Note 6. Counting III

1 Combinations and Permutations

Counting Strategy: Break the task into small pieces, like task A, task B,...

1. If disjoint, use the rules of sum and product.
2. If intersecting, draw a picture, and use the formula like:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Problem 1. *Suppose a professor needs 96 students to register for a course offered in 3 sections, A, B, and C, each with 32 students. How many ways can this be done?*

Solution 1 (Using Combinations): We can divide it into 3 steps as follows.

Step 1 Choose 32 students from 96 for section A:

$$\binom{96}{32}$$

Step 2 Choose 32 students from the remaining 64 for section B:

$$\binom{64}{32}$$

Step 3 The remaining 32 students go to section C:

$$\binom{32}{32}$$

Thus, the total number of ways is:

$$\binom{96}{32}\binom{64}{32}\binom{32}{32}$$

□

Solution 2 (Using Permutations): Consider the students as a sequence of 96 letters labeled A, B, and C (32 each). The total number of arrangements is equivalent to the number of permutations of 96 letters (32 each). Thus

$$\frac{96!}{32! \cdot 32! \cdot 32!}$$

□

Problem 2. *How many permutations of the letters in "TALLAHASSEE" exist such that no two A's are adjacent?*

We have 11 letters. $A \times 3$, $L \times 2$, $S \times 2$, $E \times 2$, T, H.

Solution 1 (main idea).

- #non-adjacent permutations = #all permutations−#adjacent permutations, where #all permutations = $\frac{11!}{3!2!2!2!}$ = 831600.
- Let adjacent labeled permutations be the set of permutations of letter "$TA_1LLA_2HA_3SSEE$" such that some A_i and A_j are adjacent for $i, j \in \{1, 2, 3\}$. Then

$$\#\text{adjacent permutations} \times 3! = \#\text{adjacent labeled permutations}.$$

  (hint: consider balls and boxes, each box has 3! balls)
- There are 3 adjacent cases:
  1. X: A_1 adjacent to A_2
  2. Y: A_2 adjacent to A_3
  3. Z: A_3 adjacent to A_1

  Note that

  $$\#\text{adjacent labeled permutations} = |X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |Y \cap Z| - |Z \cap X| + |X \cap Y \cap Z|.$$
- Calculate $|X|$: a permutation in X contains consecutive A_1A_2 or A_2A_1. View A_1A_2 as a single letter, then the number of permutations of $A_1A_2, A_3, L, L, S, S, E, E, T, H$ is $\frac{10!}{2!2!2!}$. Thus

$$|X| = 2 \times \frac{10!}{2!2!2!} = 907200.$$

  Note that $|X| = |Y| = |Z|$.
- Calculate $|X \cap Y|$: a permutation in $X \cap Y$ contains consecutive $A_1A_2A_3$ or $A_3A_2A_1$.

$$|X \cap Y| = 2 \times \frac{9!}{2!2!2!} = 90720.$$
- $X \cap Y \cap Z = \emptyset$.

Now we conclude

$$\#\text{adjacent labeled permutations} = 3 \times 907200 - 3 \times 90720 + 0 = 2721600 - 272160 = 2449440,$$

$$\#\text{adjacent permutations} = \#\text{adjacent labeled permutations}/3! = 408240,$$

and

$$\#\text{non-adjacent permutations} = 831600 - 408240 = 423360.$$

□

Solution 2. When we disregard the A's, there are

$$\frac{8!}{2!2!2!1!1!} = 5040$$

ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the upward arrows indicate nine possible locations for the three A's.

E E S T L L S H

Three of these locations can be selected in

$$\binom{9}{3} = 84$$

ways; and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are

$$5040 \times 84 = 423,360$$

arrangements of the letters in TALLAHASSEE with no consecutive A's.

$\square$

Problem 3. *Given the sets:*

$$|A| = 10, \quad |B| = 9, \quad |A \cap B| = 4$$

find the number of ways to choose (a, b) *such that* $a \in A$, $b \in B$, *and* $a \neq b$.

Solution. We have two cases:

- **Case 1:** $a \in A \setminus B$
- **Case 2:** $a \in A \cap B$

Case 1:

- Step 1: Choose a, where $a \in A \setminus B$.

$$|A \setminus B| = 6$$

- Step 2: Choose b, ensuring $b \in B$ but $b \neq a$.

$$|B| = 9$$

$$\#\text{Case } 1 = |A \setminus B| \times |B| = 6 \times 9 = 54$$

Case 2:

- Step 1: Choose a, where $a \in A \cap B$.

$$|A \cap B| = 4$$

- Step 2: Choose b, ensuring $b \in B$ and $b \neq a$.

$$|B \setminus \{a\}| = 8$$

$$\#\text{Case } 2 = |A \cap B| \times |B \setminus \{a\}| = 4 \times 8 = 32$$

Final Count:

$$\#\text{ways} = \#\text{Case } 1 + \#\text{Case } 2 = 54 + 32 = 86$$

$\square$

2 The Binomial Theorem

Theorem 2.1.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Recall that

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

Problem 4. *Find the coefficient of x^5y^7 in $(2x-3y)^{12}$. How about x^5y^6?*

Solution. Using the binomial theorem, we have

$$(2x-3y)^{12} = ((2x)+(-3y))^{12} = \sum_{k=0}^{12} \binom{12}{k} (2x)^k (-3y)^{12-k}.$$

The term containing x^5y^7 is

$$\binom{12}{5}(2x)^5(-3y)^7 = \binom{12}{5} 2^5 (-3)^7 x^5 y^7.$$

So the coefficient of x^5y^7 is $\binom{12}{5}2^5(-3)^7$.

The coefficient of x^5y^6 is 0. ☐

3 More on combinatorial proof

Problem 5. *Give a combinatorial proof that*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof. **Thought:**

We design a combinatorial problem such that $\binom{2n}{n}$ is the answer. Consider a set of $2n$ elements, split into two groups of n elements each.

RHS Interpretation: The right-hand side, $\binom{2n}{n}$, counts the number of ways to choose n elements from the $2n$ elements.

LHS Interpretation: The left-hand side, $\sum_{k=0}^{n} \binom{n}{k}^2$, not easy. But summation represents the rule of sum. Let us Breakdown Step-by-Step!

LHS: Outline cases or steps.

- Case 0: $\binom{n}{0}^2 = \#\text{Case } 0$
- Case 1: $\binom{n}{1}^2$
- $\vdots$
- Case n: $\binom{n}{n}^2$

For **Case** k**:**

$$\binom{n}{k} \cdot \binom{n}{k} = \#\text{Case } k.$$

represents the rule of product. Step-by-step breakdown again:

- Step 1: Choose k elements from the first n, leave $n-k$: $\binom{n}{k}$.

- Step 2: Choose k elements from the second n, leave $n-k$: $\binom{n}{k}$.

Find the relationship between left and right: first n+seconde n=$2n$ (RHS), and k in Step 1+ $n-k$ in Step 2= n (RHS).

Example interpretation: Consider n blue balls and n red balls.

- **RHS:** Choose n balls from the $2n$ available.
- **LHS:** For each $k \in [n]$: Choose k blue balls from n and $n-k$ red balls from n.

Thus, the combinatorial proof is established.

$\square$