Supplementary Ungraded Homework Problems

1 Unit 1: Basics

1.1 Relations, Functions, and Sets

For questions 1 and 2, let $A = \{\alpha, \beta, \gamma, \ldots, \omega\} = \{\text{alpha, beta, gamma, \ldots, omega}\}$, $B = \{+, -, \times, \div\}$, and $C = \{\text{hill, smile, jellyfish}\}$.

1. Write A and B in set builder notation. Is set builder notation a good choice for set C?
2. Choose a pair of sets from among A, B, and C, and define the following (you may need to swap the order of the sets between parts, which is fine).
   (a) A relation between the sets that is not a function.
   (b) A function between the sets that is neither injective nor surjective.
   (c) A function between the sets that is injective.
   (d) A function between the sets that is surjective.
   (e) Convince yourself that there is no bijection between any pair of these sets.
3. Suppose you have 3 sets, X, Y, and Z. Can you write down a general formula for the number of elements that belong to exactly 2 of these sets? (We'll talk about this more in a couple of weeks).

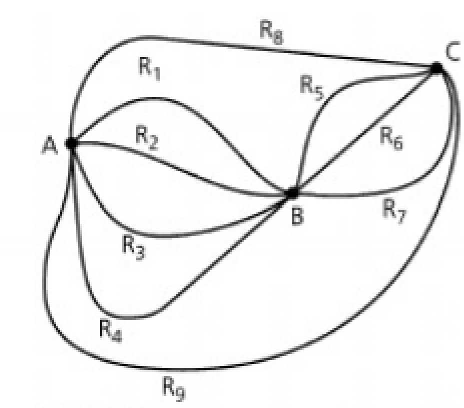
1.2 Permutations and Combinations

4. Evaluate each of the following:
   (a) P(7,2)
   (b) P(8,4)
   (c) P(10,7)
   (d) P(12,3)
5. Find the value of n such that
   (a) $P(n, 2) = 90$
   (b) $2P(n, 2) + 50 = P(2n, 2)$
6. The board of directors of a pharmaceutical corporation has 10 members, from whom a slate of 4 new company officers will be elected.
   (a) How many slates of a president, vice president, secretary, and treasurer are possible?
   (b) If 3 of the 10 board members are physicians, how many of the slates have:
      i. A physician elected as the president?
      ii. Exactly one physician elected as an officer?
      iii. At least one physician elected as an officer?

7. Hamm's Burger Parlor allows customers to order their burgers with or without any of a wide range of toppings: ketchup, mustard, mayonnaise, lettuce, tomato, onion, pickle, cheese, mushrooms, or jalapeños. How many different burger orders are possible?

8. Peter's Bakery offers 8 kinds of pastry and 6 flavors of muffins, baked fresh daily. They also sell beverages: coffee (black, or with cream, sugar, or both), tea (plain, or with milk, sugar, honey, milk and sugar, or milk and honey) hot cocoa, and orange juice, all available in small, medium, or large sizes. How many ways can Amin order
   (a) One bakery item and one medium-sized beverage?
   (b) One bakery item and a cup of coffee for himself, and a muffin and a cup of tea for Roger?
   (c) A pastry and a cup of tea for himself, a muffin and some orange juice for Jenna, and a bakery item and cup of coffee each for Steve and Marcus?

9. Anchorage (A), Butte (B), and Chugiak (C) are three towns connected by a system of two-way roads, as shown in the figure below.
   (a) In how many ways can Leilani travel from Anchorage to Chugiak?
   (b) How many different round trips can she make from Anchorage to Chugiak, and back to Anchorage?
   (c) How many of the round trips in part (b) are such that the return trip is at least partially different from the outgoing trip? *For example, if Leilani takes roads R_1 and R_6 initially, her return trip can be anything but R_6 and R_1).*



10. Use the situation of problem **??** to argue that $\binom{7}{2} \cdot 4! = \frac{1}{5}P(7,5)$ by combinatorial proof.

11. Evaluate each of the following:
   (a) $C(10,4)$
   (b) $\binom{14}{12}$
   (c) $C(12,7)$
   (d) $\binom{15}{10}$

12. Suppose we have the letters m, r, a, f, and t. With no repetition allowed,
   (a) How many permutations of 3 letters can we make?
   (b) How many combinations of 3 letters can we make?

13. Sheryl needs to choose 11 seniors to play on the varsity football team. If she can make the selection in 12,376 ways, how many students are eligible to play?

14. Express each of the following in sigma notation (assume n is a positive integer):

(a) $\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$, where $n \geq 2$

(b) $1 - 4 + 9 - 16 + 25 - 36 + 49 - \cdots$

(c) $\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$

15. In the complete expansion of $(a+b+c+d)(e+f+g+h)(u+v+w+x+y+z)$, one obtains the sum of terms such as agw, cfx, and dgv.
    (a) How many such terms appear in the complete expansion?
    (b) Which of the following terms do not appear in the expansion?
        i. afx
        ii. bvx
        iii. cgw
        iv. chz
        v. egu
        vi. dfz

16. In how many ways can 15 identical candy bars be distributed among 5 children so the youngest gets exactly one or two?

17. A certain ice cream store has 31 flavors of ice cream available. In how many ways can we order a dozen ice cream cones if
    (a) We do not want more than one of any particular flavor?
    (b) We allow every possible repetition of flavors?
    (c) No flavor can be ordered more than 11 times?

18. Determine the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 32,$$

where
    (a) $x_i > 0$ for $1 \leq i \leq 4$
    (b) $x_1, x_2 \geq 5$, $x_3, x_4 \geq 7$
    (c) $x_1, x_2, x_3 > 0, 0 < x_4 \leq 25$

19. In how many ways can Maya place 24 different poetry books on 4 distinct shelves so that there is at least one book on each shelf? *Assume the order of the books on each shelf matters too.*

20. How many pairs of nonnegative integer solutions are there to the pair of equations $x_1 + x_2 + x_3 + \cdots + x_7 = 37$ and $x_1 + x_2 + x_3 = 6$?

21. Given positive integers n and k with $n \geq k$, explain why the number of ways to distribute n identical objects into k distinct containers with no container left empty is

$$C(n-1, k-1) = C(n-1, n-k).$$

22. The sequence $(c_n) = (10, 13, 16, 19, 22, ...)$ can be described using the formula $c_n = 3n + 7$, where $n \geq 1$. Give a recursion that produces this same sequence.

23. Find the first 6 terms of the sequence produced by the recursion $a_0 = 4$, $a_1 = 5$, and $a_k = a_{k-2} + 2a_{k-1}$ for $k \geq 2$.

24. Use the Euclidean algorithm to prove that 136 and 329 are relatively prime (i.e., that their gcd is 1). Please show all of your work, including the iterations of the algorithm.

2 Unit 2: Advanced Enumeration

2.1 PIE and Rook Polynomials

25. Determine the number of positive integers n with $1 \leq n \leq 2000$ that are not divisible by 2, 3, 5, or 7.

26. Determine how many integer solutions there are to the equation $x_1 + x_2 + x_3 + x_4 = 19$ if $0 \leq x_i < 8$ for all $1 \leq i \leq 4$.

27. In how many ways can three x's, three y's, and three z's be arranged so that no letter appears 3 times in a row?

28. Let S be a finite set with $|S| = N$ and let c_1, c_2, c_3, and c_4 be conditions that the elements of S may satisfy. Argue that $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}c_4) + N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$.

29. Determine the number of positive integers n with $1 \leq n \leq 2000$ that are
    (a) Not divisible by 2, 5, or 7.
    (b) Not divisible by 2, 3, or 5, but are divisible by 7.

30. Determine the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 19$ if
    (a) $0 \leq x_i$ for $1 \leq i \leq 4$.
    (b) $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 6$, $3 \leq x_3 \leq 7$, and $3 \leq x_4 \leq 8$.

31. In how many ways can one arrange all of the letters in the word "INFORMATION" so that no pair of consecutive letters appears more than once? That is, we want to count permutations such as IINNOOFRMTA or FORTMAIINON, but not INFORINMOTA (IN appears twice) or NORTFNOIAMI (NO appears twice).

32. Dr. Gupta has just completed writing an exam for her course in advanced engineering mathematics, and needs to assign point values to each question. If the exam has 12 questions that should add up to 200 points, each question should be worth at least 10 points but no more than 25, and every question has point value that is divisible by 5, in how many ways can she assign the points?

33. In how many ways can Lindon select nine marbles from a bag of 12 identical (except for color) marbles, where there are 3 red, 3 green, 3 purple, and 3 white marbles?

34. Find the number of permutations of the alphabet in which none of the words *spin*, *game*, *net*, or *path* occurs.

35. If 8 distinct 6-sided dice are rolled, what is the probability that all 6 numbers occur? *Note: To calculate the probability, divide the number of desired outcomes by the number of total possible outcomes.*

36. In how many ways can one permute the letters of the word "CORRESPONDENTS" so that
    (a) There are no pairs of consecutive identical letters?
    (b) There are exactly 2 pairs of consecutive identical letters?
    (c) There are at least 3 pairs of consecutive identical letters?

37. In how many ways can one distribute 10 distinct prizes to 4 students with exactly 2 students getting nothing? How many ways have at least 2 students getting nothing (assuming all prizes are given out)?

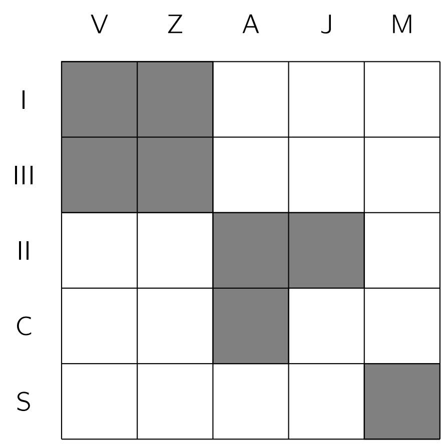
38. In how many ways can we devise a secret code by assigning each letter of the alphabet a different letter to represent it?

39. For the positive integers $1, 2, 3, \ldots, n-1, n$, there are 11,660 derangements where $1, 2, 3, 4, 5$ appear in the first 5 positions (in some order). What is the value of n?

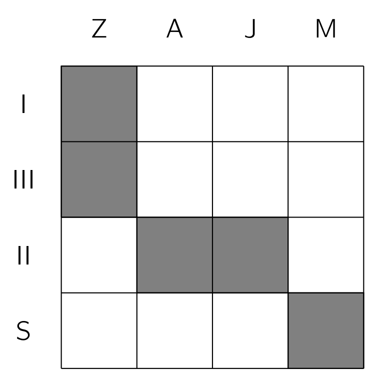
40. A certain professor requires students to turn in their phones and backpacks while they sit an exam. When they turn their exams in, the students are given a phone and backpack at random(!). In how many ways can the students' belongings be returned so that
    (a) no student receives anything that belongs to them?
    (b) no student receives both their own phone and backpack?

41. Generalize your answer to problem 6 to find the rook polynomial of a "standard" $n \times n$ chessboard.

42. Alex, Violet, Madison, Jack, and Zhiwu are deciding who will teach the classes Calculus I, Calculus II, Calculus III, Combinatorics, and Statistics. Alex will not teach Calculus II or Combinatorics, Madison can't stand Statistics, Violet and Zhiwu both refuse to teach Calculus I or Calculus III, and Jack detests Calculus II.
    (a) In how many ways can the professors be assigned to the courses without upsetting someone?



    (b) How many of the assignments in part (a) include Violet teaching Combinatorics?



2.2 Generating Functions

43. Find the ordinary generating function for each of the following sequences. Give your answer as a rational function.
    (a) $\left(\binom{8}{0}, \binom{8}{1}, \binom{8}{2}, \ldots, \binom{8}{8}, 0, 0, \ldots\right)$
    (b) $\left(\binom{8}{1}, 2\binom{8}{2}, 3\binom{8}{3}, \ldots, 8\binom{8}{8}, 0, 0, \ldots\right)$
    (c) $(1, -1, 1, -1, 1, -1, 1, \ldots)$

(d) $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$

44. Determine the sequence generated by each of the following ordinary generating functions. Be sure to give a general term of the sequence.
    (a) $(2x-3)^3$
    (b) $\frac{x^4}{1-x}$
    (c) $\frac{x^3}{1-x^2}$
    (d) $\frac{1}{1+3x}$
    (e) $\frac{1}{3-x}$

45. (a) Determine the sequence generated by the ordinary generating function $f(x) = \frac{1}{1-x} + 3x^7 - 11$.
    (b) Find the OGF for the sequence $(b_n)_{n\geq 0}$ given by $b_1 = 1$, $b_3 = 3$, $b_7 = 7$, and $b_n = 2^n + 5$ for $n \neq 1, 3, 7$.

46. Find the coefficient of x^{15} in $f(x) = x^3(1-2x)^{10}$.

47. Show that the OGF $g(x) = (1-4x)^{-1/2}$ generates the sequence $\left(\binom{2n}{n}\right)_{n\in\mathbb{N}}$.

48. Find the exponential generating function for each of the following sequences. Give your answer as a closed function (not a power series).
    (a) $(1, -1, 1, -1, 1, -1, 1, \ldots)$
    (b) $(1, 2, 2^2, 2^3, 2^4, \ldots)$
    (c) $(1, -a, a^2, -a^3, a^4, \ldots)$, where $a \in \mathbb{R}$
    (d) $(0, 1, 2(2), 3(2^2), 4(2^3), \ldots)$

49. Determine the sequence generated by each of the following exponential generating functions.
    (a) $3e^{3x}$
    (b) $6e^{5x} - 3e^{2x}$
    (c) $e^x + x^2$
    (d) $\frac{1}{1-x}$
    (e) $\frac{3}{1-2x} + e^x$

50. Determine the ordinary generating function to distribute 35 dollars (from an infinite supply) to 5 children if
    (a) there are no restrictions.
    (b) each child gets at least \$2.
    (c) the oldest child gets at least \$10.

51. (a) Explain why the ordinary generating function for the number of ways to have n cents in pennies and nickels is $(1 + x + x^2 + x^3 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)$.
    (b) Give the ordinary generating function for the number of ways to have n cents in pennies, nickels, and dimes. Can you write your answer in closed form?

52. In how many ways can 2 dozen identical robots be assigned to 4 assembly lines with
    (a) at least 3 robots assigned to each line?
    (b) at least 3, but no more than 9, robots assigned to each line?

    You may give your answer for both parts by citing a particular coefficient in a specific generating function rather than doing a full calculation.

53. Find the exponential generating function for the number of ways to arrange n letters (where $n \geq 0$) chosen from the word ISOMORPHISM.

54. Show that the convolution of the sequence $(a_n) = (1, 1, 1, 1, 1, \ldots)$ with itself is the sequence $(1, 2, 3, 4, 5, \ldots) = (n+1)_{n\geq 0}$.

55. Show that the convolution of any sequence $(a_n)_{n\geq 0}$ with the sequence $(b_n) = (1, 1, 1, 1, \ldots)$ is equal to the sequence of partial sums of (a_n), that is, the convolution is $(c_n)_{n\geq 0}$ with $c_n = \sum_{k=0}^{n} a_k$.

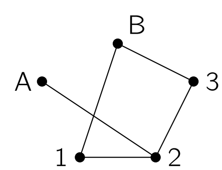
    Note: this property is the reason that $\frac{1}{1-x}$ is sometimes called the addition operator- multiplying by it always has the result of adding up the terms of any other power series.

56. Find the ordinary generating function for the sequence $(0, 1, 3, 6, 10, \ldots)$. *Hint: can you interpret this sequence as a sequence of partial sums?*

57. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ be the ordinary generating function for the sequence $(a_n)_{n\geq 0}$ and $g(x) = a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!}$ be the exponential generating function for the same sequence. Further let k be an arbitrary fixed positive integer.
    (a) Find the ordinary generating function for the sequence $(0, 0, \ldots, 0, a_0, a_1, a_2, \ldots)$ where there are k leading 0s.
    (b) Find the ordinary generating function for the sequence $(a_k, a_{k+1}, a_{k+2}, a_{k+3}, \ldots)$.
    (c) Find the exponential generating function for the sequence $(a_k, a_{k+1}, a_{k+2}, a_{k+3}, \ldots)$.

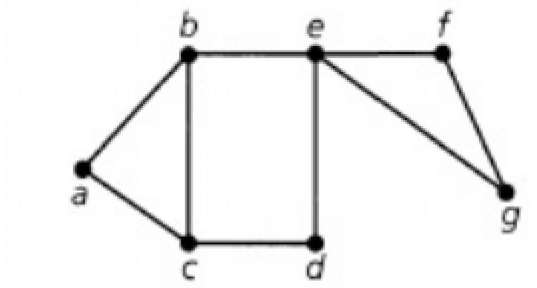
3 Unit 3: Graph Theory

58. Sketch the graph $G = (V, E)$, where $V = \{1, 2, 3, A, B\}$ and $E = \{[1, 2], [1, B], [2, 3], [2, A], [3, B]\}$.



59. (Clearly) sketch a graph $G = (V, E)$ such that all of the following hold:
    - $|V| = 7$
    - $|E| \geq 5$
    - $\kappa(G) = 3$
    - Every circuit in G is also a cycle

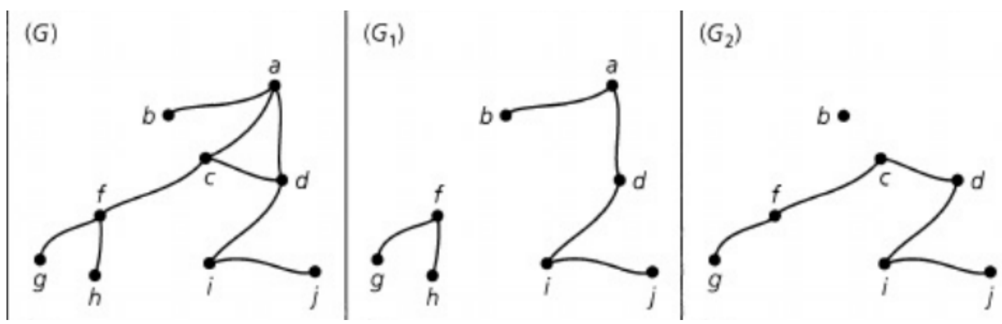
60. For the graph below, find:
    (a) a walk from b to d that is not a trail
    (b) a trail from b to d that is not a path
    (c) a path from b to d
    (d) a closed walk from b to b that is not a circuit
    (e) a circuit from b to b that is not a cycle
    (f) a cycle from b to b



61. Give an example of a connected graph G where removing any edge of G results in a disconnected graph.

62. Let $G = (V, E)$ be a loop-free connected graph, and $[a, b]$ an edge of G. Convince yourself that $[a, b]$ is part of a cycle if and only if removing the edge $[a, b]$ from G does not disconnect the graph (assume the vertices a and b remain).

63. Consider the graph G below, and its subgraphs G_1 and G_2.



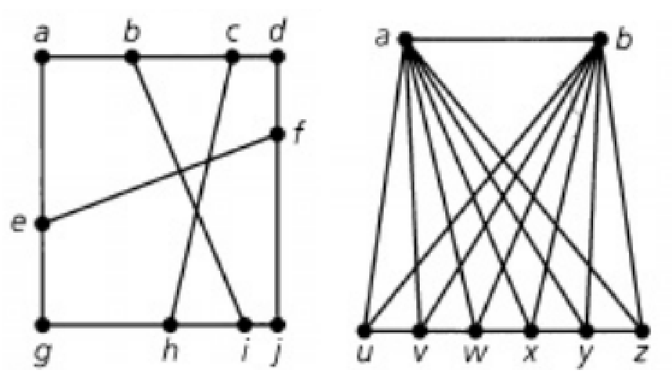
(a) How many connected subgraphs of G have 4 vertices and include a cycle?
(b) Describe the graphs G_1 and G_2 as induced subgraphs of G.
(c) Draw the subgraph of G induced by the vertex set $U = \{b, c, d, f, i, j\}$.

64. Find all loop-free undirected non-isomorphic graphs with 4 vertices. How many of them are connected?

65. How many edges are in the complete bipartite graph $K_{m,n}$? You may find it useful to do a few small examples.

66. Can a bipartite graph contain a cycle of odd length? Justify your answer.

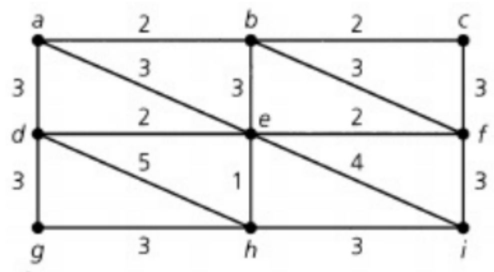
67. For each of the graphs below, determine whether the graph is planar or not. If the graph is planar, redraw it with no edges overlapping. If not, find a subgraph homeomorphic to K_5 or $K_{3,3}$.



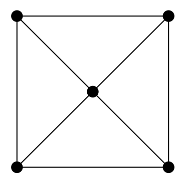
68. Let $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ be two trees where $|E_1| = 17$ and $|V_2| = 2|V_1|$. Determine $|V_1|, |V_2|$, and $|E_2|$.

69. Give an example of a graph $G = (V, E)$ where $|V| = |E| + 1$ but G is not a tree.

70. Find a minimal spanning tree for the graph below using
    (a) Kruskal's Algorithm.
    (b) Prim's Algorithm.



71. For the graph G below, assign the weights 1, 1, 2, 2, 3, 3, 4, 4 to the edges of G so that



    (a) G has a unique minimal spanning tree.
    (b) G does not have a unique minimal spanning tree.

72. If $G = (V, E)$ is a connected graph with $|E| = 17$ and $\deg(v) \geq 3$ for all $v \in V$, what is the maximum possible value of $|V|$?

73. Let $G = (V, E)$ be a loop-free undirected graph, where $|V| = 6$ and $\deg(v) = 2$ for all $v \in V$. Up to isomorphism, how many such graphs are there?

74. For the "Bridges of Königsberg" graph, how many bridges must be removed so that the resulting subgraph has an Euler trail but not an Euler circuit? Which bridges should be removed?

75. Jane and Ella attended a board game meetup with three other couples. Due to the many introductions that had to be made, a great deal of handshaking took place, but:

    - Nobody shook hands with their partner
    - Nobody shook their own hand
    - Nobody shook hands with the same person multiple times.

    Before leaving, Jane asked everyone else how many hands they had shaken, and received 7 different answers from the 7 different people. How many hands did Jane shake? How many did Ella?

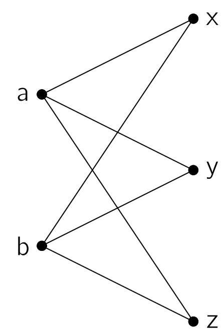
76. Describe the type of graph for which an Euler trail is also a Hamilton path. Repeat the exercise for an Euler circuit/Hamilton cycle.

77. Argue that the Petersen graph has a Hamilton path but no Hamilton cycle. *You do not need to write a very formal proof, but should try to clearly and systematically eliminate possibilities.*

78. Helen and Menelaus invite 10 friends to dinner. In this group of 12 people, each person knows at least 6 others. Prove that it's possible for everyone to be seated around a circular table to that everyone knows both people they're sitting next to.

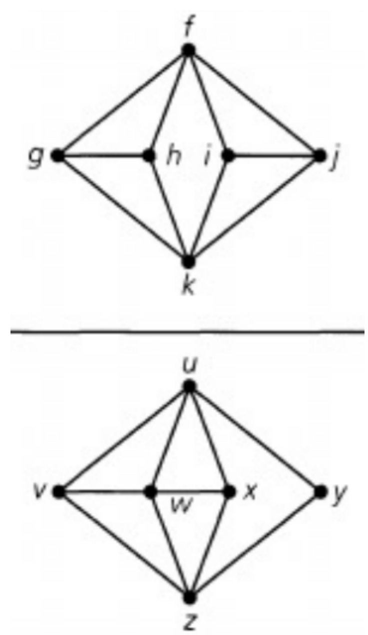
79. Give an example of a graph G such that $\chi(G) = 3$ but no subgraph of G is isomorphic to K_3. *Hint: you may find it helpful to remember that* $K_3 = C_3$.

80. Consider the complete bipartite graph $K_{2,3}$ below:



    (a) Using λ colors, how many proper colorings of $K_{2,3}$ are there if vertices a and b are the same color?
    (b) Using λ colors, how many proper colorings of $K_{2,3}$ are there if vertices a and b are different colors?
    (c) Find the chromatic polynomial $P(K_{2,3}, \lambda)$.
    (d) Generalize your answer to find the chromatic polynomial $P(K_{2,n}, \lambda)$.

81. Consider the graphs below.



    (a) Determine (with evidence) whether the two graphs are isomorphic to each other.
    (b) Find the chromatic polynomial $P(G, \lambda)$ for each graph.
    (c) Compare the results from the previous two parts. Can you make a general statement about graph isomorphisms and chromatic polynomials?

82. Give an example of a graph that proves $R(3,4) > 7$.

83. Consider the set $X = \{1,2,3,4,5\}$ and the relation $R = \{(1,1),(2,2),(3,3),(4,4),(1,3),(2,4),(2,5),(3,5),(1,5)\}$. Is the pair (X,R) a poset? If yes, explain what you needed to check to justify your answer. If no, explain what modifications would need to be made in order to make it a poset (for example, which pair(s) would you need to add to R?).