Probability, actions, and expectation

CS 3630

A Trash Sorting Robot

Remember our trash sorting robot.

Individual pieces of trash arrive to the robot's work cell on a conveyor belt.

The robot's task is to place each piece of trash in an appropriate bin:

- Glass
- Mixed paper
- Metal
- Nop (Do nothing!)

Sensors measure various characteristics of the trash, which are used to make inferences about the object type (perception).

We assume sensor uncertainty.

Over time, sensor models can be refined using machine learning methods.

Actions

For this problem, the robot either places an item of trash into one of three bins, or lets the item pass through the work cell.

This gives four possible actions:

- a_1: Glass Bin
- a_2: Metal Bin
- a_3: Paper Bin
- a_4: Nop (let object pass through the workcell)

For now, we assume that actions are executed without error, every time.

However, since we don't know with certainty the category for an item of trash in the work cell, the efficacy of an action is also uncertain.

Assessing Risk

- Because there is uncertainty in the category of a piece of trash, the robot risks making mistakes when choosing actions.
- Different mistakes have different costs.
  - Placing **paper** in the **metal bin** is unlikely to cause much harm.
  - Placing **metal** in the **paper bin** might seriously damage paper processing equipment.

Material Category

Action

COST	cardboard	paper	can	scrap metal	bottle
glass bin	2	2	4	6	0
metal bin	1	1	0	0	2
paper bin	0	0	5	10	3
nop	1	1	1	1	1

To account for these variations, we can define a table of costs for applying each action (rows) to each category (columns).

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We assign zero costs to correct actions.

The cost of Nop is due to the need for human labor to sort the item of trash.

Cost as a Random Variable

Since we only have probabilistic knowledge of an item's category, we can regard the cost of executing an action as a discrete random variable.

Consider action a_3, *place the item in the mixed paper bin*.

- Let X be the random variable that denotes the cost of applying action a_3.
- From the table of costs, we see that $X \in \{0,3,5,10\}$, since these are the only possible costs for this action.

➢What can we say about the probability distribution for X?

Computing pmf's

To compute the pmf, recall that the random variable is a mapping from outcomes to real numbers.

There are five possible outcomes. The object must be from one of five categories, each of which has a cost.

➢ Compute $p_X(x)$ for each x.

a_3

Category	P(C)	Cost
cardboard	0.20	0
paper	0.30	0
can	0.25	5
scrap metal	0.20	10
bottle	0.05	3

- $X = 0$ for cardboard and paper.
- $P(\{cardboard, paper\}) = P(\{cardboard\}) + P(\{paper\}) = 0.5$

➢ $p_X(0) = 0.5$

- $X = 5$ for can.
- $P(\{can\}) = 0.25$

➢ $p_X(5) = 0.25$

- $X = 3$ for bottle.
- $P(\{bottle\}) = 0.05$

➢ $p_X(3) = 0.05$

- $X = 10$ for scrap metal.
- $P(\{scrap\ metal\}) = 0.20$

➢ $p_X(10) = 0.20$

If we apply action a_3, *place the item in the mixed paper bin,* this is the pmf for cost!

Expectation

- Probabilities tell us something about a single outcome, but this isn't really very useful. Gamblers who make one-time bets based on probabilities can lose a lot of money.
- Most robots operate for prolonged periods of time.
- The notion of average cost over many trials seems like a useful thing to know.

➢ ***This is exactly the concept of expectation in probability theory.***

Expectation

If a random variable X takes its values from a finite set, $X \in \{x_1, \dots, x_n\}$, the ***expected value*** of X, denoted $\boldsymbol{E[X]}$, is defined by:

$$E[X] = \sum_{i=1}^{n} x_i \, p_X(x_i)$$

- Expectation is a *property of a probability distribution*.
- **$\boldsymbol{E[X]}$ is *<u>not</u>* the value you should expect to see for any specific outcome!!**

Examples

Let $X \in \{x_1, \dots, x_n\}$ be a discrete random variable that corresponds to the number of dots shown on a fair die.

- $X \in \{1,2,3,4,5,6\}$ and $p_X(x_i) = \frac{1}{6}$ for all i

➢ Compute $E[X]$.

$$E[X] = \sum_{i=1}^{n} x_i \, p_X(x_i) = \sum_{i=1}^{6} \frac{1}{6} i = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{21}{6} = \underline{3.5}$$

Examples

Let $X \in \{0,5\}$ be a discrete random variable

$$X = \begin{cases} 0 \text{ for an even roll} \\ 5 \text{ for an odd roll} \end{cases}$$

➢ Compute $E[X]$.

$$E[X] = \sum_{i=1}^{2} x_i \, p_X(x_i) = \frac{1}{2} \times 0 + \frac{1}{2} \times 5 = 2.5$$

Expectation

If a random variable X takes its values from a finite set, $X \in \{x_1, \ldots, x_n\}$, the ***expected value*** of $f(X)$, denoted $E[f(X)]$, is defined by:

$$E[f(X)] = \sum_{i=1}^{n} f(x_i)p_X(x_i)$$

- Expectation is a *property of a probability distribution*.
- **$E[f(X)]$ is *not* the value you should expect to see for any specific outcome!!**

Examples

Let $X \in \{x_1, \dots, x_n\}$ be a discrete random variable that corresponds to the number of dots shown on a fair die.

➢ Compute $E[\underline{X}^2]$.

$$E[X^2] = \sum_{i=1}^{6} \underline{x_i^2}\, p_X(x_i)$$

$$= \sum_{i=1}^{6} \frac{1}{6} i^2 = \frac{1}{6}1 + \frac{1}{6}4 + \frac{1}{6}9 + \frac{1}{6}16 + \frac{1}{6}25 + \frac{1}{6}36 = \frac{91}{6} = 15.1666$$

Trash Sorting...

Reminder:

~~Category~~	~~P(C)~~	~~Cost~~
~~cardboard~~	~~0.20~~	~~0~~
~~paper~~	~~0.30~~	~~0~~
~~can~~	~~0.25~~	~~5~~
~~scrap metal~~	~~0.20~~	~~10~~
~~bottle~~	~~0.05~~	~~3~~

We can now easily evaluate the expected cost for each action under the prior probability distribution.

$$E[f(X)] = \sum_{i=1}^{n} f(x_i)p_X(x_i)$$

COST	Card board	paper	can	scrap metal	bottle	Expected Cost	
glass bin	2	2	4	6	0	3.2	$2\times0.5 + 4\times0.25 + 6\times0.2 = 3.2$
metal bin	1	1	0	0	2	0.6	$1\times0.5 + 2\times0.05 = 0.6$
paper bin	0	0	5	10	3	3.4	$5\times0.25 + 10\times0.2 + 3\times0.05 = 3.4$
nop	1	1	1	1	1	1.0	$1\times0.5 + 1\times0.25 + 1\times0.2 + 1\times0.05 = 1.0$
$P(\omega)$	0.20	0.30	0.25	0.20	0.05		

Acting Randomly

Suppose we choose an action at random:

$$P(a_1) = P(a_2) = P(a_3) = P(a_4) = 0.25$$

What is the expected cost?

Action	Expected Cost, x_i	$p_X(x_i)$
glass bin	3.2	0.25
metal bin	0.6	0.25
paper bin	3.4	0.25
nop	1.0	0.25

Let the random variable $\mathrm{X} \in \{0.6, 1.0, 3.2, 3.4\}$ denote the expected cost for applying action a_i.

Then $p_X(x_i) = 0.25$ for each action a_i.

$$E[X] = \sum_{i=1}^{4} x_i \, p_X(x_i) = 0.25(3.2 + 0.6 + 3.4 + 1.0) = 2.05$$

Always using the metal bin (action a_2) would be a better choice than randomly choosing actions.

Simulation by sampling

Earlier, we simulated our trash sorting system using a sampling algorithm. Let's apply those ideas here.

1. Generate N samples from the prior distribution on categories.
2. Compute the cost c_i for each sample for action a_k.
3. Compute the average cost as:

$$\overline{cost_k} = \frac{1}{N}\sum_{i=1}^{N} c_i$$

4. Compare $\overline{cost_k}$ to $E[X]$for action a_k (where X is the random variable for cost).

Probability *vs* Statistics

- **Probability theory** is the study of a certain class of mathematical functions (probability distributions).

- A **statistic** is any function of data (including the identity function), and statistics is the study of such functions.

$$E[X] = \sum_{i=1}^{n} x_i \, p_X(x_i)$$

$E[X]$ is a property of $p_X(x_i)$

➢ ***Probability Theory***

$$\overline{cost_k} = \frac{1}{N} \sum_{i=1}^{N} c_i$$

$\overline{cost_k}$ is a function of data, c_i

➢ ***Statistics***

Probability Theory *and* Statistics

If it happens that certain probability distributions do a good job of describing how the world behaves, then probability theory can provide a rigorous basis for a system of inference about data.

<u>The Weak Law of Large Numbers:</u>

Consider a data set drawn from probability distribution p_X, with expected value $E[X] = \mu$.
For any $\epsilon > 0$, if $\bar{x}_N$ denotes the average of a data set of size N , then

$$\lim_{n \to \infty} P(|\bar{x}_N - \mu| < \epsilon) = 1$$

As the size of the data set increases, with probability 1 the average is arbitrarily close to the mean.

Probability Theory and Statistics

The connections between probability theory and statistics are often formalized by theorems that express variations on a simple concept:

As the size of a data set becomes large, the statistics of that data set will become increasingly good approximations for various properties of the underlying probability distribution from which the data set was generated.

- *This is one of the reasons simulation by sampling works.*
- *These theorems are important for statistical inference, machine learning, and many other problems that involve data drawn from stochastic systems.*

Probability, actions, and expectation

- Expectation of a probability distribution is computed by
$$E[X] = \sum_{i=1}^{n} x_i \, p_X(x_i)$$
- $E[X]$ is *not* the value you should expect to see for any specific outcome.

- Modeling the expected cost of the robot's actions can be used to evaluate robot performance.

- As the size of a data set becomes large, the statistics of that data set will become increasingly good approximations for various properties of the underlying probability distribution from which the data set was generated.