Note 12. Generating Function and Rook Polynomial

1 Rook polynomial

Problem 1. *Find the rook polynomial of a* 5×5 *board.*

Idea:

$$r(C, x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_5 x^5.$$

From the setup,

$$r_0 = 1, \quad r_1 = 25, \quad r_2 = ?$$

Find r_2: r_2 is the number of ways to place 2 non-attacking rooks on the board. In coordinates, we write these as

$$\{(x_1, y_1), (x_2, y_2)\}$$

subject to $x_1 \neq x_2$ and $y_1 \neq y_2$.

Counting method 1. First count *ordered* pairs of squares:

$$\#\big((x_1, y_1), (x_2, y_2)\big) = \#\big(x_1, y_1, x_2, y_2\big) = 5 \times 5 \times 4 \times 4.$$

Since the set $\{(x_1, y_1), (x_2, y_2)\}$ is unordered, we divide by 2!:

$$r_2 = \frac{5 \times 5 \times 4 \times 4}{2!} = \frac{25 \times 16}{2} = 200.$$

Counting method 2.

1. Choose which 2 rows (out of 5) will contain the rooks: $\binom{5}{2}$ ways.
2. Having fixed those 2 rows (call them x_1 and x_2 with $x_1 < x_2$), place rooks in distinct columns.
   - For y_1 there are 5 choices,
   - For y_2 there are 4 remaining choices,

   Hence $5 \times 4 = 20$ ways to pick the columns for the two rooks.

Thus

$$r_2 = \binom{5}{2} \times 20 = 10 \times 20 = 200.$$

Next, finding r_3.
Now place 3 non-attacking rooks:

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\},$$

with distinct x_1, x_2, x_3, and distinct y_1, y_2, y_3.

1. Choose which 3 rows will be used: $\binom{5}{3}$ ways.
2. In those 3 rows, choose 3 columns. There are $P(5, 3) = 5 \times 4 \times 3$ ways to pick and order 3 distinct columns out of 5.

Hence

$$r_3 = \binom{5}{3} \times P(5,3) = 10 \times 60 = 600.$$

(Equivalently, one can count ordered triples $((x_1, y_1), (x_2, y_2), (x_3, y_3))$ and then divide by 3!, etc.)

Further terms r_4 and r_5: By the same reasoning,

$$r_4 = \binom{5}{4} P(5,4) = 600,$$

$$r_5 = \binom{5}{5} P(5,5) = 120.$$

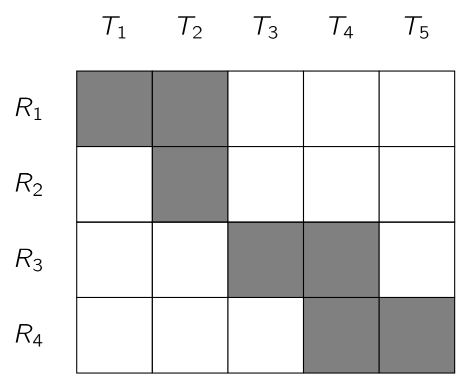
Putting it all together, the rook polynomial for the 5×5 board C is

$$r(C,x) = 1 + 25x + 200x^2 + 600x^3 + 600x^4 + 120x^5.$$

2. Rook Polynomial with Forbidden Positions

Problem 2. *Consider a 4×5 board where we wish to place 4 non-attacking rooks, but with some forbidden positions marked as follows. Count the number of valid rook arrangements.*

- *Rows:* R_1, R_2, R_3, R_4.
- *Columns:* T_1, T_2, T_3, T_4, T_5.
- *Forbidden positions:*
  - R_1 *cannot be placed in* T_1 *or* T_2.
  - R_2 *cannot be placed in* T_2.
  - R_3 *cannot be placed in* T_3 *or* T_4.
  - R_4 *cannot be placed in* T_4, T_5.



Use PIE!

Solution. Let S be the set of all unrestricted arrangements of the 4 rooks. Since each row must receive exactly one rook in a different column, the total number of ways to place the rooks without restriction is:

$$|S| = P(5,4) = 5 \times 4 \times 3 \times 2 = 120.$$

Define conditions c_i where the rook R_i is placed in its forbidden position.

By the Inclusion-Exclusion Principle, we want

$$N(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = |S| - S_1 + S_2 - S_3 + S_4.$$

or

$$N(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = |S| - \sum N(c_i) + \sum N(c_i \cap c_j) - \sum N(c_i \cap c_j \cap c_k) + N(c_1 \cap c_2 \cap c_3 \cap c_4).$$

Let us compute $N(c_1)$: $N(c_1)$ counts arrangements where R_1 is placed in T_1 or T_2. We consider two cases:

- **Case 1:** R_1 is placed in T_1. The remaining rooks R_2, R_3, R_4 are placed in the remaining 4 columns:

$$P(4,3) = 4 \times 3 \times 2 = 24.$$

- **Case 2:** R_1 is placed in T_2. The remaining rooks are placed similarly:

$$P(4,3) = 4 \times 3 \times 2 = 24.$$

Thus,

$$N(c_1) = 24 + 24 = 2 \times 24 = 48.$$

Computing $N(c_2)$:
$N(c_2)$ counts arrangements where R_2 is placed in T_2. Fixing R_2, we place R_1, R_3, R_4 in the remaining 4 columns:

$$P(4,3) = 4 \times 3 \times 2 = 24.$$

Similarly,

$$N(c_3) = 2 \times P(4,3), \quad N(c_4) = 2 \times P(4,3).$$

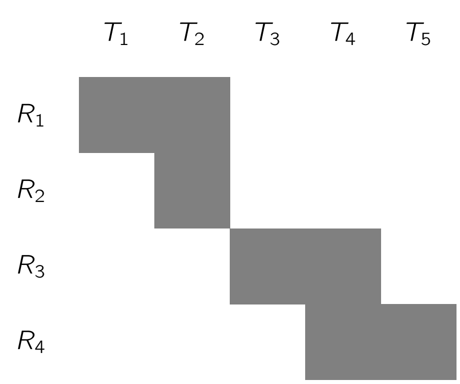
Note that

$$N(c_i) = \#\text{ways to place } R_i \text{ in its forbidden positions } \times P(4,3).$$

Summing these,

$$\begin{aligned} S_1 = \sum_{1 \le i \le 4} N(c_i) &= \left(\sum_{1 \le i \le 4} \#\text{ways to place } R_i \text{ in its forbidden positions} \right) \times P(4,3) \\ &= \#\text{ways to place one rook on the forbidden chessboard} \times P(4,3) \\ &= r_1(C_F) \times P(4,3). \end{aligned}$$

Here, C_F is the forbidden chessboard:



How about S_2? $N(c_1, c_2)$ means R_1 in T_1 or T_2, and R_2 in T_2. It implies R_1 in T_1, and R_2 in T_2. The remaining 2 rooks have $P(3,2) = 3 \times 2$ ways to be placed. Thus

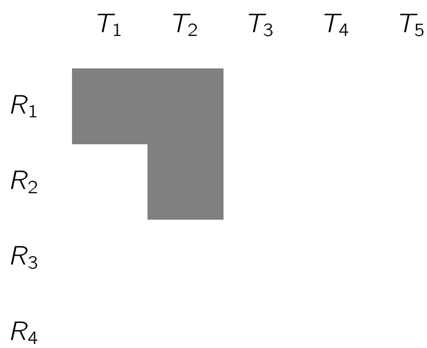
$$\begin{aligned} S_2 = \sum N(c_i c_j) &= \left(\sum \#\text{ways to place } R_i, R_j \text{ in their forbidden positions}\right) \times P(3,2) \\ &= \#\text{ways to place 2 rooks on the forbidden chessboard} \times P(3,2) \\ &= r_2(C_F) \times P(3,2). \end{aligned}$$

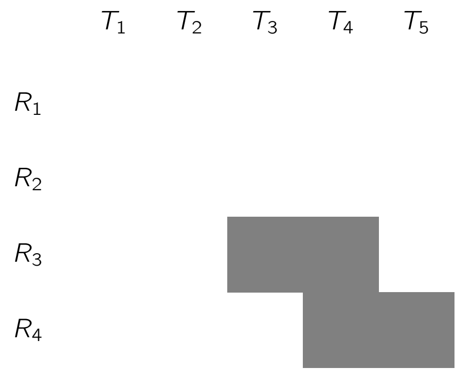
Similarly,

$$S_3 = r_3(C_F) \times P(2,1).$$

$$S_4 = r_4(C_F) \times P(1,0).$$

It remains to get the rook polynomial of C_F or $r_i(C_F)$. Note that C_F consists of two disjoint sub-boards: C_1, C_2





Thus

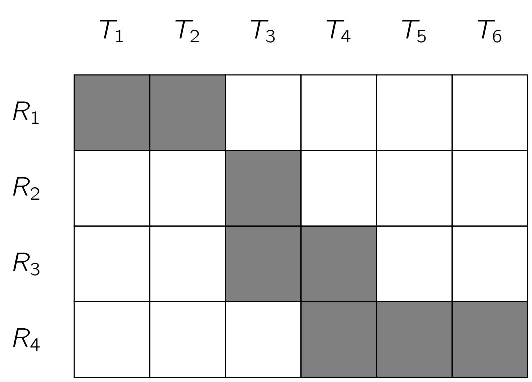
$$r(C_F, x) = r(C_1, x) r(C_2, x) = (1 + 3x + x^2)(1 + 4x + 3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4.$$

Finally, we know the totaly number of arrangements is

$$\begin{aligned} N(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) &= |S| - S_1 + S_2 - S_3 + S_4 \\ &= P(5,4) - r_1(C_F) \times P(4,3) + r_2(C_F) \times P(3,2) - r_3(C_F) \times P(2,1) + r_4(C_F) \times P(1,0) \\ &= P(5,4) - 7 \times P(4,3) + 16 \times P(3,2) - 13 \times P(2,1) + 3 \times P(1,0) = 25. \end{aligned}$$

$\square$

Problem 3. *Let $X = \{1, 2, 3, 4\}$, $Y = \{A, B, C, D, E, F\}$. How many 1-to-1 functions are there from $X \to Y$ so none of the following forbidden positions:*



Solution. Let S be the set of all 1-1 functions (like 4 rooks). Define conditions c_i where the function map i to its forbidden position. Then the number of valid functions is $N(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$.

Obviously,

$$|S| = P(6, 4).$$

Using the same reasoning as in the previous problem, we have

$$S_1 = r_1(C_F) \times P(5, 3),$$

where C_F is the forbidden chessboard. Similarly

$$S_2 = r_2(C_F) \times P(4, 2),$$

$$S_3 = r_3(C_F) \times P(3, 2),$$

$$S_4 = r_4(C_F) \times P(2, 0).$$

Next, computing the rook polynomial

$$r(C_F, x) = r(C_1, x)r(C_2, x) = \cdots$$

□

2 Generating function–closed form

Recall Maclaurin series expansions:

For any $m, n \in \mathbb{Z}$, $a \in \mathbb{R}$,

1. $$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

2. $$(1 + ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$$

3. $$(1 + x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{mn}$$

4. $$\frac{(1 - x^{n+1})}{1 - x} = 1 + x + x^2 + \cdots + x^n$$

5. $$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

6. $$\begin{aligned}\frac{1}{(1 + x)^n} &= \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots = \sum_{k=0}^{\infty} \binom{-n}{k}x^k \\ &= \sum_{k=0}^{\infty}(-1)^k\binom{n + k - 1}{k}x^k\end{aligned}$$

7.
$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

Here,

let $n, r \in \mathbb{Z}^+$ and $n \geq r > 0$, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{[n(n-1)(n-2)\cdots(n-r+1)]}{r!}.$$

If $n \in \mathbb{Z}^+$, we have

$$\begin{aligned}\binom{-n}{r} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!} \\ &= \frac{(-1)^r n(n+1)(n+2)\cdots(n+r-1)}{r!} \\ &= \frac{(-1)^r (n+r-1)!}{(n-1)!r!} = (-1)^r \binom{n+r-1}{r}.\end{aligned}$$

Problem 4. *Given $y_1 + y_2 + \cdots + y_n = k$, where $y_i \geq 0$, how many integer solutions are there?*

Solution. The generating function for the equation is

$$f(x) = \left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{i=0}^{\infty} x^i\right)\cdots\left(\sum_{i=0}^{\infty} x^i\right) = \left(\sum_{i=0}^{\infty} x^i\right)^n.$$

Then the number of solutions is the coefficient of x^k,

Since $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, we have

$$f(x) = \left(\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i.$$

Thus the coefficient of x^k is $\binom{n+k-1}{k}$. (Check this with stars-and-bars theorem) □