An explicit solution for the B-spline of degree 4

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October 15, 2004

Let $\mathcal{T} = \{t_0, t_1, \ldots, t_m\}$ be a non-decreasing sequence of real numbers such that $t_i \in [0,1]$ for $0 \leq i \leq m$ and let $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_n\}$ be a set of *control points*. Then, a *B-spline* curve of order $p \equiv m - n - 1$ is the parameterized curve defined by

$$\mathcal{C}(t) \equiv \sum_{i\,=\,0}^{n} \mathcal{P}_i \, N_{i,p}(t)$$

where $N_{i,p}(t)$ is defined by the recursive relation

$$N_{i,p}(t) = \frac{(t - t_i)}{(t_{i+p} - t_i)} \, N_{i,p-1}(t) + \frac{(t_{i+p+1} - t)}{(t_{i+p+1} - t_{i+1})} \, N_{i+1,p-1}(t) \qquad (p \geq 1, \quad i \geq 0),$$

with the base case

$$N_{i,0}(t) = \begin{cases} 1, & \text{if } t_i \leq t < t_{i+1} \text{ and } t_i < t_{i+1}\,; \\ 0, & \text{otherwise.} \end{cases} \qquad (i \geq 0)$$

Our task is to obtain a closed-form solution for $\mathcal{C}(t)$, for $p = 4$. That amounts to obtaining every $N_{i,4}(t)$ in terms of the $N_{i,0}(t)$'s, for $i \geq 0$.

$$p = 1: \quad N_{i,1}(t) \; = \; \frac{(t - t_i)}{(t_{i+1} - t_i)} \, N_{i,0}(t) + \frac{(t_{i+2} - t)}{(t_{i+2} - t_{i+1})} \, N_{i+1,0}(t)$$

$$
\begin{aligned}
p=2: \qquad & N_{i,2}(t) = \frac{(t-t_i)}{(t_{i+2}-t_i)}\,N_{i,1}(t) + \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+1})}\,N_{i+1,1}(t) \\
\text{from } p=1: \quad & N_{i+1,1}(t) = \frac{(t-t_{i+1})}{(t_{i+2}-t_{i+1})}\,N_{i+1,0}(t) + \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+2})}\,N_{i+2,0}(t) \\
\text{so}\ldots \qquad & N_{i,2}(t) = \frac{(t-t_i)}{(t_{i+2}-t_i)}\,\frac{(t-t_i)}{(t_{i+1}-t_i)}\,N_{i,0}(t) \\
& \quad + \left[\frac{(t-t_i)}{(t_{i+2}-t_i)}\,\frac{(t_{i+2}-t)}{(t_{i+2}-t_{i+1})} + \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+1})}\,\frac{(t-t_{i+1})}{(t_{i+2}-t_{i+1})}\right] N_{i+1,0}(t) \\
& \quad + \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+1})}\,\frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+2})}\,N_{i+2,0}(t)
\end{aligned}
$$

$$
\begin{aligned}
p=3: \qquad & N_{i,3}(t) = \frac{(t-t_i)}{(t_{i+3}-t_i)}\,N_{i,2}(t) + \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})}\,N_{i+1,2}(t) \\
\text{from } p=2: \quad & N_{i+1,2}(t) = \frac{(t-t_{i+1})}{(t_{i+3}-t_{i+1})}\,\frac{(t-t_{i+1})}{(t_{i+2}-t_{i+1})}\,N_{i+1,0}(t) \\
& \quad + \left[\frac{(t-t_{i+1})}{(t_{i+3}-t_{i+1})}\,\frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+2})} + \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+2})}\,\frac{(t-t_{i+2})}{(t_{i+3}-t_{i+2})}\right] N_{i+2,0}(t) \\
& \quad + \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+2})}\,\frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+3})}\,N_{i+3,0}(t) \\
\text{so}\ldots \qquad & N_{i,3}(t) = \frac{(t-t_i)}{(t_{i+3}-t_i)}\,\frac{(t-t_i)}{(t_{i+2}-t_i)}\,\frac{(t-t_i)}{(t_{i+1}-t_i)}\,N_{i,0}(t) \\
& \quad + \left[\begin{array}{c}
\frac{(t-t_i)}{(t_{i+3}-t_i)}\,\frac{(t-t_i)}{(t_{i+2}-t_i)}\,\frac{(t_{i+2}-t)}{(t_{i+2}-t_{i+1})} \quad + \\
\frac{(t-t_i)}{(t_{i+3}-t_i)}\,\frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+1})}\,\frac{(t-t_{i+1})}{(t_{i+2}-t_{i+1})} \quad + \\
\frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})}\,\frac{(t-t_{i+1})}{(t_{i+3}-t_{i+1})}\,\frac{(t-t_{i+1})}{(t_{i+2}-t_{i+1})}
\end{array}\right] N_{i+1,0}(t) \\
& \quad + \left[\begin{array}{c}
\frac{(t-t_i)}{(t_{i+3}-t_i)}\,\frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+1})}\,\frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+2})} \quad + \\
\frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})}\,\frac{(t-t_{i+1})}{(t_{i+3}-t_{i+1})}\,\frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+2})} \quad + \\
\frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})}\,\frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+2})}\,\frac{(t-t_{i+2})}{(t_{i+3}-t_{i+2})}
\end{array}\right] N_{i+2,0}(t) \\
& \quad + \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})}\,\frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+2})}\,\frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+3})}\,N_{i+3,0}(t)
\end{aligned}
$$

$$p = 4: \qquad N_{i,4}(t) = \frac{(t - t_i)}{(t_{i+4} - t_i)} N_{i,3}(t) + \frac{(t_{i+5} - t)}{(t_{i+5} - t_{i+1})} N_{i+1,3}(t)$$

$$\text{from } p = 3: \quad N_{i+1,3}(t) = \frac{(t - t_{i+1})}{(t_{i+4} - t_{i+1})} \frac{(t - t_{i+1})}{(t_{i+3} - t_{i+1})} \frac{(t - t_{i+1})}{(t_{i+2} - t_{i+1})} N_{i+1,0}(t)$$

$$+ \left[\begin{array}{cc} \frac{(t - t_{i+1})}{(t_{i+4} - t_{i+1})} \frac{(t - t_{i+1})}{(t_{i+3} - t_{i+1})} \frac{(t_{i+3} - t)}{(t_{i+3} - t_{i+2})} & + \\ \frac{(t - t_{i+1})}{(t_{i+4} - t_{i+1})} \frac{(t_{i+4} - t)}{(t_{i+4} - t_{i+2})} \frac{(t - t_{i+2})}{(t_{i+3} - t_{i+2})} & + \\ \frac{(t_{i+5} - t)}{(t_{i+5} - t_{i+2})} \frac{(t - t_{i+2})}{(t_{i+4} - t_{i+2})} \frac{(t - t_{i+2})}{(t_{i+3} - t_{i+2})} & \end{array} \right] N_{i+2,0}(t)$$

$$+ \left[\begin{array}{cc} \frac{(t - t_{i+1})}{(t_{i+4} - t_{i+1})} \frac{(t_{i+4} - t)}{(t_{i+4} - t_{i+2})} \frac{(t_{i+4} - t)}{(t_{i+4} - t_{i+3})} & + \\ \frac{(t_{i+5} - t)}{(t_{i+5} - t_{i+2})} \frac{(t - t_{i+2})}{(t_{i+4} - t_{i+2})} \frac{(t_{i+4} - t)}{(t_{i+4} - t_{i+3})} & + \\ \frac{(t_{i+5} - t)}{(t_{i+5} - t_{i+2})} \frac{(t_{i+5} - t)}{(t_{i+5} - t_{i+3})} \frac{(t - t_{i+3})}{(t_{i+4} - t_{i+3})} & \end{array} \right] N_{i+3,0}(t)$$

$$+ \frac{(t_{i+5} - t)}{(t_{i+5} - t_{i+2})} \frac{(t_{i+5} - t)}{(t_{i+5} - t_{i+3})} \frac{(t_{i+5} - t)}{(t_{i+5} - t_{i+4})} N_{i+4,0}(t)$$

$$
\begin{aligned}
\text{so}\ldots \quad N_{i,4}(t) = {} & \frac{(t-t_i)}{(t_{i+4}-t_i)} \frac{(t-t_i)}{(t_{i+3}-t_i)} \frac{(t-t_i)}{(t_{i+2}-t_i)} \frac{(t-t_i)}{(t_{i+1}-t_i)} N_{i,0}(t) \\
+ & \left[\begin{array}{r}
\frac{(t-t_i)}{(t_{i+4}-t_i)} \frac{(t-t_i)}{(t_{i+3}-t_i)} \frac{(t-t_i)}{(t_{i+2}-t_i)} \frac{(t_{i+2}-t)}{(t_{i+2}-t_{i+1})} \quad + \\
\frac{(t-t_i)}{(t_{i+4}-t_i)} \frac{(t-t_i)}{(t_{i+3}-t_i)} \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+2}-t_{i+1})} \quad + \\
\frac{(t-t_i)}{(t_{i+4}-t_i)} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+3}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+2}-t_{i+1})} \quad + \\
\frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+4}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+3}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+2}-t_{i+1})} \quad
\end{array} \right] N_{i+1,0}(t) \\
+ & \left[\begin{array}{r}
\frac{(t-t_i)}{(t_{i+4}-t_i)} \frac{(t-t_i)}{(t_{i+3}-t_i)} \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+1})} \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+2})} \quad + \\
\frac{(t-t_i)}{(t_{i+4}-t_i)} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+3}-t_{i+1})} \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+2})} \quad + \\
\frac{(t-t_i)}{(t_{i+4}-t_i)} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+2})} \frac{(t-t_{i+2})}{(t_{i+3}-t_{i+2})} \quad + \\
\frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+4}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+3}-t_{i+1})} \frac{(t_{i+3}-t)}{(t_{i+3}-t_{i+2})} \quad + \\
\frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+4}-t_{i+1})} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+2})} \frac{(t-t_{i+2})}{(t_{i+3}-t_{i+2})} \quad + \\
\frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+1})} \frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+2})} \frac{(t-t_{i+2})}{(t_{i+4}-t_{i+2})} \frac{(t-t_{i+2})}{(t_{i+3}-t_{i+2})} \quad
\end{array} \right] N_{i+2,0}(t) \\
+ & \left[\begin{array}{r}
\frac{(t-t_i)}{(t_{i+4}-t_i)} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+1})} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+2})} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+3})} \quad + \\
\frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+1})} \frac{(t-t_{i+1})}{(t_{i+4}-t_{i+1})} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+2})} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+3})} \quad + \\
\frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+1})} \frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+2})} \frac{(t-t_{i+2})}{(t_{i+4}-t_{i+2})} \frac{(t_{i+4}-t)}{(t_{i+4}-t_{i+3})} \quad + \\
\frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+1})} \frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+2})} \frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+3})} \frac{(t-t_{i+3})}{(t_{i+4}-t_{i+3})} \quad
\end{array} \right] N_{i+3,0}(t) \\
+ & \frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+1})} \frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+2})} \frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+3})} \frac{(t_{i+5}-t)}{(t_{i+5}-t_{i+4})} N_{i+4,0}(t)
\end{aligned}
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I'm sure that if one were to stare long enough at the results, one would recognize a pattern and would be able to write a closed form solution for any p. Yes, one would, but that one ain't this one here…

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