Note 1. Set Theory and Functions

Definition 0.1 *A set is a collection of objects that can be written in any order.*

Example:

$$A = \{1, 2, 3, 4, 5\}.$$
$$B = \{2, 1, 5, 4, 3\}.$$
$$C = \{x | x \text{ is an integer and } 1 \leq x \leq 5\},$$

or

$$C = \{x : x \text{ is an integer and } 1 \leq x \leq 5\}.$$

We have

$$A = B = C.$$

Notation:

- If x is an element of a set S, we write $x \in S$.
- If not, $x \notin S$.
- The empty set $\{\} = \emptyset$ has no elements.
- The **cardinality** (or the **size**) of S, written $|S|$, is the number of elements in S.

Definition 0.2 *A set B is a subset of a set A, $B \subseteq A$, if $x \in A$ for each $x \in B$.*

Notation:

- $B \subseteq A:$ B is a subset of A.
- $B \subset A:$ B is a proper subset of A, which means $B \neq A$.
- $B \not\subset A:$ B is not a subset of A.
- $A \supseteq B:$ B is a subset of A, or A contains B as a subset.

Definition 0.3 *A relation R on sets X and Y is a set of ordered pairs (x, y) where $x \in X$ and $y \in Y$.*

Properties of Relations:

- A relation R is called **reflexive** on a set S if for all $x \in S$, $(x, x) \in R$.
- A relation R is called **irreflexive** on a set S if for all $x \in S$, $(x, x) \notin R$.
- A relation R is **symmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ then $(y, x) \in R$.
- A relation R is **antisymmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.
- A relation R is **transitive** on a set S if for all $x, y, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Definition 0.4 *The Cartesion Product $X \times Y$ of sets X and Y is the relation consisting of all ordered pairs (x, y).*

$$X \times Y = \{(x, y) | x \in X \text{ and } y \in Y\}.$$

Note that every relation R on X and Y has $R \subseteq X \times Y$.
Example:
Let $A = \{2,3,4\}$ and $B = \{4,5\}$. Find $A \times B$, $B \times A$, $B^2 = B \times B$, And the sizes of them.

$$A \times B = \{(2,4),(2,5),(3,4),(3,5),(4,4),(4,5)\}.$$
$$B \times A = \{(4,2),(5,2),(4,3),(5,3),(4,4),(5,4),\}$$

,

$$B^2 = B \times B = \{(4,4),(4,5),(5,4),(5,5)\}.$$
$$|A \times B| = |A| \times |B| = |B \times A|.$$

Definition 0.5 *A relation f from X to Y is a function if each element of X appears in exactly one ordered pair of f.*

- *The domain of f is X,*
- *The codomain of f is Y,*
- *The range of f is the subset of Y containing the elements of Y that appear in ordered pairs of f.*
- *If $(x,y) \in f$, we say that f maps x to y.*

Properties:

- A function $f : X \to Y$ is **injective** (or one-to-one) if each element of the range appears in exactly one ordered pair of f.
- A function $f : X \to Y$ is **surjective** (or onto) if the codomain and range are equal.
- f is **bijective** if it's one-to-one and onto.