Transverse Normal Modes of a Vertical Heavy Chain

Wagner L. Truppel

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It is standard procedure to include in almost any book on vibrations of elastic media a discussion of the horizontally stretched string and, sometimes, also a derivation of its associated wave equation,

$$\frac{\partial^2\psi}{\partial z^2}(x,t) - \frac{\mu}{T}\,\frac{\partial^2\psi}{\partial t^2}(x,t) = 0\,,$$

where $\psi(x,t)$ is the string displacement along a direction perpendicular to the horizontal axis (the x-axis), at the position x and time t, μ is the string's linear density, that is, its mass per unit of length, and T is the magnitude of the tension stretching the string. Here, both T and μ are assumed constant.

Other interesting examples, not as simple as the stretched string but still solvable, however, are rarely discussed in any detail in undergraduate-level books. In the wonderful book *Vibrations and Waves*, [1], we find some very beautiful pictures of the first three normal modes of vibration of a heavy vertical chain, but a very limited discussion of its normal mode frequencies.

In this article we wish to discuss exactly that system, obtaining a full solution to the problem of determining its normal mode spectrum.

Consider, then, an inextensible heavy homogeneous chain hanging vertically down from some support that is made to move in a periodic fashion. This could be, for example, a uniform circular motion confined to the plane perpendicular to the vertcal direction. In such a case, and when only a single normal mode is excited, each point in the the chain moves in a circle whose plane has a *constant* vertical position relative to the support. This is the motion that corresponds to the pictures in *Vibrations and Waves*, [1].

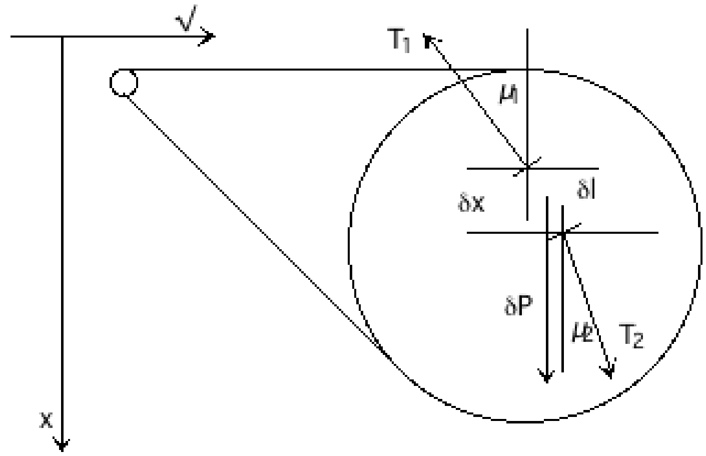
Another possible motion for the support is a simple harmonic one along a straight line in the plane perpendicular to the vertical direction. The two kinds of motion, as is well known, are dynamically equivalent but now each point of the chain has to acquire a vertical motion to prevent the chain from stretching. The resulting profile of the chain in each normal mode, therefore, looks different from the corresponding profile in the previous case. This second case is the one we will discuss in more detail.

Suppose the chain has total length ℓ and linear density μ, and that the support oscillates with a constant angular frequency ω and some small amplitude ξ along the x-axis. We will make the assumption that the amplitude ξ is so small compared with the overall transverse (horizontal) displacement of the chain that it can be considered zero at all times. The reason for such a set-up is, of course, to provide a driving force to maintain the steady-state vibration of the chain and the assumption of vanishingly small amplitude at the support is made so as to simplify the analysis.

If the external frequency ω is equal to one of the normal mode frequencies of vibration of the chain then all of its parts will vibrate at that same frequency and with definite phase differences. Under such conditions, the chain's transverse displacement, ψ, as a function of the vertical position from the top, z, and the time t, is of the form

$$\psi(z,t) = f(z)\,\cos(\omega\,t+\alpha)\,,$$

with α being a constant and $f(z)$ giving the profile of the chain in such a mode [1, 2]. Our task is to find and solve the differential equation governing the behavior of the function f. Note that our assumption of a vanishingly small amplitude at the top forces us to impose the condition $f(0) = 0$.



Thus, consider a small section of the chain, of length $\delta\ell$ and mass $\delta m = \mu\,\delta\ell$, and let us analyze the forces acting on it. As pictured in the figure, the small section we're considering is under the action of its own weight as well as the tensions on both of its sides. Although the section does have to move vertically, we'll assume its vertical acceleration to be vanishingly small compared to the acceleration of gravity (if we were considering the case where the support moves in a circle, then the vertical acceleration would be exactly zero, for each section of the chain would then move only in the transverse direction, in the steady-state situation of a single normal mode being excited).

A rough justification of this assumption follows from the fact that, for simple harmonic motion, the magnitude of the acceleration is proportional to the amplitude of the motion, $a = \omega^2\,s$. If ω is some multiple n of the chain's natural angular frequency $\omega_0 \equiv \sqrt{g/\ell}$, then $a = n^2\,g\,s/\ell$, where g is the acceleration of gravity. Since the amplitude for the motion in the vertical direction, s, is generally much smaller than the chain's total length ℓ, and if n is not too large, the conclusion follows. Newton's second law gives us, therefore,

$$\begin{aligned} &T_1\,\cos\theta_1 - T_2\,\cos\theta_2 \approx \delta P = \delta m\,g \qquad \text{and} \\ &T_2\,\sin\theta_2 - T_1\,\sin\theta_1 = \delta m\,\frac{\partial^2\psi}{\partial t^2}\,, \end{aligned}$$

with $\delta m = \mu\,\delta\ell$.

Now, if there are no kinks or other kinds of singularities with very large curvatures, the tension must vary smoothly along the chain and the angles must be very small at both ends of each small section of it. We will, therefore, make the approximations $\theta_1 \approx 0$ and $\theta_2 \approx 0$. That doesn't mean, however, that the two angles are equal. And neither are the two tensions. We then write

$$\begin{aligned} \theta_1 &= \theta(z,t)\,, & \theta_2 &= \theta(z+\delta z,t) \approx \theta_1 + \delta\theta(z,t)\,, \\ T_1 &= T(z,t)\,, \quad \text{and} & T_2 &= T(z+\delta z,t) \approx T_1 + \delta z\,\frac{\partial T}{\partial z}(z,t)\,. \end{aligned}$$

From these equations, it follows that

$$\begin{aligned} \cos\theta_2 &\approx \cos\theta_1 - \delta\theta(z,t)\,\sin\theta_1 \quad \text{and} \\ \delta m\,g &\approx T_1\,\cos\theta_1 - T_2\,\cos\theta_2 \\ &\approx T_1\,\sin\theta_1\,\delta\theta(z,t) - \frac{\partial T}{\partial z}(z,t)\,\cos\theta_1\,\delta z\,. \end{aligned}$$

Now, as we have argued above, the angle θ_1 is small, so we'll also write $\sin\theta_1 \approx 0$ and $\cos\theta_1 \approx 1$. Then,

$$\delta m\,g = \mu\,g\,\delta\ell \approx \mu\,g\,\delta z \approx -\frac{\partial T}{\partial z}(z,t)\,\delta z\,,$$

from which we find the differential equation governing the variation of the tension along the chain,

$$\frac{\partial T}{\partial z}(z,t) = -\,\mu\,g\,.$$

Considering that the tension must vanish at the very bottom of the chain, where $z = \ell$, we find that the proper solution to this equation must be

$$T(z,t) = \mu\,g\,(\ell - z)\,.$$

Notice that this is true as long as the frequency ω and the amplitude of the chain's displacement are not too large compared, respectively, to its natural frequency ω_0 and total length ℓ. *Only* under these assumptions is the tension linearly dependent on z and independent of the time. Going back to the horizontal component of the net force acting on the small section we're considering, we'll write $\sin\theta(z,t) \approx \tan\theta(z,t) = \partial\psi(z,t)\,/\,\partial z$, an approximation justified by the smallness of $\theta(z,t)$. Then,

$$\begin{aligned} \delta m\,\frac{\partial^2\psi}{\partial t^2}(z,t) &= \mu\,\delta\ell\,\frac{\partial^2\psi}{\partial t^2}(z,t) \approx \mu\,\delta z\,\frac{\partial^2\psi}{\partial t^2}(z,t) \\ &\approx T_2\,\sin\theta_2 - T_1\,\sin\theta_1 \approx T_2\,\tan\theta_2 - T_1\,\tan\theta_1 \\ &\approx T(z,t)\,\frac{\partial^2\psi}{\partial z^2}(z,t)\,\delta z + \frac{\partial T}{\partial z}(z,t)\,\frac{\partial\psi}{\partial z}(z,t)\,\delta z\,, \end{aligned}$$

from which it follows that

$$\begin{aligned} \mu\,\frac{\partial^2\psi}{\partial t^2}(z,t) &= T(z,t)\,\frac{\partial^2\psi}{\partial z^2}(z,t) + \frac{\partial T}{\partial z}(z,t)\,\frac{\partial\psi}{\partial z}(z,t)\,, \qquad \text{or} \\ \frac{\partial^2\psi}{\partial t^2}(z,t) &= g\,(\ell - z)\,\frac{\partial^2\psi}{\partial z^2}(z,t) - g\,\frac{\partial\psi}{\partial z}(z,t)\,, \end{aligned}$$

which is the desired differential equation describing the transverse vibrations of our chain.

In terms of the function $f(z)$ introduced earlier, we have

$$-\omega^2 f(z) = g\,(\ell - z)\,\frac{d^2 f}{dz^2}(z) - g\,\frac{df}{dz}(z)\,.$$

The introduction of the new variable $q = 1 - z/\ell$, as well as the function $h(q) = f(z)$, takes this equation into an alternate form,

$$q\,h''(q) + h'(q) + \lambda^2\,h(q) = 0\,,$$

with $\lambda = \omega/\omega_0$, and $\omega_0^2 = g/\ell$ in the rôle of the natural frequency of the chain.

A further transformation, $q = (s/2\,\lambda)^2$, and the identification $u(s) = h(q)$, result in a more interesting form:

$$s\,u''(s) + u'(s) + s\,u(s) = 0\,,$$

which is none other than *Bessel's equation of order zero*. Therefore, the solution to our problem, regular at the origin, is simply

$$u(s) = A\,J_0(s)\,,$$

with A being an arbitrary constant. In terms of the original function, $f(z)$, we find

$$f(z) = A\,J_0\left(2\,\frac{\omega}{\omega_0}\,\sqrt{1 - \frac{z}{\ell}}\right)\,,$$

and

$$\psi(z,t) = A\,J_0\left(2\,\frac{\omega}{\omega_0}\,\sqrt{1 - \frac{z}{\ell}}\right)\,\cos(\omega\,t + \alpha)\,.$$

Now, recall that we have imposed the condition that the displacement of the chain at its point of support must vanish. That means the frequencies of the normal modes, the *eigen-frequencies*, must be such that

$$J_0(2\,\frac{\omega}{\omega_0}) = 0\,,$$

that is,

$$\omega_n = \frac{1}{2}\,\omega_0\,r_n^{(0)} = \frac{1}{2}\,\sqrt{\frac{g}{\ell}}\,r_n^{(0)}\,,$$

where $r_n^{(0)}$ is the n-th root of J_0. The first few *eigen-frequencies* are listed below:

$$\begin{array}{lll}
\omega_1 = \;\;1.20241\,\omega_0\,, & \omega_2 = \;\;2.76004\,\omega_0\,, & \omega_3 = \;\;4.32686\,\omega_0\,, \\
\omega_4 = \;\;5.89577\,\omega_0\,, & \omega_5 = \;\;7.46546\,\omega_0\,, & \omega_6 = \;\;9.03553\,\omega_0\,, \\
\omega_7 = 10.60582\,\omega_0\,, & \omega_8 = 12.17624\,\omega_0\,, & \omega_9 = 13.74674\,\omega_0\,.
\end{array}$$

Let us see now how the solution obtained above applies to the case where the chain's support moves in a circle. In that case, we expect each point of the chain to move in a circle of constant z and so its displacement becomes a vector in the xy plane,

$$\vec{\psi}(z,t) = f(z)\;[\cos(\omega\,t + \alpha)\,\hat{\imath} + \sin(\omega\,t + \alpha)\,\hat{\jmath}]\,.$$

It's easy to see that the derivation we've followed above applies equally well to this case.

References

[1] French, A. P., *Vibration and Waves*.

[2] Crawford, *Waves*.