# Note 2. Operations on Sets and the Pigeonhole Principle

## 1 Operations on Sets

**Definition 1.1.** *The universe, denote as $U$, is the set of all elements being considered in a given context, such that every other set under discussion is a subset of $U$.*

**Definition 1.2.**

- *$A \cap B$ is the intersection of $A$ and $B$: the set containing all elements that are members of both $A$ and $B$.*
- *$A \cup B$ is the union of $A$ and $B$: the set containing all elements that are members of $A$, or $B$, or both.*
- *$A \setminus B$ is the set difference between $A$ and $B$: the set containing all elements of $A$ that are not elements of $B$.*
- *$\overline{A}$ is The complement of $A$ : the set of everything which is not an element of $A$.*

**Example:**
Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{2, 4, 6\}, C = \{1, 2, 3\}, D = \{7, 8, 9\}$. If the universe is $U = \{1, 2, \ldots, 10\}$, find:

1. $A \cup B$
2. $A \cap B$
3. $B \cap C$
4. $A \cap D$
5. $\overline{B \cup C}$
6. $A \setminus B$
7. $(D \cap \overline{C}) \cup (\overline{A \cap B})$
8. $\emptyset \cup C$
9. $\emptyset \cap C$

**Solution:**

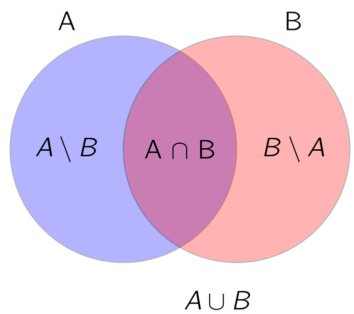
1. $A \cup B = \{1, 2, 3, 4, 5, 6\}$. Since everything in $B$ is already in $A$, the union is just $A$.
2. $A \cap B = \{2, 4, 6\}$. Since everything in $B$ is in $A$, the intersection is $B$.
3. $B \cap C = \{2\}$. The only element common to both $B$ and $C$ is 2.
4. $A \cap D = \emptyset$. $A$ and $D$ have no common elements.
5. $\overline{B \cup C} = \{5, 7, 8, 9, 10\}$.First we find that $B \cup C = \{1, 2, 3, 4, 6\}$, then we take everything not in that set.
6. $A \setminus B = \{1, 3, 5\}$. The elements $1, 3, 5$ are in $A$ but not in $B$. This is the same as $A \cap \overline{B}$.
7. $(D \cap \overline{C}) \cup (\overline{A \cap B}) = \{1, 3, 5, 7, 8, 9, 10\}$. The set contains all elements that are either in $D$ but not in $C$ (i.e., $\{7, 8, 9\}$), or not in both $A$ and $B$ (i.e., $\{1, 3, 5, 7, 8, 9, 10\}$).
8. $\emptyset \cup C = C$. The empty set does not add anything.

9. $\emptyset \cap C = \emptyset$. Nothing can be both in a set and in the empty set.

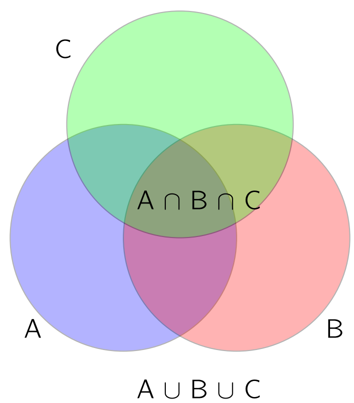
You can draw a picture!

**Sizes of sets:**

1. $$|A \cup B| = |A| + |B| - |A \cap B|.$$




2. $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$




*Proof.* Set $D = A \cup B$. Then
$$|D| = |A \cup B| = |A| + |B| - |A \cap B|,$$
and
$$|A \cup B \cup C| = |D \cup C| = |D| + |C| - |D \cap C|.$$
It remains to calculate $|D \cap C|$. We have
$$|D \cap C| = |(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|.$$
Since
$$|(A \cap C) \cap (B \cap C)| = ||A \cup B \cup C|,$$
we are now finished.

□

# 2 the Pigeonhole Principle

**Theorem 2.1** (the Pigeonhole Principle)**.** *If you must put $n+1$ pigeons into $n$ holes, then you must put two pigeons into the same hole.*

**Example 2.2.** *You have 12 pairs of socks (different colors) in a laundry bag. Drawing the socks from the bag randomly, you'll have to draw at most 13 of them to get a matched pair.*

*Proof.* Consider the 12 pairs as 12 holes. By the pigeonhole principle, you need at least 13 draws (pigeons) to ensure that at least one hole contains two pigeons. □

**Example 2.3.** *Let $S \subset Z^+$, where $|S| = 37$. Then $S$ contains two elements that have the same remainder upon division by 36*

*Proof.* Here the pigeons are the 37 positive integers in $S$. We know for any positive integer $n$, there exist a unique quotient $q$ and a remainder $r$ such that

$$n = 36q + r, \qquad 0 \le r < 36.$$

The 36 possible values of $r$ constitute the pigeonholes, and the result is now established by the pigeonhole principle. □

Generalized versions:

**Theorem 2.4.** *If you must put $2n+1$ pigeons into $n$ holes, then you must put three pigeons into the same hole.*

**Theorem 2.5.** *If you must put $kn+1$ pigeons into $n$ holes, then you must put $k+1$ pigeons into the same hole.*