Two-Dimensional Projectile Motion Subjected To Air Resistance: A Detailed Treatment

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1 Introduction

We'll attempt to answer the following questions:

- How do the position and velocity of the projectile depend on time?
- What equation describes its trajectory?
- How does the time spent in the upward motion compare with the time spent coming down?
- What are the maximum height and maximum range reached by the projectile for a given initial velocity?
- What velocity does the projectile have when it returns to its initial vertical position?

2 1-D Vertical motion, no air resistance

- How do the position and velocity of the projectile depend on time?

$$\begin{aligned} y(t) &= v_0 t - \frac{1}{2} g t^2 \\ v(t) &= v_0 - g t \end{aligned}$$

- How does the time spent in the upward motion compare with the time spent coming down? They're equal, $\Delta t_{\mathrm{up}} = \Delta t_{\mathrm{down}} = v_0/g$.
- What is the maximum height reached by the projectile for a given initial velocity? $H = v_0^2/(2g)$.

- What velocity does the projectile have when it returns to its initial vertical position? Same speed (v_0), opposite direction.

3 1-D Vertical motion, with air resistance: linear model

Equations of motion:
$\dot{v} = -g - \alpha\, v, \quad v(0) = v_0, \quad \dot{y} = v, \quad y(0) = 0.$

Dimensionless variables:
$\tau = \alpha\, t, \quad \nu = (\alpha/g)\, v, \quad \xi = (\alpha^2/g)\, y.$

Dimensionless equations of motion:
$\frac{d\nu}{d\tau} = -1 - \nu, \quad \nu(0) = \nu_0 \equiv \alpha\, v_0/g, \quad \frac{d\xi}{d\tau} = \nu, \quad \xi(0) = 0.$

- How do the position and velocity of the projectile depend on time?

$$
\begin{aligned}
\xi(\tau) &= (1+\nu_0)(1-e^{-\tau}) - \tau \\
\nu(\tau) &= (1+\nu_0)e^{-\tau} - 1
\end{aligned}
$$

- How does the time spent in the upward motion compare with the time spent coming down?

$$
\begin{aligned}
\left.\frac{d\xi}{d\tau}\right|_{\tau=\Delta\tau_{\rm up}} = \nu(\Delta\tau_{\rm up}) &= 0 \;\Rightarrow & \Delta\tau_{\rm up} &= \ln(1+\nu_0) \\
\xi(\Delta\tau_{\rm up} + \Delta\tau_{\rm down}) &= 0 \;\Rightarrow & \Delta\tau_{\rm down} + e^{-\Delta\tau_{\rm down}} &= (1+\nu_0) - \ln(1+\nu_0)
\end{aligned}
$$

  Note the peculiar relation: $e^{-\Delta\tau_{\rm down}} + \Delta\tau_{\rm down} = e^{+\Delta\tau_{\rm up}} - \Delta\tau_{\rm up}$.

- What is the maximum height reached by the projectile for a given initial velocity? $\xi_{\rm max} = \xi(\Delta\tau_{\rm up}) = \nu_0 - \ln(1+\nu_0)$.

  Note the peculiar relation: $H = \frac{g}{\alpha^2}\,\xi_{\rm max} = \frac{1}{\alpha}(v_0 - g\Delta t_{\rm up})$.

- What velocity does the projectile have when it returns to its initial vertical position? $\nu(\tau = \Delta\tau_{\rm up} + \Delta\tau_{\rm down}) = \nu_0 - (\Delta\tau_{\rm up} + \Delta\tau_{\rm down}) < \nu_0$.

4 1-D Vertical motion, with air resistance: quadratic model

Equations of motion:
$\dot{v} = -g - \beta\, v^2, \quad v(0) = v_0, \quad \dot{y} = v, \quad y(0) = 0.$

Dimensionless variables:
$\tau = (\beta g)^{1/2}\, t, \quad \nu = (\beta/g)^{1/2}\, v, \quad \xi = \beta\, y.$

Dimensionless equations of motion:
$\frac{d\nu}{d\tau} = -1 - \nu^2, \quad \nu(0) = \nu_0 \equiv (\beta/g)^{1/2}\, v_0, \quad \frac{d\xi}{d\tau} = \nu, \quad \xi(0) = 0.$

- How do the position and velocity of the projectile depend on time?

$$\begin{aligned} \xi(\tau) &= \ln\Big|\cos(\tau) + \nu_0 \sin(\tau)\Big| \\ \nu(\tau) &= \tan\Big[\arctan(\nu_0) - \tau\Big] = \frac{\nu_0 - \tan(\tau)}{1 + \nu_0\, \tan(\tau)} \end{aligned}$$

- How does the time spent in the upward motion compare with the time spent coming down?

$$\begin{array}{rclcrclcl} \left.\frac{d\xi}{d\tau}\right|_{\tau = \Delta\tau_{\rm up}} = \nu(\Delta\tau_{\rm up}) &=& 0 & \Rightarrow & \Delta\tau_{\rm up} &=& \arctan(\nu_0) && \\ \xi(\Delta\tau_{\rm up} + \Delta\tau_{\rm down}) &=& 0 & \Rightarrow & \Delta\tau_{\rm up} + \Delta\tau_{\rm down} &=& \arctan\left(\frac{2\nu_0}{1-\nu_0^2}\right) &=& 2\arctan(\nu_0) \\ && & \Rightarrow & \Delta\tau_{\rm down} &=& \arctan(\nu_0) &=& \Delta\tau_{\rm up}\,!! \end{array}$$

- What is the maximum height reached by the projectile for a given initial velocity?
  $\xi_{\rm max} = \xi(\Delta\tau_{\rm up}) = \frac{1}{2}\ln(1 + \nu_0^2).$

- What velocity does the projectile have when it returns to its initial vertical position? Remarkably, despite the effects of dissipation, the projectile returns to its initial vertical position with the same speed it had when it left, namely, ν_0:
  $\nu(\Delta\tau_{\rm up} + \Delta\tau_{\rm down}) = \tan\Big[\arctan(\nu_0) - (\Delta\tau_{\rm up} + \Delta\tau_{\rm down})\Big] = -\nu_0.$

5 1-D Vertical motion, with air resistance: general quadratic model

Equations of motion:
$\dot{v} = -g - \alpha\, v - \beta\, v^2, \quad v(0) = v_0, \quad \dot{y} = v, \quad y(0) = 0.$

Equivalent equations of motion, obtained by 'completing the squares':
$\dot{v} = -\Big(g - \frac{\alpha^2}{4\beta}\Big) - \beta\Big(v + \frac{\alpha}{2\beta}\Big)^2, \quad v(0) = v_0, \quad \dot{y} = v, \quad y(0) = 0.$

Another set of equivalent equations of motion:
$\dot{u} = -\bar{g} - \beta\, u^2, \quad u(0) = v_0 + \frac{\alpha}{2\beta}, \quad \dot{y} = u - \frac{\alpha}{2\beta}, \quad y(0) = 0,$
where $u \equiv v + \frac{\alpha}{2\beta}$ and $\bar{g} \equiv g - \frac{\alpha^2}{4\beta}$.

This is now essentially the same problem as that of the previous section.

Dimensionless variables:
$\tau = (\beta\bar{g})^{1/2}\, t, \quad \nu = (\beta/\bar{g})^{1/2}\, u, \quad \xi = \beta\, y.$

Dimensionless equations of motion:
$\frac{d\nu}{d\tau} = -1 - \nu^2, \quad \nu(0) = \nu_0 \equiv (\beta/\bar{g})^{1/2}\,(v_0 + \frac{\alpha}{2\beta}), \quad \frac{d\xi}{d\tau} = \nu - \frac{\alpha}{2}(\beta\bar{g})^{-1/2}, \quad \xi(0) = 0.$

- How do the position and velocity of the projectile depend on time?

$$\begin{aligned}
\xi(\tau) &= \ln\Big|\cos(\tau) + \nu_0 \sin(\tau)\Big| - \frac{\alpha}{2}(\beta\bar{g})^{-1/2}\,\tau \\
\nu(\tau) &= \tan\Big[\arctan(\nu_0) - \tau\Big] = \frac{\nu_0 - \tan(\tau)}{1 + \nu_0\,\tan(\tau)} \\
y(t) &= \frac{1}{\beta}\ln\Big|\cos(\tau) + \nu_0 \sin(\tau)\Big| - \frac{\alpha}{2\beta}\,t \\
v(t) &= (\beta/\bar{g})^{-1/2}\,\nu(\tau) - \frac{\alpha}{2\beta}
\end{aligned}$$

- How does the time spent in the upward motion compare with the time spent coming down?

$$\begin{aligned}
\frac{d\xi}{d\tau}\Big|_{\tau=\Delta\tau_{\rm up}} \;=\; 0 \;&\Rightarrow\; \nu(\Delta\tau_{\rm up}) \;=\; \frac{\alpha}{2}(\beta\bar{g})^{-1/2} \\
&\Rightarrow \quad \Delta\tau_{\rm up} \;=\; \arctan(\nu_0) \;-\; \arctan\Big[\frac{\alpha}{2}(\beta\bar{g})^{-1/2}\Big] \\
\xi(\Delta\tau_{\rm up} + \Delta\tau_{\rm down}) \;=\; 0 \;&\Rightarrow\; \text{ugly}\ldots
\end{aligned}$$

- What is the maximum height reached by the projectile for a given initial velocity?
$\xi_{\rm max} = \xi(\Delta\tau_{\rm up}) = \frac{1}{2}\ln\Big[\frac{\bar{g}}{g}\,(1+\nu_0^2)\Big] - \frac{\alpha}{2}(\beta\bar{g})^{-1/2}\,\Delta\tau_{\rm up}.$

- What velocity does the projectile have when it returns to its initial vertical position?
$\nu(\Delta\tau_{\rm up} + \Delta\tau_{\rm down}) = \text{ugly}\ldots$

6 2-D motion, no air resistance

- How do the position and velocity of the projectile depend on time?

$$
\begin{aligned}
x(t) &= v_0 \cos\theta_0\, t \\
y(t) &= v_0 \sin\theta_0\, t - \frac{1}{2} g t^2 \\
v_x(t) &= v_0 \cos\theta_0 \\
v_y(t) &= v_0 \sin\theta_0 - g t \\
v(t) &= v_0 \sqrt{1 - \frac{2\sin\theta_0 g t}{v_0} + (\frac{gt}{v_0})^2} \\
\tan\theta(t) &= \tan\theta_0 - \frac{g t}{v_0 \cos\theta_0}
\end{aligned}
$$

- What equation describes its trajectory? $y(x) = \tan\theta_0\, x - \frac{1}{2} \frac{g}{v_0^2 \cos^2\theta_0}\, x^2$
- How does the time spent in the upward motion compare with the time spent coming down? They're equal, $\Delta t_{\rm up} = \Delta t_{\rm down} = \frac{v_0 \sin\theta_0}{g}$
- What are the maximum height and maximum range reached by the projectile for a given initial velocity?

$$
\begin{aligned}
H &= \frac{v_0^2 \sin^2\theta_0}{2g} \\
R &= \frac{v_0^2}{2g}\, \sin(2\theta_0)
\end{aligned}
$$

7 2-D motion, with air resistance: general considerations

Equations of motion:
$\dot{\vec{v}} = \vec{g} - f(v)\, \frac{\vec{v}}{v}$, $\quad \vec{v}(0) = \vec{v}_0 = v_0\, (\cos\theta_0, \sin\theta_0)$, $\quad \vec{g} = (0, -g)$.

Let $\vec{v}(t) = v(t)\, [\cos\theta(t), \sin\theta(t)]$, with $v(0) = v_0$ and $\theta(0) = \theta_0$. We may then obtain separate equations for $v(t)$ and $\theta(t)$:

$$
\begin{aligned}
\dot{v} &= -f(v) - g \sin\theta \\
v\, \dot{\theta} &= -g \cos\theta
\end{aligned}
$$

Now let $\psi(t) = \sin\theta(t)$. Then, $\dot{\psi} = \cos\theta\, \dot{\theta}$ and we have the equivalent equations:

$$
\begin{aligned}
\dot{v} &= -f(v) - g\, \psi \\
v\, \dot{\psi} &= -g\, (1 - \psi^2)
\end{aligned}
$$

From these, we can decouple $v(t)$ from $\psi(t)$:

$$\begin{aligned} -g\,\psi &= \dot{v} + f(v) \\ \dot{\psi} &= -\frac{g}{v}\,(1-\psi^2) \\ \frac{d}{dt}[\dot{v} + f(v)] = -g\,\dot{\psi} &= \frac{g^2}{v}\,(1-\psi^2) \\ v\,\frac{d}{dt}[\dot{v} + f(v)] &= g^2 - (g\,\psi)^2 \end{aligned}$$

$$\boxed{v\,\frac{d}{dt}[\dot{v} + f(v)] = g^2 - [\dot{v} + f(v)]^2}$$

In addition, once we've solved the equation for $v(t)$, obtaining $\theta(t)$ is just a quadrature away. From $v\,\dot{\theta} = -g\,\cos\theta$, we obtain:

$$\boxed{\sin\theta(t) = \frac{\sin\theta_0 - \tanh h(t)}{1 - \sin\theta_0\,\tanh h(t)}\,, \quad \text{where} \quad h(t) \equiv g\int_0^t \frac{dt'}{v(t')}\,.}$$

8 2-D motion, with air resistance: linear model

In the linear model, $f(v) = \alpha\,v$. Expanding the equation for $v(t)$ derived in the previous section, we obtain:

$$\begin{aligned} v\,\frac{d}{dt}[\dot{v} + f(v)] &= g^2 - [\dot{v} + f(v)]^2 \\ v\,(\ddot{v} + \alpha\,\dot{v}) &= g^2 - (\dot{v} + \alpha\,v)^2 \\ (\dot{v}^2 + v\,\ddot{v}) + 3\alpha\,v\,\dot{v} &= g^2 - \alpha^2 v^2 \\ \frac{d}{dt}(v\,\dot{v}) + 3\alpha\,(v\,\dot{v}) &= g^2 - \alpha^2 v^2 \\ \frac{d}{dt}\frac{d}{dt}(\frac{v^2}{2}) + 3\alpha\,\frac{d}{dt}(\frac{v^2}{2}) &= g^2 - 2\alpha^2(\frac{v^2}{2}) \end{aligned}$$

Defining $u = v^2/2$, we arrive at a second-order linear equation on u, $\ddot{u} + 3\alpha\,\dot{u} + 2\alpha^2\,u = g^2$, whose general solution is $u(t) = A\exp(-2\alpha\,t) + B\exp(-\alpha\,t) + g^2/(2\alpha^2)$. Since $u = v^2/2$, it follows that $u(0) = v_0^2/2$ and $\dot{u}(0) = v_0\,\dot{v}(0)$. Then, since $\dot{v}(0) = -f(v_0) - g\sin\theta_0 = -(\alpha\,v_0 + g\,\sin\theta_0)$, we obtain:

$$\begin{aligned} u(0) &= A + B + \frac{g^2}{2\alpha^2} &&= \frac{v_0^2}{2} \\ -\dot{u}(0) &= \alpha\,(2A + B) &&= v_0\,(\alpha\,v_0 + g\,\sin\theta_0)\,, \end{aligned}$$

from which it follows that

$$\begin{aligned} A &= \frac{1}{2}\Big[(v_0\,\sin\theta_0 + g/\alpha)^2 + v_0^2\,\cos^2\theta_0\Big] \\ B &= -\,(\frac{g}{\alpha})^2\,(1 + \frac{\alpha\,v_0\,\sin\theta_0}{g}) \end{aligned}$$

and

$$v(t) = \sqrt{(\frac{g}{a})^2 + \Big[(v_0\,\sin\theta_0 + g/\alpha)^2 + v_0^2\,\cos^2\theta_0\Big]e^{-2\alpha\,t} - 2(\frac{g}{\alpha})^2(1 + \frac{\alpha\,v_0\,\sin\theta_0}{g})e^{-\alpha\,t}}$$

How does the time spent in the upward motion compare with the time spent coming down? Since $u = v^2/2$, we determine $\Delta t_{\rm up}$ from $u(\Delta t_{\rm up}) = 0$. Thus,

$$A\,z^2 + B\,z + g^2/(2\alpha^2) = 0\,, \quad \text{where} \quad z \equiv \exp(-\alpha\,\Delta t_{\rm up})\,,$$

so

$$\exp(-\alpha\,\Delta t_{\rm up}) = z = \frac{-B \pm \sqrt{B^2 - 2Ag^2/\alpha^2}}{2A}$$

9 2-D motion, with air resistance: quadratic model

10 2-D motion, with air resistance: general quadratic model

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