The Equilibrium of a Tilted Soda Can

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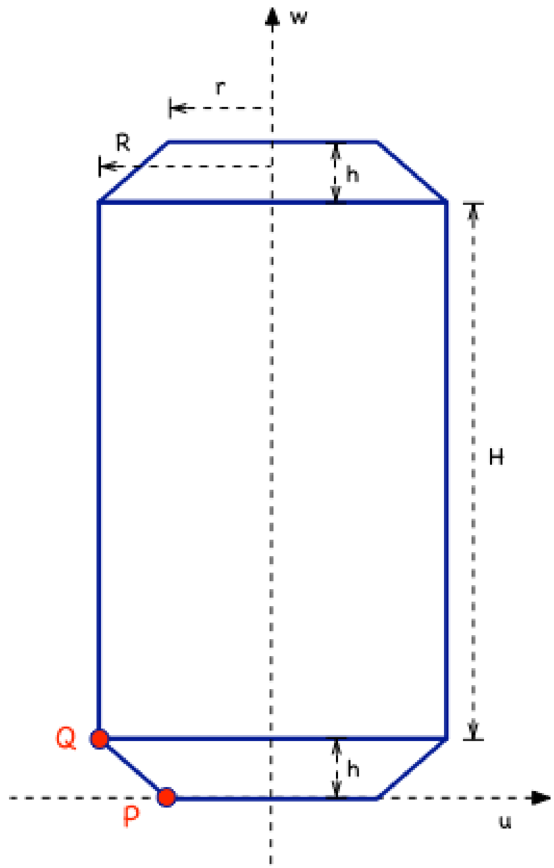
The analytical geometry of an upright soda can

We'll make two simplifying assumptions:

- The can's bottom surface is a flat circle. In reality, it's a concave 'dish'.
- The surface connecting the can's main body (a cylinder) with its bottom is a section of a uniform cone. In reality, it has a complicated shape.

Given these assumptions, our simplified soda can can be represented by the following model: a cylindrical body capped at the top and bottom by flat circles with sections of uniform cones connecting the body to its top and bottom. In the figure below, w is the symmetry axis. The v axis points into the plane of the drawing but is not shown.



The various surfaces are, therefore, described mathematically as follows:

- The main body is a cylinder of radius $R > 0$ and height $H > 0$ so its surface is described by

$$\begin{cases} u^2 + v^2 = R^2\,, & \text{with } 0 \leq u, v \leq R \\ \qquad\qquad w\,, & \text{with } h \leq w \leq H + h\,, \quad 0 < h < H\,. \end{cases}$$

- The top and bottom are flat circles of radius r, where $0 < r < R$:

$$\begin{cases} u^2 + v^2 \leq r^2\,, & \text{with } 0 \leq u, v \leq r \\ \qquad w = H + 2h\,, & \text{(top)} \\ \qquad w = 0\,. & \text{(bottom)} \end{cases}$$

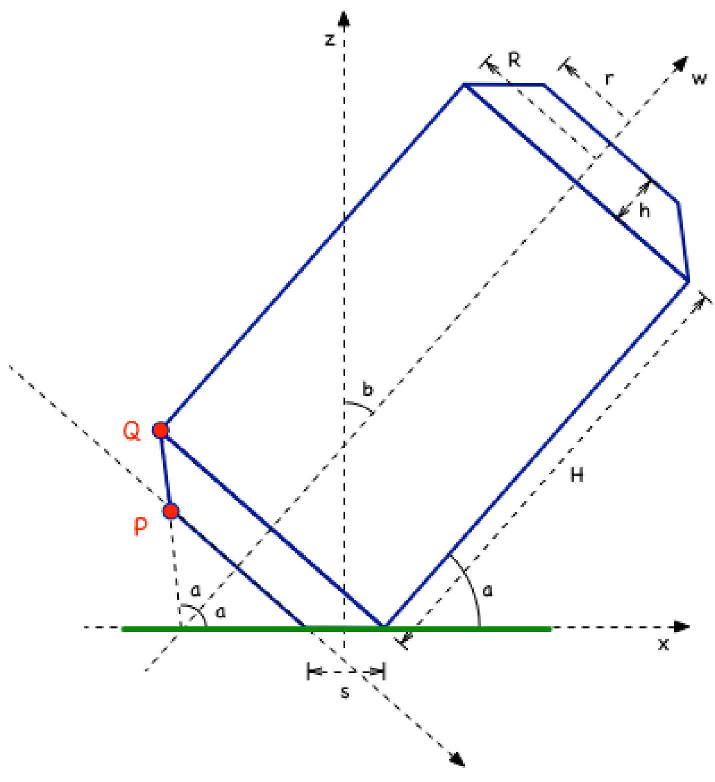
- The top and bottom surfaces connecting the main body to the top and bottom circles, respectively, are sections of uniform cones:

$$\text{top:} \quad \begin{cases} u^2 + v^2 = \left[r + \dfrac{(R-r)}{h} (H + 2h - w) \right]^2, \text{ with } 0 \le u, v \le R \\ H + h \le w \le H + 2h, \quad \text{and} \end{cases}$$

$$\text{bottom:} \quad \begin{cases} u^2 + v^2 = \left[r + \dfrac{(R-r)}{h} w \right]^2, \text{ with } 0 \le u, v \le R \\ 0 \le w \le h. \end{cases}$$

The analytical geometry of a tilted soda can

Armed with the geometry described above, we can now imagine rotating the can's coordinate system around its v axis and then translating its origin so that the bottom cone's surface stands horizontally with respect to the coordinate system we're going to use to describe the geometry of the tilted can.



These rotation and translation operations result in a relationship between the uvw coordinates and the xyz coordinates, as follows:

$$\begin{cases} x = u\cos(b) + w\sin(b) - \Big[\, s/2 + r\cos(b) \Big] \\ y = v \\ z = w\cos(b) - u\sin(b) + r\sin(b)\,, \end{cases}$$

with the corresponding inverse relationship

$$\begin{cases} u = x\cos(b) - z\sin(b) + \Big[\, r + (s/2)\cos(b) \Big] \\ v = y \\ w = z\cos(b) + x\sin(b) + (s/2)\sin(b)\,. \end{cases}$$

The a and b angles are related to the can's geometry by:

$$\begin{aligned} \sin(a) &= \cos(b) &&= \frac{R-r}{s} \\ \cos(a) &= \sin(b) &&= \frac{h}{s} \\ \tan(a) &= \frac{1}{\tan(b)} &&= \frac{R-r}{h}, \end{aligned}$$

so we can rewrite the inverse transformation equations as

$$\begin{cases} u = x\cos(b) - z\sin(b) + (R+r)/2 \\ v = y \\ w = z\cos(b) + x\sin(b) + \frac{h}{2}\,. \end{cases}$$

We also find that

$$s^2 = (R-r)^2 + h^2\,.$$

Note the coordinates of the points marked P and Q. In the uvw coordinate system, $P = (-r, 0, 0)$ and $Q = (-R, 0, h)$. Therefore, in the xyz coordinate system, it follows that

$$\begin{aligned} P_x &= -\,[\, s/2 + 2r\cos(b)\,] &&= -\frac{s}{2} - \frac{2(R-r)r}{s} \\ P_y &= 0 \\ P_z &= 2r\sin(b) &&= 2hr/s \end{aligned}$$

$$\begin{aligned} Q_x &= h\sin(b) - \Big[\, s/2 + (R+r)\cos(b) \Big] &&= -\frac{s}{2} - \frac{(R^2 - r^2 - h^2)}{s} \\ Q_y &= 0 \\ Q_z &= h\cos(b) + (R+r)\sin(b) &&= 2hR/s\,. \end{aligned}$$

We can now express the various surfaces in the xyz coordinate system. First, however, we need to express u^2+v^2 in terms of the new coordinate system:

$$u^2+v^2 = \Big[\,\frac{(s+2x)R+(s-2x)r-2hz}{2s}\,\Big]^2 + y^2\,.$$

Thus,

- **Main body**:

$$\Big[\,\frac{(s+2x)R+(s-2x)r-2hz}{2s}\,\Big]^2 + y^2 = R^2$$

and

$$\frac{h}{2} \le \frac{(R-r)z+hx}{s} \le H+\frac{h}{2}\,.$$

- **Top and bottom**:

$$\Big[\,\frac{(s+2x)R+(s-2x)r-2hz}{2s}\,\Big]^2 + y^2 \le r^2$$

and

$$\begin{aligned} \text{Top:} \quad \frac{(R-r)z+hx}{s} &= H+\frac{3h}{2} \\ \text{Bottom:} \quad \frac{(R-r)z+hx}{s} &= -\frac{h}{2}\,. \end{aligned}$$

- **Top cone section**:

$$\Big[\,\frac{(s+2x)R+(s-2x)r-2hz}{2s}\,\Big]^2 + y^2 = \left\{ r - \frac{(R-r)}{h}\Big[\,\frac{(R-r)z+hx}{s} - H - 3h/2\,\Big] \right\}^2$$

and

$$H+\frac{h}{2} \le \frac{(R-r)z+hx}{s} \le H+\frac{3h}{2}\,.$$

- **Bottom cone section**:

$$\Big[\,\frac{(s+2x)R+(s-2x)r-2hz}{2s}\,\Big]^2 + y^2 = \left\{ r + \frac{(R-r)}{h}\Big[\,\frac{(R-r)z+hx}{s} + h/2\,\Big] \right\}^2$$

and

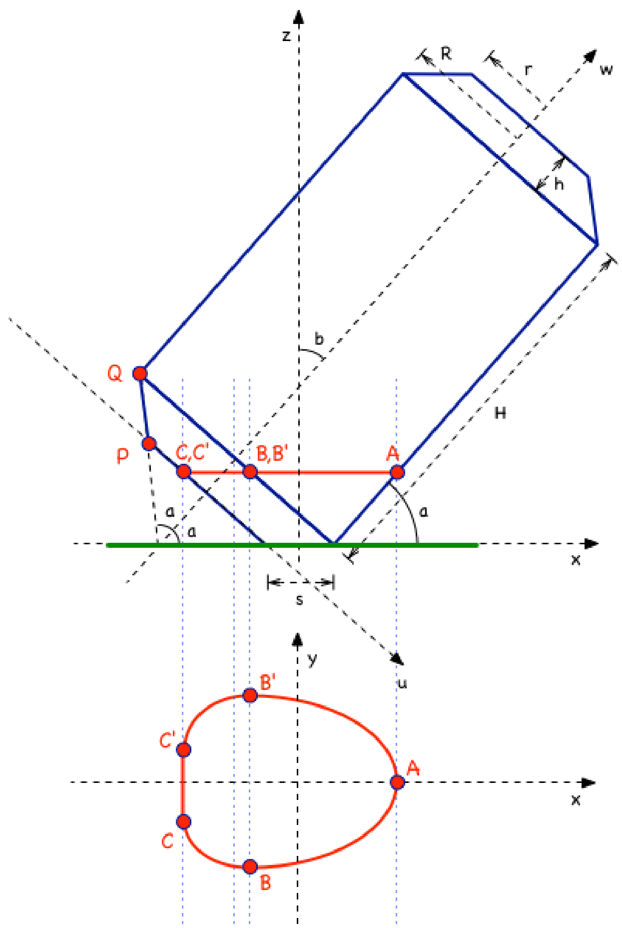
$$-\frac{h}{2} \le \frac{(R-r)z+hx}{s} \le \frac{h}{2}\,.$$

Note that in all the expressions above, x and y can have both positive as well as negative values but z is always non negative.

Computing volumes and centers of gravity

We'll be interested in computing the center of gravity of the volume of liquid inside the can, as a function of the height of the liquid surface. There are several cases to consider.

- **The liquid surface is below** P:



The intersection of the liquid surface (a plane) with the body of the can (a cylinder) results in a conic section, more specifically, an ellipse. That ellipse will be cut short at the points B and B' by the intersection of the liquid surface with the cone section at the bottom (this intersection is a parabola connecting the points B and C, and another connecting the points B' and C'). Finally, the bottom surface (also a plane) will intersect the liquid surface on a line segment, the points C and C'.

Let's now find out the equations describing the various intersections.

- **The intersection between the liquid surface and the main body**: For a fixed value of z, this intersection should be an arc of an ellipse.

$$\Big[\frac{(s+2x)R+(s-2x)r-2hz}{2s}\Big]^2+y^2\;=\;R^2$$

$$\Rightarrow\qquad \frac{\Big\{x-\Big[\dfrac{2hz-(R+r)s}{2(R-r)}\Big]\Big\}^2}{\Big[\dfrac{Rs}{(R-r)}\Big]^2}+\frac{y^2}{R^2}\;=\;1$$

where x is limited to the interval

$$\frac{h}{2}\leq\frac{(R-r)z+hx}{s}\leq H+\frac{h}{2}\,.$$

We thus find that the ellipse is centered at

$$x_0=\frac{2hz-(R+r)s}{2(R-r)}\qquad\text{and}\qquad y_0=0\,,$$

and has semi-major and semi-minor axes $R/\cos(b)$ and R, respectively. The point A has coordinates

$$A_x=\frac{R}{\cos(b)}+x_0=\frac{s}{2}+\frac{hz}{(R-r)}\,,\qquad A_y=0\,,\qquad\text{and}\qquad A_z=z\,.$$

- **The intersection between the liquid surface and the cone section at the bottom**: For a fixed value of z, this intersection should be the arc of a parabola.

$$\Big[\frac{(s+2x)R+(s-2x)r-2hz}{2s}\Big]^2+y^2\;=\;\Big\{r+\frac{(R-r)}{h}\Big[\frac{(R-r)z+hx}{s}+h/2\Big]\Big\}^2$$

$$\Rightarrow\qquad y^2-\frac{2(R-r)z}{h}\,x\;=\;\frac{(R+r)s}{h}\,z+\Big[\frac{(R-r)^2}{h^2}-1\Big]z^2$$

where x is limited to the interval

$$-\frac{h}{2}\leq\frac{(R-r)z+hx}{s}\leq\frac{h}{2}\,.$$

Imposing that this parabola intersects the ellipse we computed just above gives us the coordinates of B and B', whose values with the correct limits as $z \to 0$ turn out to be:

$$\begin{cases} B_x \;=\; B'_x \;=\; x_0 - \dfrac{z}{\sin(b)\cos(b)} + \dfrac{Rs}{R-r} \;=\; \frac{s}{2} - \dfrac{(R-r)\,z}{h} \\[2ex] -\,B_y \;=\; B'_y \;=\; \sqrt{\left(\frac{sz}{h}\right)\left(2R - \frac{sz}{h}\right)} \\[2ex] B_z \;=\; B'_z \;=\; z\,. \end{cases}$$

– **The intersection between the liquid surface and the circular bottom**: For a fixed value of z, this intersection should be a line segment.

$$\Big[\,\frac{(s+2x)R + (s-2x)r - 2hz}{2s}\,\Big]^2 + y^2 \le r^2$$

where x must satisfy

$$\frac{(R-r)z + hx}{s} \;=\; -\,\frac{h}{2}\,.$$

Thus,

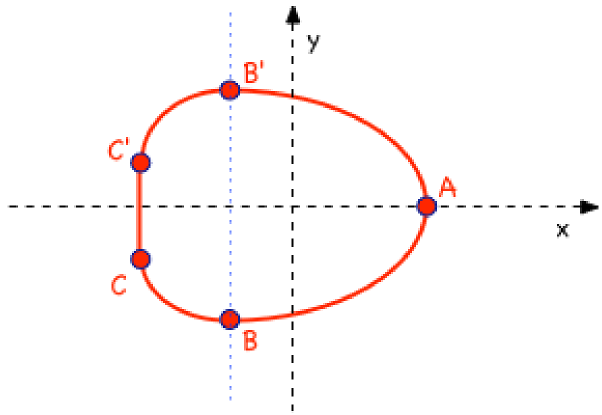
$$\begin{cases} x \;=\; -\,\Big[\,\frac{s}{2} + \dfrac{(R-r)\,z}{h}\,\Big] \\[2ex] y^2 \;\le\; \left(\frac{sz}{h}\right)\left(2r - \frac{sz}{h}\right), \end{cases}$$

from which we find the coordinates of C and C':

$$\begin{cases} C_x \;=\; C'_x \;=\; -\,\Big[\,\frac{s}{2} + \dfrac{(R-r)\,z}{h}\,\Big] \;=\; B_x - s \\[2ex] -\,C_y \;=\; C'_y \;=\; \sqrt{\left(\frac{sz}{h}\right)\left(2r - \frac{sz}{h}\right)} \\[2ex] C_z \;=\; C'_z \;=\; z\,. \end{cases}$$

Therefore, the set of equations describing the boundary of the liquid surface at height $0 \le z < P_z$ is:

$$\begin{cases} y^2 \;=\; \dfrac{2(R-r)z}{h}\,x + \dfrac{(R+r)s}{h}\,z + \Big[\,\dfrac{(R-r)^2}{h^2} - 1\,\Big]z^2\,, & \text{for} \quad B_x - s \le x \le B_x \\[2ex] y^2 \;=\; R^2 - \Big[\,\dfrac{(s+2x)R + (s-2x)r - 2hz}{2s}\,\Big]^2, & \text{for} \quad B_x \le x \le A_x\,, \end{cases}$$



where

$$A_x = \frac{s}{2} + \frac{h}{(R-r)}\, z\,, \qquad \text{and} \qquad B_x = \frac{s}{2} - \frac{(R-r)}{h}\, z\,.$$

These can be rewritten as

$$\begin{cases} y^2 \;=\; \dfrac{2}{\tan(b)}\, z\, x + \dfrac{(R+r)}{\sin(b)}\, z + \left[\, \dfrac{1}{\tan^2(b)} - 1 \,\right] z^2\,, & \text{for} \quad B_x - s \le x \le B_x \\[2ex] y^2 \;=\; R^2 - \left[\, \cos(b)\, x - \sin(b)\, z + \dfrac{(R+r)}{2} \,\right]^2, & \text{for} \quad B_x \le x \le A_x\,, \end{cases}$$

where

$$A_x = \frac{s}{2} + \tan(b)\, z \qquad \text{and} \qquad B_x = \frac{s}{2} - \frac{1}{\tan(b)}\, z\,.$$

Now, the coordinates of the center of gravity of the liquid volume whose surface is at the given value of z are:

$$\mathrm{CG}_x(z) \;\equiv\; \frac{\iiint x'\, dx'\, dy'\, dz'}{\iiint dx'\, dy'\, dz'} \;=\; \frac{1}{V(z)} \int_0^z dz' \int x'\, y(x', z')\, dx'$$

$$\mathrm{CG}_y(z) \;\equiv\; \frac{\iiint y'\, dx'\, dy'\, dz'}{\iiint dx'\, dy'\, dz'} \;=\; \frac{1}{V(z)} \int_0^z dz' \int y^2(x', z')\, dx'$$

$$\mathrm{CG}_z(z) \;\equiv\; \frac{\iiint z'\, dx'\, dy'\, dz'}{\iiint dx'\, dy'\, dz'} \;=\; \frac{1}{V(z)} \int_0^z dz' \int z'\, y(x', z')\, dx'\,,$$

where the volume at height z is

$$V(z) \equiv \iiint dx'\,dy'\,dz' = \int_0^z dz' \left\{ 2\int_{B'_x - s}^{B'_x} y(x',z')\,dx' + 2\int_{B'_x}^{A'_x} y(x',z')\,dx' \right\}$$

and the integrals on x' are to be evaluated at the appropriate boundary points for each branch of $y(x',z')$. Symmetry dictates that $\mathrm{CG}_y(z) = 0$.

The first integral on x' is

$$\begin{aligned}
I_1(z) &\equiv 2\int_{B'_x - s}^{B'_x} y(x',z)\,dx' \\
&= 2\int_{B'_x - s}^{B'_x} dx' \sqrt{\frac{2}{\tan(b)}\, z\, x' + \frac{(R+r)}{\sin(b)}\, z + \Big[\frac{1}{\tan^2(b)} - 1 \Big]\, z^2} \\
&= 2\int_{B'_x - s}^{B'_x} dx' \sqrt{F\, z\, x' + G\, z + H\, z^2} = 2\int_{B'_x - s}^{B'_x} dx' \sqrt{F\, z\, x' + J(z)} \\
&= \frac{4}{3F\, z} (F\, z\, x' + J)^{3/2} \Bigg|_{B'_x - s}^{B'_x} \\
&= \frac{2\sqrt{z}}{3\sin^2(b)\cos(b)} \Bigg[(2R\sin(b) - z)^{3/2} - (2r\sin(b) - z)^{3/2} \Bigg] .
\end{aligned}$$

The second integral on x' is

$$\begin{aligned}
I_2(z) &\equiv 2\int_{B'_x}^{A'_x} y(x',z)\,dx' \\
&= 2\int_{B'_x}^{A'_x} dx' \sqrt{R^2 - \Big[\cos(b)\, x' - \sin(b)\, z + \frac{(R+r)}{2} \Big]^2} \\
&= \frac{R^2}{\cos(b)} \Big[\frac{\pi}{2} - \arcsin(\beta) - \beta\sqrt{1-\beta^2} \Big], \quad \text{where } \beta = 1 - \frac{z}{R\sin(b)} = 1 - \frac{s}{h\,R}\, z\, .
\end{aligned}$$

Now for the z' integrals.

$$\begin{aligned}
J_1(z) &\equiv \int_0^z dz' I_1(z') = \frac{s^3}{3(R-r)h^2} \times \\
&\quad \Big[\frac{\alpha^3}{4} \arctan\sqrt{\frac{z}{\alpha - z}} - \frac{1}{12}\sqrt{z\,(\alpha - z)}\,(3\alpha^2 - 14\alpha z + 8z^2) \Big] \Bigg|_{\alpha = 2hr/s}^{\alpha = 2hR/s}
\end{aligned}$$

and

$$
\begin{aligned}
J_2(z) &\equiv \int_0^z dz' I_2(z') \\
&= R^2 \tan(b) \int_{1-z/R\sin(b)}^{1} \Big[\frac{\pi}{2} - \arcsin(\xi) - \xi \sqrt{1-\xi^2} \Big] \, d\xi \\
&= R^3 \tan(b) \left\{ [\arcsin(\beta) - \tfrac{\pi}{2}] \, \beta + (1-\beta^2)^{1/2} - \tfrac{1}{3}(1-\beta^2)^{3/2} \right\} .
\end{aligned}
$$

So, the volume at height z,

$$
V(z) = \int_0^z dz' I_1(z') + \int_0^z dz' I_2(z') \, ,
$$

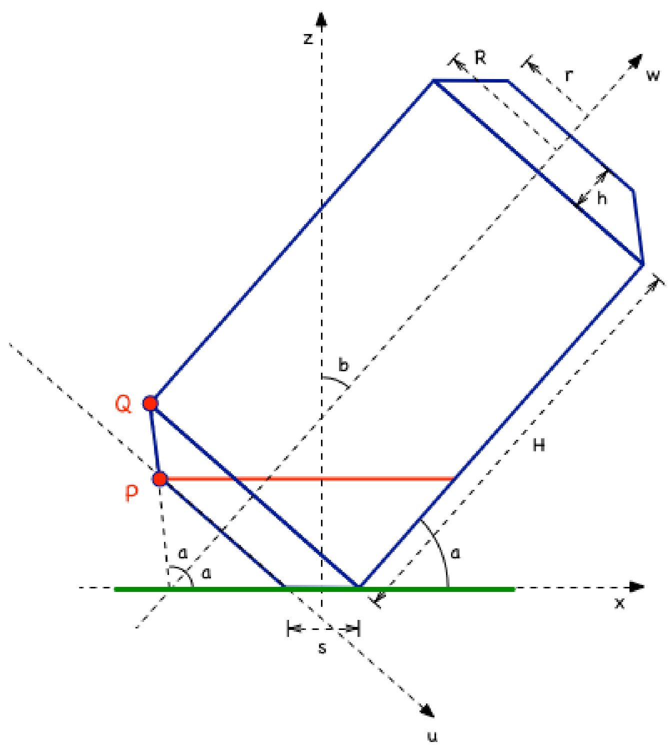
results in

$$
\begin{aligned}
V(z) \; &= \; \frac{s^3}{3(R-r)h^2} \Big[\, \frac{\alpha^3}{4} \arctan \sqrt{\frac{z}{\alpha - z}} - \frac{1}{12} \sqrt{z\,(\alpha - z)} \, (3\alpha^2 - 14\alpha z + 8 z^2) \Big] \Bigg|_{\alpha = 2hr/s}^{\alpha = 2hR/s} \\
&\quad + \; \frac{h\, R^3}{(R-r)} \left\{ [\arcsin(\beta) - \tfrac{\pi}{2}] \, \beta + (1-\beta^2)^{1/2} - \tfrac{1}{3}(1-\beta^2)^{3/2} \right\} ,
\end{aligned}
$$

where $\beta = 1 - (s\,z)/(h\,R)$ and $0 \leq z < (2h/s)\,r$.

- **The liquid surface is at P:**
- **The liquid surface is between P and Q:**
- **The liquid surface is above Q:**

$\blacksquare$



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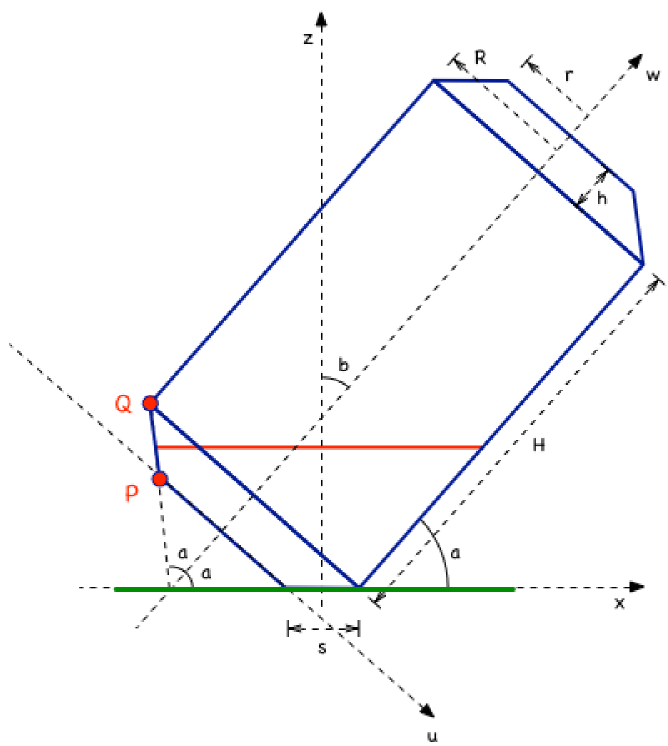

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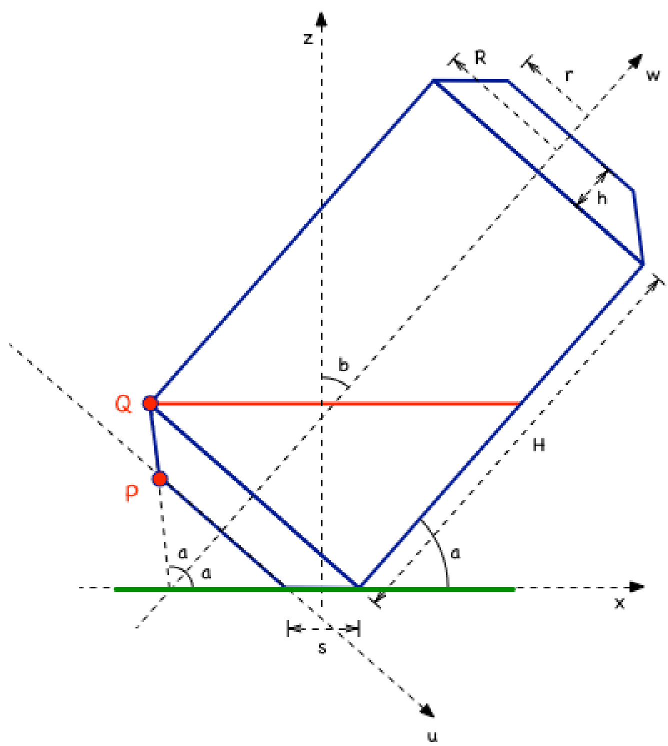

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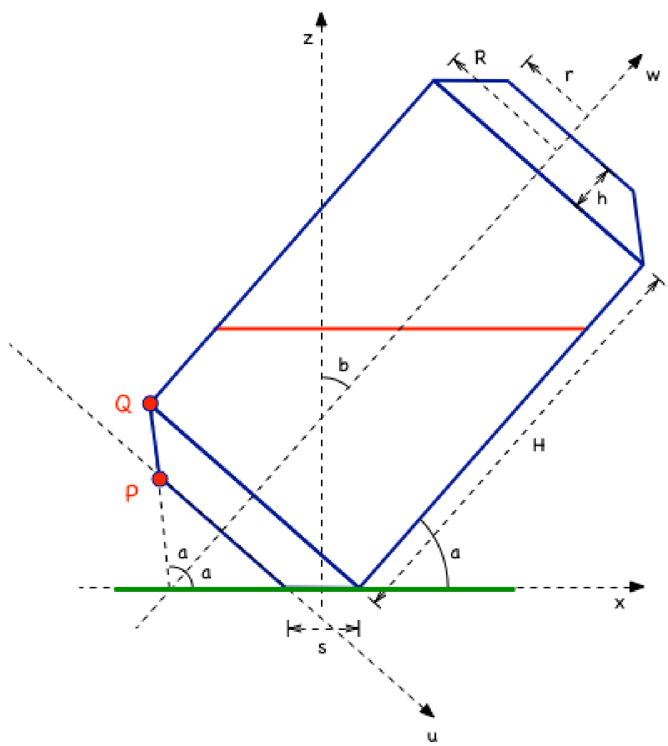
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