Note 11. Generating Function

1 Review

Problem 1. *Ten women attend a business luncheon. Each woman checks her coat and case. Upon leaving, each woman is given a coat and case at random.*

(b) In how many ways can they be distributed so that no woman gets back both of her possessions?

Here is a question: Let X be the set of arrangements of coats $\times$ the derangements of cases. Let Y be the set of derangements of coats $\times$ the arrangements of cases. Is it correct that the number of ways in (b)

$$= |X \cup Y| = |X| + |Y| - |X \cap Y| = n! \times d_{10} + d_{10} \times n! - d_{10} \times d_{10}.$$

Checking for Overcounted or Missing Cases: Some cases are missing. Suppose there are three women. The following arrangement is not in $X \cup Y$, but in (b).

women	1	2	3
coats	A	C	B
cases	c	b	a

Hence, the method is incorrect.

2 Generating function

Problem 2. *Let*

$$f(x) = (x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^3 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4 + x^5) = \sum_{i=0}^{\infty} c_i x^i,$$

determine c_{12}.

Solution. c_{12} is the number of $x^i x^j x^k$ such that $i + j + k = 12$, and $x^i \in \{x^4, x^5, x^6, x^7, x^8\}$ or we can say $i \in \{4, 5, 6, 7, 8\}$, $j \in \{2, 3, \ldots, 6\}$, $k \in \{2, 3, 4, 5\}$.

This implies that c_{12} is equivalent to the number of integer solution to $i + j + k = 12$ where $x^i \in \{x^4, x^5, x^6, x^7, x^8\}$ or we can say $i \in \{4, 5, 6, 7, 8\}$, $j \in \{2, 3, \ldots, 6\}$, $k \in \{2, 3, 4, 5\}$. $\square$

Similarly, c_{11} is equivalent to the number of integer solution to $i + j + k = 11$ where $x^i \in \{x^4, x^5, x^6, x^7, x^8\}$ or we can say $i \in \{4, 5, 6, 7, 8\}$, $j \in \{2, 3, \ldots, 6\}$, $k \in \{2, 3, 4, 5\}$.

Definition 2.1. *For a sequence $c_0, c_1, c_2, \ldots, c_n, \ldots$, the corresponding generating function $f(x)$ is the series*

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{i=0}^{\infty} c_i x^i.$$

Note that c_n is the coefficient of x^n in $f(x)$.

Problem 3. *Find a generating function (GF) that gives the number of integer solutions to*

$$y_1 + y_2 + y_3 + y_4 = N,$$

where $y_i \geq 0$, and

- *y_1 is odd,*
- *y_2 is even,*
- *$y_3 \leq 4$,*
- *$y_4 > 7$.*

Idea

$$f(x) = \left(\sum_{y_1}\right)\left(\sum_{y_2}\right)\left(\sum_{y_3}\right)\left(\sum_{y_4}\right).$$

Solution. 1. For y_1 (odd, $y_1 \geq 1$):

$$y_1 : \sum_{i=0}^{\infty} x^{2i+1} = x + x^3 + x^5 + \cdots$$

2. For y_2 (even, $y_2 \geq 0$):

$$y_2 : \sum_{i=0}^{\infty} x^{2i} = x^0 + x^2 + x^4 + x^6 + \cdots$$

3. For y_3 ($y_3 \leq 4$):

$$y_3 : x^0 + x^1 + x^2 + x^3 + x^4$$

4. For y_4 ($y_4 > 7$, so $y_4 \geq 8$):

$$y_4 : \sum_{i=8}^{\infty} x^i = x^8 + x^9 + x^{10} + \cdots$$

Thus, the generating function is

$$f(x) = \left(\sum_{i=0}^{\infty} x^{2i+1}\right)\left(\sum_{i=0}^{\infty} x^{2i}\right)\left(x^0 + x^1 + x^2 + x^3 + x^4\right)\left(\sum_{i=8}^{\infty} x^i\right).$$

□

Problem 4. *A student wants to order a total of N chicken nuggets. The restaurant sells nuggets in boxes of 2, 3, and 7. In how many ways can she order N nuggets?*

Solution. This can be modeled by the equation

$$2a + 3b + 7c = N,$$

where $a, b, c \geq 0$ represent the number of boxes of 2, 3, and 7, respectively.

- For $2a$ (where $a \geq 0$):

$$2a : 0, 2, 4, 6, \ldots$$

$$x^0 + x^2 + x^4 + x^6 + \cdots = \sum_{i=0}^{\infty} x^{2i}$$

- For $3b$ (where $b \geq 0$):

$$3b : 0, 3, 6, 9, \ldots$$

$$x^0 + x^3 + x^6 + x^9 + \cdots = \sum_{j=0}^{\infty} x^{3j}$$

- For $7c$ (where $c \geq 0$):

$$7c : 0, 7, 14, 21, \ldots$$

$$x^0 + x^7 + x^{14} + x^{21} + \cdots = \sum_{k=0}^{\infty} x^{7k}.$$

The generating function (GF) is $f(x)$, which is the product of the individual generating functions:

$$f(x) = \left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{j=0}^{\infty} x^{3j}\right)\left(\sum_{k=0}^{\infty} x^{7k}\right).$$

The coefficient of x^N in $f(x)$ gives the number of ways to order N nuggets. □

3 Rook polynomial

The rook is a piece in the game of chess. It may move any number of squares horizontally or vertically.

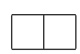
Definition 3.1. *A rook polynomial is the generating function for the number of ways (sequence) to place non-attacking*[1] *rooks on a generalized board.*

The rook polynomial $r(C, x)$ for a board C can be written as

$$r(C, x) = \sum_{k=0}^{\infty} r_k x^k,$$

where r_k is the number of ways to place k non-attacking rooks on the board.

Problem 5. *Find the rook polynomial for the* 2×1 *board.*



Solution.

- $r_0 :=$ # ways to put 0 rooks

$$r_0 = 1.$$

- $r_1 :=$ # ways to put 1 rook

$$r_1 = 2.$$

- $r_2 :=$ # ways to put 2 rooks

$$r_2 = 0.$$

- $r_3 :=$ # ways to put 3 rooks

$$r_3 = 0.$$

Sequence: $r_0, r_1, r_2, r_3, \ldots$

$$1, 2, 0, 0, \ldots$$

The rook polynomial is

$$r(C, x) = 1 + 2x.$$

□

Problem 6. *Find the rook polynomial for the* 2×2 *board.*

Solution.

- $r_0 :=$ # ways to put 0 rooks
$$r_0 = 1.$$
- $r_1 :=$ # ways to put 1 rook
$$r_1 = 4.$$
- $r_2 :=$ # ways to put 2 rooks
$$r_2 = 2.$$

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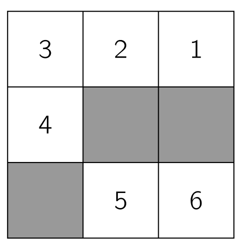
- $r_3 :=$ # ways to put 3 rooks
$$r_3 = 0.$$

The rook polynomial $r(C, x)$ for this board is
$$r(C, x) = 1 + 4x + 2x^2.$$

□

Problem 7. *Find the rook polynomial for the following board. Not allowed to place rooks in the shaded squares.*



Solution.

- r_0: Number of ways to place 0 rooks (empty board):
$$r_0 = 1.$$
- r_1: Number of ways to place 1 rook :
$$r_1 = 6.$$
- r_2: Number of ways to place 2 rooks (non-attacking):
$$r_2 = 8.$$

Possible placements (using cell numbers): $(1, 4)$, $(1, 5)$, $(2, 4)$, $(2, 6)$, $(3, 5)$, $(3, 6)$, $(4, 5)$, $(4, 6)$.

[1] Non-attacking: Each row and column has at most one rook.

- r_3: Number of ways to place 3 rooks (non-attacking, so 1 per row, in unshaded cells: 1, 2, 5, 8):

$$r_3 = 2.$$

  Possible placements:

$$\{(4, 2, 6), (4, 5, 1)\}.$$

- $$r_4 = 0, \quad r_5 = 0, \quad \dots$$

The rook polynomial for this board is

$$r(C, x) = 1 + 6x + 8x^2 + 2x^3.$$

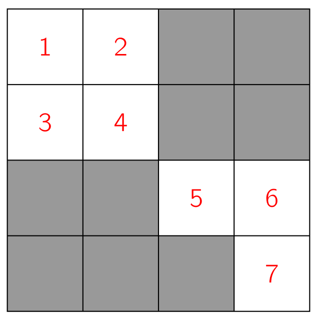
□

3.1 Disjoint sub-boards

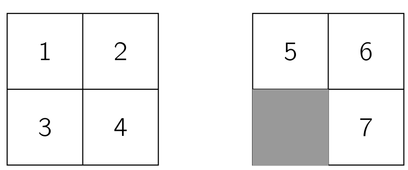
Theorem 3.2. *If a board C consists of n pairwise disjoint*[2] *sub-boards $(C_1, C_2, \dots, C_n)$, sharing no rows or columns, then*

$$r(C, x) = r(C_1, x) \cdot r(C_2, x) \cdot \dots \cdot r(C_n, x) = \prod_{i=0}^{n-1} r(C_i, x).$$

Consider a 4×4 board C, with two sub-boards C_1C_2.



Sub-Boards: We divide C into two sub-boards C_1 and C_2:



We have

$$r(C_1, x) = 1 + 4x + 2x^2, \quad r(C_2, x) = 1 + 3x + x^2$$

The composite board C is formed by C_1 and C_2.

$$r(C, x) = r_0(C) + r_1(C)x + r_2(C)x^2 + \cdots$$

$$r(C_1, x) = r_0(C_1) + r_1(C_1)x + r_2(C_1)x^2 + \cdots$$
$$r(C_2, x) = r_0(C_2) + r_1(C_2)x + r_2(C_2)x^2 + \cdots$$

[2] disjoint: sharing no rows and columns

$r_n(C)$ is the number of ways to place n rooks on C_1. It is equivalent to place k rooks on C_1 and $n-k$ rooks on C_2 for all possible k. That is

$$r_n(C) = \sum_k r_k(C_1) r_{n-k}(C_2)$$

So (similar to generating function: the number of solutions to $k + \ell = n$ is a product of two polynomials)

$$r(C, x) = r(C_1, x) \cdot r(C_2, x).$$

Given:

$$r(C_1, x) = 1 + 4x + 2x^2, \quad r(C_2, x) = 1 + 3x + x^2$$

Thus:

$$r(C, x) = (1 + 4x + 2x^2)(1 + 3x + x^2)$$

3.2 Use Recursion: divide it into several cases

Using recursion to compute the rook polynomial was too challenging, so our course administrator decided to exclude it from the exam syllabus.