Note 9. Principle of Inclusion and Exclusion

Take Attendance Today!

1 PIE

Theorem 1.1 (The Principle of Inclusion and Exclusion)**.** *Let S be a set with $|S| = N$ and suppose we have n conditions c_i, satisfied (respectively) by some of the elements of S.*

The number of elements in S satisfying **none** *of the conditions c_i is:*

$$N(\bar{c}_1\bar{c}_2 \dots \bar{c}_n) = N - \sum_{1\le i\le n} N(c_i) + \sum_{1\le i<j\le n} N(c_i, c_j) - \cdots + (-1)^n N(c_1, c_2, \dots, c_n) \tag{1}$$

or PIE on sets:

Theorem 1.2 (The Principle of Inclusion and Exclusion)**.** *Let S be a set with $|S| = N$ and suppose we have n conditions subsets $A_i \subseteq S$. Then*

$$|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| = N - \sum_{1\le i\le n} |A_i| + \sum_{1\le i<j\le n} |A_i \cap A_j| - \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_i \cap A_k \cap \cdots \cap A_n| \tag{2}$$

Recall:

$$\left|\overline{A_1 \cup A_2 \cup A_3}\right| = N - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|.$$

Proof. Although this result can be established by induction on t, we give a combinatorial argument here.

For each $x \in S$ we show that x contributes the same count, either 0 or 1, to each side of Eq (2).

If x satisfies none of the conditions, then x is counted once in $\overline{A_1 \cup A_2 \cup \cdots \cup A_n}$ and once in N, but not in any of the other terms in Eq (2). Consequently, x contributes a count of 1 to each side of the equation.

The other possibility is that x satisfies exactly r of the n sets. (For example $A_1, A_2, \cdots, A_r$). In this case, x contributes nothing to $\overline{N}$. But on the right-hand side of Eq. (2), x is counted

- One time in N.
- r times in $\sum_{1\le i\le t} |A_i|$. (Once for each of the $A_1, A_2, \cdots, A_r$.)
- $\binom{r}{2}$ times in $\sum_{0\le i<j\le n} |A_i \cap A_j|$. (Once for each pair of $A_1, A_2, \cdots, A_r$.)
- $\binom{r}{3}$ times in $\sum_{0\le i<j<k\le n} |A_i \cap A_j \cap A_k|$. (Once for each triple of $A_1, A_2, \cdots, A_r$.)
- $\vdots$
- $\binom{r}{r}$ times in $\sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}|$, where the summation is taken over all selections of size r from the n subsets.

Consequently, on the right-hand side of Eq. (2), x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0$$

times, by the binomial theorem. Therefore, the two sides of Eq. (2) count the same elements of S, and the equality is verified.

□

Problem 1. *Find the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 18$, where $0 \le x_i \le 7$, for all $1 \le i \le 4$.*

Idea:

- **Step 1: Statement** Use sets or conditions
- **Step 2: Calculation** PIE

Solution 1. Let S be the set of solutions of $x_1 + x_2 + x_3 + x_4 = 18$, with $0 \le x_i$ for all $1 \le i \le 4$. So $|S| = N = \binom{4+18-1}{18} = \binom{21}{18}$.

We say that a solution x_1, x_2, x_3, x_4 satisfies condition c_i, where $1 \le i \le 4$, if $x_i > 7$. The answer to the problem is $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$.

Here by symmetry $N(c_1) = N(c_2) = N(c_3) = N(c_4)$. To compute $N(c_1)$, we consider the integer solutions for $x_1 + x_2 + x_3 + x_4 = 10$, with each $x_i \ge 0$ for all $1 \le i \le 4$. Then we add 8 to the value of x_1 and get the solutions of $x_1 + x_2 + x_3 + x_4 = 18$ that satisfy condition c_1. Hence $N(c_i) = \binom{4+10-1}{10} = \binom{13}{10}$, for each $1 \le i \le 4$.

Likewise, $N(c_1c_2)$ is the number of integer solutions of $x_1 + x_2 + x_3 + x_4 = 2$, where $x_i \ge 0$ for all $1 \le i \le 4$. So $N(c_1c_2) = \binom{4+2-1}{2} = \binom{5}{2}$.

Since $N(c_ic_jc_k) = 0$ for any selection of three conditions, and $N(c_1c_2c_3c_4) = 0$, we have

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = N - \sum_{1\le i\le 4} N(c_i) + \sum_{1\le i<j\le 4} N(c_i, c_j) - \cdots + N(c_1, c_2, c_3, c_4) = \binom{21}{18} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\binom{5}{2} - 0 + 0 = 246.$$

□

or we can use sets:

Solution 2. Let S be the set of solutions of $x_1 + x_2 + x_3 + x_4 = 18$, with $0 \le x_i$ for all $1 \le i \le 4$. So $|S| = N = \binom{4+18-1}{18} = \binom{21}{18}$. Let $A_i \subseteq S$ be the set of solutions such that $x_i > 7$. The answer to the problem is $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}|$.

Here by symmetry $|A_1| = |A_2| = |A_3| = |A_4|$. (Similarly to the above, we omit the details). Note that $|A_i| = \binom{13}{10}$ for $0 \le i \le 4$.

Note that $|A_i \cap A_j| = \binom{13}{10}$ for $0 \le i < j \le 4$. And the intersecting of any three sets is empty. Thus we have

$$|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = N - \sum_{1\le i\le 4} |A_i| + \sum_{1\le i<j\le 4} |A_i \cap A_j| - \sum_{1\le i<j<k\le 4} |A_i \cap A_j \cap A_k| + \cdots + |A_1 \cap A_2 \cap A_3 \cap A_4| = 246.$$

□

Problem 2. *Determine the number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3, or 5.*

Idea:

- **Step 1: Statement** Use sets or conditions
- **Step 2: Calculation** PIE

Solution. Here $S = \{1, 2, 3, \ldots, 100\}$ and $N = 100$.

- Let $A_1 \subseteq S$ be the set of integers divisible by 2, or condition c_1
- Let $A_2 \subseteq S$ be the set of integers divisible by 3, or condition c_2
- Let $A_3 \subseteq S$ be the set of integers divisible by 5, or condition c_3

Then the answer to this problem is $|\overline{A_1 \cup A_2 \cup A_3}|$.

$$|A_1| = \left\lfloor \frac{100}{2} \right\rfloor = 50 \quad \text{[since the 50 positive integers } 2, 4, 6, \ldots, 100 \text{ are divisible by 2];}$$

$$|A_2| = \left\lfloor \frac{100}{3} \right\rfloor = 33 \quad \text{[since the 33 positive integers } 3, 6, 9, \ldots, 99 \text{ are divisible by 3];}$$

$$|A_3| = \left\lfloor \frac{100}{5} \right\rfloor = 20;$$

$$|A_1 \cap A_2| = \left\lfloor \frac{100}{6} \right\rfloor = 16 \quad \text{[since there are 16 elements in } S \text{ divisible by } \operatorname{lcm}(2,3) = 6\text{];}$$

$$|A_1 \cap A_3| = \left\lfloor \frac{100}{10} \right\rfloor = 10;$$

$$|A_2 \cap A_3| = \left\lfloor \frac{100}{15} \right\rfloor = 6;$$

and,

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{100}{30} \right\rfloor = 3.$$

Applying the inclusion-exclusion principle, we find that

$$\begin{aligned}\left|\overline{A_1 \cup A_2 \cup A_3}\right| &= N - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|. \\ &= 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.\end{aligned}$$

□

2 Generalizations of the Principle

What if we want the integers n where $1 \le n \le 100$ and n is divisible by exactly one of 2, 3, 5? The answer is

$$N(c_1, \overline{c_2}, \overline{c_3}) + N(\overline{c_1}, c_2, \overline{c_3}) + N(\overline{c_1}, \overline{c_2}, c_3). \tag{3}$$

Theorem 2.1. *For each $1 \le m \le n$, the number of elements in S that satisfy exactly m of the conditions $c_1, c_2, \ldots, c_n$ is given by*

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{n-m} \binom{n}{n-m} S_n,$$

where

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} N(c_{i_1} c_{i_2} \ldots c_{i_k}).$$

So (3) is

$$E_1 = S_1 - 2S_2 + 3S_3 = \cdots$$

Problem 3. *Let $X = \{x_1, x_2, \ldots, x_{10}\}$ and $Y = \{y_1, y_2, \ldots, y_7\}$. How many functions $f : X \to Y$ have exactly 4 elements in their range?*

Solution. Let S be the set of all functions $f : X \to Y$. Define conditions

$$c_i : y_i \text{ is } \textbf{not} \text{ in the range of } f \text{ (for } f \in S).$$

Then any function with exactly 4 elements in its range satisfies exactly 3 conditions. (why?) Thus our answer is E_3.

$$E_3 = S_3 - \binom{4}{1}S_4 + \binom{5}{2}S_5 - \binom{6}{3}S_6 + \binom{7}{4}S_7$$

Note that for any $i, j, k \leq 7$, $N(c_i, c_j, c_k)$ is the number of functions $f : X \to Y \backslash \{y_i, y_j, y_k\}$. Then $N(c_i, c_j, c_k) = 4^{10}$. So

$$S_3 = \binom{7}{3}N(c_i, c_j, c_k) = \binom{7}{3}4^{10}$$

Here, $\binom{7}{3}$: ways to choose i, j, k .

Similarly,

$$S_4 = \binom{7}{4}3^{10}$$

$$S_5 = \binom{7}{5}2^{10}$$

$$S_6 = \binom{7}{6}1^{10} = 7$$

$$S_7 = \binom{7}{7}(7-7)^{10} = 0$$

Plugging these values in,

$$E_3 = \left[\binom{7}{3}4^{10}\right] - \left[\binom{4}{1}\binom{7}{4}3^{10}\right] + \left[\binom{5}{2}\binom{7}{5}2^{10}\right] - \left[\binom{6}{3}\binom{7}{6}7\right] + \left[\binom{7}{4}0\right]$$

□