

Summary

1 Introduction

The midterm exam will cover

1. Injective and surjective functions
2. Pigeonhole principle
3. Induction (It was removed from the midterm exam)
4. Combinatorial proofs
5. Counting
 - Counting strings (with and without conditions)
 - Making multiple choices
 - Counting permutations (with and without conditions)
 - Circular permutations
6. Multinomial coefficients
7. Stars and Bars

2 Functions

- Definitions of functions, injective, surjective, bijective, one-to-one, onto.
- Let $f : X \rightarrow Y$. If injective, then $|X| \leq |Y|$. If surjective then $|X| \geq |Y|$.
- Give examples (see homework)

3 Pigeonhole principle

- Definition
- Apply it

It's not necessary to master the proof, but since it's quite straightforward, I included it here for reference.

Proof of the Pigeonhole principle. Assume each pigeonhole has at most 1 pigeon. Then we have at most n pigeons, which is a contradiction. Thus there exist one pigeonhole with at least 2 pigeons. \square

4 Induction

- Two steps!
- Refer to Levin's *Discrete Mathematics: An Open Introduction* for detailed examples and guidelines on writing **formal proofs**.

5 Combinatorial proofs

- Definition!
- Review our homework.

Problem 1. Provide a combinatorial proof of the identity

$$\binom{2n}{n} + \binom{2n}{n-1} = \frac{1}{2} \binom{2n+2}{n+1},$$

Hint. Simplify:

$$2\binom{2n}{n} + 2\binom{2n}{n-1} = \binom{2n+2}{n+1},$$

Let's think of the RHS: $(2n+2 \text{ choose } n+1)$ counts all subsets of size $n+1$ from $2n+2$ elements.

The LHS involves $2n$ elements, and the RHS involves $2n+2$. Maybe adding two more elements to the original $2n$ set. Consider adding two new elements, say x and y , to the original set of $2n$ elements. Now, the total number of ways to choose $n+1$ elements from $2n+2$ is $\binom{2n+2}{n+1}$. We can partition these subsets based on whether they contain x , y , both, or neither.

- Subsets containing neither x nor y : $\binom{2n}{n+1} = \binom{2n}{n-1}$
- Subsets containing x but not y : $\binom{2n}{n}$
- Subsets containing y but not x : $\binom{2n}{n}$
- Subsets containing both x and y : $\binom{2n}{n-1}$

Then we are done. □

6 Multinomial coefficients

I had left it as homework for you. Let us go over it again here.

Theorem 6.1 (Multinomial Theorem). For positive integers n, t , the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ in the expansion of

$$(x_1 + x_2 + x_3 + \cdots + x_t)^n$$

is

$$\frac{n!}{n_1! n_2! n_3! \cdots n_t!},$$

where each n_i is an integer with $0 \leq n_i \leq n$, for all $1 \leq i \leq t$, and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

It is also written as

$$\binom{n}{n_1, n_2, n_3, \dots, n_t},$$

and is called a **multinomial coefficient**.

Proof. As in the proof of the binomial theorem, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, \dots , and x_t from n_t of the last $n - n_1 - n_2 - n_3 - \cdots - n_{t-1} = n_t$ remaining factors. This can be carried out in

$$\binom{n}{n_1} \binom{n - n_1}{n_2} \binom{n - n_1 - n_2}{n_3} \cdots \binom{n - n_1 - n_2 - \cdots - n_{t-1}}{n_t}$$

ways. It is equal to

$$\frac{n!}{n_1! n_2! n_3! \cdots n_t!},$$

□

Problem 2. Determine the coefficient of $a^2b^3c^2d^5$ in the expansion of

$$(a + 2b - 3c + 2d + 5)^{16}.$$

Proof. If we replace a by v , $2b$ by w , $-3c$ by x , $2d$ by y , and 5 by z , then we can apply the multinomial theorem to

$$(v + w + x + y + z)^{16}$$

and determine the coefficient of $v^2w^3x^2y^5z^4$ as

$$\binom{16}{2, 3, 2, 5, 4}.$$

Then

$$\binom{16}{2, 3, 2, 5, 4} (a^2)(2b)^3 (-3c)^2 (2d)^5 (5)^4 = \binom{16}{2, 3, 2, 5, 4} (1^2)(2^3)(-3^2)(2^5)(5^4)(a^2b^3c^2d^5)$$

The coefficient is $\binom{16}{2, 3, 2, 5, 4} (1^2)(2^3)(-3^2)(2^5)(5^4)$.

□

7 Stars and Bars

- Apply it!

8 Counting

For more exercises, see Chapter 1.6 in Discrete and Combinatorial Mathematics by Grimaldi.

Problem 3. A machine has nine different dials, each with five settings labeled 0, 1, 2, 3, and 4.

- In how many ways can all the dials on the machine be set?
- If the nine dials are arranged in a line at the top of the machine, how many of the machine settings have no two adjacent dials with the same setting?
- How many machine settings in part (b) use only 0, 2, and 4 as dial settings?

Solution. (a) Each dial has 5 possible settings, and since there are 9 dials, the total number of settings is:

$$5^9 = 1,953,125.$$

- Each first dial has 5 choices. - Each subsequent dial has only 4 choices (it cannot be the same as the previous dial).

Thus, we have:

$$5 \times 4^8.$$

-

$$3 \times 2^8.$$

□

Problem 4. How many n -digit quaternary $(0,1,2,3)$ sequences have exactly r 1's?

Solution.

□

We proceed as follows:

1. In how many ways can Francesca line up 12 of these books on her bookshelf?
2. The remaining $n - r$ positions can be filled with any of the remaining three digits (0, 2, or 3), giving each of these positions 3 choices. Thus, the number of ways to fill these positions is:

$$3^{n-r}.$$

3. Therefore, the total number of valid sequences is:

$$\binom{n}{r} \cdot 3^{n-r}.$$

Problem 5. *Francesca has 20 different books but the shelf in her dormitory residence will hold only 12 of them.*

- (a) *In how many ways can Francesca line up 12 of these books on her bookshelf?*
- (b) *How many of the arrangements in part (a) include Francesca's three books on tennis?*

Proof. (a) The permutations of selecting r items from n is

$$P(20, 12) = \frac{20!}{8!}.$$

- (b) we first need to choose the remaining 9 books from the 17 non-tennis books. This is a combination problem, $\binom{17}{9}$. Then, we arrange all 12 books (including the three tennis books) on the shelf, which is $12!$. Thus the number of arrangements is $\binom{17}{9}12!$.

□

You can try solving the remaining exercises on your own. Feel free to discuss them on Ed Discussion or use AI tools for assistance.

You may turn to Deepseek for assistance. Here are its reflections in Deepthink mode. Take the time to understand its thought process, as it can be highly beneficial. However, in light of school policies, it is advisable not to utilize it for homework purposes.

Okay, let's see. I need to solve these two problems about Fran and her books. Let me start with part (1). The question is asking: In how many ways can Fran line up 12 of these books on her bookshelf? She has 20 different books, and the shelf can hold 12. Hmm. So, lining up books implies that the order matters, right? If order matters, that's a permutation problem.

So, permutations versus combinations. If the order didn't matter, it would be combinations, but since we're talking about arranging them in a line on a shelf, the order definitely matters. So, permutations. The formula for permutations when selecting r items from n is $P(n, r) = n! / (n - r)!$.

So in this case, n is 20, and r is 12. So plugging into the formula, that would be $20!$ divided by $(20 - 12)!$ which is $8!$. So $20! / 8!$. Let me check if that makes sense. If you have 20 books and you want to arrange 12 of them, the first position has 20 choices, the next 19, then 18, and so on down to $20 - 11 = 9$. So that would be $20 \times 19 \times 18 \times \dots \times 9$. Which is the same as $20!$ divided by $8!$ because $20!$ is $20 \times 19 \times \dots \times 1$ and dividing by $8!$ removes the last 8 terms, leaving $20 \times 19 \times \dots \times 9$. So yes, that seems right. So the answer to part (1) should be $20P12 = 20! / 8!$.

Now moving on to part (2): How many arrangements in part (a) include Fran's three books on tennis? Hmm. So now, we need to ensure that all three tennis books are included in the 12 books arranged on the shelf. So, first, we have to include those three specific books, and then choose the remaining 9 books from the remaining 17 books (since 20 total minus 3 tennis books is 17). Then, once we have those 12 books (3 tennis + 9 others), we need to arrange all 12 of them on the shelf.

So, breaking it down: First, choose the 9 additional books from the 17 non-tennis books. The number of ways to choose those is combination of 17 choose 9, which is $C(17, 9)$. Then, once we have the 12 books (3 tennis + 9 others), we need to arrange all 12 of them. Since all the books are different, the number of arrangements is $12!$.

But wait, is that all? Wait a second. Because the three tennis books are already included, so we don't need to choose them separately? Let me think. So, if we have to include all three tennis books, then the total number of books to select is 12, which must include the 3 tennis ones, so the remaining 9 are selected from the other 17. So yes, the number of ways is $C(17, 9)$ multiplied by $12!$ (to arrange all 12 books). Alternatively, another way to think about it is: first, ensure the three tennis books are included, and then calculate the permutations.

Alternatively, maybe we can compute it as: total permutations of 12 books that include the three tennis books. So, the total number of permutations where all three tennis books are present. To calculate this, we can subtract the permutations that don't include all three tennis books from the total permutations. But that might be more complicated. The direct method seems better: choose the remaining books and then arrange all.

Wait, but let's check. Let's consider that we have to have all three tennis books in the arrangement. So first, we need to choose 9 more books from the 17 non-tennis books, which is $C(17, 9)$. Then, arrange those 12 books. However, arranging those 12 books includes arranging the three tennis books among themselves and the others. So yes, that should be correct. So the total number of arrangements is $C(17, 9) \times 12!$.

Alternatively, another approach: fix that the three tennis books are included. So the total number is equal to the number of ways to arrange 12 books where 3 specific books are included. So, first, include those 3 books, then choose and arrange the remaining 9. So, perhaps, first, decide how many ways to place the three tennis books in the 12 positions, and then fill the rest. Wait, but that might be overcomplicating. Let me see.

Alternatively, think of it as: first, select 12 books including all three tennis books. The number of such selections is $C(17, 9)$. Then, for each such selection, the number of permutations is $12!$. Therefore, total is $C(17, 9) \times 12!$.

Alternatively, we can think of it as: first, arrange all 12 books, ensuring that the three tennis books are among them. To compute this, we can use the principle of inclusion. But I think the first approach is

straightforward. Let me verify.

Suppose we have 20 books, 3 are tennis. We need permutations of 12 books that include all 3 tennis books. So, the number of such permutations is equal to the number of ways to choose the remaining 9 books from 17 non-tennis books, multiplied by the number of ways to arrange all 12 books. So, yes, that would be $C(17, 9) \times 12!$.

Alternatively, if we think in terms of permutations directly: the number of permutations of 12 books that include the 3 tennis books is equal to the total permutations of 12 books (which is $20P12$) minus the permutations that exclude at least one tennis book. But that approach would require inclusion-exclusion, which is more complex. So, the direct method is better here.

Therefore, the answer to part (2) should be $C(17, 9) \times 12!$.

Wait, let me check the numbers. $C(17, 9)$ is 17 choose 9, which is $17! / (9! \times 8!)$. Then multiplied by $12!$ gives $(17! / (9! \times 8!)) \times 12! = (17! \times 12!) / (9! \times 8!)$. Alternatively, this can be written as $(17! / 8!) \times (12! / 9!) = (17P8) \times (12P9)$. Hmm, not sure if that's helpful. Alternatively, maybe there's a different way to express it.

Wait, let's see. Alternatively, we can think of it as first arranging the three tennis books and then the remaining nine. But since the three tennis books are indistinct in terms of selection (they are all required), but once selected, they are distinct. Wait, no, the books are all different. So, if we have 3 specific books that must be included, then the total number is: choose 9 from 17, then permute all 12. So, that's $C(17,9) \times 12!$.

Alternatively, another way: imagine we have to place the three tennis books somewhere in the 12 spots. The number of ways to choose positions for the tennis books is $C(12, 3)$, then arrange the three tennis books in those positions ($3!$), then arrange the remaining 9 books from the 17 non-tennis books in the remaining 9 spots (which is $P(17,9)$). So total arrangements would be $C(12,3) \times 3! \times P(17,9)$. Let's compute this: $C(12,3)$ is 220, $3!$ is 6, $P(17,9)$ is $17! / 8!$.

So $220 \times 6 \times (17! / 8!) = 220 \times 6 \times 17! / 8! = (220 \times 6) \times 17! / 8! = 1320 \times 17! / 8!$.

But let's see if this is equal to the previous expression. The previous expression was $C(17,9) \times 12! = (17! / (9!8!)) \times 12!$.

Compare with the second approach: $1320 \times 17! / 8!$.

Hmm, let's see:

$C(17,9) \times 12! = (17! / (9!8!)) \times 12! = (17! \times 12!) / (9!8!)$

Second approach: $C(12,3) \times 3! \times P(17,9) = (12! / (3!9!)) \times 3! \times (17! / 8!) = (12! / 9!) \times (17! / 8!) = (12! \times 17!) / (9!8!)$

So, same as the first method. Therefore, both approaches are equivalent. So either way, the result is $(17! \times 12!) / (9!8!)$. Which makes sense.

So, either way, the answer is the same. Therefore, both methods confirm that the number of arrangements is $C(17,9) \times 12!$.

Alternatively, if I calculate $12! / 9!$ is $12 \times 11 \times 10 \times 9! / 9! = 12 \times 11 \times 10 = 1320$. Wait, but that's not directly related here. Hmm. Maybe I'm overcomplicating.

But regardless, the answer is $C(17,9)$ multiplied by 12 factorial. So, that's part (2).

Let me verify once again. Suppose all three tennis books must be on the shelf. So, we need to choose 9 more books from the remaining 17, then arrange all 12. So yes, the number of ways is combinations of 17 choose 9 multiplied by 12 factorial. That seems correct.

Alternatively, think of the total permutations as $20P12$. To find how many of those include all three tennis books, we can calculate it as $20P12$ minus the permutations that exclude at least one tennis book. But that would require inclusion-exclusion. Let's see:

Total permutations: $20P12$.

Subtract permutations missing at least one tennis book. There are three tennis books, say T1, T2, T3.

Number of permutations missing T1: $C(19,12) \times 12! = 19P12$.

Similarly for missing T2 and T3, each $19P12$.

But then we have to add back in permutations missing both T1 and T2, which would be $C(18,12) \times 12! = 18P12$, and similarly for missing T1 and T3, T2 and T3.

Then subtract permutations missing all three T1, T2, T3: $C(17,12) \cdot 12! = 17P12$.

So by inclusion-exclusion, the number of permutations including all three tennis books is:

$$20P12 - 3 \cdot 19P12 + 3 \cdot 18P12 - 17P12.$$

Let me compute that and see if it equals the previous result.

Wait, but that seems complicated. Let's check with numbers. Let me compute both expressions symbolically.

$$\text{First expression: } C(17,9) \cdot 12! = (17! / (9!8!)) \cdot 12!.$$

$$\text{Second expression: } 20P12 - 3 \cdot 19P12 + 3 \cdot 18P12 - 17P12.$$

$$\text{Compute } 20P12 = 20! / 8!.$$

$$19P12 = 19! / 7!.$$

$$18P12 = 18! / 6!.$$

$$17P12 = 17! / 5!.$$

So,

$$20! / 8! - 3 \cdot (19! / 7!) + 3 \cdot (18! / 6!) - 17! / 5!.$$

Let me factor out $17! / 5!$.

So:

$$(17! / 5!) \left[(20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13) / (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) - 3 \cdot (19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13) / (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) + 3 \cdot (18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13) / (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) - 1 \right]$$

Wait, this might get too messy. Alternatively, let's note that both methods should give the same result, so perhaps computing both expressions numerically would confirm they are equal, but since we can't compute factorials here, maybe it's better to accept that both approaches are valid and yield the same result.

But given that the first method is straightforward and logical, I think that's the correct approach. Therefore, part (2) is $C(17,9) \cdot 12!$.

Therefore, summarizing:

1) The number of ways to line up 12 books out of 20 is $20P12 = 20! / 8!$.

2) The number of such arrangements that include all three tennis books is $C(17,9) \cdot 12!$.

So, I think these are the answers.

****Final Answer****

(1) $\frac{20!}{8!}$

(2) $\binom{17}{9} \times 12!$