A famous triangle puzzle

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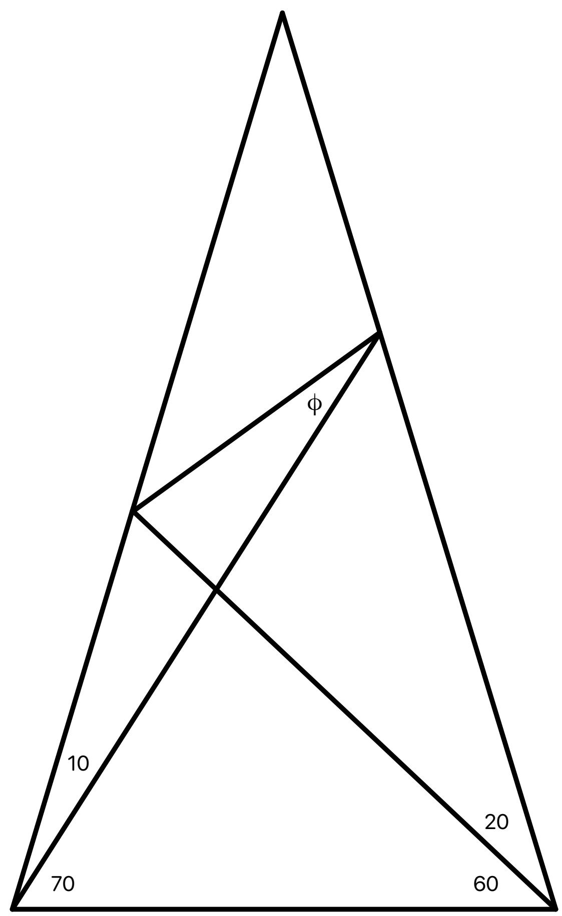
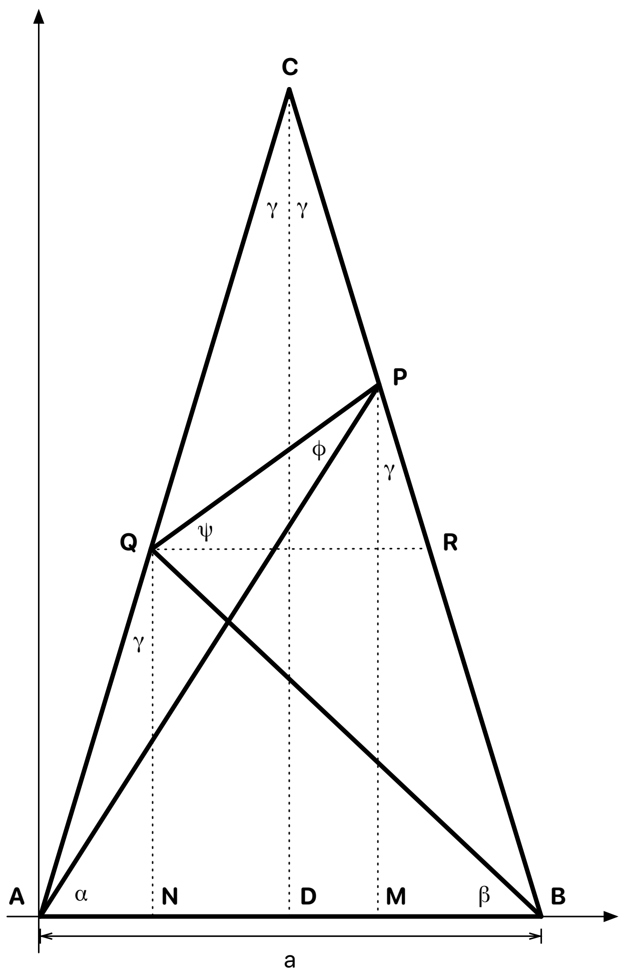


The problem is to determine the value of the angle ϕ. A repeated application of the property that the sum of the internal angles of any triangle is 180 degrees is *not* sufficient to uniquely determine ϕ but it is clear that a unique value *must* exist because, given the isosceles 80-80-20 triangle ABC, raising a segment from A at an angle of 70 degrees uniquely determines P. Likewise, raising a segment from B at the angle of 60 degrees uniquely determines Q. Therefore, the segment PQ is also uniquely determined, which makes its angle with the segment AP, namely, ϕ, also uniquely determined.

Let's use analytic geometry to solve a more general version of this puzzle, starting by setting the vertex A as the origin of a Cartesian coordinate system. Let the legnth of the segment AB have some arbitrary value a and let the segment CD bissect the angle at the vertex C. Also let PM and QN be parallel to the vertical axis, and QR be parallel to AB. The original puzzle has $\alpha = 70°$, $\beta = 60°$, and $\gamma = 10°$.



Note that

$$\tan\alpha = \frac{P_y}{P_x}\,, \qquad \tan\beta = \frac{Q_y}{a - Q_x}\,, \qquad \text{and} \qquad \tan\gamma = \frac{Q_x}{Q_y} = \frac{a - P_x}{P_y}\,.$$

The four independent equalities above give us

$$\begin{aligned} P_x &= \frac{a}{(1 + \tan\alpha\tan\gamma)} \\ P_y &= \tan\alpha\, P_x = \frac{a\,\tan\alpha}{(1 + \tan\alpha\tan\gamma)} \\ Q_x &= \tan\gamma\, Q_y = \frac{a\tan\beta\tan\gamma}{(1 + \tan\beta\tan\gamma)} \\ Q_y &= \frac{a\tan\beta}{(1 + \tan\beta\tan\gamma)} \end{aligned}$$

Next, note that $\alpha = \psi + \phi$ and

$$\tan\psi = \frac{P_y - Q_y}{P_x - Q_x}$$

so that

$$\phi = \alpha - \arctan\Big(\frac{P_y - Q_y}{P_x - Q_x}\Big)\,,$$

which gives us the angle we're after. We then need $(P_y - Q_y)$ and $(P_x - Q_x)$:

$$\begin{aligned} P_y - Q_y &= \frac{a\,(\tan\alpha - \tan\beta)}{(1 + \tan\alpha\tan\gamma)(1 + \tan\beta\tan\gamma)} \\ P_x - Q_x &= \frac{a\,(1 - \tan\alpha\tan\beta\tan^2\gamma)}{(1 + \tan\alpha\tan\gamma)(1 + \tan\beta\tan\gamma)} \end{aligned}$$

Hence,

$$\phi = \alpha - \arctan\Big(\frac{\tan\alpha - \tan\beta}{1 - \tan\alpha\tan\beta\tan^2\gamma}\Big)\,.$$

For the original problem, where $\alpha = 70°$, $\beta = 60°$, and $\gamma = 10°$, we find

$$\arctan\Big(\frac{\tan\alpha - \tan\beta}{1 - \tan\alpha\tan\beta\tan^2\gamma}\Big) = 50°$$

and, so, $\phi = 20°$.

$\blacksquare$