# Note 5. Counting II

## 1 Permutation

Warm up:

**Problem 1.** *How many permutations of the letters in $ATLANTA$ are there?*

Hint: Consider balls and boxes. Let $S$ (balls) be the set of permutations of $A_1T_1LA_2NT_1A_3$. Let $T$ be the set of permutations of the letters in $ATLANTA$. Find a map from $S$ to $T$. Then each box has $3! \times 2! \times 1! \times 1!$. It follows that

$$|S| = |T| \times 3!2! \implies |T| = \frac{|S|}{3!2!} = \frac{6!}{3!2!}$$

**Theorem 1.1.** *If there are $n$ total objects, $n_1$ of type 1, $n_2$ of type 2, ..., $n_k$ of type $k$ $[n_1 + n_2 + \cdots + n_k = n]$, where objects of each type are indistinguishable, then there are*

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

*permutations of these objects.*

Similarly, you can prove it by considering balls and boxes.

**Problem 2.** *How many 6-letter permutations of the letters in "ATLANTA" are there?*

*Solution.* **Case 1:** Leave out an $A$, permute "ATLANT":

$$\frac{6!}{2! \cdot 2!}.$$

**Case 2:** Leave out a $T$, permute "ATLANA":

$$\frac{6!}{3!}.$$

**Case 3:** Leave out $L$, permute "ATANTA" :

$$\frac{6!}{3! \cdot 2!}$$

**Case 4:** Leave out $N$, permute "ATLATA":

$$\frac{6!}{3! \cdot 2!}$$

Total number of permutations:

$$\frac{6!}{4} + \frac{6!}{3!} + 2 \cdot \frac{6!}{3! \cdot 2!}.$$

$\square$

## 2 Circular permutations

What if we want to count circular permutations rather than a line?

**Problem 3.** *Suppose 6 people, $A, B, C, D, E, F$, sit at a table. If permutations are the same when they differ only by rotation, how many permutations are possible?*

Here, permutations are the same when they differ only by rotation implies that these are the same:

$$\text{ABCDEF, BCDEFA, CDEFAB, DEFABC, EFABCD, FABCDE.}$$

So (consider balls and boxes):

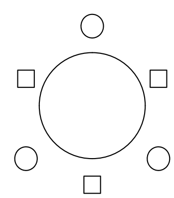
$$6 \cdot (\#\text{ of circular permutations}) = \#\text{ of permutations} = 6!.$$

Therefore, the number of circular permutations is:

$$5!.$$

**Theorem 2.1.** *The number of circular permutations of $n$ distinct objects is $(n-1)!$.*

**Problem 4.** *A baker wants to arrange a cake tasting plate where 3 flavors of cake alternate with 3 flavors of frosting. How many different arrangements are possible?*



**Solution:**
**Method 1: Iterative Placement**

1. Place a cake down and designate its position as the 1st position.
2. How many choices for the 2nd position? We have 3 different flavors of frosting. (3 ways).
3. How about the 3rd position? Another piece of cake to go next (2 ways).
4. 4th, Another frosting (2 choices).
5. 5th, (1 way).
6. 6th, (1 way).

$$\text{Total: } 3 \cdot 2 \cdot 2 = 12 \text{ permutations.}$$

**Method 2: Fix the Position of the Cake**

1. Fix the position of the cakes. There are 2! circular permutations. (for example, circular permutation $A, B, C$.
2. Place frosting in 3! ways:
   - 3 flavors to choose from in the first gap (between $A$ and $B$),
   - 2 in the second gap, (between $B$ and $C$),

- 1 in the third gap.(between $C$ and $A$),

$$\text{Total: } 2! \cdot 3! = 12 \text{ ways.}$$

**Method 3: Choose Orientations of Cake and Frosting**

1. Choose orientations of cake and frosting
   - 2! ways for cake
   - 2! ways for frosting.
2. Then, 3 ways to orient these 2 circular arrangements relative to each other. For example: cakes ABC, frostings 213. We have 3 different arrangements A2B1C3, A3B2C1, A1B3C2.

$$2! \cdot 2! \cdot 3 = 12 \text{ ways.}$$

# 3 Combinatorial proofs

**Definition 3.1.** *A combinatorial proof counts the same set of objects in two different ways, proving the numbers obtained are equal.*

**Problem 5.** *Prove combinatorially that: Let $n$ and $k$ be positive integers with $n = 2k$. Prove that $\frac{n!}{2^k}$ is an integer.*

*Proof.* Consider the $n$ letters $x_1, x_1, x_2, x_2, \dots, x_k, x_k$. The number of ways in which we can arrange all of these $n = 2k$ letters is an integer that equals

$$\frac{n!}{\underbrace{2! \cdot 2! \cdots 2!}_{k \text{ factors of } 2!}} = \frac{n!}{2^k}.$$

□

**Problem 6.** *Prove combinatorially that:*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

*Proof.* Consider the question: how many ways can you select $k$ pizza toppings from a menu containing $n$ choices? One way to do this is just $\binom{n}{k}$. Another way to answer the same question is to first decide whether or not you want the first topping.

- If you **do want** it, you still need to pick $k-1$ toppings, now from just $n-1$ choices. That can be done in $\binom{n-1}{k-1}$ ways.
- If you **do not want** it, then you still need to select $k$ toppings from $n-1$ choices (the anchovies are out). You can do that in $\binom{n-1}{k}$ ways.

Since the choices with anchovies are disjoint from the choices without anchovies, the total choices are:

$$\binom{n-1}{k-1} + \binom{n-1}{k}.$$

We answered the same question in two different ways, so the two answers must be the same. Thus:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

□

**Problem 7.** *Prove combinatorially that:*

$$\binom{n}{k} = \binom{n}{n-k}$$

*Proof.* Let's try the pizza counting example like we did above. How many ways are there to pick $k$ toppings from a list of $n$ choices? On the one hand, the answer is simply $\binom{n}{k}$. Alternatively, you could make a list of all the toppings you don't want. To end up with a pizza containing exactly $k$ toppings, you need to pick $n-k$ toppings to not put on the pizza. You have $\binom{n}{n-k}$ choices for the toppings you don't want. Both of these ways give you a pizza with $k$ toppings, which means:

$$\binom{n}{k} = \binom{n}{n-k}.$$

□

**Problem 8.** *Prove that:*

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

*Proof.* Let's do a "pizza proof" again. We need to find a question about pizza toppings that has $2^n$ as the answer. How about this:

If a pizza joint offers $n$ toppings, how many pizzas can you build using any number of toppings from no toppings to all toppings, using each topping at most once?

On one hand, the answer is $2^n$. For each topping, you can say "yes" or "no," so you have two choices for each topping.

On the other hand, divide the possible pizzas into disjoint groups: the pizzas with no toppings, the pizzas with one topping, the pizzas with two toppings, etc. If we want no toppings, there is only one pizza like that (the empty pizza, if you will), but it would be better to think of that number as $\binom{n}{0}$, since we choose 0 of the $n$ toppings. How many pizzas have 1 topping? We need to choose 1 of the $n$ toppings, so $\binom{n}{1}$. We have:

- Pizzas with 0 toppings: $\binom{n}{0}$,
- Pizzas with 1 topping: $\binom{n}{1}$,
- Pizzas with 2 toppings: $\binom{n}{2}$,
- $\vdots$
- Pizzas with $n$ toppings: $\binom{n}{n}$.

The total number of possible pizzas will be the sum of these, which is exactly the left-hand side of the identity we are trying to prove. □

Again, we could have proved the identity using subsets or bit strings.