The Goat and The Fence

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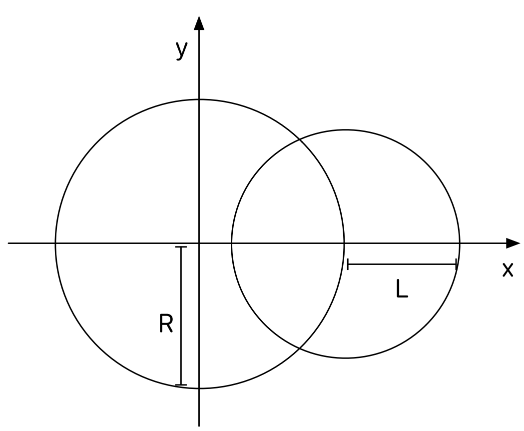
I just read an interesting mathematical puzzle. Suppose there is a goat inside of a circular fence of radius R. If the goat has a leash attached to a point inside the fence, how long does the leash have to be so that the goat can graze through exactly half the total area inside the fence?

It sounds like a reasonably easy problem to solve, so here it goes. The situation is abstracted to the figure below. The circle of radius $L \leq 2R$ is centered on a point at the edge of the circle of radius R. The area available to the goat for grazing is the intersection of the two circles, and is twice the area of the 'round triangle' above the horizontal axis. We shall now proceed to calculate that area.



The two circles have the following Cartesian equations:

$$x^2 + y^2 = R^2 \qquad \text{and} \qquad (x - R)^2 + y^2 = L^2 \,.$$

They will intersect at two points, with coordinates $(x_0, \pm\, y_0)$, where

$$x_0 = R - \frac{L^2}{2R} \qquad \text{and} \qquad y_0 = \sqrt{R^2 - x_0^2} = L\sqrt{1 - \frac{L^2}{4R^2}}\,.$$

The available grazing area is, therefore, given by

$$A = 2\left(\int_{R-L}^{x_0} dx\,\sqrt{L^2 - (x-R)^2} + \int_{x_0}^{R} dx\,\sqrt{R^2 - x^2}\right)$$

Now let $x - R = L\,\sin\phi$ in the first integral and $x = R\,\sin\psi$ in the second. Then,

$$A = 2\left(\int_{-\pi/2}^{-\arcsin(L/2R)} L^2\,\cos^2\phi\,d\phi + \int_{\arcsin(1-L^2/2R^2)}^{\pi/2} R^2\,\cos^2\psi\,d\psi\right).$$

Since $\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1$, we have

$$\int \cos^2 u\,du = \int \frac{1+\cos(2u)}{2}\,du = \frac{u}{2} + \frac{\sin(2u)}{4}$$

and, then,

$$A = 2\left\{ L^2\left(\frac{u}{2} + \frac{\sin(2u)}{4}\right)\Bigg|_{-\pi/2}^{-\arcsin(L/2R)} + R^2\left(\frac{u}{2} + \frac{\sin(2u)}{4}\right)\Bigg|_{\arcsin(1-L^2/2R^2)}^{\pi/2} \right\},$$

which amounts to

$$A = \left\{ (1+\lambda^2)\,\frac{\pi}{2} - \lambda^2\,\arcsin(\frac{\lambda}{2}) - \arcsin(1 - \frac{\lambda^2}{2}) - \lambda\sqrt{1 - \frac{\lambda^2}{4}} \right\} R^2\,,$$

where $\lambda \equiv L/R$. In order for this area to equal half the fence's area, we must have

$$\frac{\pi}{2}\,\lambda^2 = \lambda^2\,\arcsin(\frac{\lambda}{2}) + \lambda\sqrt{1 - \frac{\lambda^2}{4}} + \arcsin(1 - \frac{\lambda^2}{2})\,.$$

This is an equation for $\lambda = L/R$ that can only be solved numerically.

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