Note 14. Integer Partitions and Exponential Generating Function

1 Integer Partitions

Definition: For any positive integer n, a **partition** of n is a grouping of n into positive unordered summands. The number of partitions of n is denoted $p(n)$.

$$P(0) = 1 \quad \text{(by convention)}$$

Examples:

$$P(1) = 1 : \quad 1$$
$$P(2) = 2 : \quad 2, 1+1$$
$$P(3) = 3 : \quad 3, 2+1, 1+1+1$$
$$P(4) = 5 : \quad 4, 3+1, 2+2, 2+1+1, 1+1+1+1$$

Note that: If summands were ordered, this would be equivalent to finding the number of non-negative integer solutions to

$$x_1 + x_2 + \cdots + x_n = n.$$

But non-order makes this different.

Problem 1. *what is $p(n)$?*

Recall Note 11–generating function. We can use generating functions to find out.

Solution. It is equivalent to find the number of solutions to

$$n = 1 \times k_1 + 2 \times k_2 + 3 \times k_3 + \cdots n \times k_n.$$

For each possible summand $(1, 2, 3, \dots)$, we list all the possibilites:
Summands:

$$1 \times k_1 : \text{range } 0, 1, 2, 3, \dots; \text{function } 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$
$$2 \times k_2 : 0, 2, 4, 6, \dots; 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$
$$3 \times k_3 : 0, 3, 6, 9, \dots; 1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1-x^3}$$

The contributions to the total sum n by summands of size i can be represented by $\frac{1}{1-x^i}$. So $p(n)$ is the coefficient of x^n in:

$$f(x) = \prod_{i=1}^{n} \frac{1}{1-x^i}$$

□

This problem is also equivalent to find the number of solutions to

$$n = 1 \times k_1 + 2 \times k_2 + 3 \times k_3 + \cdots n \times k_n + n + 1 \times k_{n+1}.$$

Because k_{n+1} should be 0. In this case, the generating function is

$$g(x) = \prod_{i=1}^{n=1} \frac{1}{1 - x^i}$$

And $p(n)$ is the coefficient of x^n in $g(x)$.

Note that

$$g(x) = f(x)(1 + x^{n+1} + x^{2(n+1)} + x^{3(n+1)} + \dots) = f(x) + f(x)x^{n+1}(1 + x^{n+1} + x^{2(n+1)} + x^{3(n+1)} + \dots).$$

Since $f(x)x^{n+1}$ doesn't contribute to the coefficient of x^n, we know the coefficient of x^n in $g(x)$ is the same as $f(x)$.

Therefore, $p(n)$ is also the coefficient of x^n in:

$$F(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$$

Problem 2. *Find the number, $p_o(n)$, of partitions of n into odd summands.*

Solution. Equation:

$$n = 1 \times k_1 + 3 \times k_2 + 5 \times k_3 + \cdots .$$

Summands:

$$1 \times k_1 : 0, 1, 2, 3, \dots ; 1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}$$

$$3 \times k_2 : 0, 3, 6, \dots ; 1 + x^3 + x^6 + \cdots = \frac{1}{1 - x^3}$$

$$5 \times k_3 : 0, 5, 10, \dots ; 1 + x^5 + x^{10} + \cdots = \frac{1}{1 - x^5}$$

$$(2i + 1) \times k_i : 0, 2i + 1, 2(2i + 1), \dots ; 1 + x^{2i+1} + x^{2(2i+1)} + \cdots = \frac{1}{1 - x^{2i+1}}$$

So the GF for the sequence $(p_o(n))_{n \geq 0}$ is:

$$f(x) = \prod_{i=0}^{\infty} \frac{1}{1 - x^{2i+1}}$$

□

Check $p_o(5)$: Obviously, $p_o(5) = 3$ since $5 = 5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Next, we compute the coefficient of x^5 in $f(x)$.

$$f(x) = (1 + x + x^2 + x^3 + \dots)(1 + x^3 + x^6 + \dots)(1 + x^5 + x^10 + \dots) \dots$$

Since terms x^k for $k \geq 6$ do not contribute to the coefficient of x^5, they can be omitted. Therefore the coefficient is equivalent to the coefficient of x^5 in the following $g(x)$.

$$g(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^3)(1 + x^5) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^3 + x^5 + x^8).$$

Obviously, the coefficient 3. This matches our earlier count, so the result is verified.

Problem 3. *Find the number of partitions of n into even summands.*

Solution. Equation

$$n = 2 \times k_1 + 4 \times k_2 + 6 \times k_3 + \cdots.$$

Summands:

$$2 \times k_1 : 0, 2, 4, \ldots ; 1 + x^2 + x^4 + \cdots = \frac{1}{1 - x^2}$$

$$4 \times k_2 : 0, 4, 8, \ldots ; 1 + x^4 + x^8 + \cdots = \frac{1}{1 - x^4}$$

$$(2i) \times k_i : 0, 2i, 2(2i), \ldots ; 1 + x^{2i} + x^{2(2i)} + \cdots = \frac{1}{1 - x^{2i}}$$

So the GF is:

$$f(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i}}$$

$\square$

Problem 4. *Find the number of partitions of n into odd summands, each of which appears an odd number of times (or not at all).*

Solution. Equation

$$n = 1 \times k_1 + 3 \times k_2 + 5 \times k_3 + \cdots.$$

Each appears an odd number of times (or not at all) means $k_i = 0$ or odd. Summands:

$$1 \times k_1 : 0, 1, 3, \ldots ; 1 + x + x^3 + \cdots = 1 + \sum_{i=0}^{\infty} x^{2i+1}$$

$$3 \times k_2 : 0, 3, 9, \ldots ; 1 + x^3 + x^9 + \cdots = 1 + \sum_{i=0}^{\infty} x^{3(2i+1)}$$

$$(2m+1) \times k_i : 0, 2m+1, 3(2m+1), \ldots ; 1 + x^{2m+1} + x^{3(2m+1)} + \cdots = 1 + \sum_{i=0}^{\infty} x^{(2m+1)(2i+1)}$$

So the GF is:

$$f(x) = \prod_{m=0}^{\infty} \left(1 + \sum_{i=0}^{\infty} x^{(2m+1)(2i+1)} \right)$$

The number of such partitions is the coefficient of x^n in $f(x)$. $\square$

Problem 5. *Find the number $P_d(n)$ of partitions of n into distinct summands.*

Solution. Equation

$$n = 1 \times k_1 + 2 \times k_2 + 3 \times k_3 + \cdots n \times k_n.$$

Distinct summands means $k_i = 0$ or 1.

Summands:

$$1 \times k_1 : \qquad 0, 1; \qquad 1 + x \text{ or } = \frac{1 - x^2}{1 - x}.$$

$$2 \times k_2 : \qquad 0, 2; \qquad 1 + x^2 = \frac{1 - x^4}{1 - x^2}.$$

$$3 \times k_3 : \qquad 0, 3; \qquad 1 + x^3 = \frac{1 - x^6}{1 - x^3}.$$

So the GF is:

$$f(x) = \prod_{k=0}^{\infty} (1 + x^k)$$

$\square$

Problem 6. $p_o(n) = p_d(n)$ *(The number of partitions into odd parts equals the number of partitions into distinct parts.)*

Solution. NOTE THAT the GF of $(p_d(n))_{n\geq 0}$ is

$$\begin{aligned} f(x) &= (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots \\ &= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdots \\ &= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots \end{aligned}$$

equals the GF of $(p_o(n))_{n\geq 0}$. So $p_o(n) = p_d(n)$. $\square$

Strategy: If two sequences are generated by the same GF, they must be equal (so each term is equal).

Problem 7. *Show that the number of partitions of n where no summand appears more than twice is equal to the number of partitions of n where no summand is divisible by 3.*

Exercise.

2 Exponential Generating Function (EGF)

Recall

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Definition 2.1. *For a sequence $a_0, a_1, a_2, \ldots$, the* **exponential generating function** *(EGF) is given by:*

$$f(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

Recall: The generating function (GF) is given by:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Problem 8. *Find the EGF for the sequence* $(1, 1, 1, 1, \ldots)$.

Solution.

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

$\square$

Problem 9. *Find the EGF for the sequence* $(1, -1, 1, -1, \ldots)$.

Solution.

$$f(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = e^{-x}$$

□

Problem 10. *Find the sequence generated by the following EGF*

$$\frac{e^x + e^{-x}}{2}$$

Solution.

$$\begin{aligned}\frac{e^x + e^{-x}}{2} &= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) \\ &= \frac{1}{2} \cdot 2\left(1 + 0 + \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + \dots\right) \\ &= 1 + 0 + \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + \dots\end{aligned}$$

So the sequence is $(1, 0, 1, 0, 1, 0, \dots)$. □

Problem 11. *Find the sequence generated by the following EGF*

$$\frac{e^x - e^{-x}}{2}$$

Exercise.

Concerning EGF, you only need to understand the content presented above; all other topics are beyond the scope of the exam.