Relativistic Motion Under A Constant Rest-Frame Force

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Abstract

The goal is to obtain the solution, from the perspective of an inertial reference frame, for the relativistic motion of a particle under the influence of a force which is constant in the particle's own frame.

We begin by obtaining the form of a particle's acceleration 4-vector. Recall the definition of the particle's 4-velocity, $u^\mu \equiv dx^\mu \,/\, d\tau$, whose contravariant components in an inertial reference frame are given by $\gamma\,(c, \vec{v}\,)$. Defining the 4-acceleration in the obvious way, we obtain

$$a^\mu \equiv \frac{du^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2}\,.$$

Now recall that $d/d\tau = \gamma\, d/dt$, so

$$(a^\mu) = \left(\gamma\,\frac{d}{dt}\left[\gamma\,\frac{dx^\mu}{dt}\right]\right) = \gamma\,\frac{d}{dt}\Big[\gamma\,(c, \vec{v}\,)\Big] = \gamma\,\frac{d\gamma}{dt}\,(c, \vec{v}\,) + \gamma^2\,(0, \vec{a}\,) = \frac{1}{2}\,\frac{d\gamma^2}{dt}\,(c, \vec{v}\,) + \gamma^2\,(0, \vec{a}\,)\,.$$

Since $\gamma^2 = (1-\beta^2)^{-1}$, we have

$$\frac{1}{2}\,\frac{d\gamma^2}{dt} = (1-\beta^2)^{-2}\,\vec{\beta}\cdot\frac{d\vec{\beta}}{dt} = \gamma^4\,\frac{\vec{v}\cdot\vec{a}}{c^2}$$

and

$$(a^\mu) = \gamma^4\,\frac{\vec{v}\cdot\vec{a}}{c^2}\,(c, \vec{v}\,) + \gamma^2\,(0, \vec{a}\,)\,.$$

These are the contravariant components of the acceleration 4-vector in a given inertial reference frame (call it $\mathcal{S}$). In an inertial reference frame instantaneously at rest with respect to the accelerating particle when the particle's velocity with respect to $\mathcal{S}$ is $\vec{v}$ (call it $\mathcal{S}'_{\vec{v}}$), the contravariant components of the 4-acceleration take the form $(0, \vec{a}_{\mathrm{RF}})$.

Assuming that the motion is along the common x, x' axes and using the Lorentz transformations, we find that the particle's acceleration, measured in $\mathcal{S}$, is related to that measured in the particle's instantaneous rest frame ($\mathcal{S}'_{\vec{v}}$) by

$$\begin{aligned} a^0 &= \gamma\,(a'^0 + \beta\,a'^1) \\ a^1 &= \gamma\,(a'^1 + \beta\,a'^0) \end{aligned}$$

that is,

$$\begin{aligned}\gamma^4 \frac{v\,a}{c^2}\, c &= \gamma\,\beta\, a_{\mathrm{RF}} \\ \gamma^4 \frac{v\,a}{c^2}\, v + \gamma^2\, a &= \gamma\, a_{\mathrm{RF}}\end{aligned}$$

both of which result in

$$a = \frac{a_{\mathrm{RF}}}{\gamma^3} = a_{\mathrm{RF}}\,(1-\beta^2)^{3/2}\,.$$

This is a differential equation for the particle's motion, as observed in the inertial reference frame $\mathcal{S}$:

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = c\,\frac{d\beta}{dt} = a_{\mathrm{RF}}\,(1-\beta^2)^{3/2}\,.$$

Integrating it under the assumption of a constant a_{RF}, we find:

$$\gamma\,\beta = \gamma_0\,\beta_0 + \frac{a_{\mathrm{RF}}}{c}\,t$$

where $\beta_0 = \beta(t=0)$. Note how the non-relativistic result $v = v_0 + a_{\mathrm{RF}}\,t$ is properly recovered when $\beta \ll 1$. Isolating β, we get:

$$\beta(t) = \frac{\gamma_0\,\beta_0 + (a_{\mathrm{RF}}/c)\,t}{\sqrt{1+\Big[\gamma_0\,\beta_0 + (a_{\mathrm{RF}}/c)\,t\Big]^2}}\,.$$

Integrating it once more, we obtain:

$$x - x_0 = \frac{c^2}{a_{\mathrm{RF}}}\sqrt{1+\Big[\gamma_0\,\beta_0 + (a_{\mathrm{RF}}/c)\,t\Big]^2} - \frac{c^2}{a_{\mathrm{RF}}}\sqrt{1+\Big[\gamma_0\,\beta_0\Big]^2}$$

where $x_0 = x(t=0)$. This result, too, is easily seen to approach the non-relativistic one when $\beta \ll 1$. We can also determine how the particle's proper time relates to time as measured by an observer in the inertial frame $\mathcal{S}$. Recall that $d\tau = dt\,/\,\gamma$. Thus,

$$\tau = \int_0^t \sqrt{1-\beta^2(t')}\,dt' = \int_0^t \frac{dt'}{\sqrt{1+\Big[\gamma_0\,\beta_0 + (a_{\mathrm{RF}}/c)\,t'\Big]^2}}$$

which results in

$$\tau = \frac{c}{a_{\mathrm{RF}}}\,\ln\Big[\,\frac{a_{\mathrm{RF}}}{c}\,t + \sqrt{1+(\frac{a_{\mathrm{RF}}}{c}\,t)^2}\,\Big]\,.$$

As an application, consider the motion of a spacecraft taking off from rest near Earth, bound to α-*Centauri* (3.9 light-years away). If its proper acceleration (the acceleration in its rest frame) is constant and equal to Earth's gravity ($g = 10\ m/s^2$) during the first half of the trip but negative g during the second half (so as to arrive at the destination at rest) how long will the trip take, according to passengers on the craft and to the people left on Earth? And how fast will the craft be moving when it reaches the halfway point?

Using the result derived for position as a function of time in the inertial frame at rest with respect to the Earth, and setting $\beta_0 = 0$, we obtain:

$$L = \frac{c^2}{g}\sqrt{1 + \Big[\,(g/c)\,\Delta t\,\Big]^2} - \frac{c^2}{g} \qquad \Rightarrow \qquad \Delta t = \frac{c}{g}\sqrt{(1 + \frac{gL}{c^2})^2 - 1}$$

With $L = 1.95$ light-years and $g = 10\ m/s^2$, we find $\Delta t \approx 2.741$ years, so that the entire trip takes, with respect to an observer left behind on Earth, approximately 5.5 years. According to an observer in the spacecraft, however, using the result obtained for $\tau(t)$, each half of the journey will last approximately 1.694 years, or approximately 3.4 years total. At the midpoint, the craft's speed with respect to the Earth will be approximately $0.945\,c$.

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