Note 4. Counting I

1 Rules of Sum and Production

Problem 1. *At our school, there are two teams: the red team with 5 members and the blue team with 7 members. Each team will select one member to complete for the position of president of the school board.*

1. *How many possibilities are there for an eventual winner?*
2. *How many different pairs of candidates can be chosen?*

Solution:

1. $5 + 7 = 12$;
2. $5 * 7 = 35$.

$\square$

Let us generalize:

Theorem 1.1. *If task A can be performed in m ways, and task B can be performed in n ways. The two tasks can not be performed simultaneously. Then*

- *(The Rule of Sum) Performing A or B can be accomplished in $m + n$ ways.*
- *(The Rule of Product) Performing A and B can be accomplished in mn ways.*

Consider two disjoint sets A and B.

- The Rule of Sum means $|A \cup B| = |A| + |B|$.
- The Rule of Product means the ordered set $A \times B$ has size $|A \times B| = |A||B|$

Problem 2. *# is commonly used to indicate the number of objects.*

1. *# relations from A to B?*
2. *# functions from A to B?*

Solution: Using the rule of product!

1. Let R be a relation from A to B. Then $R \subseteq A \times B$. So # relations $= 2^{|}A||B|$.
2. $|B|$ choices for each element in A. So # functions $= |B|^{|A|}$.

$\square$

Problem 3. *How many license plates are there of the form 2 numbers (0 9) followed by 4 letters.*

Solution: $10 \times 10 \times 26 \times 26 \times 26 \times 26 = 10^2 26^4$. $\square$

Problem 4. *In a class of 10 students, five are to be chosen and stated in a row. How many linear arrangements are possible?*

Solution: $10 * 9 * 8 * 7 * 6$. $\square$

Definition 1.2. *If n is a non-negative integer, n factorial is*

$$n! = \begin{cases} 1 & if n = 0 \\ n(n-1)(n-2)\cdots 1 & if n \geq 1 \end{cases}$$

Definition 1.3. *Given a collection of n distinct objects, any (linear) arrangement of these objects is called a permutation of the collection.*

Let $P(n, r)$ be the number of permutations of size r on n distinct objects. Then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Note that $P(n, r)$ are good at counting situations where order matters. If order does not matter, use combination.

Definition 1.4. *Given a set of n distinct items, a combination is a subset of r items chosen from the set without regard to the order of the items.*

The number of combinations of r items from n items is denoted as

$$C(n, r) \text{ or } \binom{n}{r}.$$

And is calculated using the formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}. \tag{1}$$

Proof of (1). Find a map/function between permutations and combinations. Then we can get

$$P(n, r) = C(n, r)P(r, r).$$

□

Problem 5. *Determine the number of arrangements of letters:*

1. *computer (obviously, $P(8, 8) = 8!$.)*
2. *pepper*

Solution 1: The letter p appears 3 times. The letter e appears 2 times. The letter r appears 1 time. The number of distinct arrangements is (by mapping the full permutations to these arrangements.)

$$\frac{6!}{3!2!1!} = 60.$$

□

Solution 2: The letter p appears 3 times. The letter e appears 2 times. The letter r appears 1 time. We first choose 3 positions from the 6 available positions for the letter p. The number of ways is $\binom{6}{3}$. After placing the 3 p's, there are 3 remaining positions. We now choose 2 of these 3 remaining positions for the e's. The number of ways to choose 2 positions from 3 is $\binom{3}{2}$. After placing the p's and e's, there is only 1 remaining position for the letter r. There is exactly 1 way to place r in this position. Now, to find the total number of distinct permutations is

$$\binom{6}{3}\binom{3}{2}\binom{1}{1} = 60.$$

□