Solving Polynomial Equations of Degrees 3 and 4

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Solving $p(x) = x^3 + a_2 x^2 + a_1 x + a_0 = 0$

The first goal is to remove the x^2 monomial, which we can do by defining a new variable,

$$x = y + \lambda\,,$$

where λ is to be determined. In terms of y, we have:

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 = y^3 + (3\lambda + a_2)\, y^2 + (3\lambda^2 + 2a_2\lambda + a_1)\, y + p(\lambda) = 0\,.$$

Setting the coefficient of y^2 to zero allows us to determine λ,

$$\lambda = -\frac{a_2}{3}\,,$$

from which we obtain

$$y^3 + \Big(a_1 - \frac{a_2^2}{3}\Big)\, y + \Big(a_0 - \frac{a_1 a_2}{3} + \frac{2a_2^3}{27}\Big) = 0\,.$$

Now, defining new coefficients by

$$A_0 \equiv a_0 - \frac{a_1 a_2}{3} + \frac{2a_2^3}{27} \qquad \text{and} \qquad A_1 \equiv a_1 - \frac{a_2^2}{3}\,,$$

it follows that y satisfies the simpler equation

$$y^3 + A_1 y + A_0 = 0\,.$$

We can now reduce this equation to a polynomial of second degree by transforming to yet another variable, z, as follows:

$$y = z + \frac{\alpha}{z} \quad \Rightarrow \quad z^3 + (3\alpha + A_1)\,(z + \frac{\alpha}{z}) + \frac{\alpha^3}{z^3} + A_0 = 0 \quad \Rightarrow \quad (z^3)^2 + A_0 z^3 - (\frac{A_1}{3})^3 = 0\,,$$

if we choose

$$\alpha = -\frac{A_1}{3}\,.$$

Thus, we now have

$$z = \left[-\frac{A_0}{2} + \sqrt{\frac{A_0^2}{4} + \frac{A_1^3}{27}} \right]^{1/3},$$

from which we obtain

$$y = z - \frac{A_1}{3z} \qquad \text{and} \qquad x = y - \frac{a_2}{3}\,.$$

Now, it might seem that we have 6 solutions, rather than 3, since the cubic root gives us 3 solutions for each of the 2 solutions of the square root. However, it's easy to show that they're pairwise repeated, for a total of at most 3 distinct solutions.

It's possible to write a closed-form solution for each of the three solutions, as follows. Define

$$Q \equiv \frac{A_1}{3}\,, \quad R \equiv -\frac{A_0}{2}\,, \quad \text{and} \quad D \equiv R^2 + Q^3 = \frac{A_0^2}{4} + \frac{A_1^3}{27}\,.$$

Next, define

$$S \equiv [R + \sqrt{D}]^{1/3} \quad \text{and} \quad T \equiv [R - \sqrt{D}]^{1/3}\,.$$

Finally, it's possible to show that the three solutions are

$$\begin{aligned} x_1 &= -\tfrac{a_2}{3} + (S+T)\,, \\ x_2 &= -\tfrac{a_2}{3} - \tfrac{1}{2}\,(S+T) + i\,\tfrac{\sqrt{3}}{2}\,(S-T)\,, \quad \text{and} \\ x_3 &= -\tfrac{a_2}{3} - \tfrac{1}{2}\,(S+T) - i\,\tfrac{\sqrt{3}}{2}\,(S-T)\,. \end{aligned}$$

Note that x_1 is always real while x_2 and x_3 are the complex conjugate of one another.

Solving $p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$

The first goal is the same as in the previous case, namely, to remove the second-highest monomial, a_3x^3 in this case, and the procedure is the same as before, that is, we define a new variable y by means of

$$x = y + \lambda\,,$$

from which we obtain

$$y^4 + (4\lambda + a_3)\,y^3 + (6\lambda^2 + 3\lambda a_3 + a_2)\,y^2 + (4\lambda^3 + 3\lambda^2 a_3 + 2\lambda a_2 + a_1)\,y + p(\lambda) = 0\,,$$

resulting in

$$y^4 + \left(a_2 - \frac{3a_3^2}{8}\right) y^2 + \left(a_1 - \frac{a_2 a_3}{2} + \frac{a_3^3}{8}\right) y + \left(a_0 - \frac{a_1 a_3}{4} + \frac{a_2 a_3^2}{16} - \frac{3a_3^4}{256}\right) = 0\,,$$

after the choice

$$\lambda = -\frac{a_3}{4}\,.$$

Once again, we define new coefficients,

$$A_0 \equiv a_0 - \frac{a_1 a_3}{4} + \frac{a_2 a_3^2}{16} - \frac{3a_3^4}{256}\,, \quad A_1 \equiv a_1 - \frac{a_2 a_3}{2} + \frac{a_3^3}{8}\,, \quad \text{and} \quad A_2 \equiv a_2 - \frac{3a_3^2}{8}\,,$$

so that

$$y^4 + A_2 y^2 + A_1 y + A_0 = 0\,.$$

We now add and subtract the quantity $2\alpha y^2$, where α is yet to be determined, resulting in:

$$y^4 + 2\alpha y^2 + (A_2 - 2\alpha)\, y^2 + A_1 y + A_0 = (y^2+\alpha)^2 - \Big[\,(2\alpha - A_2)\, y^2 - A_1 y + (\alpha^2 - A_0)\Big] = 0\,.$$

Next, let $P \equiv y^2 + \alpha$. If we can find an α such that the term within square brackets is a perfect square, call it Q^2, then we'll have

$$P^2 - Q^2 = (P+Q)\,(P-Q) = 0\,,$$

and we'll then have to solve $P \pm Q = 0$, possibly a simpler equation. In order for this to work, then, we need to set α such that

$$(2\alpha - A_2)\, y^2 - A_1 y + (\alpha^2 - A_0) = Q^2\,.$$

We do this as follows:

$$Q^2 = (2\alpha - A_2)\, y^2 - A_1 y + (\alpha^2 - A_0) = (2\alpha - A_2)\left[\, y^2 - \frac{A_1}{2\alpha - A_2}\, y + \frac{\alpha^2 - A_0}{2\alpha - A_2}\,\right].$$

In order for this to be a perfect square, the term within square brackets must equal $(y-\beta)^2 = y^2 - 2\beta y + \beta^2$, for some β, so

$$2\beta = \frac{A_1}{2\alpha - A_2} \qquad \text{and} \qquad \beta^2 = \frac{\alpha^2 - A_0}{2\alpha - A_2}\,,$$

which is satisfied only if

$$\beta^2 = \left[\,\frac{A_1}{2(2\alpha - A_2)}\,\right]^2 = \frac{\alpha^2 - A_0}{2\alpha - A_2}\,,$$

thus determining α. Simplifying the equation above results in

$$\alpha^3 - \frac{A_2}{2}\alpha^2 - A_0\alpha - \Big(\frac{A_2 A_0}{2} + \frac{A_1^2}{8}\Big) = 0\,.$$

This equation in α is called the **resolvent cubic** of the original polynomial equation of degree 4. Once we have solved for α, we then write

$$Q = (2\alpha - A_2)^{1/2}\Big[y - \frac{A_1}{2(2\alpha - A_2)}\Big]\,,$$

from which we can finally solve for y by solving the quadratic equation

$$P \pm Q = (y^2 + \alpha)\ \pm\ (2\alpha - A_2)^{1/2}\Big[y - \frac{A_1}{2(2\alpha - A_2)}\Big] = 0\,.$$

Once again, a naïve root count would lead us to believe that there are at most 12 distinct roots (3 cubic roots times 2 square roots per cubic root times 2 for the two signs) but, in fact, these roots appear multiple times and we have at most 4 distinct roots, as we should have for a polynomial equation of degree 4.

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