Frank's lunch problem

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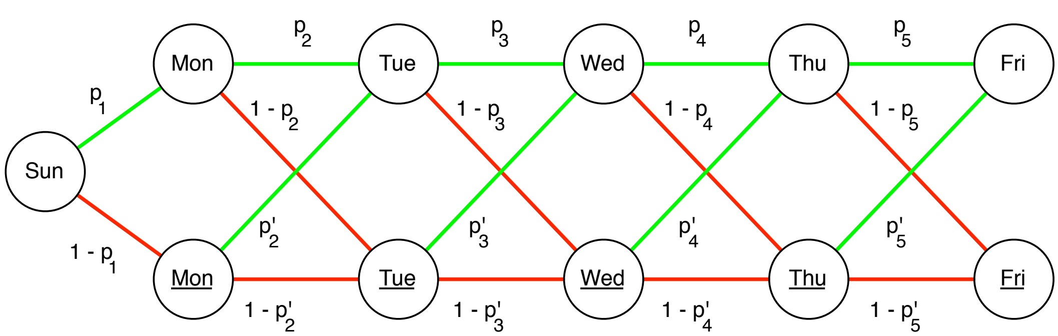
The problem

Frank eats lunch 4 out of every 5 working days and spends a minimum of 5 pounds, a maximum of 15 pounds, with a most likely value of 8 pounds, per lunch. What is the distribution of possible lunch costs at the end of a work week?

Soft-rule approach

If the rule that Frank eats lunch 4 out of 5 days is a soft one then in some weeks he may have lunch on fewer than 4 days while in other weeks he may have lunch all week long. One way to model this is to use a Markov chain with 9 parameters:



Come Monday, there's a probability p_1 that he'll have lunch and a corresponding probability $1 - p_1$ that he'll skip lunch that day. If he does have lunch on Monday, then there's a probability p_2 that he'll have lunch on Tuesday and a corresponding probability $1 - p_2$ that he'll skip lunch that day, and so on. The primed probabilities (p'_1, p'_2, etc) are the probabilities that he will have lunch on a certain day given that he skipped lunch the day before, and need not be equal to the probabilities that he will have lunch on a certain day given that he *did* have lunch the day before (the unprimed probabilities).

There are multiple paths to having lunch on any given day, except Monday, for which there's only one path. If $p(\mathrm{d})$ is the probability of having lunch on day d and $p(\neg\mathrm{d})$ is the probability of *not* having lunch on day d, then:

$$
\begin{aligned}
p(\mathrm{Mon}) &= p_1 \\
p(\neg\mathrm{Mon}) &= 1 - p_1 \\[1em]
p(\mathrm{Tue}) &= p_2\, p(\mathrm{Mon}) + p'_2\, p(\neg\mathrm{Mon}) \\
p(\neg\mathrm{Tue}) &= (1 - p_2)\, p(\mathrm{Mon}) + (1 - p'_2)\, p(\neg\mathrm{Mon}) \\[1em]
p(\mathrm{Wed}) &= p_3\, p(\mathrm{Tue}) + p'_3\, p(\neg\mathrm{Tue}) \\
p(\neg\mathrm{Wed}) &= (1 - p_3)\, p(\mathrm{Tue}) + (1 - p'_3)\, p(\neg\mathrm{Tue}) \\[1em]
p(\mathrm{Thu}) &= p_4\, p(\mathrm{Wed}) + p'_4\, p(\neg\mathrm{Wed}) \\
p(\neg\mathrm{Thu}) &= (1 - p_4)\, p(\mathrm{Wed}) + (1 - p'_4)\, p(\neg\mathrm{Wed}) \\[1em]
p(\mathrm{Fri}) &= p_5\, p(\mathrm{Thu}) + p'_5\, p(\neg\mathrm{Thu}) \\
p(\neg\mathrm{Fri}) &= (1 - p_5)\, p(\mathrm{Thu}) + (1 - p'_5)\, p(\neg\mathrm{Thu})
\end{aligned}
$$

We may think of the decision of having lunch on a given day as a Bernoulli trial where the probability of success is the probability of having lunch and the probability of failure is the probability of skipping lunch. If these Bernoulli trials are *independent* then the primed probabilities must equal the corresponding unprimed ones because independent trials do not 'remember' the results of previous trials. Moreover, if the trials are *identical* then all the unprimed probabilities must equal the same value p. Then, as expected,

$$
\begin{aligned}
p(\mathrm{Mon}) &= p(\mathrm{Tue}) &&= p(\mathrm{Wed}) &&= p(\mathrm{Thu}) &&= p(\mathrm{Fri}) &&= p \\
p(\neg\mathrm{Mon}) &= p(\neg\mathrm{Tue}) &&= p(\neg\mathrm{Wed}) &&= p(\neg\mathrm{Thu}) &&= p(\neg\mathrm{Fri}) &&= 1 - p\,.
\end{aligned}
$$

We may now ask for the probability of having lunch on any k days out of n days, regardless of which days those are. This is just the Binomial distribution

$$
p(k, n) \equiv p(\text{lunch on any } k \text{ days out of } n \text{ days}) = \frac{n!}{k!\,(n-k)!}\, p^k\, (1-p)^{n-k}\,,
$$

where $n > 0$ and $0 \leq k \leq n$. In particular, the probability of having lunch on any 4 out of 5 working days is

$$
p(k = 4, n = 5) = 5\, p^4\, (1 - p)\,.
$$

See figure 1 on the next page for graphs of $p(k, n)$ as functions of p, for $n = 5$.

Now, on any given day when Frank does have lunch, he spends a minimum of 5 pounds and a maximum of 15 pounds. That means there is a probability distribution for the amount he spends on lunch on a given day. Let $p(x)$ be that probability distribution, that is, $p(x)\,dx$ is the probability that he spends at lunch on a given day an amount in the interval dx surrounding x, where $\min \le x \le \max$. Note that $p(x)$ is *not* the probability of spending x. Note also that $p(x) = 0$ if x is outside the interval $[\min, \max]$.

Now, If Frank does not have lunch on any day of some week (the $k = 0$ case), his total lunch cost for that week will be zero so the probability that he spends nothing on lunch in a given week is $(1-p)^n$ (for a week with n days). Since $p(x)$ is not the probability of spending x, we can't say that $p(x=0) = (1-p)^n$. Technically, $p(x) = (1-p)^n\,\delta(x)$, where $\delta(x)$ is the so-called *Dirac distribution*. Moreover, if he has lunch on k out of n days then the probability distribution of spending x_1 on day 1, x_2 on day 2, and so on, is

$$p(x_1, x_2, \ldots, x_k) = \frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k}\,p(x_1)\,p(x_2)\ldots p(x_k)\,.$$

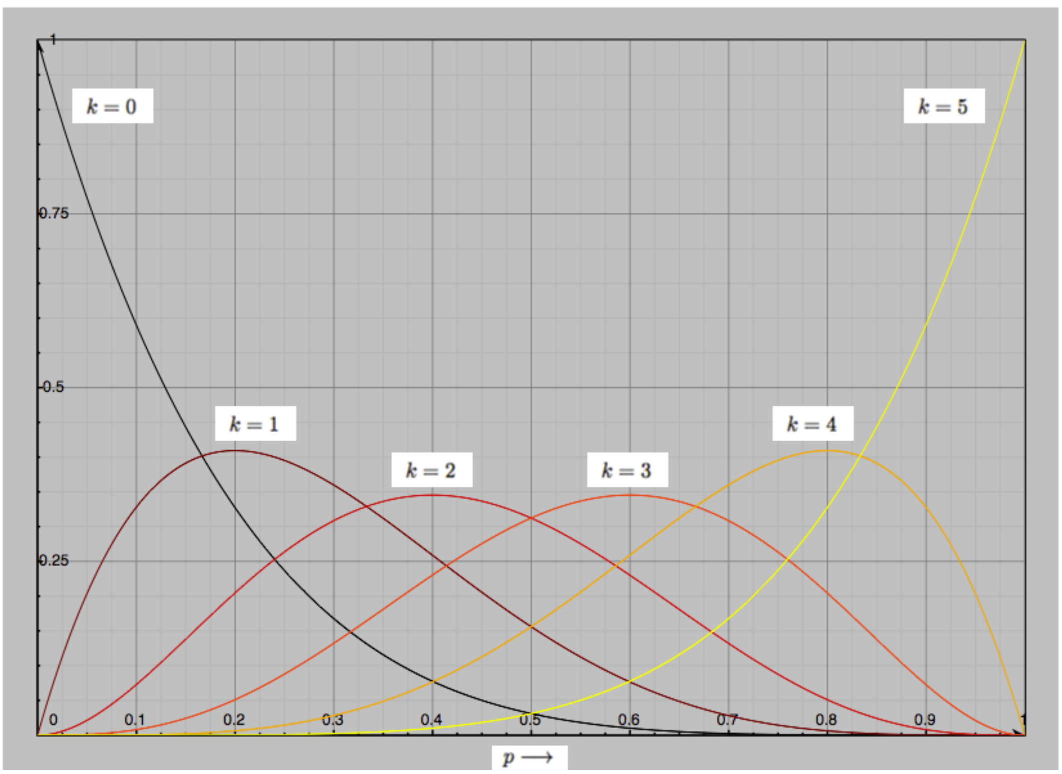


Figure 1: Probability of having lunch on any k days out of $n = 5$ days as functions of k and the probability of having lunch on any given day, p.

Now, with this particular arrangement of expenditures the total lunch cost is $x = x_1 + x_2 + \ldots + x_k$ and it's at least k times the per-day minimum of 5 and at most k times the per-day maximum of 15. There are many other arrangements that result in the same total cost x and their probabilities need to be added together to find the probability distribution of spending a given total amount x regardless of how much is spent on any given day. But these expenditures are continuous variables so sums over the probabilities turn into integrals of the probability distributions over the individual amounts.

Of dice and men, at lunch

One way to see this is to look at a discrete example. Suppose you have 3 dice and you ask for the probability that you'll get a total of 15 when you throw all of them once. There are only 10 ways you get 15 out of 3 dice:

$$\{(3,6,6),(4,5,6),(4,6,5),(5,4,6),(5,5,5),(5,6,4),(6,3,6),(6,4,5),(6,5,4),(6,6,3)\}\,.$$

Thus, the probability of getting 15 in one throw of 3 dice is:

$$\begin{aligned}
p(15) &= p(3,6,6)+p(4,5,6)+p(4,6,5)+p(5,4,6)+p(5,5,5) \\
&+ p(5,6,4)+p(6,3,6)+p(6,4,5)+p(6,5,4)+p(6,6,3) \\
&= p(3)\,p(6)\,p(6)+p(4)\,p(5)\,p(6)+p(4)\,p(6)\,p(5)+p(5)\,p(4)\,p(6)+p(5)\,p(5)\,p(5) \\
&+ p(5)\,p(6)\,p(4)+p(6)\,p(3)\,p(6)+p(6)\,p(4)\,p(5)+p(6)\,p(5)\,p(4)+p(6)\,p(6)\,p(3) \\
&= p(3)\,p(6)\,p(15-3-6)+p(4)\,p(5)\,p(15-4-5)+p(4)\,p(6)\,p(15-4-6) \\
&+ p(5)\,p(4)\,p(15-5-4)+p(5)\,p(5)\,p(15-5-5)+p(5)\,p(6)\,p(15-5-6) \\
&+ p(6)\,p(3)\,p(15-6-3)+p(6)\,p(4)\,p(15-6-4)+p(6)\,p(5)\,p(15-6-5) \\
&+ p(6)\,p(6)\,p(15-6-6) \\
&= \sum_{i=1}^{i=6}\sum_{j=1}^{j=6} p(i)\,p(j)\,p(15-i-j)\,.
\end{aligned}$$

Note that in the last line there will be contributions that are, in fact, zero, such as $p(1)\,p(2)\,p(15-1-2) = p(1)\,p(2)\,p(12)$ because $p(12) = 0$ since you can't get a 12 from a single die. That's fine. It all sorts itself out automatically. Note also that there are 2 sums for 3 dice. In general, for k dice, you'll have $k-1$ sums.

Back to Frank's lunch problem

So, the probability distribution that Frank will spend x pounds having lunch on k out of n days, *irrespective* of how much he spends on each day, or which k days those are, is given by

$$p_k(k\min \le x \le k\max) = \frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k}\times$$
$$\underbrace{\int_{\min}^{\max}\cdots\int_{\min}^{\max}}_{(k-1)\text{ integrals}} p(x_1)\,p(x_2)\ldots p(x_{k-1})\,p(\underbrace{x-x_1-x_2-\ldots-x_{k-1}}_{x_k})\,dx_1\,dx_2\ldots dx_{k-1}\,.$$

Note that we're integrating (adding) the *probabilities* $p(x_j)\,dx_j$ to obtain the probability *distribution* of spending x, $p(x)$. There is no dx in the expression above, so we end up with a probability *distribution* for x, as we should. Note also that the $k=1$ case has no integrals:

$$p_1(\min \le x \le \max) = \frac{n!}{1!\,(n-1)!}\,p^1\,(1-p)^{n-1}\,p(x) = n\,p\,(1-p)^{n-1}\,p(x)\,.$$

Let's look at those integrals in more detail. First, define

$$I_k(x) \equiv \underbrace{\int_{\min}^{\max}\cdots\int_{\min}^{\max}}_{(k-1)\text{ integrals}} p(x_1)\,p(x_2)\ldots p(x_{k-1})\,p(x-x_1-x_2-\ldots-x_{k-1})\,dx_1\,dx_2\ldots dx_{k-1}$$

where $k>1$, so that the probability distribution of spending x pounds having lunch on k out of n days, *irrespective* of how much Frank spends on each day, or which k days those are, is given by

$$p_k(k\min \le x \le k\max) = \frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k} I_k(x)\,.$$

Now, let $z_{k-2} \equiv x-x_1-x_2-\ldots-x_{k-2}$, for $k>2$. Then,

$$I_k(x) = \underbrace{\int_{\min}^{\max}\cdots\int_{\min}^{\max}}_{(k-1)\text{ integrals}} p(x_1)\,p(x_2)\ldots p(x_{k-1})\,p(z_{k-2}-x_{k-1})\,dx_1\,dx_2\ldots dx_{k-1}\,.$$

Since z_{k-2} does not depend on x_{k-1}, we can compute the integral on x_{k-1} first:

$$I_k(x) = \underbrace{\int_{\min}^{\max}\cdots\int_{\min}^{\max}}_{(k-2)\text{ integrals}} p(x_1)p(x_2)\ldots p(x_{k-2})dx_1dx_2\ldots dx_{k-2}\underbrace{\int_{\min}^{\max} p(x_{k-1})p(z_{k-2}-x_{k-1})dx_{k-1}}_{f(z_{k-2})}\,.$$

Note that $f(z_{k-2}) = f(x - x_1 - x_2 - \ldots - x_{k-2}) = f(z_{k-3} - x_{k-2})$ and z_{k-3} does not depend on x_{k-2} so we can then do the integral on x_{k-2}:

$$I_k(x) = \underbrace{\int_{\min}^{\max} \ldots \int_{\min}^{\max}}_{(k-3)\text{ integrals}} p(x_1)p(x_2)\ldots p(x_{k-3})dx_1dx_2\ldots dx_{k-3} \underbrace{\int_{\min}^{\max} p(x_{k-2})f(z_{k-3} - x_{k-2})dx_{k-2}}_{g(z_{k-3})} \,.$$

Now note that $g(z_{k-3}) = g(x - x_1 - x_2 - \ldots - x_{k-3}) = g(z_{k-4} - x_{k-3})$ and z_{k-4} does not depend on x_{k-3} so we can then do the integral on x_{k-3}, and so on. The bottom line is that we can peel one integral at a time by performing only integrals of the form

$$\int_{\min}^{\max} p(u)\, F(z-u)\, du \,,$$

for appropriately defined z variables and F functions. Now, since $p(u) = 0$ for u not in the interval $\min \leq u \leq \max$, we can extend the integration interval to $(-\infty, +\infty)$ without changing the value of the integral. The result is that we'll need to compute integrals of the form

$$\int_{-\infty}^{+\infty} p(u)\, F(z-u)\, du \,,$$

which are known as *convolution integrals*:

$$(p \star F)(z) \equiv \int_{-\infty}^{+\infty} p(u)\, F(z-u)\, du \,.$$

Let's take a step back and look at specific examples. We already saw that the probability distribution of spending x pounds on lunch when *not* having lunch over the course of an entire week of n days is[1]

$$\boxed{p_0(x) = (1-p)^n\, \delta(x)}$$

and the probability distribution of spending x pounds having lunch on only 1 out of n days is

$$p_1(\min \leq x \leq \max) = n\, p\, (1-p)^{n-1}\, p(x) \,,$$

where p is the probability that he will have lunch on any given day and $p(x)$ is the probability distribution of lunch costs on a given lunch. We also saw that the probability distribution of spending x pounds having lunch on 2 out of n days is

$$p_2(2\min \leq x \leq 2\max) = \frac{n!}{2!\,(n-2)!}\, p^2\, (1-p)^{n-2} \int_{\min}^{\max} p(x_1)\, p(x - x_1)\, dx_1 \,.$$

[1]We're adding an index 0 to p to indicate that this corresponds to the $k = 0$ out of n case.

This is essentially the convolution integral of $p(x)$ with itself:

$$p_2(2\min \leq x \leq 2\max) = \frac{n\,(n-1)}{2}\, p^2\,(1-p)^{n-2}\ (p \star p)(x)\,.$$

The Convolution Theorem and Fourier Transforms

Let

$$\mathcal{F}\{f\} = \hat{f}(z) = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i\, xz} dx \quad \text{and} \quad \mathcal{F}^{-1}\{\hat{f}\} = f(x) = \int_{-\infty}^{+\infty} \hat{f}(z)\, e^{+2\pi i\, xz} dz$$

be the Fourier transform of $f(x)$ and the inverse Fourier transform of $\hat{f}(z)$, respectively. Then, the *Convolution Theorem* states that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of those functions (provided that the Fourier transforms exist in the first place),

$$\mathcal{F}\{f \star g\} = \mathcal{F}\{f\}\,\mathcal{F}\{g\}\,,$$

so the convolution can be computed as the inverse Fourier transform of the product of the Fourier transforms:

$$f \star g = \mathcal{F}^{-1}\{\,\mathcal{F}\{f\}\,\mathcal{F}\{g\}\,\}\,.$$

As a specific example, the convolution of $p(x)$ with itself can be computed as such:

$$p \star p = \mathcal{F}^{-1}\{\,\mathcal{F}\{p\}\,\mathcal{F}\{p\}\,\} = \mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^2\,\}\,.$$

To be clear, $p(x)$ (the probability distribution of lunch costs on a given lunch) is a function of x (the cost of lunch). Its Fourier transform is a function of the variable z (*not* related to the z_{k-2} we defined earlier). We then square that function and take its inverse Fourier transform, resulting in a function of x again. That is the convolution we're looking to compute.

Back to computing $I_k(x)$

Now that we know that $I_k(x)$ is essentially an iterated convolution integral and that we can use Fourier transforms to compute the individual convolution integrals, let's get a closed-form solution for $I_k(x)$. Start with the definition of $I_k(x)$:

$$I_k(x) = \underbrace{\int_{\min}^{\max} \cdots \int_{\min}^{\max}}_{(k-1)\text{ integrals}} p(x_1)\,p(x_2)\ldots p(x_{k-1})\,p(x - x_1 - x_2 - \ldots - x_{k-1})\,dx_1\,dx_2\ldots dx_{k-1}\,.$$

As we saw, this can be written as

$$I_k(x) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{(k-2)\text{ integrals}} p(x_1)p(x_2)\dots p(x_{k-2})dx_1 dx_2 \dots dx_{k-2} \int_{-\infty}^{+\infty} p(x_{k-1})p(z_{k-2}-x_{k-1})dx_{k-1}\,.$$

The integral on the right is, however, the convolution of $p(x)$ with itself, evaluated at z_{k-2}, so:

$$I_k(x) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{(k-2)\text{ integrals}} p(x_1)\,p(x_2)\dots p(x_{k-2})\,dx_1\,dx_2\dots dx_{k-2}\,\mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^2\,\}(z_{k-2})\,.$$

Now we integrate on x_{k-2} to get:

$$\begin{aligned} I_k(x) \;=\; & \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{(k-3)\text{ integrals}} p(x_1)\,p(x_2)\dots p(x_{k-3})\,dx_1\,dx_2\dots dx_{k-3}\times \\ & \int_{-\infty}^{+\infty} p(x_{k-2})\,\mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^2\,\}(\underbrace{z_{k-3}-x_{k-2}}_{z_{k-2}})\,dx_{k-2}\,. \end{aligned}$$

The integral on the right, as expected, is another convolution so we use the convolution theorem again to get

$$\begin{aligned} \int_{-\infty}^{+\infty} p(x_{k-2})\,\mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^2\,\}(z_{k_3}-x_{k-2})\,dx_{k-2} &= p\star\mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^2\,\} \\ &= \mathcal{F}^{-1}\{\,\mathcal{F}\{p\}\,\mathcal{F}\{\,\mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^2\,\}\,\}\,\} \\ &= \mathcal{F}^{-1}\{\,\mathcal{F}\{p\}\,(\mathcal{F}\{p\})^2\,\} \\ &= \mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^3\,\}(z_{k-3})\,. \end{aligned}$$

Thus,

$$I_k(x) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{(k-3)\text{ integrals}} p(x_1)\,p(x_2)\dots p(x_{k-3})\,dx_1\,dx_2\dots dx_{k-3}\,\mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^3\,\}(z_{k-3})\,.$$

The pattern is clear: every convolution integral in $I_k(x)$ adds a new power of $\mathcal{F}\{p\}$, so the final result is

$$I_k(x) = \mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^k\,\}(x)$$

and the probability distribution of spending x pounds having lunch on k out of n days, *irrespective* of how much Frank spends on each day, or which k days those are, is given by

$$\boxed{p_k(k\min \leq x \leq k\max) = \frac{n!}{k!\,(n-k)!}\, p^k\,(1-p)^{n-k}\,\mathcal{F}^{-1}\{\,(\mathcal{F}\{p\})^k\,\}(x) \quad (1 \leq k \leq n)\,.}$$

This is a *general* result, valid for *any* $p(x)$ that is zero outside the interval $[\min, \max]$. In fact, it's so general a result that it's even valid for $k = 0$ since $\mathcal{F}^{-1}\{\,1\,\}(x)$ is precisely the Dirac distribution $\delta(x)$, and we recover the previous result that

$$\boxed{p_0(x) = (1-p)^n\,\delta(x)\,.}$$

The lunch cost probability distribution $p(x)$: Normal approximation

Suppose we use a normal distribution for $p(x)$,

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\exp\Big[-\frac{(x-\mu)^2}{2\sigma^2}\,\Big]}$$

where x now takes values in the interval $(-\infty, +\infty)$, $\boxed{\mu = \mathrm{mlv}}$, and σ must be chosen such that $p(x = \mathrm{mlv})$ has the appropriate value, that is,

$$\boxed{\sigma = \frac{1}{\sqrt{2\pi}\,p(\mathrm{mlv})}\,.}$$

Note that $p(\mathrm{mlv})$ is not the probability of the most likely value but simply the value of the distribution $p(x)$ at $x = \mathrm{mlv}$. Anyway, the normal distribution may not be the most appropriate choice because the cost of lunch should never be negative, unless Frank is being paid to have lunch, but it is a choice that makes it easy to obtain a complete solution since the Fourier transform of a normal distribution is also a normal distribution.

The Fourier transform of $p(x)$: Normal approximation

It's relatively straight-forward to show that, in this case,

$$\mathcal{F}\{\,p\,\}(z) = \exp\big[-(2\pi^2\sigma^2z^2 + 2\pi i\mu z)\big]$$

so that

$$\big(\mathcal{F}\{\,p\,\}\big)^k(z) = \exp\big[-(2\pi^2\,k\sigma^2z^2 + 2\pi i\,k\mu\,z)\big]\,.$$

But this is exactly the same as $\mathcal{F}\{\,p\,\}(z)$ with μ replaced with $k\mu$ and σ^2 replaced with $k\sigma^2$. Therefore, the inverse Fourier transform of $\big(\mathcal{F}\{\,p\,\}\big)^k$ should be $p(x)$ with those replacements in place:

$$\mathcal{F}^{-1}\{\,\big(\mathcal{F}\{\,p\,\}\big)^k\,\}(x) = \frac{1}{\sqrt{2\pi\,k\sigma^2}}\,\exp\big[-\frac{(x-k\mu)^2}{2k\sigma^2}\,\big]\,.$$

As a result, we now have a complete and very simple solution to Frank's lunch problem. The probability distribution of spending x pounds having lunch on k out of n days, *irrespective* of how much he spends on each day, or which k days those are, is given by

$$\boxed{p_k(x) = \begin{cases} (1-p)^n\,\delta(x)\,, & \text{for } k=0\,,\\[2ex] \dfrac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k}\,\dfrac{1}{\sqrt{2\pi\,k\sigma^2}}\,\exp\big[-\dfrac{(x-k\mu)^2}{2k\sigma^2}\,\big]\,, & \text{for } 1\le k\le n\,,\end{cases}}$$

where $-\infty < x < +\infty$, $\mu = \text{mlv}$, and $\sigma = [\,\sqrt{2\pi}\,p(\text{mlv})\,]^{-1}$. Moreover, the probability distribution of having spent x pounds on lunch at the end of an n-day week, regardless of how many days Frank had lunch on that week, is given by the sum of the probability distributions for all k (including $k=0$):

$$\boxed{P(x) = (1-p)^n\,\delta(x) + \sum_{k=1}^{k=n}\frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k}\,\frac{1}{\sqrt{2\pi\,k\sigma^2}}\,\exp\big[-\frac{(x-k\mu)^2}{2k\sigma^2}\,\big]\,.}$$

Note that $P(x)$ is properly normalized:

$$\begin{aligned}\int_{-\infty}^{+\infty} P(x)\,dx \;&=\; (1-p)^n\,\underbrace{\int_{-\infty}^{+\infty}\delta(x)\,dx}_{=1}\\ &\quad+\sum_{k=1}^{k=n}\frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k}\,\underbrace{\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\,k\sigma^2}}\,\exp\big[-\frac{(x-k\mu)^2}{2k\sigma^2}\,\big]\,dx}_{=1}\\ &=\;(1-p)^n+\sum_{k=1}^{k=n}\frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k}\\ &=\;\sum_{k=0}^{k=n}\frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k} = p+(1-p) = 1\,.\end{aligned}$$

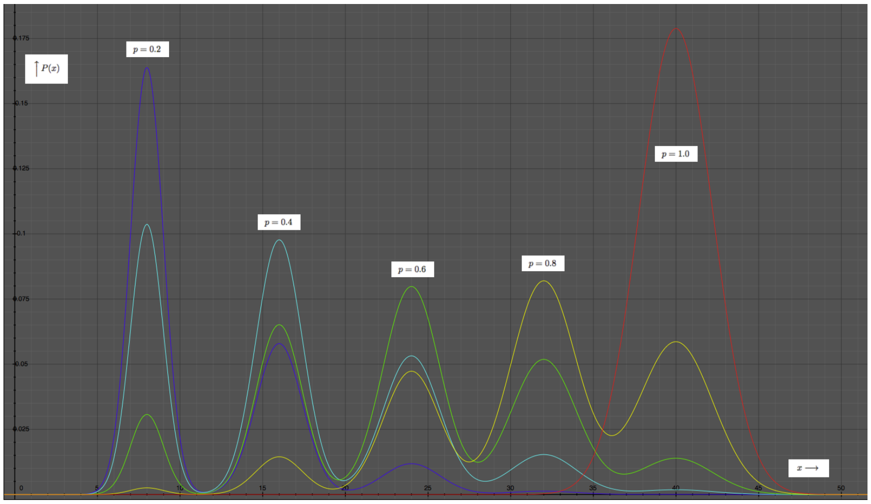


Figure 2: The probability distribution $P(x)$ of the total lunch cost x for a week of $n = 5$ days where the probability of having lunch on any given day, p, is a member of the set $\{0.2, 0.4, 0.6, 0.8, 1.0\}$ and the most-likely lunch cost on any given lunch day is $\mathrm{mlv} = 8$ pounds, with $p(\mathrm{mlv}) = 0.4$. For small (large) values of p, the most likely scenario is to have lunch on fewer (more) days.

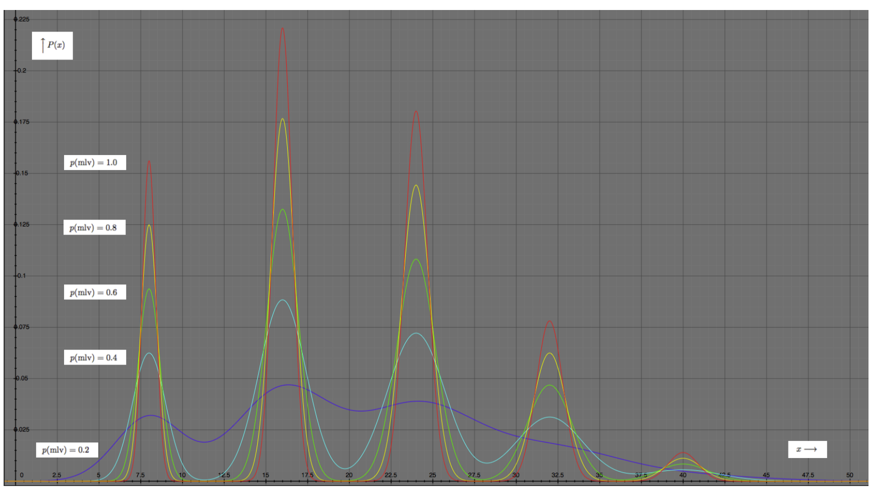


Figure 3: The probability distribution $P(x)$ of the total lunch cost x for a week of $n = 5$ days where the probability of having lunch on any given day is $p = 0.8$ and the most-likely lunch cost on any given lunch day is $\mathrm{mlv} = 8$ pounds, with $p(\mathrm{mlv})$ in the set $\{0.2, 0.4, 0.6, 0.8\}$. Smaller (larger) values of $p(\mathrm{mlv})$ make the distribution broader (sharper) around its peaks.

From $P(x)$ we can obtain actual probabilities:

- The probability of having spent a total of *at most* a pounds ($a \geq 0$) at the end of an n-day week, regardless of the details of how that money is spent:

$$\boxed{p(0 \leq x \leq a) = \int_0^a P(x)\,dx = (1-p)^n + \sum_{k=1}^{k=n} \frac{n!}{k!\,(n-k)!}\, p^k\,(1-p)^{n-k}\left[\,\Phi_k(a) - \Phi_k(0)\,\right].}$$

- The probability of having spent a total of *at least* a pounds ($a > 0$) at the end of an n-day week, regardless of the details of how that money is spent:

$$\boxed{p(x \geq a > 0) = \int_a^{+\infty} P(x)\,dx = \sum_{k=1}^{k=n} \frac{n!}{k!\,(n-k)!}\, p^k\,(1-p)^{n-k}\left[\,1 - \Phi_k(a)\,\right].}$$

- The probability of having spent a total of x pounds between a and b (with $0 < a \leq b$), at the end of an n-day week, regardless of the details of how that money is spent:

$$\boxed{p(0 < a \leq x \leq b) = \int_a^b P(x)\,dx = \sum_{k=1}^{k=n} \frac{n!}{k!\,(n-k)!}\, p^k\,(1-p)^{n-k}\left[\,\Phi_k(b) - \Phi_k(a)\,\right].}$$

In all three expressions, the function $\Phi_k(x)$ is defined by

$$\boxed{\Phi_k(x) \equiv \int_{-\infty}^{x} \frac{du}{\sqrt{2\pi\, k\sigma^2}}\, \exp\left[-\frac{(u-k\mu)^2}{2k\sigma^2}\right] = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-k\mu}{\sqrt{2k\sigma^2}}\right)\right],}$$

where erf(x) is the so-called *Error Function*. Unfortunately, there's no closed-form expression for the Error Function in terms of known functions. However, there are many useful and accurate approximations. See http://en.wikipedia.org/wiki/Error_function for details. In particular, a numerical approximation that has a maximum error of 1.2×10^{-7} over the complete domain of erf(x) is given by:

$$\boxed{\operatorname{erf}(x) = \begin{cases} 1-\tau\,, & \text{for } x \geq 0 \\ \tau - 1\,, & \text{for } x < 0 \end{cases}}$$

where

$$\boxed{\begin{aligned} \tau = t\,\exp(-x^2 - 1.26551223 &+ 1.00002368\,t + 0.37409196\,t^2 + 0.09678418\,t^3 \\ &- 0.18628806\,t^4 + 0.27886807\,t^5 - 1.13520398\,t^6 \\ &+ 1.48851587\,t^7 - 0.82215223\,t^8 + 0.17087277\,t^9\,) \end{aligned}}$$

and

$$\boxed{t = \frac{1}{1 + 0.5\,|x|}\,.}$$

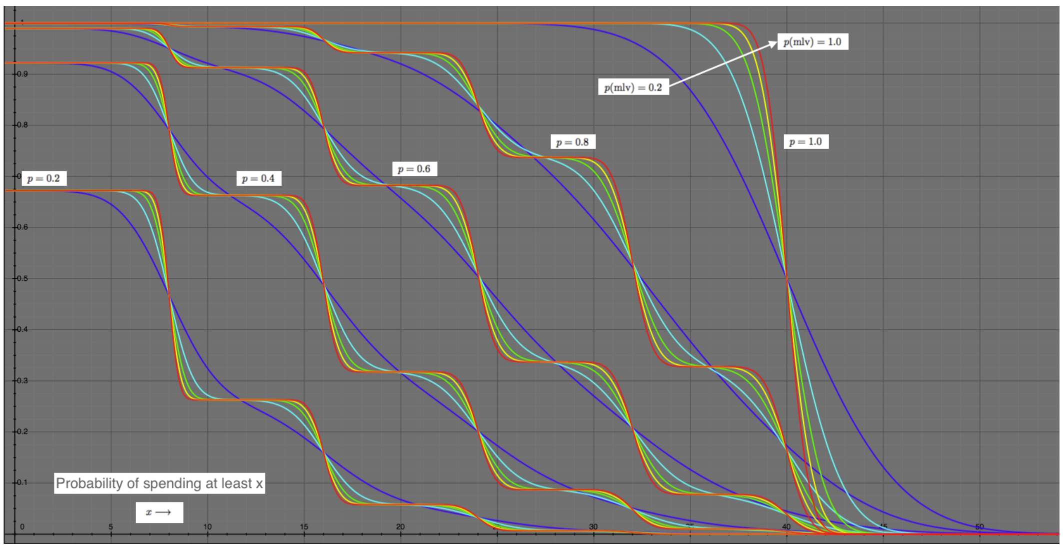


Figure 4: The probability of spending a total amount of *at least* x pounds for a week of $n = 5$ days where the probability of having lunch on any given day, p, is a member of the set $\{0.2, 0.4, 0.6, 0.8, 1.0\}$ and the most-likely lunch cost on any given lunch day is mlv $= 8$ pounds, with $p(\text{mlv})$ in the set $\{0.2, 0.4, 0.6, 0.8, 1.0\}$. $p(\text{mlv})$ increases in the direction of the white arrow.

The lunch cost probability distribution $p(x)$: Log-Normal approximation

The normal distribution has a domain of values that extends to the negative side of the real line so it's not really the best approximation for strictly non-negative expenditues. Suppose, instead, that we use a log-normal distribution for $p(x)$,

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\frac{1}{x}\,\exp\big[-\frac{(\ln x - \mu)^2}{2\sigma^2}\,\big]}$$

where x now takes values in the open interval $(0, +\infty)$. μ and σ are the *location* and *scale* parameters, respectively, and are *not* the mean and standard deviation of the log-normal distribution. Instead, the mean, mode, median, and variance (the square of the standard deviation) of the log-normal distribution are

$$\boxed{\begin{array}{rl} \text{mean:} & e^{\mu+\frac{\sigma^2}{2}} \\ \text{mode:} & e^{\mu-\sigma^2} \\ \text{median:} & e^{\mu} \\ \text{variance:} & (e^{\sigma^2}-1)\,e^{2\mu+\sigma^2}\,. \end{array}}$$

μ and σ can be computed numerically, given the desired mode mlv and the value of $p(x)$ at that point:

$$\boxed{\begin{aligned} \frac{e^{-\sigma^2/2}}{\sigma} &= \sqrt{2\pi}\,\mathrm{mlv}\,p(\mathrm{mlv}) \\ \mu &= \sigma^2 + \ln(\mathrm{mlv})\,. \end{aligned}}$$

The Fourier transform of $p(x)$: Log-Normal approximation

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The lunch cost probability distribution $p(x)$: Triangular approximation

On any given lunch day, Frank spends a minimum of 5 pounds, a maximum of 15 pounds, and a most-likely value of 8 pounds. In the absence of any other information (say, historical data), we could make the simplest possible assumption, namely, that the probability distribution of lunch costs is triangular:

$$p(x) = \begin{cases} 0\,, & \text{if } x < \min\,, \\ \dfrac{p(\mathrm{mlv}) - p(\min)}{\mathrm{mlv} - \min}\,(x - \min) + p(\min)\,, & \text{if } \min \le x \le \mathrm{mlv}\,, \\ \dfrac{p(\mathrm{mlv}) - p(\max)}{\max - \mathrm{mlv}}\,(\max - x) + p(\max)\,, & \text{if } \mathrm{mlv} \le x \le \max\,, \\ 0\,, & \text{if } x > \max\,, \end{cases}$$

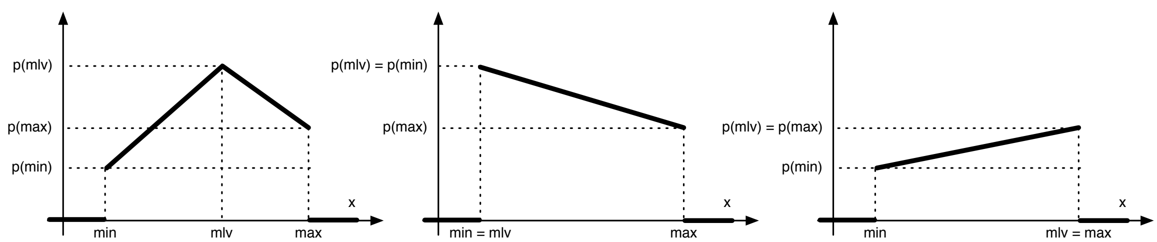
assuming $\mathrm{mlv} \neq \mathrm{min}$, $\mathrm{mlv} \neq \mathrm{max}$, $p(\mathrm{min}) \geq 0$, $p(\mathrm{max}) \geq 0$, $p(\mathrm{mlv}) \geq \max[\,p(\mathrm{min}), p(\mathrm{max})\,]$. If $\mathrm{mlv} = \mathrm{min}$ or $\mathrm{mlv} = \mathrm{max}$ then we have degenerate triangular distributions:

$$p(x) = \begin{cases} 0\,, & \text{if } x < \mathrm{min}\,, \\ \dfrac{p(\mathrm{min}) - p(\mathrm{max})}{\mathrm{max} - \mathrm{min}}\,(\mathrm{max} - x) + p(\mathrm{max})\,, & \text{if } \mathrm{min} \leq x \leq \mathrm{max}\,, \\ 0\,, & \text{if } x > \mathrm{max} \end{cases}$$

if $\mathrm{mlv} = \mathrm{min}$ and

$$p(x) = \begin{cases} 0\,, & \text{if } x < \mathrm{min}\,, \\ \dfrac{p(\mathrm{max}) - p(\mathrm{min})}{\mathrm{max} - \mathrm{min}}\,(x - \mathrm{min}) + p(\mathrm{min})\,, & \text{if } \mathrm{min} \leq x \leq \mathrm{max}\,, \\ 0\,, & \text{if } x > \mathrm{max} \end{cases}$$

if $\mathrm{mlv} = \mathrm{max}$.



We want $p(x)$ to be a probability distribution so it needs to be normalized:

$$\int_{\mathrm{min}}^{\mathrm{max}} p(x)\,dx = 1\,.$$

The integral is just the area under the curve so

$$\int_{\mathrm{min}}^{\mathrm{max}} p(x)\,dx = \frac{[\,p(\mathrm{mlv}) + p(\mathrm{min})\,]\,(\mathrm{mlv} - \mathrm{min})}{2} + \frac{[\,p(\mathrm{mlv}) + p(\mathrm{max})\,]\,(\mathrm{max} - \mathrm{mlv})}{2}\,,$$

an expression that is valid in all three cases. So, if we define a normalization constant by

$$\boxed{\mathcal{N} \equiv \frac{[\,p(\mathrm{mlv}) + p(\mathrm{min})\,]\,(\mathrm{mlv} - \mathrm{min})}{2} + \frac{[\,p(\mathrm{mlv}) + p(\mathrm{max})\,]\,(\mathrm{max} - \mathrm{mlv})}{2}}$$

we can redefine $p(x)$ to be the previous definition divided by $\mathcal{N}$. That will guarantee that $p(x)$ is normalized to 1:

$$\boxed{p(x) = \frac{1}{\mathcal{N}} \begin{cases} 0\,, & \text{if } x < \min\,, \\ \dfrac{p(\text{mlv}) - p(\min)}{\text{mlv} - \min}\,(x - \min) + p(\min)\,, & \text{if } \min \leq x \leq \text{mlv}\,, \\ \dfrac{p(\text{mlv}) - p(\max)}{\max - \text{mlv}}\,(\max - x) + p(\max)\,, & \text{if } \text{mlv} \leq x \leq \max\,, \\ 0\,, & \text{if } x > \max\,, \end{cases}}$$

if mlv $\neq$ min and mlv $\neq$ max,

$$\boxed{p(x) = \frac{1}{\mathcal{N}} \begin{cases} 0\,, & \text{if } x < \min\,, \\ \dfrac{p(\min) - p(\max)}{\max - \min}\,(\max - x) + p(\max)\,, & \text{if } \min \leq x \leq \max\,, \\ 0\,, & \text{if } x > \max\,, \end{cases}}$$

if mlv $=$ min, and

$$\boxed{p(x) = \frac{1}{\mathcal{N}} \begin{cases} 0\,, & \text{if } x < \min\,, \\ \dfrac{p(\max) - p(\min)}{\max - \min}\,(x - \min) + p(\min)\,, & \text{if } \min \leq x \leq \max\,, \\ 0\,, & \text{if } x > \max\,, \end{cases}}$$

if mlv $=$ max. Note that, in all cases, we must still have $p(\min) \geq 0$, $p(\max) \geq 0$, $p(\text{mlv}) \geq \max[\,p(\min), p(\max)\,]$, and, of course, $\min \leq \text{mlv} \leq \max$ (but $\min \neq \max$).

The Fourier transform of $p(x)$: Triangular approximation

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