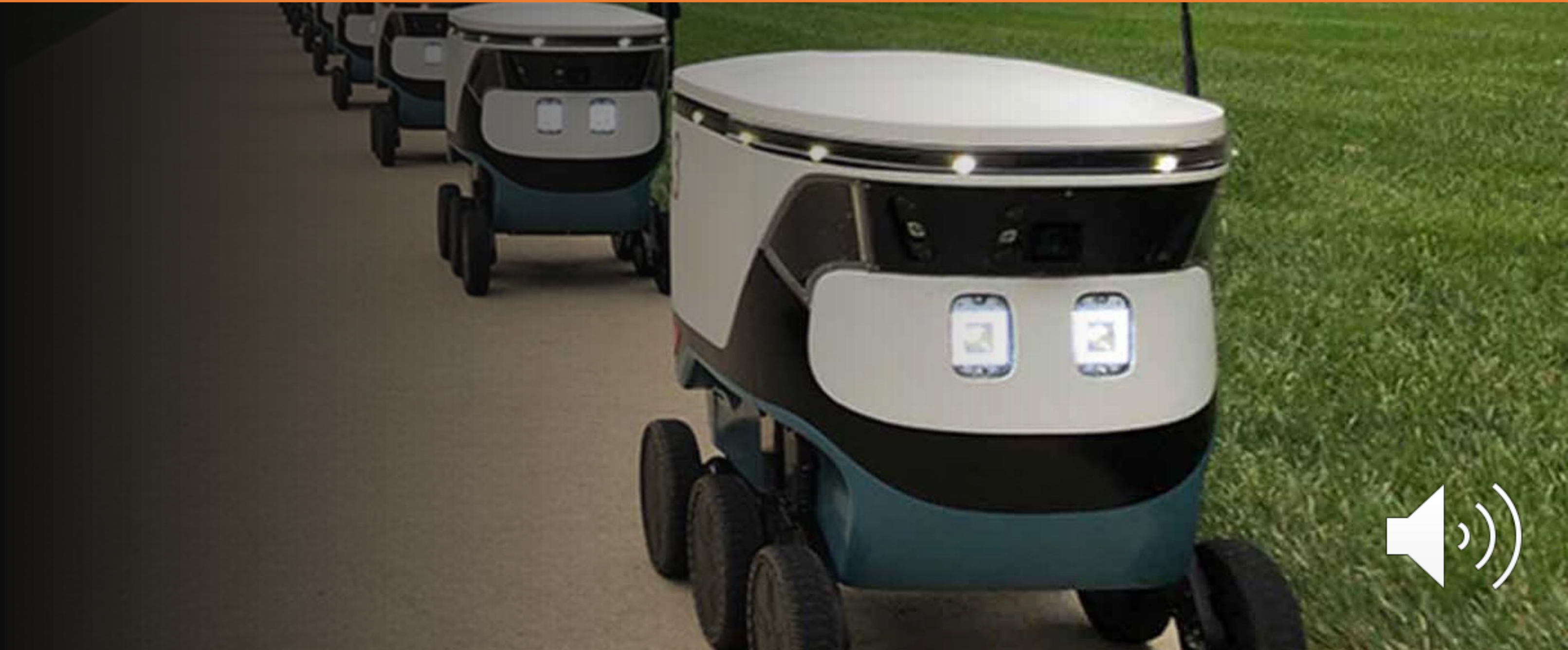
Joint and Conditional Probability



CS 3630

Some concepts from probability theory

Before we begin describing sensors and how we model uncertainty in sensing, we’ll need a few new concepts from probability theory:

- Joint Distributions
- Conditional Probability
- Independence

We’ll introduce these concepts with simple examples before describing how to model the sensors for our trash sorting robot.

Joint Probability

Consider two events, $X, Y \subset \Omega$. The joint probability of $X\ and\ Y$ is the probability that both events occur.

- When we talk about a joint probability distribution, we use the notation $\mathrm{P}(X, Y)$, indicating that $X\ and\ Y$ are random events.

- When we talk about the joint probability for two specific events, we write

$$P(X = x \text{ and } Y = y) = P(x, y)$$

✓ Recall, upper case denotes a random event, and lower case denotes a specific value.

An Example

Roll two dice, observe x_1 and x_2.

We know that there are 36 possible outcomes, all of which are equally likely (assuming the dice are *fair*).

It's easy to compute probabilities by simply counting outcomes:

- Probability $x_1 = 6$:

$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \rightarrow P = \frac{6}{36} = \frac{1}{6}$$

- Probability x_1 is even:

$$\begin{matrix} (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{matrix} \quad \rightarrow \quad P = \frac{18}{36} = \frac{1}{2}$$

An Example

Roll two dice, observe x_1 and x_2.

Now suppose we want to know the probability that two events occur.

Again, we can compute probabilities simply by counting outcomes (since all outcomes are equally probable).

- Probability $x_1 = 6$ ***<u>and</u>*** x_2 is even:

$$(6,2),(6,4),(6,6) \rightarrow P = \frac{3}{36} = \frac{1}{12}$$

- Probability x_1 is even and $x_1 > 3$:

(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) $\rightarrow \quad P = \frac{12}{36} = \frac{1}{3}$

Conditional Probability

- When two events are related to one another, observing the occurrence of one of the events can influence what we believe about the other.
- In this case, we talk about the conditional probability of x ***given*** y, denoted $P(x \mid y)$.

- This conditional probability is defined in terms of the joint probability of $x \ and \ y$:

$$P(x \mid y) = \frac{P(x, y)}{P(y)}$$

Assuming $P(y) \neq 0$

- *We can rewrite this expression as:*

$$P(x, y) = P(x \mid y)\, P(y)$$

This form will come in handy a bit later in the class

Independence

- If X and Y are independent, then

$$P(x,y) = P(x)P(y)$$

Definition of Independence

- If X and Y are independent, then

$$P(x \mid y) = \underbrace{\frac{P(x,y)}{P(y)}}_{\text{From previous slide}} = \underbrace{\frac{P(x)P(y)}{P(y)}}_{\text{By substitution from above}} = P(x)$$

Let's apply rules of conditional and joint probabilities:

$$P(x \mid y) = \frac{P(x,y)}{P(y)}$$

$$P(x,y) = P(x \mid y)\, P(y)$$

If x and y are independent:

$$P(x,y) = P(x)P(y)$$
$$P(x \mid y) = P(x)$$

Define events:

A: x_1 is even $\quad$ B: $\mathrm{x}_1 = 6$ $\quad$ C: x_2 is even $\quad$ D: $\mathrm{x}_2 = 5$

From the previous examples, we easily compute the following:

$$P(A) = \frac{1}{2}, \qquad P(B) = \frac{1}{6}, \qquad P(C) = \frac{1}{2}, \qquad P(D) = \frac{1}{6}.$$

Let's look at some combinations of events:

- A and B are not independent

$$P(A,B) = \frac{1}{6} \qquad \neq \qquad P(A)P(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$$

- A and C are independent

$$P(A,C) = \frac{9}{36} = \frac{1}{4} \qquad = \qquad P(A)P(C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Let's apply rules of conditional and joint probabilities:

Define events:

A: x_1 is even $\quad\quad$ B: $\mathrm{x}_1 = 6$

- $P(B|\underline{A}) = \frac{P(A,B)}{P(A)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$

This agrees with our intuition, since $\mathrm{x}_1 = 6$ in one third of the cases of x_1 being even:

(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)

$$P(x \mid y) = \frac{P(x,y)}{P(y)}$$

$$P(x,y) = P(x \mid y)\, P(y)$$

If x and y are independent:

$$P(x,y) = P(x)P(y)$$
$$P(x \mid y) = P(x)$$

Independence

- If X and Y are independent, then

$$P(x, y) = P(x)P(y)$$

- If X and Y are independent, then

$$P(x \mid y) = \frac{P(x,y)}{P(y)} = \frac{P(x)P(y)}{P(y)} = P(x)$$

- *Sensors are useful because their measurements depend on the world state.*
- *However, if we have multiple sensors, **quite often there are independence properties for various combinations of sensors**.*
    - *E.g., a color sensor might give a measurement that is independent of the weight sensor*