The eigenvalues and eigenvectors of a tridiagonal *nearly* Toeplitz matrix

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Introduction

An $N \times N$ matrix A_N is **tridiagonal** when only the main diagonal and the two diagonals immediately surrounding it have non-zero elements, that is, a matrix for which

$$(A_N)_{ij} = 0 \quad \text{if} \quad |i - j| > 1 \quad \text{for} \quad 1 \le i \le N \quad \text{and} \quad 1 \le j \le N\,.$$

An $N \times N$ tridiagonal matrix has at most $(3N - 2)$ non-zero elements. For example, the 4×4 matrix below is tridiagonal:

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & e & 0 \\ 0 & f & g & h \\ 0 & 0 & i & j \end{pmatrix} .$$

An $N \times N$ matrix A_N is **Toeplitz** when elements in each diagonal are all the same, although possibly different between diagonals, that is, a matrix for which

$$(A_N)_{j+k,\,j} = c_k \quad \text{for} \quad |k| < N \quad \text{and} \quad 1 \le j + k \le N\,, \quad 1 \le j \le N\,,$$

where the c_k values are arbitrary. It has at most $(2N - 1)$ non-zero elements. For example, the 4×4 matrix below is Toeplitz:

$$\begin{pmatrix} a & b & c & d \\ e & a & b & c \\ f & e & a & b \\ g & f & e & a \end{pmatrix} .$$

An $N \times N$ matrix A_N that is both tridiagonal **and** Toeplitz is then fully characterized by at most 3 non-zero values so it stands to reason that its eigenvalues and eigenvectors should be relatively easy to find. The 4×4 matrix below is an example of a matrix that is both tridiagonal and Toeplitz:

$$\begin{pmatrix} a & b & 0 & 0 \\ c & a & b & 0 \\ 0 & c & a & b \\ 0 & 0 & c & a \end{pmatrix} .$$

Tridiagonal *nearly* Toeplitz matrices

We'll define a tridiagonal *nearly* Toeplitz matrix to be a matrix that is tridiagonal Toeplitz except for the element in the first row and first column, an example of which is

$$\begin{pmatrix} w & b & 0 & 0 \\ c & a & b & 0 \\ 0 & c & a & b \\ 0 & 0 & c & a \end{pmatrix} .$$

The eigenvalues of a tridiagonal *nearly* Toeplitz matrix

We want to solve the eigenvalue problem for a matrix A_N that is tridiagonal but nearly Toeplitz as defined above, that is, we want to find the set of constants $\lambda_{N,n}$ and column vectors $u_{N,n}$ such that

$$A_N u_{N,n} = \lambda_{N,n} u_{N,n} ,$$

for $1 \leq n \leq N$. The eigenvalues are the N roots of A_N's *characteristic polynomial*, a polynomial of order N obtained by setting the determinant of $(A_N - \lambda\, I_N)$ to zero, where I_N is the $N \times N$ identity matrix. Now,

$$\Delta_N(\lambda) \equiv \det(A_N - \lambda I_N) = \begin{vmatrix} (w-\lambda) & b & 0 & 0 & \cdots & 0 \\ c & (a-\lambda) & b & 0 & \cdots & 0 \\ 0 & c & (a-\lambda) & b & \cdots & 0 \\ 0 & 0 & c & (a-\lambda) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & c & (a-\lambda) \end{vmatrix}$$

is an $N \times N$ determinant. We can compute it by using Laplace's method of expansion in cofactors. Defining

$$\alpha \equiv (a - \lambda)$$

and using the last column as our base column, we have:

$$\Delta_N = \boxed{\alpha}\begin{vmatrix} (w-\lambda) & b & 0 & \cdots & 0 & \mathbf{0} \\ c & \alpha & b & \cdots & 0 & \mathbf{0} \\ 0 & c & \alpha & \cdots & 0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & \boldsymbol{b} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{c} & \boxed{\boldsymbol{\alpha}} \end{vmatrix} - \boxed{b}\begin{vmatrix} (w-\lambda) & b & 0 & \cdots & 0 & \mathbf{0} \\ c & \alpha & b & \cdots & 0 & \mathbf{0} \\ 0 & c & \alpha & \cdots & 0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\alpha} & \boxed{\boldsymbol{b}} \\ 0 & 0 & 0 & \cdots & c & \boldsymbol{\alpha} \end{vmatrix},$$

where bold rows and columns are to be removed from the matrix whose determinant is being computed. The first determinant above is clearly just Δ_{N-1}. Thus,

$$\Delta_N = \alpha\,\Delta_{N-1} - \boxed{b}\begin{vmatrix} (w-\lambda) & b & 0 & \cdots & 0 & \mathbf{0} \\ c & \alpha & b & \cdots & 0 & \mathbf{0} \\ 0 & c & \alpha & \cdots & 0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\alpha} & \boxed{\boldsymbol{b}} \\ 0 & 0 & 0 & \cdots & c & \boldsymbol{\alpha} \end{vmatrix} = \alpha\,\Delta_{N-1} - b\begin{vmatrix} (w-\lambda) & b & 0 & \cdots & 0 & 0 \\ c & \alpha & b & \cdots & 0 & 0 \\ 0 & c & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & b \\ 0 & 0 & 0 & \cdots & 0 & c \end{vmatrix},$$

The remaining $(N-1)\times(N-1)$ determinant can be computed by another pass of Laplace's method, so:

$$\Delta_N = \alpha\,\Delta_{N-1} - b\boxed{c}\begin{vmatrix} (w-\lambda) & b & 0 & \cdots & 0 & \mathbf{0} \\ c & \alpha & b & \cdots & 0 & \mathbf{0} \\ 0 & c & \alpha & \cdots & 0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & \boldsymbol{b} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boxed{\boldsymbol{c}} \end{vmatrix} + b\boxed{b}\begin{vmatrix} (w-\lambda) & b & 0 & \cdots & 0 & \mathbf{0} \\ c & \alpha & b & \cdots & 0 & \mathbf{0} \\ 0 & c & \alpha & \cdots & 0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\alpha} & \boxed{\boldsymbol{b}} \\ 0 & 0 & 0 & \cdots & 0 & \boldsymbol{c} \end{vmatrix}.$$

The first determinant shown explicitly is clearly Δ_{N-2} and the second is clearly zero, since its last row contains only zeros. Therefore,

$$\Delta_N = \alpha\,\Delta_{N-1} - bc\,\Delta_{N-2}\,.$$

This recurrence relation lets us find Δ_N given its previous values but we need two consecutive starting values. Clearly,

$$\Delta_1 = (w-\lambda) \qquad \text{and} \qquad \Delta_2 = (w-\lambda)\alpha - bc\,.$$

In fact, the recurrence is also valid for $N = 0$ if we set $\Delta_0 = 1$. Thus,

$$\Delta_N = \begin{cases} 1 & \text{if } N = 0 \\ (w - \lambda) & \text{if } N = 1 \\ \alpha\,\Delta_{N-1} - bc\,\Delta_{N-2} & \text{if } N > 1\,. \end{cases}$$

Generating functions

Suppose we define a function $f(\xi)$ by a formal power series with coefficients equal to the Δ_N values:

$$f(\xi) \equiv \sum_{n=0}^{\infty} \Delta_n\,\xi^n\,.$$

If we can somehow find a closed form for this function, then we can recover the Δ_n values by expanding $f(\xi)$ in a Taylor series around $\xi = 0$:

$$f(\xi) = f(0) + f'(0)\,\xi + \frac{f''(0)}{2}\,\xi^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,\xi^n$$

which shows that $\Delta_n = f^{(n)}(0)/n!$. Such a function $f(\xi)$ is called a *generating function* for the recurrence relation in question. In order to find a closed form for it, we start with its definition and make use of the recurrence relation for Δ_n:

$$\begin{aligned} f(\xi) &= \sum_{n=0}^{\infty} \Delta_n\,\xi^n = 1 + \Delta_1\,\xi + \sum_{n=2}^{\infty} \Delta_n\,\xi^n = 1 + \Delta_1\,\xi + \sum_{n=0}^{\infty} \Delta_{n+2}\,\xi^{n+2} \\ &= 1 + \Delta_1\,\xi + \sum_{n=0}^{\infty} \Big[\alpha\,\Delta_{n+1} - bc\,\Delta_n\Big]\,\xi^{n+2} \\ &= 1 + \Delta_1\,\xi + \alpha\,\xi \sum_{n=0}^{\infty} \Delta_{n+1}\,\xi^{n+1} - bc\,\xi^2 \sum_{n=0}^{\infty} \Delta_n\,\xi^n \\ &= 1 + \Delta_1\,\xi + \alpha\,\xi\,[\,f(\xi) - 1\,] - bc\,\xi^2 f(\xi)\,. \end{aligned}$$

Thus,

$$f(\xi) = \frac{1 + (\Delta_1 - \alpha)\,\xi}{bc\,\xi^2 - \alpha\,\xi + 1} = \frac{1 + (w - a)\,\xi}{bc\,\xi^2 - \alpha\,\xi + 1}\,.$$

Therefore,

$$\Delta_n = \frac{1}{n!}\,\frac{d^n}{d\xi^n}\left[\,\frac{1 + (w - a)\,\xi}{bc\,\xi^2 - \alpha\,\xi + 1}\,\right]\Bigg|_{\xi=0}\,.$$

Expanding this n-th derivative is far too complicated as it stands but we can use yet another trick. Suppose we can write $f(\xi)$ as follows:

$$f(\xi) = \frac{1 + (w-a)\,\xi}{bc\,\xi^2 - \alpha\,\xi + 1} = \frac{1}{bc}\left[\frac{R}{\xi - r} - \frac{S}{\xi - s}\right]$$

for some as-yet-undetermined constants R, r, S, and s. The n-th derivative then becomes far simpler. Finding the constants is very simple:

$$\frac{1}{bc}\left[\frac{R}{\xi - r} - \frac{S}{\xi - s}\right] = \frac{(R-S)\,\xi - (Rs - Sr)}{bc\,(\xi - r)(\xi - s)}\,,$$

which require

$$\begin{aligned} bc\,\xi^2 - \alpha\,\xi + 1 &= bc\,(\xi - r)(\xi - s) \\ Rs - Sr &= -1 \\ R - S &= w - a\,. \end{aligned}$$

Thus

$$R = \frac{1 + (w-a)\,r}{(r-s)} \qquad \text{and} \qquad S = \frac{1 + (w-a)\,s}{(r-s)}$$

provided that r and s are the roots of $\xi^2 - \alpha\,(bc)^{-1}\xi + (bc)^{-1} = 0$, that is,

$$r = \frac{1}{2bc}\,(\alpha + \sqrt{\alpha^2 - 4bc}) \qquad \text{and} \qquad s = \frac{1}{2bc}\,(\alpha - \sqrt{\alpha^2 - 4bc})\,.$$

Next, we have

$$\frac{1}{n!}\,\frac{d^n}{d\xi^n}\left[(\xi - r)^{-1}\right]\Bigg|_{\xi=0} = \frac{1}{n!}\left[(-1)^n\,n!\,(\xi - r)^{-(n+1)}\right]\Bigg|_{\xi=0} = -\,\frac{1}{r^{n+1}}$$

so

$$\begin{aligned} \Delta_n &= \frac{1}{n!}\,\frac{d^n f(\xi)}{d\xi^n}\Bigg|_{\xi=0} = \frac{1}{bc}\,\frac{1}{n!}\,\frac{d^n}{d\xi^n}\left[\frac{R}{\xi - r} - \frac{S}{\xi - s}\right]\Bigg|_{\xi=0} = \frac{1}{bc}\left[\frac{S}{s^{n+1}} - \frac{R}{r^{n+1}}\right] \\ &= \frac{(bc)^n}{(r-s)}\left[\left(r^{n+1} - s^{n+1}\right) + \frac{w-a}{bc}\left(r^n - s^n\right)\right] \end{aligned}$$

since $rs = (bc)^{-1}$.

We've implicitly assumed that $bc \neq 0$, so the eigenvalues of A_N are then given by the roots of $\delta_N(\lambda) = 0$ where

$$\delta_N(\lambda) = \frac{\Delta_N(\lambda)}{(bc)^N} = \frac{1}{(r-s)}\left[\left(r^{N+1} - s^{N+1}\right) + \frac{w-a}{bc}\left(r^N - s^N\right)\right].$$

Note that neither r nor s is ever equal to zero. Also, $(r-s)=0$ only if $\alpha = \pm 2\sqrt{bc}$, that is, only when $\lambda = a \mp 2\sqrt{bc}$, in which case $r = s = \pm 1/\sqrt{bc}$. Since

$$\frac{bc}{r^N} \lim_{s\to r} \delta_N(\lambda) = \left[\, (N+1)\, bc\, r + N(w-a) \,\right],$$

we conclude that $\lambda = a \pm 2\sqrt{bc}$ is a root of $\Delta_N(\lambda)=0$ only when

$$w - a = \mp \frac{N+1}{N} \sqrt{bc}\,.$$

Otherwise, we may assume $r \neq s$. The roots of $\Delta_N(\lambda)=0$ then require that

$$\left(r^{N+1} - s^{N+1}\right) + \frac{w-a}{bc}\left(r^N - s^N\right) = 0\,.$$

Now recall the definitions of r and s:

$$\begin{aligned} r \;&=\; \frac{1}{2bc}\,(\alpha + \sqrt{\alpha^2 - 4bc}) \;=\; \frac{1}{\sqrt{bc}}\left[\frac{\alpha}{2\sqrt{bc}} + \sqrt{(\frac{\alpha}{2\sqrt{bc}})^2 - 1}\right] \\ s \;&=\; \frac{1}{2bc}\,(\alpha - \sqrt{\alpha^2 - 4bc}) \;=\; \frac{1}{\sqrt{bc}}\left[\frac{\alpha}{2\sqrt{bc}} - \sqrt{(\frac{\alpha}{2\sqrt{bc}})^2 - 1}\right]. \end{aligned}$$

Defining ψ by

$$\cos\psi \equiv \frac{\alpha}{2\sqrt{bc}} = \frac{a-\lambda}{2\sqrt{bc}}$$

we have

$$\begin{aligned} r \;&=\; \frac{1}{\sqrt{bc}}\,(\cos\psi + i\sin\psi) \;=\; \frac{e^{+i\psi}}{\sqrt{bc}} \\ s \;&=\; \frac{1}{\sqrt{bc}}\,(\cos\psi - i\sin\psi) \;=\; \frac{e^{-i\psi}}{\sqrt{bc}}\,. \end{aligned}$$

Plugging that into our equation for the roots of $\Delta_N(\lambda)=0$, we find

$$\frac{1}{(\sqrt{bc})^{N+1}}\left(e^{i\,(N+1)\psi} - e^{-i\,(N+1)\psi}\right) + \frac{1}{(\sqrt{bc})^N}\,\frac{w-a}{bc}\left(e^{i\,N\psi} - e^{-i\,N\psi}\right) = 0$$

or, more simply, the eigenvalues we're searching for are the N values of λ such that

$$\lambda_n = a - 2\sqrt{bc}\,\cos(\psi_n) \quad \text{with} \quad \sin\left[(N+1)\,\psi_n\right] = \frac{a-w}{\sqrt{bc}}\,\sin(N\psi_n)\,, \quad (1 \le n \le N)\,.$$

We may also write this result as follows, which turns out to be more convenient. It amounts to choosing the negative value of $\sqrt{bc}$ in the equations above.

$$\lambda_n = a + 2\sqrt{bc}\,\cos(\psi_n) \quad \text{with} \quad \sin\left[(N+1)\,\psi_n\right] = \frac{w-a}{\sqrt{bc}}\,\sin(N\psi_n)\,, \quad (1 \le n \le N)\,.$$

There doesn't seem to be a way to solve for ψ_n in closed analytical form when $a \neq w$ and $N > 1$. When $a = w$, we recover the known result for tridiagonal pure Toeplitz matrices, namely, $\psi_n = n\pi/(N+1)$.

The eigenvectors of A_N

We now want to find the set of eigenvectors associated with A_N's eigenvalues, that is, we want to find the set of (column) vectors $\{u_n \,|\, 1 \leq n \leq N\}$ such that

$$A_N \, u_n = \lambda_n \, u_n \,,$$

for each n. Since multiples of eigenvectors are also eigenvectors, we can arbitrarily choose one component of each vector, so let's define $u_n \equiv (u_{n,1}, u_{n,2}, u_{n,3}, \ldots, 1)^t$, that is, $u_{n,N} \equiv 1$. Now, starting from the bottom row,

$$A_N \, u_n = \lambda_n \, u_n \qquad \Rightarrow \qquad \begin{array}{rcl} c \, u_{n,N-1} & = & (\lambda_n - a) \, u_{n,N} \\ c \, u_{n,N-2} & = & (\lambda_n - a) \, u_{n,N-1} - b \, u_{n,N} \\ c \, u_{n,N-3} & = & (\lambda_n - a) \, u_{n,N-2} - b \, u_{n,N-1} \\ & \vdots & \\ c \, u_{n,2} & = & (\lambda_n - a) \, u_{n,3} - b \, u_{n,4} \\ c \, u_{n,1} & = & (\lambda_n - a) \, u_{n,2} - b \, u_{n,3} \\ (\lambda_n - w) \, u_{n,1} & = & b \, u_{n,2} \,. \end{array}$$

Excluding the last equation, which can be ignored because the procedure to obtain the eigenvalues has already accounted for it, these are nearly identical in form to the recurrence relation obtained earlier, that is,

$$u_{n,k} = \begin{cases} 1 & \text{if } k = N \\ \alpha_n / c & \text{if } k = N-1 \\ (\alpha_n / c) \, u_{n,k+1} - (b/c) \, u_{n,k+2} & \text{if } N-2 \geq k \geq 1 \end{cases} \qquad \text{where} \quad \alpha_n = (\lambda_n - a) \,.$$

Note that w does not appear explicitly in the equations above but we should keep in mind that λ_n does depend on it.

We now proceed just as before, defining a generating function for each eigenvector, except that we now must run the indices backwards, as it were:

$$f_n(\xi) = u_{n,N} \, \xi + u_{n,N-1} \, \xi^2 + \ldots + u_{n,2} \, \xi^{N-1} + u_{n,1} \, \xi^N = \sum_{k=1}^{N} u_{n,N-k+1} \, \xi^k$$

because the initial values of the recurrence are the *last* two components of the eigenvector. Then,

$$f_n(\xi) = \frac{\xi + \big[(b/c)\, u_{n,2} - (\alpha_n/c)\, u_{n,1}\big]\, \xi^{N+1} + (b/c)\, u_{n,1}\, \xi^{N+2}}{(b/c)\, \xi^2 - (\alpha_n/c)\, \xi + 1}\,.$$

Now, it might appear that we're in trouble because of the terms depending on $u_{n,1}$ and $u_{n,2}$ which we don't know and, in fact, are trying to find but they appear multiplied by powers of ξ above N. Therefore, any derivative of those terms of order no greater than N vanishes when $\xi = 0$. As a result, we might as well use

$$f_n(\xi) = \frac{\xi}{(b/c)\, \xi^2 - (\alpha_n/c)\, \xi + 1}$$

instead. Next, we write

$$f_n(\xi) = \frac{(c/b)\, \xi}{\xi^2 - (\alpha_n/b)\, \xi + (c/b)} = \frac{(c/b)}{(r-s)} \left[\frac{r}{\xi - r} - \frac{s}{\xi - s} \right],$$

provided that r and s are now the roots of $\xi^2 - (\alpha_n/b)\, \xi + (c/b) = 0$, that is,

$$r = \frac{1}{2b}\, (\alpha_n + \sqrt{\alpha_n^2 - 4bc}) \qquad \text{and} \qquad s = \frac{1}{2b}\, (\alpha_n - \sqrt{\alpha_n^2 - 4bc})\,.$$

Note that

$$\alpha_n = \lambda_n - a = 2\sqrt{bc}\, \cos(\psi_n)$$

so

$$\alpha_n^2 - 4bc = -4bc\, \sin^2(\psi_n)$$

and

$$r = \sqrt{\frac{c}{b}}\, e^{+i\,\psi_n} \qquad \text{and} \qquad s = \sqrt{\frac{c}{b}}\, e^{-i\,\psi_n}\,.$$

Then,

$$u_{n,N-k+1} = \frac{f_n^{(k)}(0)}{k!} = \frac{(c/b)}{(rs)^k} \left(\frac{r^k - s^k}{r - s} \right) = \left(\frac{b}{c} \right)^{(k-1)/2} \frac{\sin(k\psi_n)}{\sin(\psi_n)}\,,$$

from which we find

$$u_{n,k} = \left(\frac{b}{c} \right)^{(N-k)/2} \frac{\sin[(N+1-k)\, \psi_n]}{\sin(\psi_n)}\,.$$

We may now normalize these eigenvectors, resulting in

$$u_{n,k} = \frac{\left(\dfrac{b}{c}\right)^{(N-k)/2} \sin[(N+1-k)\,\psi_n]}{\sqrt{\displaystyle\sum_{j=1}^{N} \left(\frac{b}{c}\right)^{(N-j)} \sin^2[(N+1-j)\,\psi_n]}} = \frac{\left(\dfrac{b}{c}\right)^{(N-k)/2} \sin[(N+1-k)\,\psi_n]}{\sqrt{\displaystyle\sum_{j=1}^{N} \left(\frac{b}{c}\right)^{(j-1)} \sin^2(j\psi_n)}}$$

where $1 \le n \le N$ and $1 \le k \le N$. Let $\beta \equiv b/c$. Then,

$$\begin{aligned}
\sum_{j=1}^{N} \left(\frac{b}{c}\right)^{(j-1)} \sin^2(j\psi_n) &= \sum_{j=1}^{N} \beta^{\,j-1} \sin^2(j\psi_n) \\
&= \sum_{j=1}^{N} \beta^{\,j-1} \Big(\frac{e^{i\psi_n j} - e^{-i\psi_n j}}{2i}\Big)^2 = \sum_{j=1}^{N} \beta^{\,j-1} \Big(\frac{2 - e^{2i\psi_n j} - e^{-2i\psi_n j}}{4}\Big) \\
&= \frac{1}{2\beta} \sum_{j=1}^{N} \beta^{\,j} - \frac{1}{4\beta} \sum_{j=1}^{N} \beta^{\,j} \left(e^{2i\psi_n j} + e^{-2i\psi_n j}\right) \\
&= \frac{1}{2}\Big(\frac{\beta^N - 1}{\beta - 1}\Big) - \frac{1}{4\beta} \sum_{j=1}^{N} \left(\beta e^{+2i\psi_n}\right)^j - \frac{1}{4\beta} \sum_{j=1}^{N} \left(\beta e^{-2i\psi_n}\right)^j
\end{aligned}$$

Now,

$$\sum_{j=1}^{N} \left(\beta e^{2i\psi_n}\right)^j = \left(\beta e^{2i\psi_n}\right) \left[\frac{\beta^N e^{2iN\psi_n} - 1}{\beta e^{2i\psi_n} - 1}\right]$$

so

$$\begin{aligned}
&\sum_{j=1}^{N} \left(\frac{b}{c}\right)^{(j-1)} \sin^2(j\psi_n) = \frac{1}{2}\Big(\frac{\beta^N - 1}{\beta - 1}\Big) - \frac{1}{4\beta} \sum_{j=1}^{N} \left(\beta e^{+2i\psi_n}\right)^j - \frac{1}{4\beta} \sum_{j=1}^{N} \left(\beta e^{-2i\psi_n}\right)^j \\
&= \frac{1}{2}\Big(\frac{\beta^N - 1}{\beta - 1}\Big) - \frac{1}{4} \left\{ \frac{e^{+2i\psi_n}\,(\beta^N e^{+2iN\psi_n} - 1)}{(\beta e^{+2i\psi_n} - 1)} + \frac{e^{-2i\psi_n}\,(\beta^N e^{-2iN\psi_n} - 1)}{(\beta e^{-2i\psi_n} - 1)} \right\} \\
&= \frac{1}{2}\Big(\frac{\beta^N - 1}{\beta - 1}\Big) - \frac{1}{2} \left\{ \frac{\beta^{N+1}\cos(2N\psi_n) - \beta^N \cos[2(N+1)\psi_n] - \beta + \cos(2\psi_n)}{\beta^2 - 2\beta\cos(2\psi_n) + 1} \right\}
\end{aligned}$$

The **normalized eigenvectors** are, then, given by

$$
\begin{aligned}
u_{n,k} &\equiv \frac{1}{\sqrt{\Phi_n}}\, \beta^{(N-k)/2}\, \sin[(N+1-k)\,\psi_n] \\
\Phi_n &\equiv \frac{1}{2}\Big(\frac{\beta^N - 1}{\beta - 1}\Big) - \frac{1}{2}\left\{\frac{\beta^{N+1}\cos(2N\psi_n) - \beta^N\cos[2(N+1)\psi_n] - \beta + \cos(2\psi_n)}{\beta^2 - 2\beta\cos(2\psi_n) + 1}\right\} \\
\beta &\equiv \frac{b}{c}\,, \quad 1 \le n \le N\,, \quad \text{and} \quad 1 \le k \le N\,.
\end{aligned}
$$

Note that, when the matrix is symmetric, $b/c = 1$, and

$$
\begin{aligned}
\Phi_n &\equiv \frac{N}{2} + \frac{1}{4} + \frac{\cos[2(N+1)\psi_n] - \cos(2N\psi_n)}{8\sin^2(\psi_n)} \\
&= \frac{N}{2} + \frac{1}{4} + \frac{\sin^2(N\psi_n) - \sin^2[(N+1)\psi_n]}{4\sin^2(\psi_n)}\,.
\end{aligned}
$$

Recall, however, that

$$
\sin\big[(N+1)\,\psi_n\big] = \frac{w-a}{\sqrt{bc}}\, \sin(N\psi_n)
$$

so, when the matrix is **symmetric**, the normalized eigenvectors are

$$
\begin{aligned}
u_{n,k} &= \frac{1}{\sqrt{\Phi_n}}\, \sin[(N+1-k)\,\psi_n] \qquad \text{with} \\
\Phi_n &\equiv \frac{N+1}{2} + \frac{1}{4}\Big[\frac{\sin^2(N\psi_n)}{\sin^2(\psi_n)} - 1\Big] - \frac{1}{4}\frac{(w-a)^2}{bc}\,\frac{\sin^2(N\psi_n)}{\sin^2(\psi_n)}\,.
\end{aligned}
$$

Summary

Eigenvalues and normalized eigenvectors for a tridiagonal nearly Toeplitz matrix

$$
A = \begin{pmatrix}
\boldsymbol{w} & \boldsymbol{b} & 0 & \cdots & 0 & 0 & 0 \\
\boldsymbol{c} & \boldsymbol{a} & \boldsymbol{b} & \cdots & 0 & 0 & 0 \\
0 & \boldsymbol{c} & \boldsymbol{a} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \boldsymbol{a} & \boldsymbol{b} & 0 \\
0 & 0 & 0 & \cdots & \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{b} \\
0 & 0 & 0 & \cdots & 0 & \boldsymbol{c} & \boldsymbol{a}
\end{pmatrix}.
$$

$$
\begin{aligned}
\lambda_n &= a + 2\sqrt{bc}\,\cos(\psi_n) \qquad \text{where} \qquad \sin\big[(N+1)\,\psi_n\big] = \frac{w-a}{\sqrt{bc}}\,\sin(N\psi_n) \\
u_{n,k} &= \frac{1}{\sqrt{\Phi_n}}\,\beta^{(N-k)/2}\,\sin[(N+1-k)\,\psi_n] \\
\Phi_n &\equiv \frac{1}{2}\Big(\frac{\beta^N-1}{\beta-1}\Big) - \frac{1}{2}\left\{\frac{\beta^{N+1}\cos(2N\psi_n) - \beta^N\cos[2(N+1)\psi_n] - \beta + \cos(2\psi_n)}{\beta^2 - 2\beta\cos(2\psi_n)+1}\right\}
\end{aligned}
$$

where $\beta \equiv \dfrac{b}{c}\,, \quad 1 \le n \le N\,, \quad \text{and} \quad 1 \le k \le N\,.$

Eigenvalues and normalized eigenvectors for a symmetric tridiagonal nearly Toeplitz matrix

$$
A = \begin{pmatrix}
\boldsymbol{w} & \boldsymbol{b} & 0 & \cdots & 0 & 0 & 0 \\
\boldsymbol{b} & \boldsymbol{a} & \boldsymbol{b} & \cdots & 0 & 0 & 0 \\
0 & \boldsymbol{b} & \boldsymbol{a} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \boldsymbol{a} & \boldsymbol{b} & 0 \\
0 & 0 & 0 & \cdots & \boldsymbol{b} & \boldsymbol{a} & \boldsymbol{b} \\
0 & 0 & 0 & \cdots & 0 & \boldsymbol{b} & \boldsymbol{a}
\end{pmatrix}.
$$

$$
\begin{aligned}
\lambda_n &= a + 2\,|b|\cos(\psi_n) \qquad \text{where} \qquad \sin\big[(N+1)\,\psi_n\big] = \frac{w-a}{|b|}\,\sin(N\psi_n) \\
u_{n,k} &= \frac{1}{\sqrt{\Phi_n}}\,\sin[(N+1-k)\,\psi_n] \\
\Phi_n &\equiv \frac{N+1}{2} + \frac{1}{4}\Big[\frac{\sin^2(N\psi_n)}{\sin^2(\psi_n)} - 1\Big] - \frac{1}{4}\frac{(w-a)^2}{b^2}\,\frac{\sin^2(N\psi_n)}{\sin^2(\psi_n)}
\end{aligned}
$$

where $\qquad 1 \le n \le N \quad \text{and} \quad 1 \le k \le N\,.$

Eigenvalues and normalized eigenvectors for a tridiagonal pure Toeplitz matrix

$$A = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} & 0 & \cdots & 0 & 0 & 0 \\ \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{b} & \cdots & 0 & 0 & 0 \\ 0 & \boldsymbol{c} & \boldsymbol{a} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \boldsymbol{a} & \boldsymbol{b} & 0 \\ 0 & 0 & 0 & \cdots & \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{b} \\ 0 & 0 & 0 & \cdots & 0 & \boldsymbol{c} & \boldsymbol{a} \end{pmatrix} .$$

$$\boxed{\begin{aligned} \lambda_n \;&=\; a + 2\sqrt{bc}\,\cos(\psi_n) \qquad \text{where} \qquad \psi_n = \frac{n\,\pi}{N+1} \\ u_{n,k} \;&=\; \frac{1}{\sqrt{\Phi_n}}\,\beta^{(N-k)/2}\,\sin(k\,\psi_n) \\ \Phi_n \;&\equiv\; \frac{1}{2}\left(\frac{\beta^N - 1}{\beta - 1}\right) - \frac{1}{2}\left\{\frac{(\beta^{N+1}+1)\cos(2\psi_n) - \beta(\beta^{N-1}+1)}{\beta^2 - 2\beta\cos(2\psi_n) + 1}\right\} \\ &\text{where } \beta \equiv \frac{b}{c}\,, \quad 1 \le n \le N\,, \quad \text{and} \quad 1 \le k \le N\,. \end{aligned}}$$

Eigenvalues and normalized eigenvectors for a symmetric tridiagonal pure Toeplitz matrix

$$A = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} & 0 & \cdots & 0 & 0 & 0 \\ \boldsymbol{b} & \boldsymbol{a} & \boldsymbol{b} & \cdots & 0 & 0 & 0 \\ 0 & \boldsymbol{b} & \boldsymbol{a} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \boldsymbol{a} & \boldsymbol{b} & 0 \\ 0 & 0 & 0 & \cdots & \boldsymbol{b} & \boldsymbol{a} & \boldsymbol{b} \\ 0 & 0 & 0 & \cdots & 0 & \boldsymbol{b} & \boldsymbol{a} \end{pmatrix} .$$

$$\boxed{\begin{aligned} \lambda_n \;&=\; a + 2\,|b|\cos(\psi_n) \qquad \text{where} \qquad \psi_n = \frac{n\,\pi}{N+1} \\ u_{n,k} \;&=\; \sqrt{\frac{2}{N+1}}\,\sin(k\,\psi_n) \\ &\text{where} \qquad 1 \le n \le N \quad \text{and} \quad 1 \le k \le N\,. \end{aligned}}$$

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