Note 13. Generating Function–Calculational Techniques

1 Shift and Subtract

Problem 1. *Find the closed-form generating function for the sequence* $1, 1, 1, 1, \ldots$

Solution 1. Maclaurin Series Expansion

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

□

Solution 2. Let

$$S = 1 + x + x^2 + x^3 + \ldots$$

$$xS = x + x^2 + x^3 + x^4 + \ldots$$

$$(1-x)S = 1 \Rightarrow S = \frac{1}{1-x}$$

□

Question: why do we use xS?
The quotient of the i-th term divided by the $(i-1)$-th term is x.

Problem 2. *Find the closed-form generating function for the sequence* $1, -1, 1, -1, 1, -1, \ldots$

Solution. Let

$$S = 1 - x + x^2 - x^3 + \ldots$$

$$xS = x - x^2 + x^3 - x^4 + \ldots$$

$$(1+x)S = 1 \Rightarrow S = \frac{1}{1+x}$$

□

Problem 3.

$$S = 1 + 3x + 9x^2 + 27x^3 + \cdots =?$$

Solution.

$$3xS = 3x + 9x^2 + 27x^3 + \ldots$$

$$S - 3xS = 1 \Rightarrow S = \frac{1}{1-3x}$$

□

Problem 4.

$$S = 3 + 3 \cdot 3x + 3 \cdot 9x^2 + 3 \cdot 27x^3 + \ldots$$

Solution.

$$S = \frac{3}{1-3x}$$

□

Problem 5.

$$S = 2 + 4x + 10x^2 + 28x^3 + \cdots = ?$$

. *Here* $a_n = 1 + 3^n$

Solution.

$$S = \sum_{n=0}^{\infty}(1+3^n)x^n. = \sum_{n=0}^{\infty} 1 \cdot x^n + \sum_{n=0}^{\infty} 3^n \cdot x^n$$

Define

$$A = \sum_{n=0}^{\infty} x^n, \quad B = \sum_{n=0}^{\infty} 3^n x^n$$

Then

$$A = \frac{1}{1-x}, \quad B = \frac{1}{1-3x}$$

Hence,

$$S = A + B = \frac{1}{1-x} + \frac{1}{1-3x}.$$

□

Problem 6. *Find the closed-form generating function for the sequence* $1, 0, 1, 0, 1, 0, \dots$

Solution.

$$S = 1 + x^2 + x^4 + x^6 + \dots$$

$$x^2 S = x^2 + x^4 + x^6 + \dots$$

$$S - x^2 S = 1 \Rightarrow S = \frac{1}{1-x^2}$$

□

Problem 7. *Find the closed-form generating function for the sequence* $0, 1, 0, 1, 0, 1, \dots$.

Solution.

$$S = x + x^3 + x^5 + \cdots = x \cdot (1 + x^2 + x^4 + \dots) = x \cdot \frac{1}{1-x^2}$$

Or

$$x^2 S = x^3 + x^5 + \dots$$

$$S - x^2 S = x \Rightarrow S = \frac{x}{1-x^2}$$

□

Problem 8. *Find the closed-form generating function for the sequence* $1, 2, 3, 4, \dots$.

Solution.

$$S = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$xS = x + 2x^2 + 3x^3 + 4x^4 + \dots$$

$$S - xS = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

$$(1-x)S = \frac{1}{(1-x)^2} \Rightarrow S = \frac{1}{(1-x)^2}$$

□

Problem 9. *Find the closed-form generating function for the sequence* $1, 3, 5, 7, 9, \dots$.

Solution.

$$S = 1 + 3x + 5x^2 + 7x^3 + \dots$$

$$xS = x + 3x^2 + 5x^3 + \dots$$

$$(1-x)S = 1 + 2x + 2x^2 + 2x^3 + \dots$$

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$(1-x)S = 2 \cdot \frac{1}{1-x} - 1 = \frac{2-1+x}{1-x} = \frac{1+x}{1-x}$$

$$\Rightarrow S = \frac{1+x}{(1-x)^2}$$

□

Problem 10.

$$S = 1 + 4x + 9x^2 + 16x^3 + \dots = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots = ?$$

Solution.

$$xS = x + 4x^2 + 9x^3 + \dots$$

$$S - xS = 1 + 3x + 5x^2 + 7x^3 + \dots = \frac{1+x}{(1-x)^2}$$

$$\Rightarrow S = \frac{1+x}{(1-x)^3}$$

□

Problem 11.

$$S = 1 + 4x + 9x^2 + 16x^3 + \dots = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots = ?$$

2 Differentiating

Problem 12.

$$S = 1 + 4x + 9x^2 + 16x^3 + \cdots = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = ?$$

Solution.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Differentiating both sides with respect to x:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \dots$$

$$x \cdot \left(\frac{1}{(1-x)^2}\right) = \sum_{n=0}^{\infty} nx^n$$

Differentiating again,

$$\frac{1+x}{(1-x)^3} = \sum_{n=0}^{\infty} n^2x^{n-1} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$$

$$\frac{x(1+x)}{(1-x)^3} = \sum n^2x^n = S$$

□

3 convolution

Addition / Subtraction

Theorem 3.1. *If $f(x)$ is a GF for the sequence $(a_n)_{n\geq 0}$ and $g(x)$ is a GF for $(b_n)_{n\geq 0}$, then $f(x) + g(x)$ is a GF for $(a_n + b_n)_{n\geq 0}$.*

Multiplication

Theorem 3.2. *If $f(x)$ is a GF for $(a_n)_{n\geq 0}$ and $g(x)$ is a GF for $(b_n)_{n\geq 0}$, then the sequence $(c_n)_{n\geq 0}$ generated by $f(x) \cdot g(x)$ is the* **convolution** *of (a_n) and (b_n). In particular, $f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_nx^n$, where*

$$c_n = \sum_{i=0}^{n} a_i \cdot b_{n-i} = \sum_{i=0}^{n} a_{n-i} \cdot b_i.$$

Proof. Idea: Find the coefficient of x^n in $(a_0 + a_1x + a_2x^2 + \cdots) \cdot (b_0 + b_1x + b_2x^2 + \cdots)$:

$$c_0 = a_0 \cdot b_0$$

$$c_1x = a_0 \cdot b_1x + a_1x \cdot b_0 = (a_0b_1 + a_1b_0)x$$

$$c_2x^2 = a_0 \cdot b_2x^2 + a_1x \cdot b_1x + a_2x^2 \cdot b_0$$

$$c_2 = a_0b_2 + a_1b_1 + a_2b_0$$

Therefore

$$c_n = a_0b_{n-1} + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0 = \sum_{i=0}^{n} a_ib_{n-i}$$

□

Problem 13.

$$f(x) = \frac{1}{1-x} \quad \Rightarrow \quad (a_n) = (1, 1, 1, 1, \ldots)$$

$$g(x) = \frac{1}{1+x} \quad \Rightarrow \quad (b_n) = (1, -1, 1, -1, \ldots)$$

Find (c_n): the convolution of (a_n) and (b_n).

$$c_0 = a_0 b_0 = 1$$

$$c_1 = a_0 b_1 + a_1 b_0 = 1 \cdot (-1) + 1 \cdot 1 = -1 + 1 = 0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 1 = 1 - 1 + 1 = 1$$

Guess: $\{c_n\} = 1, 0, 1, 0, 1, 0 \cdots$

Solution 1.

$$c_n = \sum_{i=0}^{n} a_{n-i} b_i = \sum_{i=0}^{n} b_i$$

Case 1: When n is even,

$$c_n = \sum_{i=0}^{n} b_i = 0$$

Case 2: When n is odd

$$c_n = \sum_{i=0}^{n} b_i = -1$$

□

Solution 2.

$$f(x) \cdot g(x) = \frac{1}{1-x} \cdot \frac{1}{1+x} = \frac{1}{1-x^2}$$

Recall

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$$

Replace $y = x^2$:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots$$

Then $(c_n)_{n \geq 0} = (1, 0, 1, 0, 1, 0, \ldots)$. □

Problem 14.

$$f(x) = 1 + x + x^2 + x^3 \quad \Rightarrow \quad (a_n) = (1, 1, 1, 1, 0, 0, \ldots)$$

$$g(x) = \frac{1}{1-3x} \quad \Rightarrow \quad (b_n) = (1, 3, 3^2, 3^3, \ldots)$$

Find c_n, which is the convolution of (a_n) and (b_n).

Solution.

$$a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_n = 0 \text{ for } n \geq 4$$

$$b_n = 3^n.$$

$$c_0 = a_0 b_0 = 1$$

$$c_1 = a_0 b_1 + a_1 b_0 = 3 + 1 = 4$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 9 + 3 + 1 = 13$$

$$c_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0$$

$$c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + a_3b_{n-3} + \cdots = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + a_3b_{n-3} + 0$$

So

$$c_n = 3^n + 3^{n-1} + 3^{n-2} + 3^{n-3} \quad \text{for } n \geq 3$$

□

Note that it is not easy to expand $f(x) \cdot g(x)$.

$$f(x) = 1 + x + x^2 + x^3 = \frac{1-x^5}{1-x}$$

$$g(x) = \frac{1}{1-3x}$$

$$f(x) \cdot g(x) = \frac{1-x^5}{1-x} \cdot \frac{1}{1-3x} = \quad ?$$

4 Partial Fraction

Problem 15. *Determine the coefficients of x^8 in*

$$f(x) = \frac{1}{(x-3)(x-2)^2}$$

Solution. Use the partial fraction decomposition:

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

This decomposition implies that

$$1 = A(x-2)^2 + B(x-2)(x-3) + C(x-3),$$

or

$$0 \cdot x^2 + 0 \cdot x + 1 = (A+B)x^2 + (-4A - 5B + C)x + (4A + 6B - 3C).$$

We find that $A + B = 0$, $-4A - 5B + C = 0$, and $4A + 6B - 3C = 1$. Solving these equations yields $A = 1$, $B = -1$, and $C = -1$. Hence

$$\frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}.$$

Recall that

$$\frac{1}{x-3} = \left(-\frac{1}{3}\right) \frac{1}{1-(x/3)} = \left(-\frac{1}{3}\right) \sum_{i=0}^{\infty} \left(\frac{x}{3}\right)^i,$$

$$\frac{1}{x-2} = \left(\frac{1}{2}\right) \frac{1}{1-(x/2)} = \left(\frac{1}{2}\right) \sum_{i=0}^{\infty} \left(\frac{x}{2}\right)^i,$$

$$\frac{1}{(x-2)^2} = \left(-\frac{1}{4}\right) \frac{1}{(1-(x/2))^2} = \left(-\frac{1}{4}\right) \sum_{i=0}^{\infty} (i+1) \left(\frac{x}{2}\right)^i.$$

Hence, the coefficient of x^8 is

$$-\frac{1}{3^9} + \frac{1}{2^9} - \frac{9}{2^{10}} = -\frac{1}{3^9} - \frac{7}{2^{10}}$$

□