Note 7. Counting IV

1 Review

Sometimes a problem is too complex to handle all at once. In such cases, we divide the problem into simpler, distinct cases and count each case separately. After counting the cases, we combine the results. Ensures that no situation is **missed** or **overcounted**.

Problem 1. *Given the sets:*

$$|A| = 10, \quad |B| = 9, \quad |A \cap B| = 4$$

find the number of ways to choose $\{a, b\}$ such that $a \in A$, $b \in B$. (Note that $a \neq b$ by the definition of set.)

Solution. For each element $a \in A$, we have two cases: And for $b \in B$, we have another two cases. Consequently, we have four cases as follows.

- **Case 1:** $a \in A \setminus B$
  - **Case 1.1:** $b \in B \setminus A$
  - **Case 1.2:** $b \in A \cap B$
- **Case 2:** $a \in A \cap B$
  - **Case 2.1:** $b \in B \setminus A$
  - **Case 2.2:** $b \in A \cap B$

(no need to write, but ensure to verify there is no omission or overcounting.)

In Case 1.1, our goal is to determine the number of ways to choose $\{a, b\}$ such that $a \in A \setminus B$, $b \in B \setminus A$. That is $|A \setminus B||B \setminus A| = 6 \times 5 = 30$. Similarly, in Case 1.2, there are 24 ways. In Case 2.1, there are 20 ways. However, in Case 2.2, our goal is to determine the number of ways to choose $\{a, b\}$ such that $a, b \in A \cap B$. That is, choose 2 elements from $|A \cap B|$, resulting in $\binom{4}{2} = 6$ ways.

Finally, by combining all cases, we obtain a total of 80 ways. □

2 Combinations with Repetition

Problem 2. *Find the number of non-negative integer solutions to the equation:*

$$x_1 + x_2 = 5$$

Obviously, 6.

Problem 3. *Find the number of non-negative integer solutions to the equation:*

$$x_1 + x_2 + x_3 = 5$$

Idea: Stars and Bars Representation:

- Represent the total sum 5 as 5 stars: $\star\star\star\star\star$.
- Use 2 bars ($|$) to partition these stars into 3 groups, corresponding to x_1, x_2, x_3.

Examples:

$$\star\star\,|\,\star\,|\,\star\star \quad (\text{Solution: } x_1 = 2, x_2 = 1, x_3 = 2)$$

$$||\,\star\star\star\star\star \quad (\text{Solution: } x_1 = 0, x_2 = 0, x_3 = 5)$$

This establishes a bijection between the solutions and the arrangements.

Counting Arrangements:

- Total symbols: 5 stars $+2$ bars $= 7$ symbols.
- Choose 2 positions for bars out of 7 total positions.

$$\binom{7}{2} = \frac{7!}{2! \cdot 5!} = \frac{7 \times 6}{2 \times 1} = 21.$$

□

Problem 4. *On their way home from track practice, seven high-school freshmen stop at a fast-food restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible?*

Solution. Consider the number of cheeseburgers, hot dogs, tacos and fish sandwiches as variables x_1, x_2, x_3, x_4, respectively. Then our goal is to find the number of non-negative integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 = 7$$

Using the stars and bars method, we get $= \binom{10}{3} = 120$. □

Let's generalize:

Theorem 2.1. *The number of ways to distribute n identical items into k distinct groups is*

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

The following are equivalent:

1. The number of selections, with repetition, of n things from a set of size k.
2. The number of ways n identical objects can be distributed into k distinct containers.
3. The number of non-negative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

2.1 More examples

Problem 5. *Find the number of solutions to the equation*

$$x_1 + x_2 + x_3 = 17$$

where $x_1, x_2 \geq 0$, $x_3 \geq 4$, and $x_1, x_2, x_3 \in \mathbb{Z}$.

Solution. Since $x_3 \geq 4$, we perform the change of variables:

$$x_3 = 4 + y_3, \quad \text{where } y_3 \geq 0.$$

Substituting this into the equation, we get:

$$x_1 + x_2 + (4 + y_3) = 17.$$

Simplifying, this reduces to:

$$x_1 + x_2 + y_3 = 13,$$

where $x_1, x_2, y_3 \geq 0$.

Applying our theorem, the total number of solutions is:

$$\binom{15}{2} = \frac{15 \times 14}{2} = 105.$$

□

Problem 6. *Find the number of non-negative integer solutions to the equation*

$$x_1 + x_2 + x_3 \leq 17$$

Solution. This problem is equivalent to determining the number of non-negative integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 \leq 17$$

By our theorem, the total number of solutions is $\binom{20}{3} = 1140$. □

Problem 7. *In how many ways can we distribute seven apples and six oranges among four children so that each child receives at least one apple?*

Solution. First, ensure each child receives at least one apple. Since there are four children and seven apples, we give each child one apple. This leaves us with $7 - 4 = 3$ apples to distribute freely among the four children. The number of ways to distribute these three apples is: $\binom{3+4-1}{3} = 20$.

Next, distribute the six oranges among the four children with no restrictions. The number of ways to distribute the oranges is: $\binom{6+4-1}{6} = 84$.

By the rule of product, the total number of ways to distribute the fruit under the stated conditions is:

$$20 \times 84 = 1680.$$

□

In the binomial expansion for $(x + y)^n$, each term is of the form $\binom{n}{k}x^k y^{n-k}$. The total number of terms in the expansion is equal to the number of nonnegative integer solutions to the equation:

$$n_1 + n_2 = n,$$

where n_1 is the exponent for x and n_2 is the exponent for y. This number is given by:

$$C(2 + n - 1, n) = n + 1.$$

Problem 8. *Determine the number of terms in the expansion of $(x + y + z)^{10}$.*

Solution. The number of nonnegative integer solutions to this equation:

$$n_1 + n_2 + n_3 = 10.$$

□

Can you determine the coefficient of each term?