Laplace's Equation In Bi-Spherical Coordinates And The Potential Between Two Spherical Conductors

Wagner L. Truppel

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Bi-Spherical Coordinates

The most common definition of the bi-spherical coordinate system (μ, ν, φ) is as depicted below. μ measures the ratio of the distances from the point under consideration, P, to each of the two foci, F_1 and F_2, which are located at the points with Cartesian coordinates $x = y = 0$ and $z = \pm a$, ν is the angle between the two segments connecting P to the foci, and φ measures the angle from the xz plane to the plane containing those two segments.

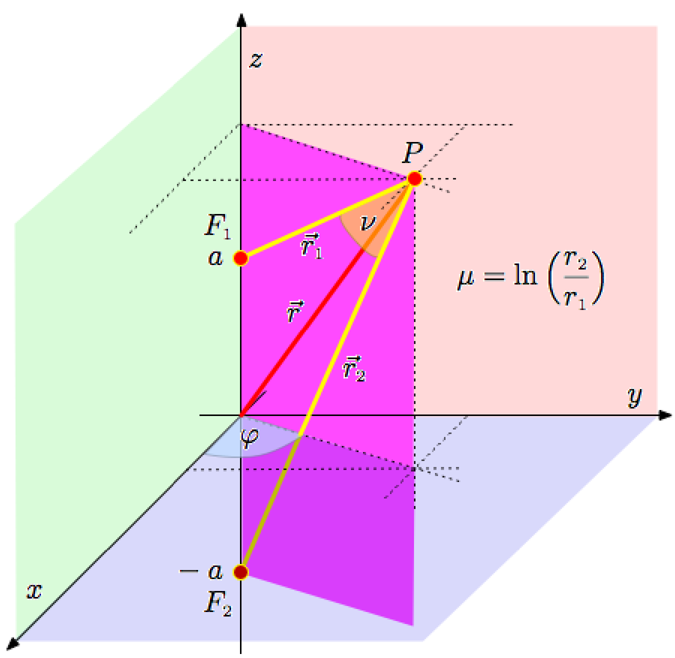


Figure 1: The definition of the bi-spherical coordinates (μ, ν, φ).

From the picture, we deduce the following relations between $\vec{r}$, $\vec{r}_1$, and $\vec{r}_2$,

$$\vec{r} = \vec{r}_1 + a\hat{z} = \vec{r}_2 - a\hat{z}\,, \tag{1}$$

from which we find

$$\begin{aligned} r_1^2 &= r^2 + a^2 - 2az \\ r_2^2 &= r^2 + a^2 + 2az \\ \vec{r}_1 \cdot \vec{r}_2 &= r^2 - a^2\,. \end{aligned} \tag{2}$$

Since $\mu \equiv \ln(r_2/r_1)$ and $\vec{r}_1 \cdot \vec{r}_2 = r_1 r_2 \cos\nu$, it then follows that

$$\mu = \frac{1}{2}\ln\left[\frac{r^2 + a^2 + 2az}{r^2 + a^2 - 2az}\right] \qquad\qquad \cos\nu = \frac{r^2 - a^2}{\sqrt{(r^2 + a^2)^2 - (2az)^2}}\,. \tag{3}$$

These, along with

$$\tan\varphi = \frac{y}{x} \tag{4}$$

and $r^2 = x^2 + y^2 + z^2$, give us the bi-spherical coordinates in terms of the Cartesian coordinates (x, y, z). Note that ν and φ are constrained to the ranges $0 \le \nu \le \pi$ and $0 \le \varphi < 2\pi$, respectively. Note also that μ, while unconstrained, carries the same sign as z, that is, $\mu > 0$ when $z > 0$, $\mu < 0$ when $z < 0$, and $\mu = 0$ when $z = 0$. From the first of equations (3), we can obtain an alternative expression for μ in terms of the Cartesian coordinates,

$$\tanh\mu = \frac{2az}{r^2 + a^2}\,. \tag{5}$$

Then, using this and the second of equations (3), we find

$$\frac{\sinh\mu}{\cos\nu} = \frac{2az}{r^2 - a^2}\,.$$

The last two equations, in turn, can be used to isolate r^2 and z,

$$\frac{r^2}{a^2} = \frac{\cosh\mu + \cos\nu}{\cosh\mu - \cos\nu} \qquad\qquad \frac{z}{a} = \frac{\sinh\mu}{\cosh\mu - \cos\nu}\,, \tag{6}$$

and these can be used to compute $x^2 + y^2$,

$$x^2 + y^2 = \left(\frac{a\,\sin\nu}{\cosh\mu - \cos\nu}\right)^2\,. \tag{7}$$

We can finally write the complete set of inverse transformation equations as

$$\begin{aligned} x &= a\ \xi(\mu,\nu)\ \sin\nu\ \cos\varphi \\ y &= a\ \xi(\mu,\nu)\ \sin\nu\ \sin\varphi \\ z &= a\ \xi(\mu,\nu)\ \sinh\mu\,, \end{aligned} \tag{8}$$

where

$$\xi(\mu,\nu) \equiv \frac{1}{\cosh\mu - \cos\nu}\,. \tag{9}$$

Iso-surfaces

We can rewrite equation (5) as follows,

$$x^2 + y^2 + \left[z - \frac{a}{\tanh\mu} \right]^2 = \left[\frac{a}{\sinh\mu} \right]^2. \tag{10}$$

which shows that the surfaces of constant μ, for $\mu \neq 0$, are spherical shells centered around points located on the z axis. For each value of μ, the corresponding iso-surface is a spherical shell centered on $x = y = 0$ and $z = a/\tanh\mu$, with a radius $R = a/|\sinh\mu|$. It's easy to see from the first of equations (3) that the iso-surface for which $\mu = 0$ is the $z = 0$ plane. Figure 2 below shows projections of these iso-surfaces onto the xz plane.

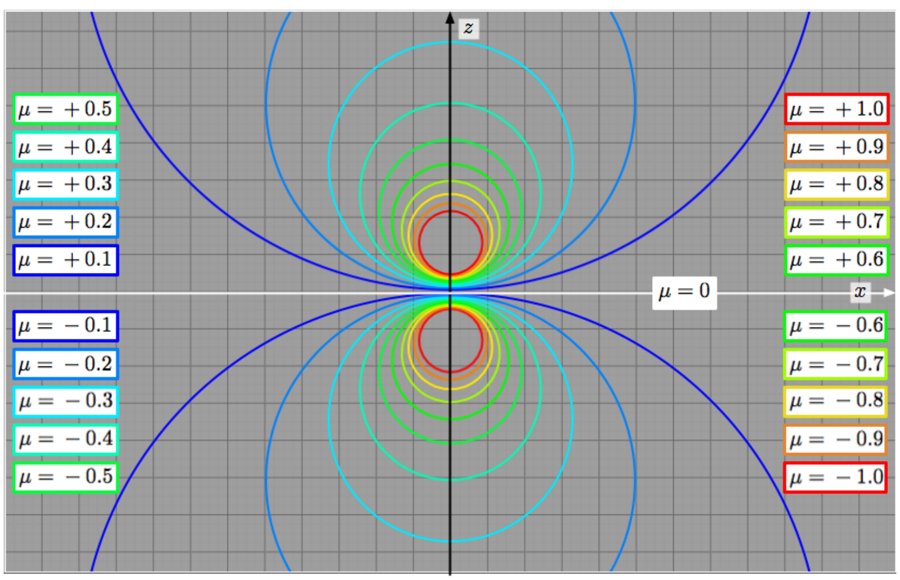


Figure 2: Regions of constant μ, projected onto the xz plane.

From the second of equations (3), it's easy to show that the surfaces of constant ν, for $\nu \neq \{0, \pi\}$, must satisfy

$$\left[\sqrt{x^2+y^2} - \frac{a}{\tan\nu}\right]^2 + z^2 = \left[\frac{a}{\sin\nu}\right]^2. \tag{11}$$

For $\nu = \pi/2$, this is a spherical shell centered at the origin of the coordinate system, with a radius equal to a. For $\nu \neq \pi/2$, the projections onto the xz plane of the ν iso-surfaces satisfy

$$\left[|x| - \frac{a}{\tan\nu}\right]^2 + z^2 = \left[\frac{a}{\sin\nu}\right]^2,$$

and are circumferences on the xz plane, of radii $a/\sin\nu$, centered on points on the x axis with coordinates $x = \pm\, a/\tan\nu$. If $\nu < \pi/2$, these circumferences intersect in such a way that the iso-surfaces with $0 < \nu < \pi/2$ resemble tori whose symmetry axes all coincide with the z axis. For $\pi/2 < \nu < \pi$, those circumferences on the xz plane now intersect in a way so that the iso-surfaces with $\pi/2 < \nu < \pi$ resemble 'lenses,' with symmetry axes which also all coincide with the z axis. The surfaces with $\nu = \{0, \pi\}$ are actually not surfaces: the region with $\nu = \pi$ is the segment of the z axis for which $|z| < a$ and the region with $\nu = 0$ is the union of the sections of the z axis for which $|z| > a$. Refer to figure 3 below for a depiction of all these ν iso-surfaces. Finally, the surfaces of constant φ are half-planes containing the z axis, as illustrated in figure 1.

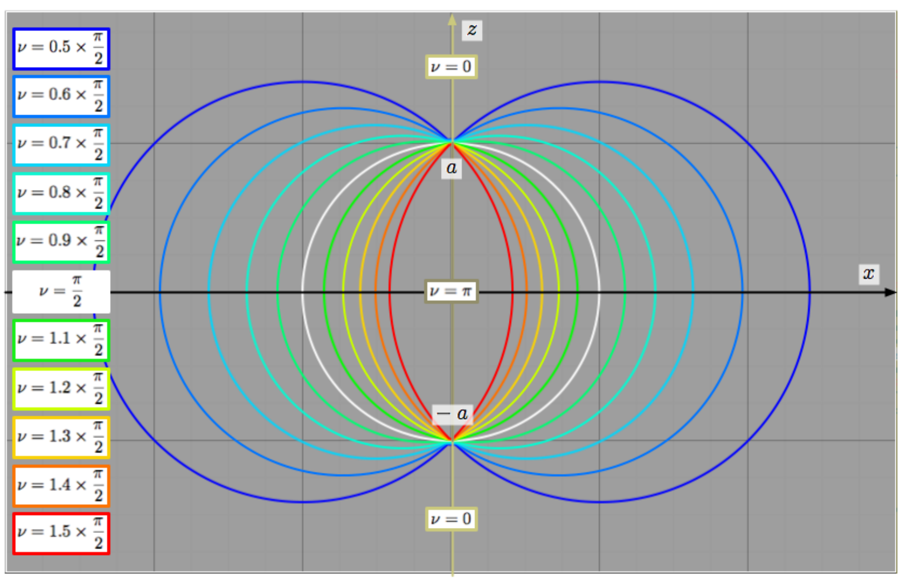


Figure 3: Regions of constant ν, projected onto the xz plane. The region in white is a circumference of radius a centered at the origin, and corresponds to $\nu = \pi/2$.

Combining the two pictures, as shown in figure 4 below, we see that this coordinate system appears to be *orthogonal*, since lines of constant μ cross lines of constant ν at what appear to be 90-degree angles. We will verify shortly that this is indeed the case.

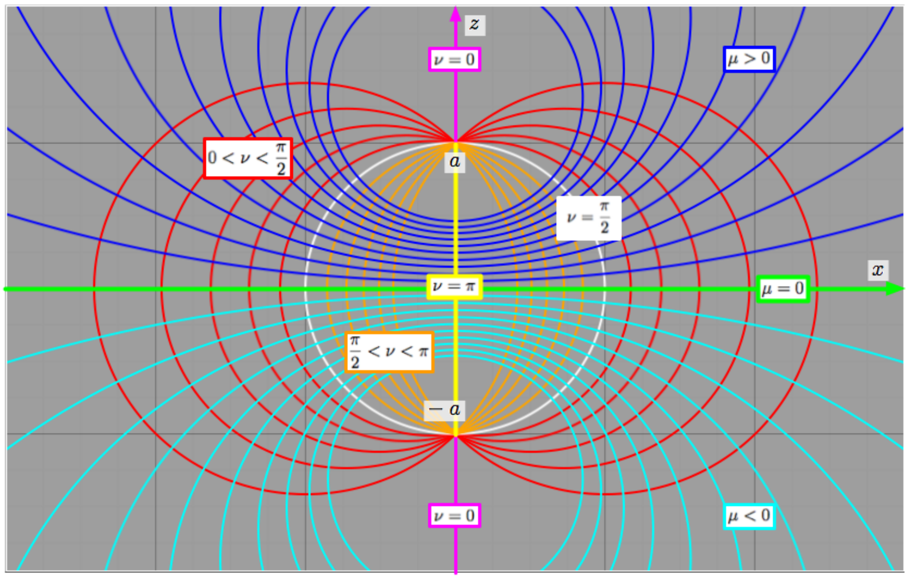


Figure 4: Regions of constant μ and constant ν, projected onto the xz plane.

Metric Coefficients

The squared line element in Cartesian coordinates $(u^1, u^2, u^3) \equiv (x, y, z)$ is given by

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \eta_{mn}\, du^m\, du^n \,,$$

where the last equality makes use of Einstein's summation convention and where (η_{mn}) is the metric tensor for such coordinates, whose components equal those of the Kronecker delta, $\eta_{mn} = \delta^m_n$. The same squared line element, in a more general coordinate system (v^1, v^2, v^3), takes on the form

$$(d\ell)^2 = g_{ik}\, dv^i\, dv^k \,,$$

where (g_{ik}) is the metric tensor for such coordinates. Since the two expressions must yield the same result, it follows that

$$g_{ik}\, dv^i\, dv^k = \eta_{mn}\, du^m\, du^n = \eta_{mn} \frac{\partial u^m}{\partial v^i} \frac{\partial u^n}{\partial v^k}\, dv^i\, dv^k$$

and, therefore,

$$g_{ik} = \eta_{mn} \frac{\partial u^m}{\partial v^i} \frac{\partial u^n}{\partial v^k} = \frac{\partial u^m}{\partial v^i} \frac{\partial u^m}{\partial v^k} .$$

Note that the metric tensor is symmetric in its indices. With $(v^1, v^2, v^3) \equiv (\mu, \nu, \varphi)$, we obtain

$$g_{11} = (\frac{\partial x}{\partial \mu})^2 + (\frac{\partial y}{\partial \mu})^2 + (\frac{\partial z}{\partial \mu})^2 \qquad g_{22} = (\frac{\partial x}{\partial \nu})^2 + (\frac{\partial y}{\partial \nu})^2 + (\frac{\partial z}{\partial \nu})^2$$

$$g_{33} = (\frac{\partial x}{\partial \varphi})^2 + (\frac{\partial y}{\partial \varphi})^2 + (\frac{\partial z}{\partial \varphi})^2 \qquad g_{12} = \frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \nu} + \frac{\partial z}{\partial \mu}\frac{\partial z}{\partial \nu}$$

$$g_{13} = \frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial \mu}\frac{\partial z}{\partial \varphi} \qquad g_{23} = \frac{\partial x}{\partial \nu}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \nu}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial \nu}\frac{\partial z}{\partial \varphi} .$$

Then, using

$$\begin{aligned}
&\frac{\partial x}{\partial \mu} = \frac{x}{\xi}\frac{\partial \xi}{\partial \mu} && \frac{\partial x}{\partial \nu} = \frac{x}{\xi}\frac{\partial \xi}{\partial \nu} + \frac{x}{\tan\nu} && \frac{\partial x}{\partial \varphi} = -\,x\,\tan\varphi \\
&\frac{\partial y}{\partial \mu} = \frac{y}{\xi}\frac{\partial \xi}{\partial \mu} && \frac{\partial y}{\partial \nu} = \frac{y}{\xi}\frac{\partial \xi}{\partial \nu} + \frac{y}{\tan\nu} && \frac{\partial y}{\partial \varphi} = \frac{y}{\tan\varphi} \\
&\frac{\partial z}{\partial \mu} = \frac{z}{\xi}\frac{\partial \xi}{\partial \mu} + \frac{z}{\tanh\mu} && \frac{\partial z}{\partial \nu} = \frac{z}{\xi}\frac{\partial \xi}{\partial \nu} && \frac{\partial z}{\partial \varphi} = 0\,,
\end{aligned} \tag{12}$$

along with

$$\frac{\partial \xi}{\partial \mu} = -\,\xi^2\,\sinh\mu \qquad \frac{\partial \xi}{\partial \nu} = -\,\xi^2\,\sin\nu\,, \tag{13}$$

we find

$$\begin{aligned}
g_{11} = g_{22} = (a\,\xi)^2 &= \frac{a^2}{(\cosh\mu - \cos\nu)^2} \\
g_{33} = (a\,\xi\sin\nu)^2 &= \frac{a^2\,\sin^2\nu}{(\cosh\mu - \cos\nu)^2} \\
g_{12} = g_{13} = g_{23} &= 0\,.
\end{aligned} \tag{14}$$

Note that, since $g_{ik} = 0$ for $i \neq k$, this coordinate system is indeed orthogonal. Once we have the metric coefficients, we proceed to compute the metric's determinant. Since the metric is diagonal, the determinant is trivial to compute and results in

$$g \equiv ||\,g_{ik}\,|| = (a\,\xi)^6\,\sin^2\nu = \frac{a^6\,\sin^2\nu}{(\cosh\mu - \cos\nu)^6}\,. \tag{15}$$

Next, we need the *contravariant* components of the metric tensor, g^{ik}, defined such that $g^{ij}g_{jk} = \delta^i_k$. This amounts to inverting the matrix form of the metric tensor obtained above. Because the metric is diagonal, inverting it is entirely trivial, and we get

$$\begin{aligned} g^{11} &= \frac{1}{g_{11}} = (a\,\xi)^{-2} = \frac{(\cosh\mu - \cos\nu)^2}{a^2} \\ g^{22} &= \frac{1}{g_{22}} = (a\,\xi)^{-2} = \frac{(\cosh\mu - \cos\nu)^2}{a^2} \\ g^{33} &= \frac{1}{g_{33}} = (a\,\xi\sin\nu)^{-2} = \frac{(\cosh\mu - \cos\nu)^2}{a^2\,\sin^2\nu} \\ g^{12} &= g^{13} = g^{23} = 0\,. \end{aligned} \tag{16}$$

Laplace's Equation

Armed with the results obtained so far, we can now express Laplace's equation in bi-spherical coordinates. A general result of the theory of curvilinear coordinates is that the Laplacian of a scalar field Φ, expressed in some general coordinate system, is given by the contraction

$$\nabla^2\Phi = \frac{1}{\sqrt{g}}\,\partial_i\,(\sqrt{g}\;g^{ik}\,\partial_k\Phi)\,,$$

where $\partial_i \equiv \partial/\partial v^i$. Thus, Laplace's equation in such a system can be written as

$$\partial_i\,(\sqrt{g}\;g^{ik}\,\partial_k\Phi) = 0\,.$$

In terms of the bi-spherical coordinates, the contraction above expands into

$$\frac{\partial}{\partial\mu}\Big[\sqrt{g}\;g^{11}\,\frac{\partial\Phi}{\partial\mu}\Big] + \frac{\partial}{\partial\nu}\Big[\sqrt{g}\;g^{22}\,\frac{\partial\Phi}{\partial\nu}\Big] + \frac{\partial}{\partial\varphi}\Big[\sqrt{g}\;g^{33}\,\frac{\partial\Phi}{\partial\varphi}\Big] = 0\,,$$

which results in

$$\boxed{\frac{\sin^2\nu}{\xi}\,\frac{\partial}{\partial\mu}\Big(\xi\,\frac{\partial\Phi}{\partial\mu}\Big) + \frac{\sin\nu}{\xi}\,\frac{\partial}{\partial\nu}\Big(\xi\,\sin\nu\,\frac{\partial\Phi}{\partial\nu}\Big) + \frac{\partial^2\Phi}{\partial\varphi^2} = 0}\,. \tag{17}$$

Separating and Solving Laplace's Equation

Evidently, the angular dependence can be easily separated, so we write

$$\Phi(\mu,\nu,\varphi) \equiv R(\mu,\nu)\,\Theta(\varphi)\,,$$

which leads to the following two equations,

$$\frac{\sin^2\nu}{\xi}\,\frac{\partial}{\partial\mu}\Big(\xi\,\frac{\partial R}{\partial\mu}\Big)+\frac{\sin\nu}{\xi}\,\frac{\partial}{\partial\nu}\Big(\xi\,\sin\nu\,\frac{\partial R}{\partial\nu}\Big)-m^2\,R=0$$

$$\frac{d^2\Theta}{d\varphi^2}+m^2\,\Theta=0\,,$$

where m is a constant. The solution for the angular equation is trivial,

$$\Theta(\varphi)=A_m\sin(m\varphi)+B_m\cos(m\varphi)\,,$$

where A_m and B_m are constants that depend on m but not on any coordinate. Due to the angular solution's periodicity, m must be an integer. Of course, because of the linear nature of Laplace's equation, we can combine solutions with different values of m,

$$\Phi(\mu,\nu,\varphi)=\sum_{m\,\in\mathcal{N}}\Big[A_m\sin(m\varphi)+B_m\cos(m\varphi)\Big]\,R_m(\mu,\nu)\,, \tag{18}$$

where $R_m(\mu,\nu)$ must satisfy

$$\frac{\sin^2\nu}{\xi}\,\frac{\partial}{\partial\mu}\Big(\xi\,\frac{\partial R_m}{\partial\mu}\Big)+\frac{\sin\nu}{\xi}\,\frac{\partial}{\partial\nu}\Big(\xi\,\sin\nu\,\frac{\partial R_m}{\partial\nu}\Big)-m^2\,R_m=0\,. \tag{19}$$

We shall attempt to solve this equation by again employing the method of separation of variables. That is a non-trivial task, however, because ξ is a function of both μ and ν. To get around this problem, we will attempt to factor some common functional behavior that depends on both variables. To that end, we propose the following solution

$$R_m(\mu,\nu)=F(\mu,\nu)\,M(\mu)\,N(\nu)\,, \tag{20}$$

and hope to be able to choose an appropriate function $F(\mu,\nu)$ so that the resulting equation is separable. Inserting this candidate solution into equation (19), we obtain

$$\Big[\frac{M''}{M}+\frac{M'}{M}\frac{\partial}{\partial\mu}\ln(\xi\,F^2)\Big]+\Big[\frac{N''}{N}+\frac{N'}{N}\frac{\partial}{\partial\nu}\ln(\xi\,\sin\nu\,F^2)\Big]+$$
$$\frac{1}{\xi F}\,\frac{\partial}{\partial\mu}(\xi\,\frac{\partial F}{\partial\mu})+\frac{1}{\xi\,\sin\nu\,F}\,\frac{\partial}{\partial\nu}(\xi\,\sin\nu\,\frac{\partial F}{\partial\nu})-\frac{m^2}{\sin^2\nu}=0\,, \tag{21}$$

where a prime indicates ordinary differentiation with respect to the appropriate variable.

Now, two necessary conditions for equation (21) to be separable are that the factor multiplying M'/M must be a function only of μ and that the factor multiplying N'/N must be a function only of ν, that is, we must impose

$$\frac{\partial}{\partial\nu}\Big[\frac{\partial}{\partial\mu}\ln(\xi\,F^2)\Big]=0\qquad\qquad\frac{\partial}{\partial\mu}\Big[\frac{\partial}{\partial\nu}\ln(\xi\,\sin\nu\,F^2)\Big]=0\,.$$

Switching the order in which the derivatives are applied, we get

$$\frac{\partial}{\partial \mu}\Big[\frac{\partial}{\partial \nu}\ln(\xi\, F^2)\Big] = 0$$

$$\frac{\partial}{\partial \nu}\Big[\frac{\partial}{\partial \mu}\ln(\xi\,\sin\nu\, F^2)\Big] = \frac{\partial}{\partial \nu}\Big[\frac{\partial}{\partial \mu}\ln(\xi\, F^2)\Big] = 0\,.$$

Thus, we conclude that $\ln(\xi\, F^2) = u(\mu) + v(\nu)$, which is to say,

$$\xi\, F^2 = U(\mu)\, V(\nu)\,,$$

where $U = \exp(u)$ and $V = \exp(v)$ are functions yet to be determined. Given the specific dependence of ξ on μ and ν, the only possible solutions for U and V are constants, that is, $\xi\, F^2$ must be a constant. Therefore, we conclude that

$$F(\mu,\nu) \equiv \xi^{-1/2} = \sqrt{\cosh\mu - \cos\nu}\,,$$

where the constant is set to 1 since we can absorb any other value into the functions M and N. We still need to verify that this choice of F makes the remaining terms of equation (21) separable. It turns out that, for $F = \xi^{-1/2}$,

$$\frac{1}{\xi F}\frac{\partial}{\partial \mu}(\xi\,\frac{\partial F}{\partial \mu}) + \frac{1}{\xi\,\sin\nu\, F}\frac{\partial}{\partial \nu}(\xi\,\sin\nu\,\frac{\partial F}{\partial \nu}) = -\frac{1}{4}\,.$$

Thus, this choice for F turns equation (21) into

$$\frac{M''}{M} + \Big[\frac{N''}{N} + \frac{1}{\tan\nu}\frac{N'}{N} - \frac{m^2}{\sin^2\!\nu} - \frac{1}{4}\Big] = 0\,.$$

This is clearly separable. Setting $M''/M = \alpha^2 =$ constant, we get

$$M(\mu) = C(\alpha)\, e^{\alpha\mu} + D(\alpha)\, e^{-\alpha\mu}$$

and

$$N'' + \frac{N'}{\tan\nu} + \Big[\alpha^2 - \frac{1}{4} - \frac{m^2}{\sin^2\!\nu}\Big]N = 0\,, \tag{22}$$

where $C(\alpha)$ and $D(\alpha)$ are arbitrary functions of α. Equation (22) is none other than the *associated Legendre equation*, with $\ell(\ell+1) = \alpha^2 - 1/4$, whose solutions that are regular in the interval $[-1,1]$ are the associated Legendre functions $P_\ell^m(\cos\nu)$, where ℓ and m must be integers satisfying $\ell \geq 0$ and $0 \leq m \leq \ell$. Since ℓ must be a non-negative integer and $\alpha^2 - 1/4 = \ell(\ell+1)$, it follows that the only acceptable values for α are $\pm(\ell + 1/2)$.

Collecting together all these results, we have that Laplace's equation in bi-spherical coordinates has the general solution

$$\Phi(\mu,\nu,\varphi) = \sqrt{\cosh\mu - \cos\nu}\ \times \sum_{\ell\geq 0}\ \sum_{0\leq m\leq \ell} \Big[A_m \sin(m\varphi) + B_m \cos(m\varphi) \Big] \times \Big[C_\ell\, e^{(\ell+1/2)\mu} + D_\ell\, e^{-(\ell+1/2)\mu} \Big]\, P_\ell^m(\cos\nu)\,,$$

where C_ℓ and D_ℓ are constants. A more compact expression can be written by combining the trigonometric functions with the associated Legendre functions into the *spherical harmonics* $Y_{\ell m}(\nu,\varphi)$,

$$\boxed{\Phi(\mu,\nu,\varphi) = \sqrt{\cosh\mu - \cos\nu} \sum_{\ell\geq 0}\ \sum_{|m|\leq \ell} \Big[A_{\ell m}\, e^{(\ell+1/2)\mu} + B_{\ell m}\, e^{-(\ell+1/2)\mu} \Big]\, Y_{\ell m}(\nu,\varphi)}\,, \tag{23}$$

where $A_{\ell m}$ and $B_{\ell m}$ are constants. Note that this is now a complex-valued function, because of the spherical harmonics' $\exp(i\,m\varphi)$ factor. Note also that the sum over m now extends to negative values.

The Electrostatic Potential Between Two Spherical Conductors

As an application of the results obtained in the previous sections, consider the situation where we have two non-intersecting spherical conductors, of radii R_1 and R_2, set to electrostatic potentials Φ_1 and Φ_2, respectively, and separated by a distance s_{12} between their centers. We would like to find the electrostatic potential everywhere in the region between the two conductors.

First, note that because the conductors are spherical in shape and because we can choose the z axis to pass through their centers, we expect the situation to have complete azimuthal symmetry, so the electrostatic potential everywhere should not depend on φ.

Both conductors are equipotential surfaces whose potentials depend on neither of the two angular coordinates of their points. Thus, in matching the solution of Laplace's equation to the conductors' given potentials, we need only consider the μ coordinate. As we have seen earlier, equation (10) shows that iso-surfaces of constant μ are spherical shells of radii $a/|\sinh\mu|$ centered on points on the z axis located at $z = a/\tanh\mu$. Therefore, each conductor is associated with a unique μ value, given by

$$\mu_1 = \operatorname{sign}(z_1)\operatorname{arccosh}\!\left(\frac{|z_1|}{R_1}\right) \qquad\qquad \mu_2 = \operatorname{sign}(z_2)\operatorname{arccosh}\!\left(\frac{|z_2|}{R_2}\right). \tag{24}$$

It may seem from the above that only combinations of z_k and R_k such that $|z_k| \geq R_k$ are physically possible, since $\cosh\mu \geq 1$. However, it's always possible to redefine our coordinate system, repositioning the origin so as to make $|z_1| \geq R_1$ and $|z_2| \geq R_2$. More specifically, if s_{12} is the distance between the conductors' centers, then choosing the origin of the coordinate system such that

$$z_1 = \frac{(R_1 - R_2) + s_{12}}{2} \qquad\qquad z_2 = \frac{(R_1 - R_2) - s_{12}}{2} \tag{25}$$

will guarantee that $|z_1| \geq R_1$ and $|z_2| \geq R_2$, provided that $s_{12} \geq (R_1 + R_2)$. Incidentally, from equation (24), we can determine a by a variety of means,

$$a = z_1 \tanh(\mu_1) = z_2 \tanh(\mu_2) = R_1 |\sinh(\mu_1)| = R_2 |\sinh(\mu_2)| \,. \tag{26}$$

From equation (23), the potential everywhere in the region between the two conductors is given by

$$\Phi(\mu, \nu) = \sqrt{\cosh\mu - \cos\nu} \sum_{\ell \geq 0} \Big[A_{\ell 0}\, e^{(\ell+1/2)\mu} + B_{\ell 0}\, e^{-(\ell+1/2)\mu} \Big] Y_{\ell 0}(\nu) \,,$$

where the φ dependence and all $m > 0$ terms have been dropped because of the expected azimuthal symmetry. Dropping the zero index in $A_{\ell 0}$ and $B_{\ell 0}$ and reverting back to using Legendre polynomials, we have

$$\Phi(\mu, \nu) = \sqrt{\cosh\mu - \cos\nu} \sum_{\ell \geq 0} \Big[A_\ell\, e^{(\ell+1/2)\mu} + B_\ell\, e^{-(\ell+1/2)\mu} \Big] P_\ell(\cos\nu) \,. \tag{27}$$

The potentials on both conductors are, then, as follows

$$\Phi_1 = \sqrt{\cosh\mu_1 - v} \sum_{\ell \geq 0} \Big[A_\ell\, e^{(\ell+1/2)\mu_1} + B_\ell\, e^{-(\ell+1/2)\mu_1} \Big] P_\ell(v)$$
$$\Phi_2 = \sqrt{\cosh\mu_2 - v} \sum_{\ell \geq 0} \Big[A_\ell\, e^{(\ell+1/2)\mu_2} + B_\ell\, e^{-(\ell+1/2)\mu_2} \Big] P_\ell(v) \,,$$

Neither of these two expressions can actually depend on $v \equiv \cos\nu$, because the entire surface of each conductor is an equipotential surface. Therefore, we need to find a set of A_ℓ and B_ℓ values such that the dependence on v cancels out. The easiest way to do so is to rewrite these expressions as

$$\frac{\Phi_1}{\sqrt{\cosh\mu_1 - v}} = \sum_{\ell \geq 0} \Big[A_\ell\, e^{(\ell+1/2)\mu_1} + B_\ell\, e^{-(\ell+1/2)\mu_1} \Big] P_\ell(v)$$
$$\frac{\Phi_2}{\sqrt{\cosh\mu_2 - v}} = \sum_{\ell \geq 0} \Big[A_\ell\, e^{(\ell+1/2)\mu_2} + B_\ell\, e^{-(\ell+1/2)\mu_2} \Big] P_\ell(v) \,,$$

then expand the left–hand sides in terms of Legendre polynomials. Let

$$\frac{1}{\sqrt{u-v}} = \sum_{\ell \geq 0} S_\ell(u)\, P_\ell(v)\,. \tag{28}$$

Then,

$$\sum_{\ell \geq 0} \Phi_1\, S_\ell(\cosh\mu_1)\, P_\ell(v) = \sum_{\ell \geq 0} \Big[\, A_\ell\, e^{(\ell+1/2)\mu_1} + B_\ell\, e^{-(\ell+1/2)\mu_1} \,\Big]\, P_\ell(v)$$
$$\sum_{\ell \geq 0} \Phi_2\, S_\ell(\cosh\mu_2)\, P_\ell(v) = \sum_{\ell \geq 0} \Big[\, A_\ell\, e^{(\ell+1/2)\mu_2} + B_\ell\, e^{-(\ell+1/2)\mu_2} \,\Big]\, P_\ell(v)\,,$$

and, due to the linearly independent nature of the Legendre polynomials, it follows that

$$\Phi_1\, S_\ell(\cosh\mu_1) = A_\ell\, e^{(\ell+1/2)\mu_1} + B_\ell\, e^{-(\ell+1/2)\mu_1}$$
$$\Phi_2\, S_\ell(\cosh\mu_2) = A_\ell\, e^{(\ell+1/2)\mu_2} + B_\ell\, e^{-(\ell+1/2)\mu_2}\,,$$

which we can solve for A_ℓ and B_ℓ,

$$A_\ell = \frac{e^{-(\ell+1/2)\mu_2} S_\ell(\cosh\mu_1)\,\Phi_1 - e^{-(\ell+1/2)\mu_1} S_\ell(\cosh\mu_2)\,\Phi_2}{2\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]} \tag{29}$$

$$B_\ell = \frac{e^{+(\ell+1/2)\mu_1} S_\ell(\cosh\mu_2)\,\Phi_2 - e^{+(\ell+1/2)\mu_2} S_\ell(\cosh\mu_1)\,\Phi_1}{2\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]}\,. \tag{30}$$

Note that $A_\ell = \pm\, B_\ell$ when $\mu_2 = -\mu_1$ and $\Phi_2 = \pm\,\Phi_1$. Plugging these back into equation (27), we find

$$\Phi(\mu,\nu) = \sqrt{\cosh\mu - \cos\nu}\, \sum_{\ell \geq 0} P_\ell(\cos\nu)\, \times$$
$$\left\{ \frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]\, S_\ell(\cosh\mu_1)\,\Phi_1 - \sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]\, S_\ell(\cosh\mu_2)\,\Phi_2}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]} \right\}.$$

Analytically, this is the final answer to the problem we set out to solve but, computationally, it's not a very numerically stable expression. This is more easily apparent if we rewrite the result above as follows,

$$\Phi(\mu,\nu) = \Phi_1\sqrt{\cosh\mu - \cos\nu}\,\sum_{\ell\geq 0} P_\ell(\cos\nu)\,S_\ell(\cosh\mu_1)\,\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]}$$

$$+\,\Phi_2\sqrt{\cosh\mu - \cos\nu}\,\sum_{\ell\geq 0} P_\ell(\cos\nu)\,S_\ell(\cosh\mu_2)\,\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]}\,.$$

If we now set, say, $\mu = \mu_1$, then the second term vanishes and we find

$$\Phi(\mu_1,\nu) = \Phi_1\sqrt{\cosh\mu_1 - \cos\nu}\,\sum_{\ell\geq 0} P_\ell(\cos\nu)\,S_\ell(\cosh\mu_1)\,.$$

Of course, by the very definition of $S_\ell(u)$ — equation (28) — this equals Φ_1, as it should. The problem is that, numerically, it's very difficult to guarantee this result. For starters, any numerical evaluation of $\Phi(\mu,\nu)$ will be limited to a *finite* number of terms, whereas the result above is true only in the limit of an infinite series. Moreover, numerical experiments have already shown that any finite series will have values that oscillate wildly around the limiting value.

So, a possibly better-behaved way to express the solution we have found is to extract the values of the potential when $\mu = \mu_1$ and when $\mu = \mu_2$, guaranteeing that the numerical evaluation of the finite series does not have to make sure the limiting values agree with Φ_1 and Φ_2. We can do this extraction by adding and subtracting 1 to each ratio of hyperbolic sines. This results in

$$\Phi(\mu,\nu) = \Phi_1\sqrt{\frac{\cosh\mu-\cos\nu}{\cosh\mu_1-\cos\nu}} + \Phi_1\sqrt{\cosh\mu-\cos\nu}\,\times$$

$$\sum_{\ell\geq 0} P_\ell(\cos\nu)\,S_\ell(\cosh\mu_1)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]} - 1\right] \tag{31}$$

$$+\,\Phi_2\sqrt{\frac{\cosh\mu-\cos\nu}{\cosh\mu_2-\cos\nu}} + \Phi_2\sqrt{\cosh\mu-\cos\nu}\,\times$$

$$\sum_{\ell\geq 0} P_\ell(\cos\nu)\,S_\ell(\cosh\mu_2)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]} - 1\right].$$

However, we now have inadvertently introduced the same problem at the other end. If, again, we set $\mu = \mu_1$, the expression above gives us

$$\Phi(\mu_1, \nu) = \Phi_1 + \Phi_2 \sqrt{\frac{\cosh\mu_1 - \cos\nu}{\cosh\mu_2 - \cos\nu}} - \Phi_2 \sqrt{\cosh\mu_1 - \cos\nu} \sum_{\ell \geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_2)\,,$$

and now we have the problem of numerically guaranteeing that the second and third terms above add to zero. The best compromise, then, is probably to have more than one expression for the potential, one that's numerically more stable near $\mu = \mu_1$ and another that is more stable near $\mu = \mu_2$. For a given value of μ, we may define an interpolation variable α by

$$\alpha \equiv \frac{\mu - \mu_1}{\mu_2 - \mu_1}\,. \tag{32}$$

Then, we choose an expression for the potential depending on the value of α:

- If $\alpha \approx 0$, then $\mu \approx \mu_1$ and

$$\begin{aligned} \Phi(\mu,\nu) = \Phi_1 \sqrt{\frac{\cosh\mu - \cos\nu}{\cosh\mu_1 - \cos\nu}} &+ \Phi_1 \sqrt{\cosh\mu - \cos\nu}\, \times \\ &\sum_{\ell \geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_1) \left[\frac{\sinh\Big[(\ell + 1/2)(\mu - \mu_2)\Big]}{\sinh\Big[(\ell + 1/2)(\mu_1 - \mu_2)\Big]} - 1 \right] \\ + \Phi_2 \sqrt{\cosh\mu - \cos\nu} &\sum_{\ell \geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_2) \frac{\sinh\Big[(\ell + 1/2)(\mu - \mu_1)\Big]}{\sinh\Big[(\ell + 1/2)(\mu_2 - \mu_1)\Big]}\,. \end{aligned} \tag{33}$$

- If $\alpha \approx 1$, then $\mu \approx \mu_2$ and

$$\begin{aligned} \Phi(\mu,\nu) = \Phi_2 \sqrt{\frac{\cosh\mu - \cos\nu}{\cosh\mu_2 - \cos\nu}} &+ \Phi_2 \sqrt{\cosh\mu - \cos\nu}\, \times \\ &\sum_{\ell \geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_2) \left[\frac{\sinh\Big[(\ell + 1/2)(\mu - \mu_1)\Big]}{\sinh\Big[(\ell + 1/2)(\mu_2 - \mu_1)\Big]} - 1 \right] \\ + \Phi_1 \sqrt{\cosh\mu - \cos\nu} &\sum_{\ell \geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_1) \frac{\sinh\Big[(\ell + 1/2)(\mu - \mu_2)\Big]}{\sinh\Big[(\ell + 1/2)(\mu_1 - \mu_2)\Big]}\,. \end{aligned} \tag{34}$$

- If α is far from either end of the range $[0, 1]$, then we can safely use equation (31).

Equations (31), (33), and (34) then represent the final soluton for the electrostatic potential anywhere in the region between the two conductors. The only remaining task is to compute $S_\ell(u)$. From the expansion (28), and using the orthonormality properties of the Legendre polynomials, we obtain

$$S_\ell(u) = (\ell + 1/2) \int_{-1}^{+1} \frac{P_\ell(v)}{\sqrt{u-v}}\, dv\,. \tag{35}$$

The Electrostatic Field Between The Conductors

Once we have the electrostatic potential, obtaining the electrostatic field is simply a gradient away, since $\vec{E} = -\vec{\nabla}\Phi$. We find that:

$$E_\mu = \frac{\cosh\mu - \cos\nu}{a}\,\frac{\partial\Phi}{\partial\mu} \qquad E_\nu = \frac{\cosh\mu - \cos\nu}{a}\,\frac{\partial\Phi}{\partial\nu} \qquad E_\varphi = \frac{\cosh\mu - \cos\nu}{a}\,\frac{1}{\sin\nu}\,\frac{\partial\Phi}{\partial\varphi}\,.$$

Due to the azimuthal symmetry of Φ, and because of

$$\frac{\partial\Phi}{\partial\nu} = \frac{\partial v}{\partial\nu}\,\frac{\partial\Phi}{\partial v} = (-\sin\nu)\,\frac{\partial\Phi}{\partial v}\Big|_{v\,=\,\cos\nu}\,,$$

it follows from equation (27) that

$$\begin{aligned}
E_\mu &= \frac{\sinh\mu}{2a}\,\Phi(\mu,\nu) + \frac{(\cosh\mu - \cos\nu)^{3/2}}{a}\,\frac{\partial}{\partial\mu}\Big[\,\frac{\Phi(\mu,\nu)}{\sqrt{\cosh\mu - \cos\nu}}\,\Big] \\
E_\nu &= \frac{\sin\nu}{2a}\,\Phi(\mu,\nu) - \sin\nu\,\frac{(\cosh\mu - \cos\nu)^{3/2}}{a}\,\frac{\partial}{\partial v}\Big[\,\frac{\Phi(\mu,\nu)}{\sqrt{\cosh\mu - v}}\,\Big]\Bigg|_{v\,=\,\cos\nu} \\
E_\varphi &= 0\,.
\end{aligned} \tag{36}$$

All of equations (31), (33), and (34) then give us the same result for $E_\mu(\mu,\nu)$, namely,

$$\begin{aligned}
E_\mu(\mu,\nu) = \frac{\sinh\mu}{2a}\,\Phi(\mu,\nu) + \frac{(\cosh\mu - \cos\nu)^{3/2}}{a} \times& \\
\Bigg\{\,\Phi_1 \sum_{\ell\,\geq\,0} (\ell + 1/2)\,P_\ell(\cos\nu)\,S_\ell(\cosh\mu_1)\,\frac{\cosh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]} &+ \\
\Phi_2 \sum_{\ell\,\geq\,0} (\ell + 1/2)\,P_\ell(\cos\nu)\,S_\ell(\cosh\mu_2)\,\frac{\cosh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]}\,\Bigg\}&.
\end{aligned} \tag{37}$$

Using equation (33), we obtain

$$E_\nu(\mu,\nu) = \frac{\sin\nu}{2a}\left[\Phi(\mu,\nu) - \Big(\frac{\cosh\mu-\cos\nu}{\cosh\mu_1-\cos\nu}\Big)^{3/2}\Phi_1\right] - \sin\nu\,\frac{(\cosh\mu-\cos\nu)^{3/2}}{a}\times$$

$$\left\{\Phi_1\sum_{\ell\geq 0} P'_\ell(v)\,S_\ell(\cosh\mu_1)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]}-1\right]+\right.$$

$$\left.\left.\Phi_2\sum_{\ell\geq 0} P'_\ell(v)\,S_\ell(\cosh\mu_2)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]}\right]\right\}\right|_{v=\cos\nu}.$$

However, Legendre polynomials satisfy the property

$$P'_\ell(v) = \frac{\ell}{v^2-1}\Big[v\,P_\ell(v) - P_{\ell-1}(v)\Big],$$

and, so, we get

$$E_\nu(\mu,\nu) = \frac{\sin\nu}{2a}\left[\Phi(\mu,\nu) - \Big(\frac{\cosh\mu-\cos\nu}{\cosh\mu_1-\cos\nu}\Big)^{3/2}\Phi_1\right] + \frac{(\cosh\mu-\cos\nu)^{3/2}}{a\,\sin\nu}\times$$

$$\left\{\Phi_1\sum_{\ell\geq 0}\ell\Big[v\,P_\ell(v)-P_{\ell-1}(v)\Big]\,S_\ell(\cosh\mu_1)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]}-1\right]+\right.$$

$$\left.\left.\Phi_2\sum_{\ell\geq 0}\ell\Big[v\,P_\ell(v)-P_{\ell-1}(v)\Big]\,S_\ell(\cosh\mu_2)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]}\right]\right\}\right|_{v=\cos\nu}.$$

An entirely analogous derivation gives us $E_\nu(\mu,\nu)$ when $\mu\approx\mu_2$ or when μ is far from both μ_1 and μ_2. Summarizing these results, we have:

- If $\alpha \approx 0$, then $\mu \approx \mu_1$ and

$$E_\nu(\mu,\nu) = \frac{\sin\nu}{2a}\left[\Phi(\mu,\nu) - \left(\frac{\cosh\mu - \cos\nu}{\cosh\mu_1 - \cos\nu}\right)^{3/2}\Phi_1\right] + \frac{(\cosh\mu - \cos\nu)^{3/2}}{a\,\sin\nu} \times$$

$$\left\{\Phi_1 \sum_{\ell\geq 0} \ell\left[v\,P_\ell(v) - P_{\ell-1}(v)\right] S_\ell(\cosh\mu_1)\left[\frac{\sinh\left[(\ell+1/2)(\mu-\mu_2)\right]}{\sinh\left[(\ell+1/2)(\mu_1-\mu_2)\right]} - 1\right] + \right. \tag{38}$$

$$\left.\Phi_2 \sum_{\ell\geq 0} \ell\left[v\,P_\ell(v) - P_{\ell-1}(v)\right] S_\ell(\cosh\mu_2)\left[\frac{\sinh\left[(\ell+1/2)(\mu-\mu_1)\right]}{\sinh\left[(\ell+1/2)(\mu_2-\mu_1)\right]}\right]\right\}\Bigg|_{v=\cos\nu}.$$

- If $\alpha \approx 1$, then $\mu \approx \mu_2$ and

$$E_\nu(\mu,\nu) = \frac{\sin\nu}{2a}\left[\Phi(\mu,\nu) - \left(\frac{\cosh\mu - \cos\nu}{\cosh\mu_2 - \cos\nu}\right)^{3/2}\Phi_2\right] + \frac{(\cosh\mu - \cos\nu)^{3/2}}{a\,\sin\nu} \times$$

$$\left\{\Phi_2 \sum_{\ell\geq 0} \ell\left[v\,P_\ell(v) - P_{\ell-1}(v)\right] S_\ell(\cosh\mu_2)\left[\frac{\sinh\left[(\ell+1/2)(\mu-\mu_1)\right]}{\sinh\left[(\ell+1/2)(\mu_2-\mu_1)\right]} - 1\right] + \right. \tag{39}$$

$$\left.\Phi_1 \sum_{\ell\geq 0} \ell\left[v\,P_\ell(v) - P_{\ell-1}(v)\right] S_\ell(\cosh\mu_1)\left[\frac{\sinh\left[(\ell+1/2)(\mu-\mu_2)\right]}{\sinh\left[(\ell+1/2)(\mu_1-\mu_2)\right]}\right]\right\}\Bigg|_{v=\cos\nu}.$$

- If α is far from either end of the range $[0,1]$, then

$$E_\nu(\mu,\nu) = \frac{\sin\nu}{2a}\left[\Phi(\mu,\nu) - \left(\frac{\cosh\mu - \cos\nu}{\cosh\mu_1 - \cos\nu}\right)^{3/2}\Phi_1 - \left(\frac{\cosh\mu - \cos\nu}{\cosh\mu_2 - \cos\nu}\right)^{3/2}\Phi_2\right]$$

$$+ \frac{(\cosh\mu - \cos\nu)^{3/2}}{a\,\sin\nu} \times \tag{40}$$

$$\left\{\Phi_1 \sum_{\ell\geq 0} \ell\left[v\,P_\ell(v) - P_{\ell-1}(v)\right] S_\ell(\cosh\mu_1)\left[\frac{\sinh\left[(\ell+1/2)(\mu-\mu_2)\right]}{\sinh\left[(\ell+1/2)(\mu_1-\mu_2)\right]} - 1\right] + \right.$$

$$\left.\Phi_2 \sum_{\ell\geq 0} \ell\left[v\,P_\ell(v) - P_{\ell-1}(v)\right] S_\ell(\cosh\mu_2)\left[\frac{\sinh\left[(\ell+1/2)(\mu-\mu_1)\right]}{\sinh\left[(\ell+1/2)(\mu_2-\mu_1)\right]} - 1\right]\right\}\Bigg|_{v=\cos\nu}.$$

The Cartesian components of the electric field are easily obtained, but the expressions are too complicated to write in full. We start with

$$
\begin{aligned}
-\,E_x &= \frac{\partial \Phi}{\partial x} = \frac{\partial \mu}{\partial x}\,\frac{\partial \Phi}{\partial \mu} + \frac{\partial \nu}{\partial x}\,\frac{\partial \Phi}{\partial \nu} + \frac{\partial \varphi}{\partial x}\,\frac{\partial \Phi}{\partial \varphi} \\
&= \frac{a}{\cosh\mu - \cos\nu}\left[\,\frac{\partial \mu}{\partial x}\,E_\mu(\mu,\nu) + \frac{\partial \nu}{\partial x}\,E_\nu(\mu,\nu) + \frac{\partial \varphi}{\partial x}\,\sin\nu\,E_\varphi(\mu,\nu)\right]
\end{aligned}
$$

$$
\begin{aligned}
-\,E_y &= \frac{\partial \Phi}{\partial y} = \frac{\partial \mu}{\partial y}\,\frac{\partial \Phi}{\partial \mu} + \frac{\partial \nu}{\partial y}\,\frac{\partial \Phi}{\partial \nu} + \frac{\partial \varphi}{\partial y}\,\frac{\partial \Phi}{\partial \varphi} \\
&= \frac{a}{\cosh\mu - \cos\nu}\left[\,\frac{\partial \mu}{\partial y}\,E_\mu(\mu,\nu) + \frac{\partial \nu}{\partial y}\,E_\nu(\mu,\nu) + \frac{\partial \varphi}{\partial y}\,\sin\nu\,E_\varphi(\mu,\nu)\right]
\end{aligned}
$$

$$
\begin{aligned}
-\,E_z &= \frac{\partial \Phi}{\partial z} = \frac{\partial \mu}{\partial z}\,\frac{\partial \Phi}{\partial \mu} + \frac{\partial \nu}{\partial z}\,\frac{\partial \Phi}{\partial \nu} + \frac{\partial \varphi}{\partial z}\,\frac{\partial \Phi}{\partial \varphi} \\
&= \frac{a}{\cosh\mu - \cos\nu}\left[\,\frac{\partial \mu}{\partial z}\,E_\mu(\mu,\nu) + \frac{\partial \nu}{\partial z}\,E_\nu(\mu,\nu) + \frac{\partial \varphi}{\partial z}\,\sin\nu\,E_\varphi(\mu,\nu)\right],
\end{aligned}
$$

which, taking into account the azimuthal symmetry, can be simplified to

$$
\begin{aligned}
E_x(\mu,\nu) &= \frac{(-a)}{\cosh\mu - \cos\nu}\left[\,\frac{\partial \mu}{\partial x}\,E_\mu(\mu,\nu) + \frac{\partial \nu}{\partial x}\,E_\nu(\mu,\nu)\right] \\
E_y(\mu,\nu) &= \frac{(-a)}{\cosh\mu - \cos\nu}\left[\,\frac{\partial \mu}{\partial y}\,E_\mu(\mu,\nu) + \frac{\partial \nu}{\partial y}\,E_\nu(\mu,\nu)\right] \qquad (41) \\
E_z(\mu,\nu) &= \frac{(-a)}{\cosh\mu - \cos\nu}\left[\,\frac{\partial \mu}{\partial z}\,E_\mu(\mu,\nu) + \frac{\partial \nu}{\partial z}\,E_\nu(\mu,\nu)\right],
\end{aligned}
$$

where μ and ν are expressed in terms of the Cartesian coordinates (x, y, z) by virtue of equations (3).

Summary Of Results

- Inverse square-root expansion in terms of Legendre polynomials:

$$
S_\ell(u) = (\ell + 1/2)\int_{-1}^{+1} \frac{P_\ell(v)}{\sqrt{u - v}}\,dv\,.
$$

- Electrostatic potential ($\mu \approx \mu_1$):

$$\Phi(\mu,\nu) = \Phi_1 \sqrt{\frac{\cosh\mu - \cos\nu}{\cosh\mu_1 - \cos\nu}} + \Phi_1 \sqrt{\cosh\mu - \cos\nu} \times$$

$$\sum_{\ell\geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_1) \left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]} - 1 \right] +$$

$$\Phi_2 \sqrt{\cosh\mu - \cos\nu} \sum_{\ell\geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_2) \frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]} .$$

- Electrostatic potential ($\mu \approx \mu_2$):

$$\Phi(\mu,\nu) = \Phi_2 \sqrt{\frac{\cosh\mu - \cos\nu}{\cosh\mu_2 - \cos\nu}} + \Phi_2 \sqrt{\cosh\mu - \cos\nu} \times$$

$$\sum_{\ell\geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_2) \left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]} - 1 \right] +$$

$$\Phi_1 \sqrt{\cosh\mu - \cos\nu} \sum_{\ell\geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_1) \frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]} .$$

- Electrostatic potential (μ far from μ_1 and μ_2):

$$\Phi(\mu,\nu) = \Phi_1 \sqrt{\frac{\cosh\mu - \cos\nu}{\cosh\mu_1 - \cos\nu}} + \Phi_1 \sqrt{\cosh\mu - \cos\nu} \times$$

$$\sum_{\ell\geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_1) \left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]} - 1 \right] +$$

$$\Phi_2 \sqrt{\frac{\cosh\mu - \cos\nu}{\cosh\mu_2 - \cos\nu}} + \Phi_2 \sqrt{\cosh\mu - \cos\nu} \times$$

$$\sum_{\ell\geq 0} P_\ell(\cos\nu)\, S_\ell(\cosh\mu_2) \left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]} - 1 \right] .$$

- Electrostatic field, μ-component:

$$E_\mu(\mu,\nu) = \frac{\sinh\mu}{2a}\,\Phi(\mu,\nu) + \frac{(\cosh\mu-\cos\nu)^{3/2}}{a}\times$$

$$\Bigg\{\Phi_1\sum_{\ell\geq 0}(\ell+1/2)\,P_\ell(\cos\nu)\,S_\ell(\cosh\mu_1)\,\frac{\cosh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]}+$$

$$\Phi_2\sum_{\ell\geq 0}(\ell+1/2)\,P_\ell(\cos\nu)\,S_\ell(\cosh\mu_2)\,\frac{\cosh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]}\Bigg\}.$$

- Electrostatic field, ν-component ($\mu\approx\mu_1$):

$$E_\nu(\mu,\nu) = \frac{\sin\nu}{2a}\left[\Phi(\mu,\nu) - \Big(\frac{\cosh\mu-\cos\nu}{\cosh\mu_1-\cos\nu}\Big)^{3/2}\Phi_1\right] + \frac{(\cosh\mu-\cos\nu)^{3/2}}{a\,\sin\nu}\times$$

$$\Bigg\{\Phi_1\sum_{\ell\geq 0}\ell\,\Big[v\,P_\ell(v)-P_{\ell-1}(v)\Big]\,S_\ell(\cosh\mu_1)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]}-1\right]+$$

$$\Phi_2\sum_{\ell\geq 0}\ell\,\Big[v\,P_\ell(v)-P_{\ell-1}(v)\Big]\,S_\ell(\cosh\mu_2)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]}\right]\Bigg\}\Bigg|_{v=\cos\nu}.$$

- Electrostatic field, ν-component ($\mu\approx\mu_2$):

$$E_\nu(\mu,\nu) = \frac{\sin\nu}{2a}\left[\Phi(\mu,\nu) - \Big(\frac{\cosh\mu-\cos\nu}{\cosh\mu_2-\cos\nu}\Big)^{3/2}\Phi_2\right] + \frac{(\cosh\mu-\cos\nu)^{3/2}}{a\,\sin\nu}\times$$

$$\Bigg\{\Phi_2\sum_{\ell\geq 0}\ell\,\Big[v\,P_\ell(v)-P_{\ell-1}(v)\Big]\,S_\ell(\cosh\mu_2)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_1)\Big]}{\sinh\Big[(\ell+1/2)(\mu_2-\mu_1)\Big]}-1\right]+$$

$$\Phi_1\sum_{\ell\geq 0}\ell\,\Big[v\,P_\ell(v)-P_{\ell-1}(v)\Big]\,S_\ell(\cosh\mu_1)\left[\frac{\sinh\Big[(\ell+1/2)(\mu-\mu_2)\Big]}{\sinh\Big[(\ell+1/2)(\mu_1-\mu_2)\Big]}\right]\Bigg\}\Bigg|_{v=\cos\nu}.$$

- Electrostatic field, ν-component (μ far from μ_1 and μ_2):

$$E_\nu(\mu,\nu) = \frac{\sin\nu}{2a}\left[\Phi(\mu,\nu) - \left(\frac{\cosh\mu - \cos\nu}{\cosh\mu_1 - \cos\nu}\right)^{3/2}\Phi_1 - \left(\frac{\cosh\mu - \cos\nu}{\cosh\mu_2 - \cos\nu}\right)^{3/2}\Phi_2\right]$$

$$+ \frac{(\cosh\mu - \cos\nu)^{3/2}}{a\,\sin\nu}\times$$

$$\Bigg\{\Phi_1\sum_{\ell\geq 0}\ell\left[v\,P_\ell(v) - P_{\ell-1}(v)\right]S_\ell(\cosh\mu_1)\left[\frac{\sinh\left[(\ell+1/2)(\mu-\mu_2)\right]}{\sinh\left[(\ell+1/2)(\mu_1-\mu_2)\right]} - 1\right] +$$

$$\Phi_2\sum_{\ell\geq 0}\ell\left[v\,P_\ell(v) - P_{\ell-1}(v)\right]S_\ell(\cosh\mu_2)\left[\frac{\sinh\left[(\ell+1/2)(\mu-\mu_1)\right]}{\sinh\left[(\ell+1/2)(\mu_2-\mu_1)\right]} - 1\right]\Bigg\}\Bigg|_{v=\cos\nu}.$$

- Electrostatic field, φ-component: $E_\varphi(\mu,\nu) = 0$.

- Electrostatic field, Cartesian components:

$$E_x(\mu,\nu) = \frac{(-a)}{\cosh\mu - \cos\nu}\left[\frac{\partial\mu}{\partial x}E_\mu(\mu,\nu) + \frac{\partial\nu}{\partial x}E_\nu(\mu,\nu)\right]$$

$$E_y(\mu,\nu) = \frac{(-a)}{\cosh\mu - \cos\nu}\left[\frac{\partial\mu}{\partial y}E_\mu(\mu,\nu) + \frac{\partial\nu}{\partial y}E_\nu(\mu,\nu)\right]$$

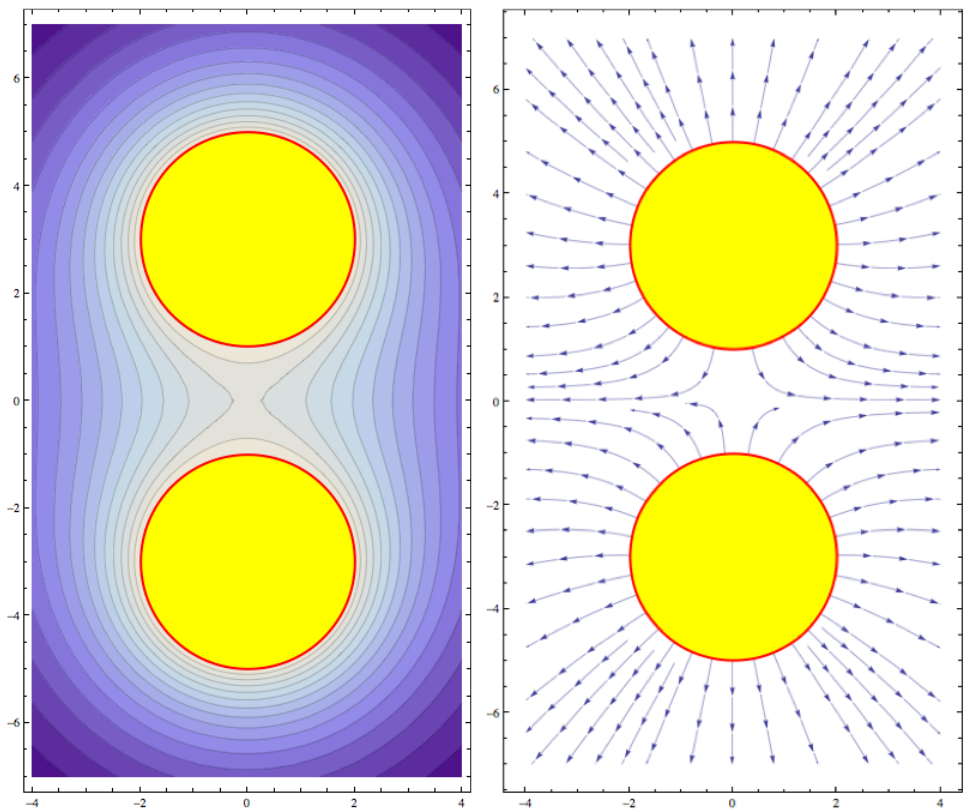
$$E_z(\mu,\nu) = \frac{(-a)}{\cosh\mu - \cos\nu}\left[\frac{\partial\mu}{\partial z}E_\mu(\mu,\nu) + \frac{\partial\nu}{\partial z}E_\nu(\mu,\nu)\right],$$

  where μ and ν are expressed in terms of the Cartesian coordinates (x,y,z) by virtue of equations (3).

Numerical simulations

The figures that follow display the equipotential lines (well, equipotential surfaces, really, projected onto a plane) and electric field lines, for a few configurations of sphere sizes, positions, and potentials.

■


6
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2
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