Square Roots And Irrational Numbers

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Let $N > 1$ be an integer. We want to find out which values of N are such that $\sqrt{N}$ is a rational number, that is, a number that can be written as the ratio of two integers.

Let's start by assuming that $\sqrt{N}$ is expressable as the ratio of two integers, a and b. Furthermore, assume that a and b are both positive and relatively prime since we can always divide them both by their common factor otherwise. Thus, assume that

$$\sqrt{N} = \frac{a}{b}$$

where $\gcd(a, b) = 1$.

Given the assumption above, it follows that $a^2 = Nb^2$. Now, if b is even, then its square is even and it follows that a^2 is even, implying that a is also even. This would contradict the assumption that $\gcd(a, b) = 1$ because any two even integers have 2 as a common divisor. Therefore, b must be and odd number. If b is odd, however, then so is b^2. What this means, then, is that the parity of a^2, and hence that of a, is the same as the parity of N: a is even (odd) if N is even (odd).

To summarize what we have so far, if $\sqrt{N} = a/b$ then b is an odd integer and the parity of a must be the same as that of N, that is, both odd or both even.

Suppose, then, that $N = 2n + s$, $a = 2m + s$, and $b = 2k + 1$, where n, m, and k are non-negative integers and s is either 0 or 1. When $s = 0$, N and a are both even; when $s = 1$, they're both odd. Note that $s^2 = s$. Combining these, we find

$$a^2 = Nb^2 \quad \Rightarrow \quad 2m(m + s) = 2k(k + 1)(2n + s) + n\,.$$

Note that every term is even, except possibly the last one. Therefore, if n is odd, $n = 2u + 1$, the above equation cannot be satisfied. We thus conclude that $N = 2n + s = 2(2u + 1) + s$ is such that $\sqrt{N}$ cannot be expressed as a rational number:

$$\sqrt{2(2u + 1) + s} = \sqrt{4u + (s + 2)} \text{ is irrational for } u \in \mathcal{N} \text{ and } s \in \{0, 1\}.$$

What if n is even? Say that $n = 2p$. Then $N = 2n + s = 4p + s$ and

$$a^2 = Nb^2 \quad \Rightarrow \quad m(m+s) = k(k+1)(4p+s) + p\,.$$

For $s = 1$, the left–hand side is always an even number. $k(k+1)(4p+s)$ is also always an even number, for either value of s. Thus, if p is odd, $p = 2u+1$, we once again cannot satisfy the condition imposed by the equation, which means that

$$\sqrt{4(2u+1)+1} = \sqrt{8u+5} \text{ is irrational for } u \in \mathcal{N}.$$

Summarizing, we've shown that $\sqrt{4u+2}$, $\sqrt{4u+3}$, and $\sqrt{8u+5}$, for $u \in \mathcal{N}$, are all irrational numbers. Here are some examples:

$$\begin{array}{rl}
\sqrt{4u+2}: & \sqrt{2}, \sqrt{6}, \sqrt{10}, \sqrt{14}, \sqrt{18}, \sqrt{22}, \sqrt{26}, \sqrt{30}, \ldots \\
\sqrt{4u+3}: & \sqrt{3}, \sqrt{7}, \sqrt{11}, \sqrt{15}, \sqrt{19}, \sqrt{23}, \sqrt{27}, \sqrt{31}, \ldots \\
\sqrt{8u+5}: & \sqrt{5}, \sqrt{13}, \sqrt{21}, \sqrt{29}, \sqrt{37}, \sqrt{45}, \sqrt{53}, \sqrt{61}, \ldots
\end{array}$$

Absent are the perfect squares ($\sqrt{4}, \sqrt{9}, \sqrt{16}, \sqrt{25}, \ldots$), numbers that can be expressed as integer multiples of irrational roots ($\sqrt{8} = 2\sqrt{2}$, $\sqrt{12} = 2\sqrt{3}$, $\sqrt{20} = 2\sqrt{5}, \ldots$), and some special cases ($\sqrt{17}, \sqrt{33}, \ldots$).

As it turns out, it's possible to prove that *only* the perfect squares have rational (in fact, integer) square roots and the proof of that statement is much easier than what we've done above!

First, let's prove that if $\gcd(a,b) = 1$ then $\gcd(a^2,b^2)$ must also equal 1. Let a and b be written as products of powers of prime numbers, so that

$$\frac{a}{b} = \frac{p_1^{m_1}\, p_2^{m_2}\, p_3^{m_3}\, \ldots}{q_1^{n_1}\, q_2^{n_2}\, q_3^{n_3}\, \ldots}$$

where the p's and q's are prime numbers and the m's and n's are positive integers. Since, by assumption, $\gcd(a,b) = 1$, it follows that the set of p values and the set of q values have no values in common. That goes in the other direction as well: if the two sets are disjoint then $\gcd(a,b) = 1$. Now consider the square of a/b:

$$\frac{a^2}{b^2} = \frac{p_1^{2m_1}\, p_2^{2m_2}\, p_3^{2m_3}\, \ldots}{q_1^{2n_1}\, q_2^{2n_2}\, q_3^{2n_3}\, \ldots}\,.$$

Clearly, the sets of p and q values are still disjoint, so $\gcd(a^2,b^2)$ is also 1.

Going back to our original question, let $\sqrt{N}$ be expressed as the ratio a/b. Then, $Nb^2 = a^2$. Now, since a, b, and N are all integers, this says that b^2 divides a^2. Since b^2 is obviously the largest integer that divides b^2, we see that $\gcd(a^2,b^2) = b^2$.

However, if $\gcd(a, b) = 1$, as we're assuming, then $\gcd(a^2, b^2)$ must also equal 1. Therefore, we conclude that b^2 must equal 1, which gives us $N = a^2$. Since a is an integer, this says that the only integers whose square roots are rational numbers are the perfect squares.

A Related Puzzle

An interesting related problem is to prove that there exist two irrational numbers a and b such that a^b is a rational number. The interesting part of this problem is that we can prove the statement but we need not show what the two numbers actually are.

Consider the expression

$$[(\sqrt{2}\,)^{(\sqrt{2}\,)}]^{(\sqrt{2}\,)} = (\sqrt{2}\,)^{[(\sqrt{2}\,)(\sqrt{2}\,)]} = (\sqrt{2}\,)^2 = 2\,,$$

which is clearly a rational result. This can be decomposed as:

$$\underbrace{[(\sqrt{2}\,)^{(\sqrt{2}\,)}]}_{a}^{\overbrace{(\sqrt{2}\,)}^{b}} = 2\,.$$

Now, a is either a rational number or an irrational number. If it is an irrational number then we're done; we've found two irrational numbers, namely $a = [(\sqrt{2}\,)^{(\sqrt{2}\,)}]$ and $b = \sqrt{2}$, such that a^b is rational (namely, 2). On the other hand, if a is rational, then we're also done since, then, $x = y = \sqrt{2}$ are such that x^y is a rational number (namely, a). Either way, we have shown that there exist two numbers satisfying the conditions of the puzzle; we just don't know what those numbers are!

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