The Goat and The Fence

Wagner L. Truppel

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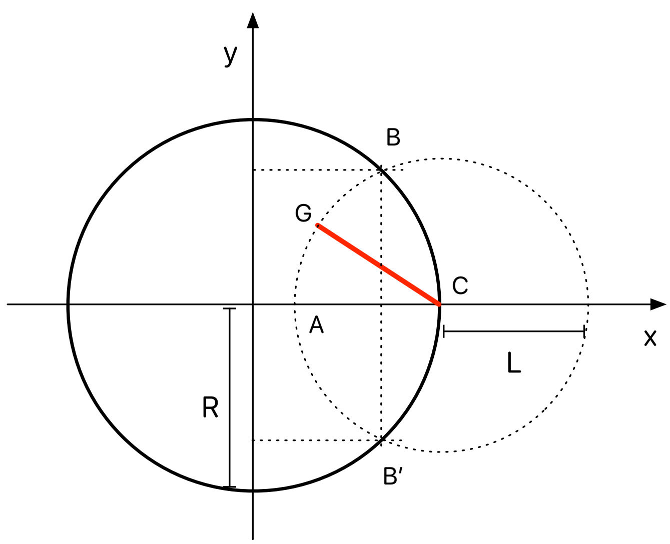
I just read an interesting mathematical puzzle. Suppose there is a goat inside of a circular fence of radius R. If the goat has a leash attached to a point inside the fence, how long does the leash have to be so that the goat can graze through exactly half the total area inside the fence?

It sounds like a reasonably easy problem to solve, so here it goes. The situation is abstracted to the figure below. The circle of radius $L \leq 2R$ is centered on a point (C) at the edge of the circle of radius R. The area available to the goat for grazing is the intersection of the two circles, and is twice the area of the region ABC above the horizontal axis. We shall now proceed to calculate that area.



The two circles have the following Cartesian equations:

$$x^2 + y^2 = R^2 \qquad \text{and} \qquad (x - R)^2 + y^2 = L^2 \,.$$

They will intersect at two points (B and B'), with coordinates $(x_0, \pm y_0)$, where

$$x_0 = R - \frac{L^2}{2R} \qquad \text{and} \qquad y_0 = \sqrt{R^2 - x_0^2} = L\sqrt{1 - \frac{L^2}{4R^2}}\,.$$

The available grazing area is, therefore, given by

$$\mathcal{A} = 2\Big(\int_{R-L}^{x_0} dx\,\sqrt{L^2-(R-x)^2} + \int_{x_0}^{R} dx\,\sqrt{R^2-x^2}\Big)\,.$$

Now let $R - x = L\sin\phi$ in the first integral and $x = R\sin\psi$ in the second. Then,

$$\frac{\mathcal{A}}{2} = -\int_{\pi/2}^{\arcsin(L/2R)} L^2\,\cos^2\phi\,d\phi + \int_{\arcsin(1-L^2/2R^2)}^{\pi/2} R^2\,\cos^2\psi\,d\psi\,.$$

Since $\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1$, we have

$$\int \cos^2 u\,du = \int \frac{1+\cos(2u)}{2}\,du = \frac{u}{2} + \frac{\sin(2u)}{4} = \frac{1}{2}\Big(u + \sin(u)\cos(u)\Big)$$

and, then,

$$\mathcal{A} = -L^2\Big(u + \sin(u)\cos(u)\Big)\Big|_{\pi/2}^{\arcsin(L/2R)} + R^2\Big(u + \sin(u)\cos(u)\Big)\Big|_{\arcsin(1-L^2/2R^2)}^{\pi/2}\,,$$

which amounts to

$$\frac{\mathcal{A}}{R^2} = (1+\lambda^2)\,\frac{\pi}{2} - \lambda^2\,\arcsin(\frac{\lambda}{2}) - \arcsin(1-\frac{\lambda^2}{2}) - \lambda\sqrt{1-\frac{\lambda^2}{4}}\,,$$

where $\lambda \equiv L/R$. In order for this area to equal half the fence's area, we must have

$$\frac{\mathcal{A}}{R^2} = \frac{\pi}{2} \quad \Rightarrow \quad \frac{\pi}{2}\,\lambda^2 = \lambda^2\,\arcsin(\frac{\lambda}{2}) + \lambda\sqrt{1-\frac{\lambda^2}{4}} + \arcsin(1-\frac{\lambda^2}{2})\,.$$

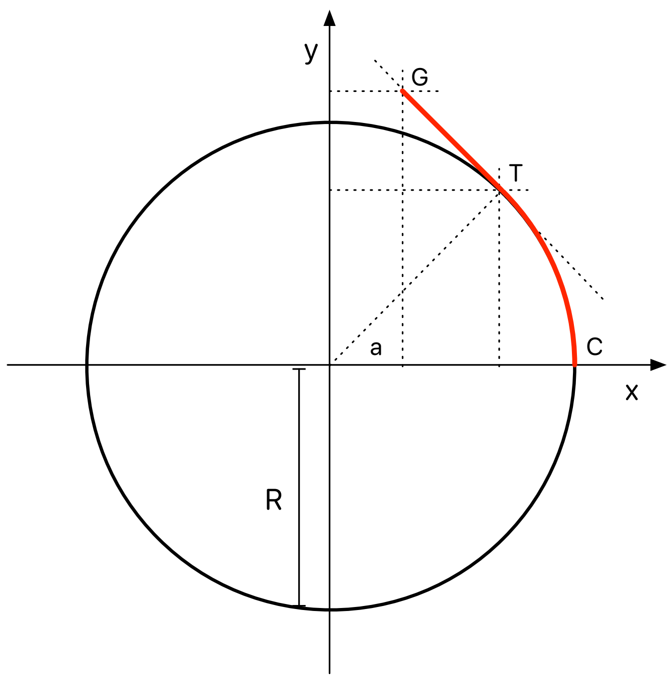
This is an equation for $\lambda = L/R$ that can only be solved numerically. Wolfram Alpha tells me that the only positive solution is

$$\lambda = \frac{L}{R} \approx 1.15873\,.$$

What if the goat jumped the fence?

But, wait. What if the leash is attached to the same point but the goat is now on the *outside* of the fence? Naively, one might think that the available grazing area is the remainder of the circle of radius L, but that's not the case.

In fact, it should be smaller than that remainder, because as the goat gets closer to the fence, keeping the leash as taut as possible, the leash has to 'hug' the fence for a certain extent, impairing the goat's ability to go as far as it could go otherwise, if the leash was always straight.



Let's figure out the equation describing the position of the goat (point G) as a function of the angle a between points C and T. T is the point where the leash ceases to hug the fence, which happens when the direction of the remainder of the leash ($\overline{TG}$) is tangent to the fence.

The Cartesian coordinates of T are easy to obtain,

$$T_x = R\cos(a) \qquad \text{and} \qquad T_y = R\sin(a)\,.$$

A line $y = m\,x + b$ passing by T must satisfy

$$T_y = m\,T_x + b \quad \Rightarrow \quad R\sin(a) = m\,R\cos(a) + b$$

so that $b = R\,(\sin(a) - m\,\cos(a))$ and

$$y = m\,(x - R\,\cos(a)) + R\,\sin(a)\,.$$

But that line must be tangent to the fence, hence perpendicular to a line from the center of the circle to the point T. That means

$$m = -\frac{1}{\tan(a)}\,,$$

and

$$y = \frac{R\,\cos(a) - x}{\tan(a)} + R\,\sin(a)\,.$$

Now, we know that the distance between the points T and G must equal the length of the part of the leash that is not hugging the fence. The total length of the leash is L and the part that *is* hugging the fence has length $R\,|a|$, where a is in radians. Therefore,

$$\overline{TG} = L - R\,|a| \quad \Rightarrow \quad \overline{TG}^{\,2} = (G_x - T_x)^2 + (G_y - T_y)^2 = (L - R\,|a|)^2\,.$$

Now let $G_x = x$. Then,

$$G_y = \frac{R\,\cos(a) - x}{\tan(a)} + R\,\sin(a)$$

and

$$\Big(x - R\,\cos(a)\Big)^2 + \Big(\frac{R\,\cos(a) - x}{\tan(a)}\Big)^2 = (L - R\,|a|)^2\,.$$

This gives us the x-coordinate of G, G_x, as a function of the angle a, which we can simplify to

$$\Big(x - R\,\cos(a)\Big)^2 = (L - R\,|a|)^2\,\sin^2(a)\,.$$

The question now is which sign to choose in

$$x - R\,\cos(a) = \pm\,(L - R\,|a|)\,\sin(a)\,.$$

Whichever sign we end up selecting, we can insert this value of x into the equation for G_y. The result, then, is that the coordinates of the goat's position G as a function of a are:

$$G_x = R\,\cos(a) \pm (L - R\,|a|)\,\sin(a) \quad \text{and} \quad G_y = R\,\sin(a) \mp (L - R\,|a|)\,\cos(a)\,.$$

Now, consider $a = 0$. In the figure, we see that G_y must be positive. Therefore,

$$G_x = R\,\cos(a) - (L - R\,|a|)\,\sin(a) \quad \text{and} \quad G_y = R\,\sin(a) + (L - R\,|a|)\,\cos(a)\,.$$

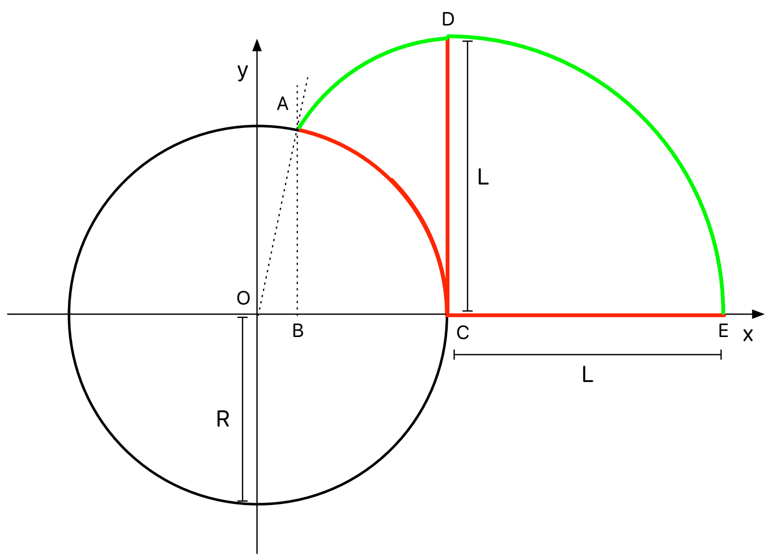
What are the limits of a? Clearly, the upper limit happens when the entire leash is hugging the fence, in which case $R\,|a| = L$. Considering the symmetry of the situation, we may as well consider $a > 0$ values only and double the resulting area. In that case, $a = 0$ is a lower limit, and

$$G_x = R\,\cos(a) - (L - R\,a)\,\sin(a) \quad \text{and} \quad G_y = R\,\sin(a) + (L - R\,a)\,\cos(a)\,,$$

where $0 \le a \le L/R$.

We're now ready to compute the area of the grazing region available to the goat. In the figure below, the total grazing area is *twice* the combined area of the regions ACD and CDE. CDE is just a quarter of the circle of radius equal to the length of the leash, so

$$\mathcal{A}_{CDE} = \frac{\pi}{4}\,L^2\,.$$



Computing the area of the region ACD is a bit complicated. We'll write it as

$$\begin{aligned}
\mathcal{A}_{ACD} &= \mathcal{A}_{ABCD} - \mathcal{A}_{ABC} \\
&= \mathcal{A}_{ABCD} - (\mathcal{A}_{OAC} - \mathcal{A}_{OAB}) \\
&= \mathcal{A}_{ABCD} - \mathcal{A}_{OAC} + \mathcal{A}_{OAB}
\end{aligned}$$

We'll need the coordinates of points A and B. A is the point G when a has its upper value, so

$$A_x = B_x = R\,\cos(L/R) \qquad \text{and} \qquad A_y = R\,\sin(L/R)\,, \quad B_y = 0\,.$$

$\mathcal{A}_{OAB}$ is the area of a triangle, so

$$\mathcal{A}_{OAB} = \frac{1}{2}\, B_x\, A_y = \frac{1}{4}\, R^2 \sin\left(\frac{2L}{R}\right).$$

$\mathcal{A}_{OAC}$ is the area of a circular section with angle $a = L/R$, so

$$\mathcal{A}_{OAC} = \frac{1}{2}\, a\, R^2 = \frac{1}{2}\, \frac{L}{R}\, R^2 \,.$$

Lastly, $\mathcal{A}_{ABCD}$ can be expressed as an integral,

$$\mathcal{A}_{ABCD} = \int_{B_x}^{C_x} dx\, G_y(x)\,.$$

But, $x = R\cos(a)$ so

$$\begin{aligned}
\mathcal{A}_{ABCD} &= \int_0^{L/R} R\sin(a)\, da\, G_y(a) \\
&= \int_0^{L/R} R\sin(a)\, da \Big(R\, \sin(a) + (L - R\, a)\, \cos(a)\Big) \\
&= \int_0^{L/R} R^2 \sin^2(a)\, da + \int_0^{L/R} R^2\, \frac{L}{R}\, \sin(a)\cos(a)\, da - \int_0^{L/R} R^2\, a\, \sin(a)\, \cos(a)\, da\,,
\end{aligned}$$

which amounts to

$$\begin{aligned}
\frac{\mathcal{A}_{ABCD}}{R^2} &= \frac{L}{2R} - \frac{\sin(2L/R)}{4} + \frac{L}{2R}\,(\frac{L}{R})^2 - \frac{1}{8}\Big(\sin(2L/R) - 2\frac{L}{R}\, \cos(2L/R)\Big) \\
&= \frac{L}{2R}\Big(1 + \frac{L^2}{R^2}\Big) + \frac{L}{4R}\, \cos(\frac{2L}{R}) - \frac{3}{8}\, \sin(\frac{2L}{R})\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{A}_{ACD} &= \mathcal{A}_{ABCD} - \mathcal{A}_{OAC} + \mathcal{A}_{OAB} \\
&= \left\{ \frac{L}{2R}\Big(1 + \frac{L^2}{R^2}\Big) + \frac{L}{4R}\, \cos(\frac{2L}{R}) - \frac{3}{8}\, \sin(\frac{2L}{R}) - \frac{1}{2}\, \frac{L}{R} + \frac{1}{4}\sin\left(\frac{2L}{R}\right) \right\} R^2 \\
&= \left\{ \frac{L^3}{2R^3} + \frac{L}{4R}\, \cos(\frac{2L}{R}) - \frac{1}{8}\, \sin(\frac{2L}{R}) \right\} R^2\,.
\end{aligned}$$

The total grazing area $\mathcal{A}$ is, therefore, such that

$$\frac{1}{2}\, \mathcal{A} = \mathcal{A}_{ACD} + \mathcal{A}_{CDE} = \left\{ \frac{L^3}{2R^3} + \frac{L}{4R}\, \cos(\frac{2L}{R}) - \frac{1}{8}\, \sin(\frac{2L}{R}) \right\} R^2 + \frac{\pi}{4}\, L^2\,,$$

or, in terms of $\lambda = L/R$,

$$\frac{\mathcal{A}}{R^2} = \lambda^3 + \frac{\pi}{2}\,\lambda^2 + \frac{\lambda}{2}\,\cos(2\lambda) - \frac{1}{4}\,\sin(2\lambda)\,.$$

In order for that to be half the area of the fence, we must have

$$\frac{\pi}{2} = \lambda^3 + \frac{\pi}{2}\,\lambda^2 + \frac{\lambda}{2}\,\cos(2\lambda) - \frac{1}{4}\,\sin(2\lambda)\,,$$

whose approximate solution is

$$\lambda = \frac{L}{R} \approx 0.880\,.$$

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