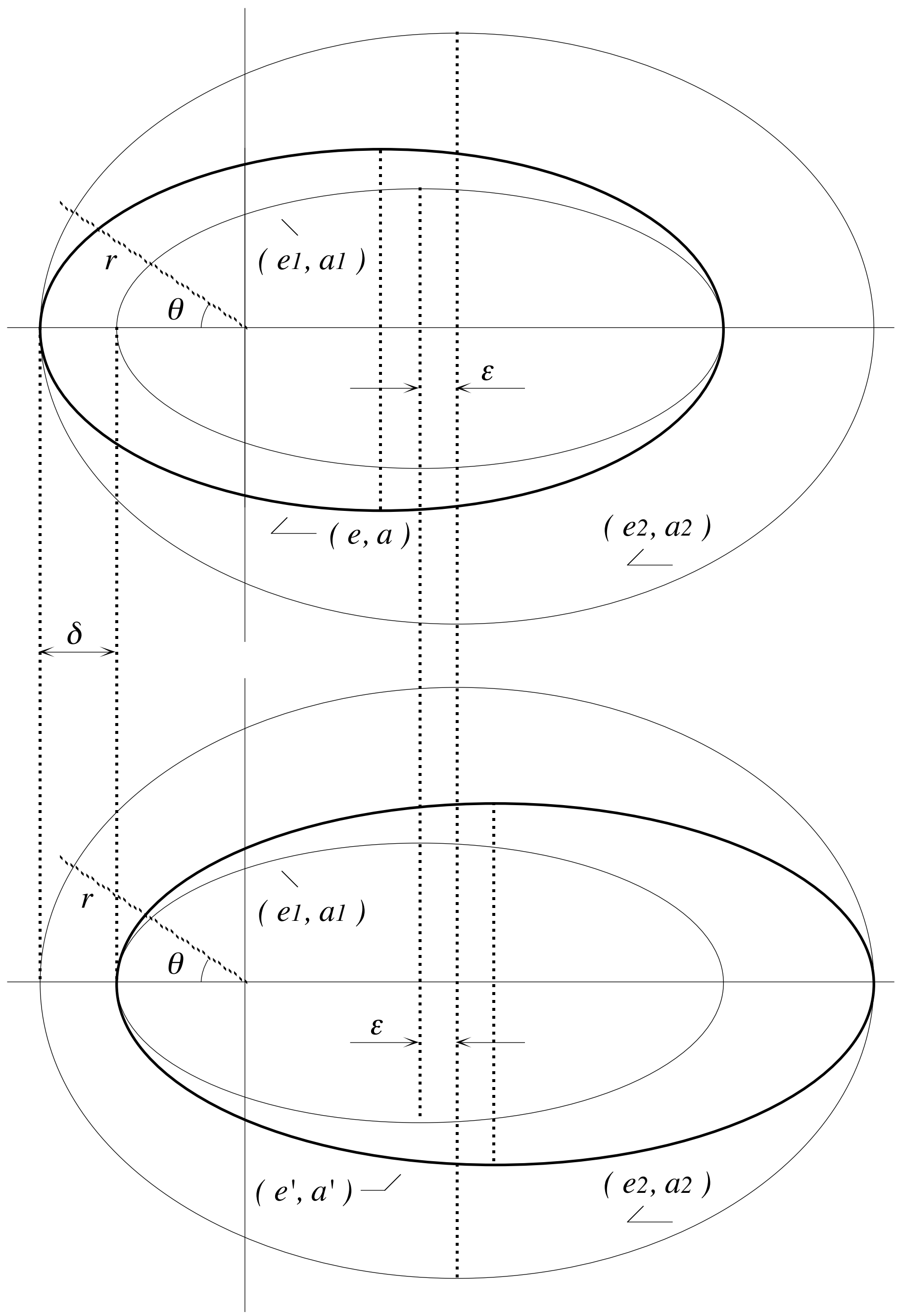
Planetary Transfer Orbits: coplanar, parallel and non intersecting case



When the source and target trajectories are both in the same plane, share the major axis, and don't intersect each other, the most fuel-efficient way to accomplish a transfer from one orbit to the other is to follow a trajectory that is tangent to both orbits at their extreme ends. In general, there are two possible such trajectories, usually different in size and shape. The one with smaller semi-major axis length gives the smallest time of travel as well as the smallest total energy for the spacecraft being transferred.

In the figure, notice that the distance between the Sun and the geometrical center of each ellipse is the focal distance of the ellipse under consideration, which we'll denote by f. We then have $\varepsilon = f_2 - f_1 = e_2 a_2 - e_1 a_1$. We also have $\varepsilon = a_2 - a_1 - \delta$, from which it follows

$$\begin{aligned} \delta &= (1 - e_2)\, a_2 - (1 - e_1)\, a_1 \\ &= (a_2 - a_1) - (e_2 a_2 - e_1 a_1). \end{aligned}$$

But the equalities $\delta = 2\,(a - a_1) = 2\,(a_2 - a')$ also hold, and so

$$\boxed{\begin{aligned} a &= \frac{1}{2}\left[(a_1 + a_2) + (e_1 a_1 - e_2 a_2)\right] \text{ and} \\ a' &= \frac{1}{2}\left[(a_1 + a_2) - (e_1 a_1 - e_2 a_2)\right]. \end{aligned}}$$

Now, to determine the eccentricities e, e' we have to recall that the general equation describing an ellipse, in polar coordinates, is

$$r = \frac{a\,(1 - e^2)}{1 + e\cos\theta}.$$

Thus, the *perihelion* position, that is, the position closest to the Sun, is given by $\theta = 0$, whereas the *aphelion*, or position of largest separation from the Sun, corresponds to $\theta = \pi$. In terms of r, these positions are thus:

$$\begin{aligned} r &= a\,(1 - e), \text{ for perihelion, and} \\ r &= a\,(1 + e), \text{ for aphelion.} \end{aligned}$$

Notice, now, in the top figure, that the perihelion of the transfer orbit coincides with the perihelion of the *outer* orbit, whereas in the bottom figure, the perihelion of the transfer orbit coincides with the perihelion of the *inner* orbit. Notice, also, that δ equals the distance between the perihelia of the outer and inner trajectories. Putting all this information together, we may write:

$$\begin{aligned} \delta &= a\,(1 - e) - a_1\,(1 - e_1) \\ &= a_2\,(1 - e_2) - a'\,(1 - e'). \end{aligned}$$

Replacing a and a' by the expressions we derived above and isolating e and e' results in

$$\boxed{\begin{array}{c} e = \dfrac{a_1\,(\,1 + e_1\,) - a_2\,(\,1 - e_2\,)}{a_1\,(\,1 + e_1\,) + a_2\,(\,1 - e_2\,)} \\ \text{and} \\ e' = \dfrac{a_2\,(\,1 + e_2\,) - a_1\,(\,1 - e_1\,)}{a_2\,(\,1 + e_2\,) + a_1\,(\,1 - e_1\,)}\;. \end{array}}$$

Now, let's try to determine which solution has the smallest semi-major axis length. From the expressions derived above, it follows that

$$a' - a = e_2\,a_2 - e_1\,a_1$$

is the relevant factor in determining such a solution. Let's assume, for the sake of being definite, that $(\,a, e\,)$ is the solution we seek. We may now determine what change in speed is required to actually accomplish the transfer.

Recall that the total energy of a particle (of mass m) orbiting another and neglecting the recoil of the latter (supposed to have a much larger mass M), is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}\;.$$

So, the difference in speed of the outer orbit and the transfer orbit, at the perihelion of the outer orbit, is

$$\begin{aligned} v_2^2 - v_-^2 &= -\,GM\,\left(\,\frac{1}{a_2} - \frac{1}{a}\,\right) = -\,\frac{GM}{a_2}\,\left(\,\frac{a - a_2}{a}\,\right) \\ &= \frac{GM}{a_2}\,\left[\,\frac{(\,a_2 - a_1\,) + (\,e_2\,a_2 - e_1\,a_1\,)}{(\,a_2 + a_1\,) - (\,e_2\,a_2 - e_1\,a_1\,)}\,\right]. \end{aligned}$$

Similarly, the difference in speed of the transfer orbit and the inner orbit, at the aphelion of the inner orbit, is

$$\begin{aligned} v_+^2 - v_1^2 &= -\,GM\,\left(\,\frac{1}{a} - \frac{1}{a_1}\,\right) = \frac{GM}{a_1}\,\left(\,\frac{a - a_1}{a}\,\right) \\ &= \frac{GM}{a_1}\,\left[\,\frac{(\,a_2 - a_1\,) - (\,e_2\,a_2 - e_1\,a_1\,)}{(\,a_2 + a_1\,) - (\,e_2\,a_2 - e_1\,a_1\,)}\,\right]. \end{aligned}$$