The Green function for the Helmholtz operator

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The problem is to solve for the Green function $G(\vec{r}-\vec{r}\,')$ associated with the Helmholtz equation:

$$(\nabla^2 + m^2)\, G(\vec{r}-\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')\,.$$

We start by noticing that the Fourier transform of the delta function is a constant:

$$\mathcal{F}\{\delta(\vec{r}\,)\} = (\frac{1}{\sqrt{2\pi}})^3 \int d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, \delta(\vec{r}\,) = (\frac{1}{\sqrt{2\pi}})^3\,.$$

Therefore, the delta function in the Helmholtz equation above has an inverse-Fourier representation as follows:

$$\delta(\vec{r}-\vec{r}\,') = \mathcal{F}^{-1}\{(\frac{1}{\sqrt{2\pi}})^3\} = (\frac{1}{\sqrt{2\pi}})^3 \int d\vec{k}\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, (\frac{1}{\sqrt{2\pi}})^3\,.$$

Assuming a similar representation for the Green function,

$$G(\vec{r}-\vec{r}\,') = \mathcal{F}^{-1}\{g(\vec{k}\,)\} = (\frac{1}{\sqrt{2\pi}})^3 \int d\vec{k}\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, g(\vec{k}\,)\,,$$

and substituting into the Helmholtz equation, we conclude that

$$g(\vec{k}\,) = -\,(\frac{1}{\sqrt{2\pi}})^3\, \frac{1}{|\vec{k}|^2 - m^2}\,,$$

from which we obtain

$$G(\vec{r}-\vec{r}\,') = -\,(\frac{1}{2\pi})^3 \int d\vec{k}\, \frac{e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{|\vec{k}|^2 - m^2}\,.$$

In order to compute the integral above, let $\vec{a} \equiv \vec{r} - \vec{r}\,'$ and choose the coordinate system in $\vec{k}$-space such that its z-axis is oriented along $\vec{a}$. Then, using spherical coordinates (k, θ, φ) in $\vec{k}$-space, it follows:

$$G(\vec{a}) = -\,(\frac{1}{2\pi})^3 \int_0^\infty \frac{dk\, k^2}{k^2 - m^2} \int_0^\pi d\theta\, \sin\theta\, e^{-ika\,\cos\theta} \int_0^{2\pi} d\varphi = -\,(\frac{1}{2\pi})^2\, \frac{2}{a} \int_0^\infty \frac{k\, \sin(ka)}{k^2 - m^2}\, dk\,.$$

The integrand is an even function of k, so we can extend the integral to the negative real axis by cutting the result in half:

$$G(\vec{a}) = -\,(\frac{1}{2\pi})^2\, \frac{1}{a} \int_{-\infty}^\infty \frac{k\, \sin(ka)}{k^2 - m^2}\, dk = -\,(\frac{1}{2\pi})^2\, \frac{1}{a} \int_{-\infty}^\infty \frac{k\, \sin(k)}{k^2 - (am)^2}\, dk\,.$$

Now we write this result as follows:

$$G(\vec{a}) = -\,(\frac{1}{2\pi})^2\, \frac{1}{2ia} \int_{-\infty}^\infty \frac{k\,(e^{ik} - e^{-ik})}{k^2 - (am)^2}\, dk = -\,(\frac{1}{2\pi})^2\, \frac{1}{2ia}\, [I_+(am) - I_-(am)]\,,$$

where

$$I_+(b) \equiv \int_{-\infty}^\infty \frac{k\, e^{ik}}{k^2 - b^2}\, dk\,, \qquad I_-(b) \equiv \int_{-\infty}^\infty \frac{k\, e^{-ik}}{k^2 - b^2}\, dk\,.$$

Next, note that the transformation $k \to -k$ in the second integral results in minus the first integral. Thus, $I_-(b) = -I_+(b)$ and we obtain

$$G(\vec{a}) = -\,(\frac{1}{2\pi})^2\, \frac{1}{ia}\, I_+(am)\,.$$

The remaining integral, $I_+(b)$, is easily computed using results from complex analysis. Consider the integral in the complex-z plane given by

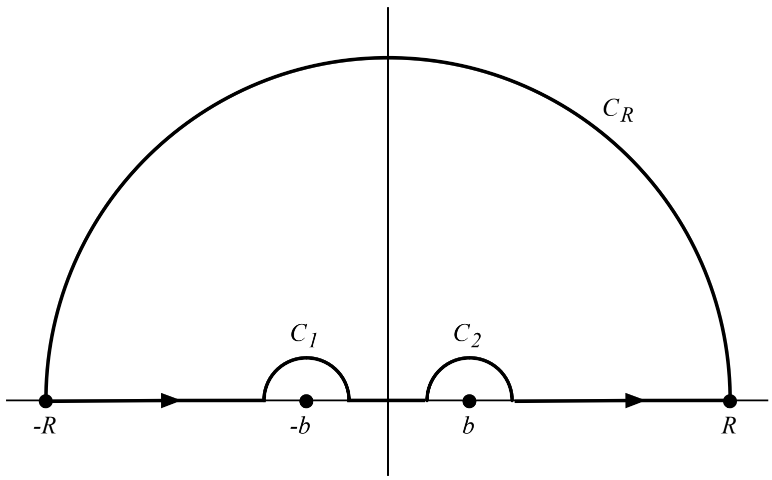
$$\oint_C \frac{z\, e^{iz}}{z^2 - b^2}\, dz\,,$$

where C is some closed contour. If we choose the contour appropriately, we may be able to use the Cauchy-Goursat theorem or the Residue theorem to compute this integral and if part of the contour coincides with the real axis, we'll be able to relate the result to the integral of interest, $I_+(b)$. Let C be the contour shown in the figure below. In other words,

$$\begin{aligned}\oint_C \frac{z\, e^{iz}}{z^2 - b^2}\, dz \;&=\; \int_{-R}^{-b-\epsilon} \frac{x\, e^{ix}}{x^2 - b^2}\, dx + \int_{C_1} \frac{z\, e^{iz}}{z^2 - b^2}\, dz + \int_{-b+\epsilon}^{b-\epsilon} \frac{x\, e^{ix}}{x^2 - b^2}\, dx \\ &+\; \int_{C_2} \frac{z\, e^{iz}}{z^2 - b^2}\, dz + \int_{b+\epsilon}^{R} \frac{x\, e^{ix}}{x^2 - b^2}\, dx + \int_{C_R} \frac{z\, e^{iz}}{z^2 - b^2}\, dz\,,\end{aligned}$$

where

$$\begin{array}{rccll} C_1: & z & = & -b+\epsilon\, e^{i\theta} & \theta \text{ from } \pi \text{ to } 0 \\ C_2: & z & = & b+\epsilon\, e^{i\theta} & \theta \text{ from } \pi \text{ to } 0 \\ C_R: & z & = & R\, e^{i\theta} & \theta \text{ from } 0 \text{ to } \pi\,. \end{array}$$



Note that the poles at $\pm\, b$ lie outside of C. Thus, the integrand is analytic inside and on C, and it follows from the Cauchy-Goursat theorem that

$$\oint_C \frac{z\, e^{iz}}{z^2-b^2}\, dz = 0\,.$$

We could, however, have chosen to include inside C either pole, or even both of them, in which case we would have to use the Residue theorem instead, which states that

$$\oint_C f(z)\, dz = 2\pi\, i \sum_{\text{poles}} \text{Residue}\{f\}\,.$$

Either way, the result is the same. The integral of interest, $I_+(b)$, is obtained when we take the limits $R \to \infty$ and $\epsilon \to 0$, so

$$\begin{aligned} \lim_{R\to\infty} \lim_{\epsilon\to 0} \oint_C \frac{z\, e^{iz}}{z^2-b^2}\, dz \quad &= \quad I_+(b) + \lim_{\epsilon\to 0} \int_{C_1} \frac{z\, e^{iz}}{z^2-b^2}\, dz + \lim_{\epsilon\to 0} \int_{C_2} \frac{z\, e^{iz}}{z^2-b^2}\, dz \\ &+ \quad \lim_{R\to\infty} \int_{C_R} \frac{z\, e^{iz}}{z^2-b^2}\, dz = 0\,. \end{aligned}$$

Since the integrand is analytic everywhere on the respective sub-contours, we may perform the limits first and then integrate:

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{C_{1,2}} \frac{z\,e^{iz}}{z^2-b^2}\,dz &= \int_{C_{1,2}} \lim_{\epsilon\to 0}\frac{z\,e^{iz}}{z^2-b^2}\,dz = \int_\pi^0 \lim_{\epsilon\to 0}\frac{(\mp b+\epsilon\,e^{i\theta})\,e^{i(\mp b+\epsilon\,e^{i\theta})}}{(\mp b+\epsilon\,e^{i\theta})^2-b^2}\,i\,\epsilon\,e^{i\theta}\,d\theta \\
&= \frac{i}{2}\,e^{\mp ib}\int_\pi^0 d\theta = -\,\frac{i\,\pi}{2}\,e^{\mp ib}\,,
\end{aligned}$$

where the top (bottom) sign is obtained when the integration is performed along the contour C_1 (C_2). Similarly,

$$\begin{aligned}
\lim_{R\to\infty}\int_{C_R} \frac{z\,e^{iz}}{z^2-b^2}\,dz &= \int_{C_R} \lim_{R\to\infty}\frac{z\,e^{iz}}{z^2-b^2}\,dz = \int_0^\pi \lim_{R\to\infty}\frac{R\,e^{i\theta}\,e^{iR\,e^{i\theta}}}{(R\,e^{i\theta})^2-b^2}\,i\,R\,e^{i\theta}\,d\theta \\
&= \int_0^\pi \lim_{R\to\infty} i\,e^{iR\,e^{i\theta}}\,d\theta = \lim_{R\to\infty}\int_0^\pi i\,e^{iR\,\cos\theta}\,e^{-\,R\,\sin\theta}\,d\theta = 0\,,
\end{aligned}$$

since $e^{iR\,\cos\theta}$ is bounded (it has magnitude 1) and, along C_R, $\sin\theta > 0$. We then obtain that

$$I_+(b) = \frac{i\,\pi}{2}\,e^{ib} + \frac{i\,\pi}{2}\,e^{-ib} = i\,\pi\,\cos(b)\,.$$

Thus,

$$G(\vec{a}) = -\,(\frac{1}{2\pi})^2\,\frac{1}{ia}\,I_+(am) = -\,\frac{1}{4\pi}\,\frac{\cos(am)}{a}\,.$$

Or, in terms of $\vec{r}$ and $\vec{r}\,'$,

$$G(\vec{r}-\vec{r}\,') = -\,\frac{1}{4\pi}\,\frac{\cos(m\,|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}\,.$$

Note that when $m\to 0$, we recover the Green function for the Laplace operator,

$$G(\vec{r}-\vec{r}\,') = -\,\frac{1}{4\pi}\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,.$$

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