The visual appearance of relativistic objects

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Single object of negligible size

Suppose an object of negligible size is moving uniformly, at relativistic speeds, first towards us and then away from us. How close is the object, as a function of time (measured in our reference frame) and where does it *appear* to be when we see it?

In order to answer these questions, let's set up two inertial reference frames, one attached to ourselves ($\mathcal{S}$) and the other attached to the object ($\mathcal{S}'$). Moreover, let's choose the two frames to have respectively parallel axes and such that their origins coincide when $t = t' = 0$. Let's also assume that $\mathcal{S}'$ is moving with velocity $\vec{u}$ along the positive z/z' axes. These choices allow us to write the Lorentz transformations connecting observations on both frames as follows:

$$\begin{aligned} c\,t &= \gamma_u\,(c\,t' + \beta_u\,z') & \qquad c\,t' &= \gamma_u\,(c\,t - \beta_u\,z) \\ x &= x' & x' &= x \\ y &= y' & y' &= y \\ z &= \gamma_u\,(z' + \beta_u\,c\,t') & z' &= \gamma_u\,(z - \beta_u\,c\,t)\,, \end{aligned}$$

where, as usual, $\beta_u = u/c$ and $\gamma_u = (1-\beta_u^2)^{-1/2}$. Since, with this choice of axes, the x and y coordinates don't change between frames, we'll simply ignore those coordinates in what follows. Also, it will be useful later on to have these transformation equations written in a "mixed" form:

$$\begin{aligned} c\,t &= \frac{c\,t'}{\gamma_u} + \beta_u\,z & \qquad c\,t' &= \frac{c\,t}{\gamma_u} - \beta_u\,z' \\ z &= \frac{z'}{\gamma_u} + u\,t & z' &= \frac{z}{\gamma_u} - u\,t'\,. \end{aligned}$$

Assuming the object to be located at the origin of $\mathcal{S}'$, its motion, as observed in our reference frame, is then described by setting $z' = 0$: $z = u\,t$.

Note that, for $t < 0$, the object is approaching us whereas, for $t > 0$, it's moving away from us, having "collided" with us at $t = 0$. This answers our first question above; the location of the object, in our frame, is $z = u\,t$. But is that where we ***see*** the object?

Suppose that at time t_e (measured in our frame $\mathcal{S}$), a short light pulse is emitted from the object. Let's call that the *time of emission*. That pulse has to travel a certain distance to reach us but, during that time, the object itself has also moved. Let's consider the two cases $t_e < 0$ and $t_e > 0$ separately:

- $t_e < 0$: At time t_e, when the pulse is emitted, the object's position is $z(t_e) = u\,t_e$, which is negative since the object is approaching us. During the amount of time $\Delta t = -\,z(t_e)/c = -\,\beta_u\,t_e$ that it takes for the signal to reach us, the object has moved towards us a distance $u\,\Delta t$ so its actual position at the time we receive the signal (t_a, for *time of arrival*) is $z(t_a) = z(t_e) + u\,\Delta t = (1-\beta_u)\,u\,t_e = (1-\beta_u)\,z(t_e)$, with $t_a = t_e + \Delta t$. Note that, since $0 \leq \beta_u < 1$,
$$0 < \frac{z(t_a)}{z(t_e)} = 1 - \beta_u \leq 1\,.$$
- $t_e > 0$: The object's position is still given by $z(t_e) = u\,t_e$, but now $z(t_e)$ is positive since the object is moving away from us. During the amount of time $\Delta t = z(t_e)/c = \beta_u\,t_e$ that it takes for the signal to reach us, the object has moved away from us an additional distance $u\,\Delta t$ so its actual position at the time we receive the signal is now $z(t_a) = z(t_e) + u\,\Delta t = (1+\beta_u)\,u\,t_e = (1+\beta_u)\,z(t_e)$. Note, then, that
$$1 \leq \frac{z(t_a)}{z(t_e)} = 1 + \beta_u < 2\,.$$

Let's now interpret these results. Recall that t_e is the time at which the signal left the object, so $z(t_e)$ is the position of the object at the moment of emission. On the other hand, t_a is the moment when we observe the signal, so $z(t_a)$ is the object's position when we *see* the object. But what we see is the image of the object when it was located at $z(t_e)$, not when it's located at $z(t_a)$. In other words, we ***see*** the object as if it were at $z(t_e)$ but it's actually located at $z(t_a)$.

Given the inequalities derived above, we conclude that **the object is actually *closer* than it appears to be, when it's approaching us, and it's actually *farther* than it appears to be, when it's moving *away* from us**. Moreover, since $z(t_a)/z(t_e) < 2$ when it's receding from us, we also conclude that it cannot, in that case, be farther than twice its apparent distance. These are all physically intuitive results.

For later reference, let's compute the moments of emission and arrival of the signal, measured in the object's reference frame. Since the emission event takes place at $z' = 0$ and the reception event at $z = 0$, it follows from the "mixed" transformations:
$$t'_e = t_e/\gamma_u \qquad \text{and} \qquad t'_a = \gamma_u\,t_a\,.$$

Notice that t_e and t'_e, as well as t_a and t'_a, have the same sign. Moreover, since $t_a > t_e$, we can show that causality is preserved, as it must be:

$$\frac{t'_a}{t'_e} = \frac{\gamma_u\, t_a}{t_e/\gamma_u} = \gamma_u^2\, \frac{t_a}{t_e} > 1\,,$$

since $\gamma_u > 1$ for $u \neq 0$. In words, the arrival of the signal always occurs *after* the emission of the signal, according to both reference frames.

Also for later reference, note that we can write the time of arrival, measured in either frame, as follows:

$$t_a = [\,1 + \beta_u\, \varepsilon(t_e)\,]\, t_e \qquad \text{and} \qquad t'_a = \gamma_u^2\, [\,1 + \beta_u\, \varepsilon(t'_e)\,]\, t'_e\,,$$

where $\varepsilon(t) = -1$ when $t < 0$ and $\varepsilon(t) = +1$ when $t > 0$. Moreover, from these results, it also follows that t_e and t_a have the same sign. This makes sense physically for, when the object emits a signal before reaching the origin of $\mathcal{S}$ ($t_e < 0$), the signal (being faster than the object) will reach the origin first. Since the object reaches the origin at $t = 0$, we conclude that $t_a < 0$. Naturally, if $t_e > 0$, so is t_a. Combining the results above, we conclude that all four quantities (t_e, t_a, t'_e, and t'_a) have the same sign.

A collection of objects

Suppose now that we have a *collection* of objects, $\{o_i \,|\, 1 \leq i \leq n\}$, all of which at rest in the frame $\mathcal{S}'$. Moreover, assume that object o_i is located at position $\{x'_i, y'_i, z'_i\}$ when, at time $t_e^{(i)\prime}$, it emits a light signal.

The event corresponding to the emission of this light signal from o_i has the space-time coordinates $\{x'_i, y'_i, z'_i, t_e^{(i)\prime}\}$ in $\mathcal{S}'$. The corresponding coordinates in $\mathcal{S}$ are given by

$$\begin{aligned} x_i &= x'_i \\ y_i &= y'_i \\ z_i &= \gamma_u\, (z'_i + \beta_u\, c\, t_e^{(i)\prime}) \\ c\, t_e^{(i)} &= \gamma_u\, (c\, t_e^{(i)\prime} + \beta_u\, z'_i)\,. \end{aligned}$$

Now, recall that we are located at the origin of our frame $\mathcal{S}$. How long does the signal from o_i take to reach us, according to our clocks? Well, that time is the distance between the point of emission and our position, both measured in our frame, divided by the speed of light. The situation is analogous to that of the previous section, except that now the events in question are no longer constrained to the z-axis. Thus,

$$c\, \Delta t_i = +\sqrt{x_i^2 + y_i^2 + z_i^2}\,,$$

and the moment (times c) at which the light signal from o_i arrives at our position is:

$$c\,t_a^{(i)} = c\,t_e^{(i)} + c\,\Delta t_i = c\,t_e^{(i)} + \sqrt{x_i^2 + y_i^2 + z_i^2}\,.$$

Next, consider what it means for us to observe that collection of objects. We observe only the light signals arriving from the various o_i's but — and this is the key idea — whatever image we make of these signals is made of their arrival *at the same time* (measured in $\mathcal{S}$, of course). In other words, the image of the various objects is a collection of points at coordinates $\{x_i, y_i, z_i\}$ such that light coming from those points all arrive at our position at the same time, t_a. But that also means the light signals must have left their sources at different times, since the signals traveled different distances. The bottom line is that we observe a *distorted* image of the collection of objects.

Imposing that all signals arrive at our position at the same time, t_a, we have

$$c\,t_a = c\,t_e^{(i)} + \sqrt{x_i^2 + y_i^2 + z_i^2}\,.$$

Note that this is a function of the emission event's space-time coordinates, as measured in $\mathcal{S}'$ and, for a given object (that is, a given location in $\mathcal{S}'$), it fixes the time (in $\mathcal{S}'$) at which the pulse must have left, to reach us at time t_a (measured in $\mathcal{S}$). In fact, farther away objects must have emmitted their pulses farther in the past, if they are to be visible at the same time.

This lets us deduce the moments, according to $\mathcal{S}'$, at which each object emitted its light signal, as a function of their spatial coordinates in that frame, by solving the equation above for $t_e^{(i)\prime}$:

$$c\,t_a = \gamma_u\,(c\,t_e^{(i)\prime} + \beta_u\,z_i') + \sqrt{(x_i')^2 + (y_i')^2 + \gamma_u^2\,(z_i' + \beta_u\,c\,t_e^{(i)\prime})^2}\,.$$

The physically meaningful solution, that is, the solution that agrees with the result of the previous section when we consider an object at the origin of $\mathcal{S}'$, namely, $t_e' = t_e/\gamma_u$, is

$$c\,t_e^{(i)\prime} = c\,\gamma_u\,t_a - \sqrt{(x_i')^2 + (y_i')^2 + (z_i' + \beta_u\,\gamma_u\,c\,t_a)^2}\,.$$

Once again, the interpretation of this result is that the $t_e^{(i)\prime}$ value computed above is the time, in $\mathcal{S}'$, at which object o_i must emit its signal so that the signal arrives at our position at the same time (according to *our* notion of time) as the signals of all other objects.

Armed with this result, we can now tell what the spatial coordinates of every object are at the times when they emitted their signals. In $\mathcal{S}'$, where the objects are at rest, we know that those coordinates are $\{x_i', y_i', z_i'\}$ but we're interested in their counterparts in $\mathcal{S}$ because those are what we in $\mathcal{S}$ see as the objects' images. Using the Lorentz transformations once again, we find that

$$z_i(t_e^{(i)}) = \frac{z_i'}{\gamma_u} + u\, t_e^{(i)}$$
$$c\, t_e^{(i)} = \gamma_u \left(\beta_u\, z_i' + c\, \gamma_u\, t_a\right) - \gamma_u \sqrt{(x_i')^2 + (y_i')^2 + (z_i' + \beta_u\, \gamma_u\, c\, t_a)^2}\,,$$

or, more succintly,

$$z_i(t_e^{(i)}) = \gamma_u\, z_i' + \beta_u\, \gamma_u^2\; c\, t_a - \beta_u\, \gamma_u \sqrt{(x_i')^2 + (y_i')^2 + (z_i' + \beta_u\, \gamma_u\, c\, t_a)^2}\,.$$

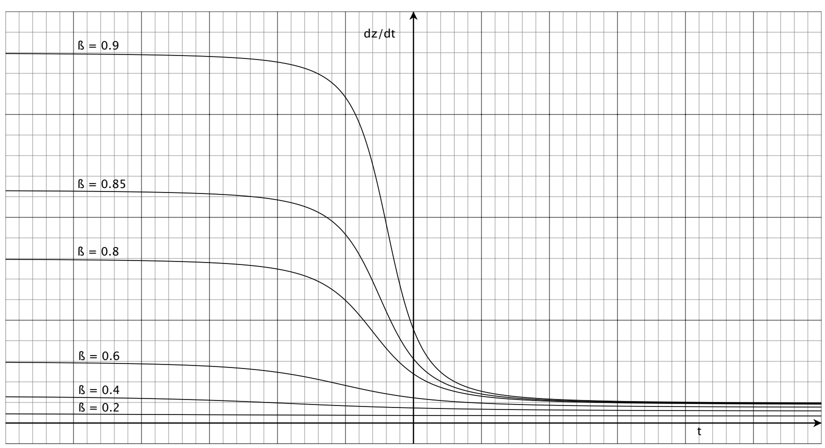
Moreover, we know that at the moment we see those images, the objects are actually located at

$$z_i(t_a) = \frac{z_i'}{\gamma_u} + u\, t_a\,.$$

Interestingly enough, the apparent speed with which the observed image moves changes significantly as time goes by. From the expression for $z_i(t_e^{(i)})$ above, we obtain:

$$\frac{dz_i(t_e^{(i)})}{dt_a} = \beta_u\, \gamma_u^2 \left[1 - \frac{\beta_u\, (z_i' + \beta_u\, \gamma_u\, c\, t_a)}{\sqrt{(x_i')^2 + (y_i')^2 + (z_i' + \beta_u\, \gamma_u\, c\, t_a)^2}}\right] c\,,$$

a plot of which is shown below for $x_i' = y_i' = z_i' = 2$.



Note how the image slows down as the object approaches the origin and then maintains a low speed after it's passed us by. The speed of the actual object, of course, remains constant. The limiting apparent speeds, for moments in time far into the past or far into the future, are:

$$\frac{1}{c}\frac{dz_i(t_e^{(i)})}{dt_a} = \begin{cases} \dfrac{\beta_u}{1-\beta_u}\,, & \text{as } t_a \to -\infty \\ \dfrac{\beta_u}{1+\beta_u}\,, & \text{as } t_a \to +\infty\,. \end{cases}$$

It's also interesting to note that, for $\beta_u > 1/2$, the image is approaching us at superluminal speeds. Why does the image slow down and how is it possible that it may move faster than light itself?

Application: approaching wall

As an application, consider the surface of a flat wall at rest in $\mathcal{S}'$, described by the coordinates

$$\{|x'| \leq a, |y'| \leq b, z' = 0\}\,.$$

What do we see as the wall approaches us? When a point on the wall is actually located at

$$\begin{aligned} x_i(t_a) &= x_i' \\ y_i(t_a) &= y_i' \\ z_i(t_a) &= u\, t_a\,, \end{aligned}$$

we see an image of that point as if it was located at

$$\begin{aligned} x_i(t_e^{(i)}) &= x_i' \\ y_i(t_e^{(i)}) &= y_i' \\ z_i(t_e^{(i)}) &= \gamma_u^2\, u\, t_a - \beta_u\, \gamma_u \sqrt{(x_i')^2 + (y_i')^2 + (\gamma_u\, u\, t_a)^2}\,. \end{aligned}$$

Note, however, that

$$\frac{z_i(t_e^{(i)})}{\gamma_u^2} = z_i(t_a) - \frac{\beta_u}{\gamma_u} \sqrt{(x_i')^2 + (y_i')^2 + [\,\gamma_u\, z_i(t_a)\,]^2} \leq z_i(t_a) - \beta_u\, |z_i(t_a)|\,.$$

Let's interpret this result by looking at the two cases, $t_a < 0$ and $t_a > 0$, separately.

- $t_a < 0$: In this case, $z_i(t_a) < 0$, so

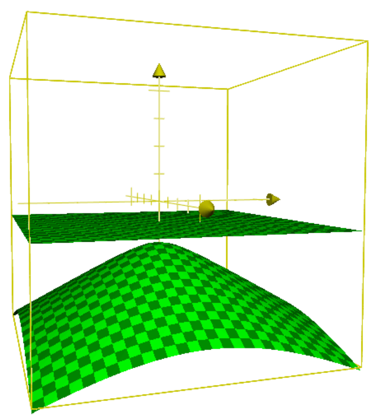
$$\frac{z_i(t_e^{(i)})}{\gamma_u^2} \leq (1 + \beta_u)\, z_i(t_a) \quad \Rightarrow \quad \frac{z_i(t_a)}{z_i(t_e^{(i)})} \leq 1 - \beta_u\,.$$

- $t_a > 0$: Now $z_i(t_a) > 0$ and

$$\frac{z_i(t_e^{(i)})}{\gamma_u^2} \leq (1 - \beta_u)\, z_i(t_a) \quad \Rightarrow \quad \frac{z_i(t_a)}{z_i(t_e^{(i)})} \geq 1 + \beta_u\,.$$

Once again, we conclude that when the wall is approaching us, it's actually closer than it appears to be and when it's receding from us, it's actually farther away than it appears to be.

More interesting, however, is the distortion effect caused by the finite speed of light. Each point of the image we perceive is the image of that point as it was sometime in the past. In fact, different points produce images from different moments in time, so what we perceive to be the complete wall is a bizarre superposition of portions of the wall from different moments in the past. It's almost as if the wall lived simultaneously in different moments in the timeline.



Pictured above is a plot of the actual current position of the wall (the flat texture) at some moment in time prior to its collision with the origin. The distorted figure below the wall is what the wall looks like at that moment, from the point of view of someone located at the origin. Yet again, it's important to emphasize that that distorted image is not the image of the entire wall at any single time but a mosaic of parts of the wall at different moments in time. In particular, it's not the image of the current position of the wall. The wall is moving at half the speed of light, so $\beta_u = 0.5$.

Application: approaching cube

Consider now a (transparent) cube at rest in $\mathcal{S}'$, whose vertices have the following coordinates with respect to that reference frame:

$$\{|x'| = \ell, |y'| = \ell, |z'| = \ell\},$$

where ℓ is some arbitrary, but fixed, length. A point $P = (x'_i, y'_i, z'_i)$ on an edge of the cube, whose endpoints are two of the vertices, say, A and B, is easily located given a real number

$0 \le \lambda \le 1$:

$$P = A + \lambda\,(B - A)\,.$$

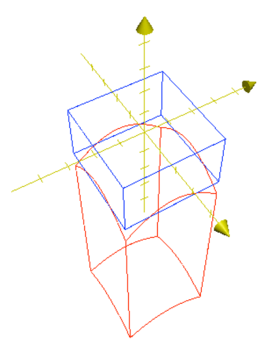
When any such point is actually located at

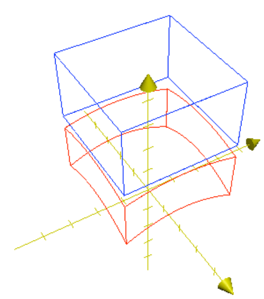
$$\begin{aligned} x_i(t_a) &= x_i' \\ y_i(t_a) &= y_i' \\ z_i(t_a) &= \frac{z_i'}{\gamma_u} + u\,t_a\,, \end{aligned}$$

we see it as if it was located at

$$\begin{aligned} x_i(t_e^{(i)}) &= x_i' \\ y_i(t_e^{(i)}) &= y_i' \\ z_i(t_e^{(i)}) &= \gamma_u\,z_i' + \beta_u\,\gamma_u^2\;c\,t_a - \beta_u\,\gamma_u\,\sqrt{(x_i')^2 + (y_i')^2 + (z_i' + \beta_u\,\gamma_u\,c\,t_a)^2}\,. \end{aligned}$$

The following two pictures show the situation when a cube is approaching us at 80% of the speed of light ($\beta_u = 0.8$). The first one shows the cube in its actual position (blue) one unit of time prior to passing us and what we see at that time (red). The second picture shows the same thing, but five units of time after the cube has passed us.





Application: approaching sphere

Consider now a sphere of radius r, at rest in $\mathcal{S}'$. An arbitrary point on its surface is described by the following coordinates:

$$\begin{aligned} x_i' &= r\,\sin\theta_i\,\cos\varphi_i \\ y_i' &= r\,\sin\theta_i\,\sin\varphi_i \\ z_i' &= r\,\cos\theta_i\,, \end{aligned}$$

where $0 \le \theta_i \le \pi$ and $0 \le \varphi_i \le 2\pi$. When any such point is actually located at

$$\begin{aligned} x_i(t_a) &= r\,\sin\theta_i\,\cos\varphi_i \\ y_i(t_a) &= r\,\sin\theta_i\,\sin\varphi_i \\ z_i(t_a) &= \frac{r\,\cos\theta_i}{\gamma_u} + u\,t_a\,, \end{aligned}$$

we see it as if it was located at

$$\begin{aligned} x_i(t_e^{(i)}) &= r\,\sin\theta_i\,\cos\varphi_i \\ y_i(t_e^{(i)}) &= r\,\sin\theta_i\,\sin\varphi_i \\ z_i(t_e^{(i)}) &= \gamma_u\,r\,\cos\theta_i + \beta_u\,\gamma_u^2\,c\,t_a - \beta_u\,\gamma_u\,\sqrt{r^2\,\sin^2\theta_i + (r\,\cos\theta_i + \beta_u\,\gamma_u\,c\,t_a)^2}\,. \end{aligned}$$

The following two pictures show the situation when such a sphere is approaching us at 80% of the speed of light ($\beta_u = 0.8$). The first one shows what the sphere looks like two units of time prior to passing us, while the second picture shows the same thing, but five units of time after the sphere has passed us. The actual position of the sphere isn't shown, for clarity.

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