy to compute unit vectors normal to each ellipse at each intersection point.

3 Tangent and normal unit vectors

Given the parametric equations of an arbitrary ellipse,

$$\begin{aligned} x &= \bar{x} + a\cos\alpha\cos\theta - b\sin\alpha\sin\theta \\ y &= \bar{y} + b\sin\alpha\cos\theta + a\cos\alpha\sin\theta \end{aligned}$$

it's easy to obtain the unit vector tangent to the curve at a given point. The vector with coordinates $(dx/d\alpha, dy/d\alpha)$ is tangent to the curve at the point parameterized by α:

$$\begin{aligned} v_x &= -a\sin\alpha\cos\theta - b\cos\alpha\sin\theta \\ v_y &= b\cos\alpha\cos\theta - a\sin\alpha\sin\theta \end{aligned}$$

Since its magnitude squared is

$$v_x^2 + v_y^2 = a^2\sin^2\alpha + b^2\cos^2\alpha$$

it's easy to obtain a unit vector tangent to the ellipse at the point parameterized by the angle α.

How about a unit vector normal to the curve at that same point? The vector

$$\vec{v} \times \hat{z} = (v_y, -v_x)$$

where $\times$ represents the vector cross product operation and $\hat{z}$ is the unit vector along the z-direction, is guaranteed to be normal to the curve, pointing outwards from the interior of the curve, and is also perpendicular to the tangent vector. However, it's not normalized. Its magnitude is clearly the magnitude of $\vec{v}$.

4 Ellipse-ellipse intersection detection in practice

Armed with the results derived above, we can use the following algorithm for detecting intersections between two arbitrary ellipses on the same plane.

- Compute the bounding rectangles of both ellipses.
- Perform a simple rectangle-rectangle intersection test. If the two bounding rectangles do not intersect, then the two ellipses do not intersect.
- If the two bounding rectangles intersect, then perform a detailed ellipse-ellipse intersection test by testing whether the condition

  $$X^2 + Y^2 - \epsilon_2^2\left(X\cos\theta_2 + Y\sin\theta_2\right)^2 - b_2^2 \leq 0$$

  is satisfied for any value of α_1.

- If the two ellipses intersect, compute the coordinates of the intersection point or points by using in the equations

$$
\begin{aligned}
x &= \bar{x}_1 + a_1 \cos\alpha_1 \cos\theta_1 - b_1 \sin\alpha_1 \sin\theta_1 \\
y &= \bar{y}_1 + b_1 \sin\alpha_1 \cos\theta_1 + a_1 \cos\alpha_1 \sin\theta_1
\end{aligned}
$$

  the value or values of α_1 for which the condition above is an equality, where

$$
\begin{aligned}
X &\equiv \bar{x}_1 - \bar{x}_2 + a_1 \cos\alpha_1 \cos\theta_1 - b_1 \sin\alpha_1 \sin\theta_1 \\
Y &\equiv \bar{y}_1 - \bar{y}_2 + b_1 \sin\alpha_1 \cos\theta_1 + a_1 \cos\alpha_1 \sin\theta_1
\end{aligned}
$$

  Note that $x = X + \bar{x}_2$ and $y = Y + \bar{y}_2$.

- Optionally compute normal and tangent vectors at the intersection point or points.

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