Note 10. Derangements

1 Recall PIE (Principle of Inclusion-Exclusion)

When applying the principle of inclusion-exclusion, be flexible.

$$|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = ?PIE$$

The following formula is also PIE.

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|,$$

where

$$E_1 = S_1 - \binom{k+1}{1} S_2 + \dots$$

$$S_1 = \sum |A_i|, \quad S_2 = \sum |A_i \cap A_j|$$

Problem 1.

$$|A_1 \setminus (A_2 \cup A_3 \cup A_4)| = ?$$

Solution 1:

$$|A_1 \setminus (A_2 \cup A_3 \cup A_4)| = |A_1 \cup A_2 \cup A_3 \cup A_4| - |A_2 \cup A_3 \cup A_4| = \text{use PIE}$$

□

Solution 2:

$$A_1 \setminus (A_2 \cup A_3 \cup A_4) = (A_1 \setminus A_2) \cup (A_1 \setminus A_3) \cup (A_1 \setminus A_4)$$

Let $B_1 = A_1 \setminus A_2$, $B_2 = A_1 \setminus A_3$, $B_3 = A_1 \setminus A_4$. Then

$$|A_1 \setminus (A_2 \cup A_3 \cup A_4)| = |B_1 \cup B_2 \cup B_3| = \sum |B_i| - \sum |B_i \cap B_j| + |B_1 \cap B_2 \cap B_3| = \cdots$$

□

2 Derangements: nothing is in its right place.

Problem 2. *Alex, Beth, and Clyde drop off their hats at a hat check when they go to a club. In how many ways can a negligent attendant return their hats so nobody gets their own?*

List all cases:

Hats	A	B	C
1	A	B	C
2	A	C	B
3	B	A	C
4	B	C	A
5	C	A	B
6	C	B	A

Answer: 2.

Solution. Let S be the set of all arrangements of hats. An arrangement is said to satisfy condition c_i if the i-th hat is in the i-th position (right place) for $i = 1, 2$ or 3.

We want $N(\overline{c_1}, \overline{c_2}, \overline{c_3})$.

By the Principle of Inclusion-Exclusion (PIE):

$$N(\overline{c_1}, \overline{c_2}, \overline{c_3}) = |S| - \sum N(c_i) + \sum N(c_i c_j) - \sum N(c_1 c_2 c_3).$$

$$|S| = 3!$$

$$N(c_i) = (3-1)! \quad (\text{fix } i)$$

$$N(c_i c_j) = (3-2)! \quad (\text{fix } i, j)$$

$$N(c_1 c_2 c_3) = (3-3)!$$

Finally, we have:

$$N(\overline{c_1}, \overline{c_2}, \overline{c_3}) = 3! - \binom{3}{1} 2! + \binom{3}{2} 1! - \binom{3}{3} 0! = 2.$$

□

Notation:

$$[n] = \{1, 2, 3, \ldots, n\}.$$

Definition 2.1. *Fix a positive integer n and let X denote the set of all permutations on $[n]$. In other words, X consists of all bijections from $[n]$ to $[n]$.*

A permutation $\sigma \in X$ is called a derangement if:

$$\sigma(i) \neq i \quad \forall i = 1, 2, \ldots, n.$$

Theorem 2.2. *For each positive integer n, the number of derangements of $[n]$ is:*

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Proof. Let S be the set of all arrangements. For each $i \in [n]$, an arrangement is said to satisfy condition c_i if the i is in the i-th position (right place).

We want $N(\overline{c_1}, \overline{c_2}, \ldots, \overline{c_n})$.

By PIE: (Calculation details omitted)

$$\begin{aligned}
N(\overline{c_1}, \ldots, \overline{c_n}) &= |S| - \sum N(c_i) + \sum N(c_i c_j) - \cdots \\
&= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}(n-n)! \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} (n-k)! \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\end{aligned}$$

□

Recall the Taylor series expansion of e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

We know

$$\lim_{n\to\infty} d_n = n! \cdot e^{-1} = \frac{n!}{e}.$$

3 More problems

Problem 3. *Give a combinatorial argument to prove that for $n \in \mathbb{Z}^+$:*

$$n! = \binom{n}{0} d_0 + \binom{n}{1} d_1 + \binom{n}{2} d_2 + \cdots + \binom{n}{n} d_n,$$

with $d_0 = 1$.

Idea.

- LHS: $n!$ represents the number of permutations of the set $[n]$.
- RHS: For each k, the term $\binom{n}{k} d_k$ counts permutations with exactly k fixed elements and the remaining $n-k$ elements forming a derangement.

□

Problem 4. *In how many ways can the integers $1, 2, \ldots, 10$ be arranged in a line so that no even integer is in its natural position?*

Solution. Let S be the set of all permutations of the integers $1, 2, \ldots, 10$. For $i \in [10]$, an arrangement $\sigma \in S$ is said to satisfy condition c_i if

$$\sigma(i) = i.$$

We want:

$$N(\overline{c_2}, \overline{c_4}, \overline{c_6}, \overline{c_8}, \overline{c_{10}}).$$

(note that it is not d_5)

It suffices to calculate

- $|S| = 10!$
- $N(c_i) = (10-1)!$ when exactly one condition is met.
- $N(c_i c_j) = (10-2)!$ when exactly two conditions are met,
- and so on.

By PIE:

$$N(\overline{c_2}, \overline{c_4}, \overline{c_6}, \overline{c_8}, \overline{c_{10}}) = 10! - \binom{5}{1}(9!) + \binom{5}{2}(8!) - \binom{5}{3}(7!) + \binom{5}{4}(6!) - \binom{5}{5}(5!).$$

□

Problem 5. *Count the derangements of the set $\{1,2,3,4,5\}$ where the first three elements are $\{1,2,3\}$ in some order.*

Solution.

$$d_3 \times d_2.$$

□

Problem 6. *How many permutations of $\{1,2,\ldots,7\}$ are not derangements?*

Solution.

$$= 7! - d_7.$$

□

Problem 7. *In how many ways can Mrs. Ford distribute ten distinct books to her ten children (one book to each child) and then collect and redistribute the books so that each child has the opportunity to peruse two different books?*

Solution. **First round:** Distribute the 10 distinct books to the 10 children. There are 10! ways to do this.

Second round: Redistribute the books such that no child receives the same book as in the first round. This is equivalent to finding the number of derangements of 10 items: d_{10}.

Total ways: The total number of ways is the product of the two rounds, giving $10! \times d_{10}$. □

Problem 8. *Ten women attend a business luncheon. Each woman checks her coat and case. Upon leaving, each woman is given a coat and case at random.*

(a) In how many ways can the coats and cases be distributed so that no woman gets either of her possessions?

(b) In how many ways can they be distributed so that no woman gets back both of her possessions?

Solution. (a). Distribute the coats such that no woman receives her own coat. This is a derangement of 10 items, and the number of ways is d_{10}. Independently, distribute the cases such that no woman receives her own case. The number of ways is d_{10}. Since the distributions of coats and cases are independent, the total number of ways is $d_{10} \times d_{10}$.

(b). Let S be the set of all permutations of the integers $1, 2, \ldots, 10$. For $i \in [10]$, an arrangement S is said to satisfy condition c_i if the receives both her own coat and case. We want $N(\overline{c_1}, \overline{c_2}, \cdots, \overline{c_{10}})$. By PIE......

□