# Note 8. Principle of Inclusion and Exclusion

## 1 Warm-up problems

**Problem 1.** *Three kids, Alberto, Bernadette, and Carlos, decide to share 11 cookies. They wonder how many ways they could split the cookies up provided that none of them receive more than 4 cookies (someone receiving no cookies is for some reason acceptable to these kids).*

*Solution.* (You can rephrase the problem as follows: Find the number of integer solutions to the equation $x_1+x_2+x_3 = 11$, where $0 \le x_i \le 4$ for $i \in \{1,2,3\}$.)

Without the "no more than 4" restriction, the answer would be

$$\binom{13}{2}$$

using 11 stars and 2 bars (separating the three kids).

Now count the number of ways that one or more of the kids violates the condition, i.e., gets at least 5 cookies.

Let $A$ be the set of outcomes in which Alberto gets more than 4 cookies. Let $B$ be the set of outcomes in which Bernadette gets more than 4 cookies. Let $C$ be the set of outcomes in which Carlos gets more than 4 cookies.

We are looking (for the sake of subtraction) for the size of the set $A \cup B \cup C$. Using the Principle of Inclusion-Exclusion (PIE), we must find the sizes of $|A|, |B|, |C|, |A \cap B|$ and so on. Here is what we find:

- $|A| = \binom{8}{2}$. First give Alberto 5 cookies, then distribute the remaining 6 to the three kids without restrictions, using 6 stars and 2 bars.
- $|B| = \binom{8}{2}$. Just like above, only now Bernadette gets 5 cookies at the start.
- $|C| = \binom{8}{2}$. Carlos gets 5 cookies first.
- $|A \cap B| = \binom{3}{2}$. Give Alberto and Bernadette 5 cookies each, leaving 1 star to distribute to the three kids (2 bars).
- $|A \cap C| = \binom{3}{2}$. Alberto and Carlos get 5 cookies first.
- $|B \cap C| = \binom{3}{2}$. Bernadette and Carlos get 5 cookies first.
- $|A \cap B \cap C| = 0$. It is not possible for all three kids to get 4 or more cookies.

Combining all of these, we see

$$|A \cup B \cup C| = \binom{8}{2} + \binom{8}{2} + \binom{8}{2} - \binom{3}{2} - \binom{3}{2} - \binom{3}{2} + 0 = 75.$$

Thus, the answer to the original question is

$$\binom{13}{2} - 75 = 78 - 75 = 3.$$

This makes sense now that we see it. The only way to ensure that no kid gets more than 4 cookies is to give two kids 4 cookies and one kid 3; there are three choices for which kid that should be. ☐

**Problem 2.** *Let $X$ and $Y$ be sets, with $|X| = k$ and $|Y| = n$.*

1. *How many functions are there from $X$ to $Y$?*

2. *How many one-to-one functions are there from $X$ to $Y$?*
3. *How many onto functions are there from $X \to Y$?*

*Solution.* 1. **How many functions are there from $X$ to $Y$?**

For each $x \in X$, we can choose any of the $n$ elements of $Y$ to map it to. Thus, there are $n^k$ possibilities.

2. **How many one-to-one functions are there from $X$ to $Y$?**

- We need $n \geq k$.
- Think of permutations! First $x$: $n$ choices. Second $x$: $n-1$ choices. ...
- This is essentially a length $k$ permutation on $n$ elements, so there are

$$P(n,k) = \frac{n!}{(n-k)!}$$

options.

3. **How many onto functions are there from $X \to Y$?**

**Condition:** $k \geq n$ for an onto function to exist.

**Idea:** Can we count the functions that are not onto?

$\square$

# 2 Principle of Inclusion/Exclusion

## Notation:

Let $A$ be a set, and for each $i \in n$, let $c_i$ be a condition that could be satisfied by the elements of $A$. Let $A_i$ be the subset of $A$ of elements satisfying $c_i$, i.e.,

$$A_i \subseteq A \quad \text{and} \quad A_i = \{a \in A \mid a \text{ satisfies condition } c_i\}.$$

$$|A_i| = N(c_i) = \# \text{ of elements satisfying } c_i$$

$$N(\bar{c}_i) = |A| - |A_i| = \# \text{ of elements not satisfying } c_i$$

We can combine these: For example, $N(c_1, c_2, \bar{c}_3)$ is the number of elements satisfying $c_1$ and $c_2$, but not $c_3$.

**Theorem 2.1** (The Principle of Inclusion and Exclusion)**.** *Let $A$ be a set with $|A| = N$ and suppose we have $n$ conditions $c_i$, satisfied (respectively) by some of the elements of $A$.*

*The number of elements in $A$ satisfying* **none** *of the conditions $c_i$ is:*

$$N(\bar{c}_1\bar{c}_2 \ldots \bar{c}_n) = N - \sum_{1 \leq i \leq n} N(c_i) + \sum_{1 \leq i < j \leq n} N(c_i, c_j) - \cdots + (-1)^n N(c_1, c_2, \ldots, c_n)$$

We will skip the proof here, as it will be provided next week. Instead, let's focus on how to apply it

**How many onto functions are there from $X \to Y$?**

Let $A$ be the set of functions from $X$ to $Y$ (then $|A| = n^k$). Suppose $Y = \{y_1, y_2, \ldots, y_n\}$, then define for each $i \in n$ the condition $c_i$ to be that $y_i$ is not in the range of $f$.

A function is onto if and only if it satisfies none of these conditions, so the number of onto functions is:

$$\# \text{ onto functions} = N(\bar{c}_1\bar{c}_2 \ldots \bar{c}_n).$$

**Note:** This problem is nice because $N(c_i) = N(c_j)$ and $N(c_i c_j) = N(c_m c_l)$, etc., so there are fewer calculations.
1. Condition: $N(c_i)$ is the number of functions that don't have $y_i$ in their range:

$$N(c_i) = (n-1)^k$$

2. Condition:

$$N(c_i c_j) = (n-2)^k \quad \text{for any } i, j$$

...
The number of functions that exclude $r$ elements from their range is $(n-r)^k$.

$$N(\bar{c}_1 \bar{c}_2 \dots \bar{c}_n) = |A| - \sum_{i \in S_n} (n-1)^k + \sum_{1 \le i < j \le n} (n-2)^k + \dots + (-1)^n (n-n)^k$$

Since this does not depend on $i$ due to symmetry:

$$= n^k - \binom{n}{1}(n-1)^k + \binom{n}{2}(n-2)^k - \dots + (-1)^n \binom{n}{n}(n-n)^k$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$$

This is the number of onto functions $X \to Y$, which is also the number of ways to group $k$ distinct objects into $n$ (non-empty) distinct sets. What if the sets weren't labeled?

We know:
- $k$ distinct objects $\to$ $n$ non-empty labeled groups

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$$

- $n!$ ways to label $n$ groups.
- So the number of ways to put $k$ unique objects into $n$ non-empty indistinguishable groups is

$$\frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$$

See **Note 6-Problem 1-Solution 2**