Bayes Filter

CS 3630

Bayes Filters: Framework

- Let x be the state of the robot (e.g. its location)
- **Given:**
  - Stream of observations z and action data u:

$$d_t = \{u_1, z_2 \ldots, u_{t-1}, z_t\}$$

  - Sensor model $P(z|x)$.
  - Action model $P(x_t|u, x_{t-1})$.
  - Prior probability of the system state $P(x)$.
- **Wanted:**
  - Estimate of the state X of a dynamical system.
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_2 \ldots, u_{t-1}, z_t)$$

Bayes Filters

z = observation
u = action
x = state

$$\boxed{Bel(x_t) =} P(x_t \mid u_1, z_2 \ldots, u_{t-1}, z_t)$$

$$= \eta \; P(z_t \mid x_t, u_1, z_2, \ldots, u_{t-1}) \; P(x_t \mid u_1, z_2, \ldots, u_{t-1})$$ **Bayes**

$$= \eta \; P(z_t \mid x_t) \; P(x_t \mid u_1, z_2, \ldots, u_{t-1})$$ **Measurement Model**

$$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_1, z_2, \ldots, u_{t-1}, x_{t-1}) \; P(x_{t-1} \mid u_1, z_2, \ldots, u_{t-1}) \; dx_{t-1}$$ **Total prob.**

$$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1}) \; P(x_{t-1} \mid u_1, z_2, \ldots, u_{t-1}) \; dx_{t-1}$$

Markov

$$= \boxed{\eta \; P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}}$$

Bayes Filters

Belief that robot is in state $X = x_t$ at time step t

If I was in state x_{t-1} and I executed action u_{t-1} what is the probability that I arrive to state x_t

Weight this probability by the belief that I was actually in state x_{t-1}

$$Bel(x_t) = \eta\, P(z_t|x_t) \int P(x_t|u_{t-1}, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

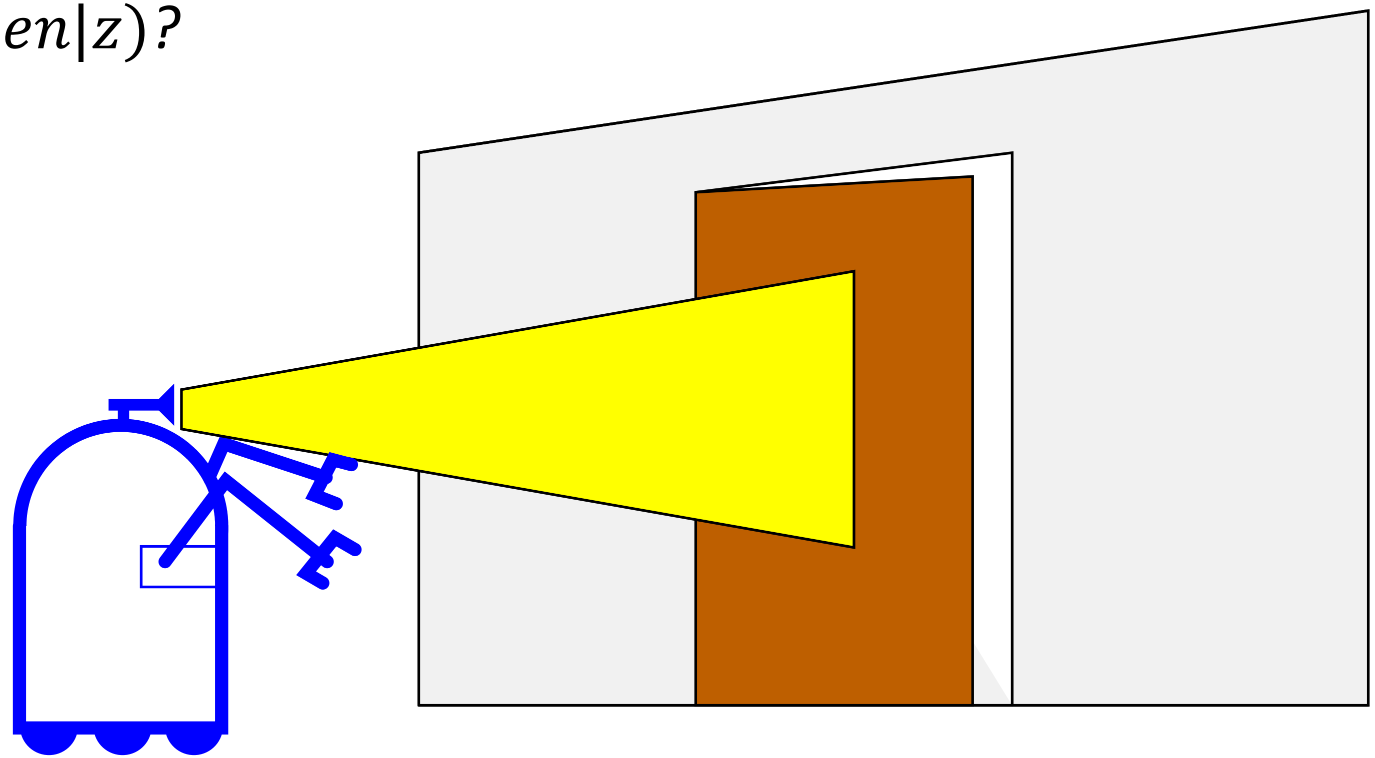
If I'm in state x_t, what is the probability I see observation z_t

Integrate over all possible previous states, x_{t-1}

z = observation
u = action
x = state

Simple Example of State Estimation

- Suppose a robot obtains measurement z *(e.g., distance sensor reports an obstacle 40cm in front of the robot)*
- What is $P(open|z)$?



Causal vs. Diagnostic Reasoning

- $P(open|z)$ is diagnostic.
- $P(z|open)$ is causal.
- Often causal knowledge is easier to obtain*
- Bayes rule allows us to use causal knowledge:

Comes from sensor model.

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}$$

Use law of total probability: $P(z) = \sum_y P(z|y)P(y)$

* If we understand the underlying causal relationship

Example

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}$$

For some observed z value:

- $P(z|open) = 0.6 \qquad P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\,p(open) + P(z \mid \neg open)\,p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

Here, observing z has raised the probability that the door is open.

Normalization Coefficient

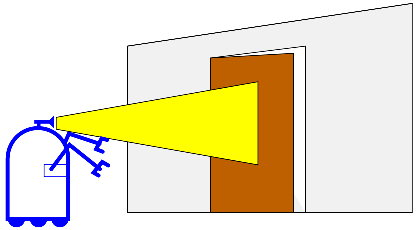
$$P(x|z) = \frac{P(z|x)P(x)}{P(z)}$$

Note that **the denominator is independent of $\boldsymbol{x}$,** and as a result will be the same for any value x in the posterior $P(x|z)$. As a result, typically this term is replaced by the coefficient η, as in:

$$P(x|z) = \eta P(z|x)P(x)$$

as a reminder that the result must be normalized to 1 when performing the calculation.

Let's try the measurement again…



$$P(x|z) = \eta\, P(z|x)P(x)$$

Given information:

$P(z_1|open) = 0.6$
$P(z_1|closed) = 0.3$
$P(open) = 0.5$
$P(closed) = 0.5$

$$P(open|z_1) = \eta\, P(z_1|open)P(open)$$

$$P(open|z_1) = \eta\, 0.6 * 0.5 = \overset{\eta}{\cancel{\alpha}}\, 0.3$$

Unlike before, we don't yet have the answer because we still have the unknown term η that indicates that we need to normalize to get the true probability.

$$P(closed|z_1) = \eta\, P(z_1|closed)P(closed)$$

$$P(closed|z_1) = \eta\, 0.3 * 0.5 = \eta\, 0.15$$

$$\eta = (0.3 + 0.15)^{-1} = 2.22$$

$$\mathbf{P(open|z_1) = 0.67}$$

Combining Evidence

- Suppose our robot obtains another observation z_2. *e.g. we made a second sensor reading with the same sensor, and it reports an obstacle 35cm away*

- How can we integrate this new information?

- More generally, how can we estimate $P(x|\, z_1 \ldots z_n)$?

Recursive Bayesian Updating

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1})\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

We can assume z_n is independent of $z_1, \ldots, z_{n-1}$ if we know *x (Markov assumption)*

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

Recursive Bayesian estimation (a.k.a. Bayes Filter)

Timestep 1 $\quad P(x|z_1) = \eta\, P(z_1|x)P(x)$

This is just **Bayes Rule**.

Timestep 2 $\quad P(x|z_1, z_2) = \eta\, P(z_2|x)P(x|z_1)$

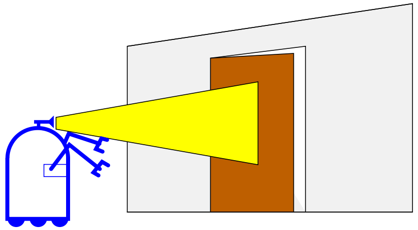
Bayes Rule applied recursively to incorporate **one more measurement**. The prior comes from the previous time step. The Markov assumption is used to simplify the likelihood term.

…

Timestep $n-1$ $\quad P(x|z_{1:n-1}) = \eta\, P(z_{n-1}|x)P(x|z_{1:n-2})$

Timestep n $\quad P(x|z_{1:n}) = \eta\, P(z_n|x)P(x|z_{1:n-1})$

Generalized to **arbitrary number of observations**. Note that this can only be calculated **incrementally** (recursively), as step n relies on the posterior from step $n-1$.

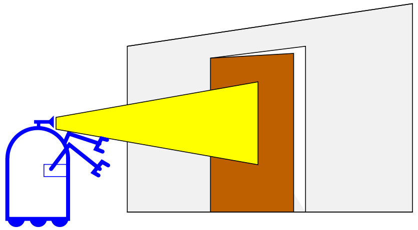


Lets go back to our example of a robot sensing a door and apply the new formulation we just defined:

$$P(x|z_{1:n}) = \eta \, P(z_n|x)P(x|z_{1:n-1})$$

to see what happens when we incorporate two sensor measurements.

Door Measurement Example #2



$$P(x_n|z_{1:n}) = \eta\, P(z_n|x_n)P(x_n|z_{1:n-1})$$

Timestep 1

<u>Originally Given information:</u>

$P(z_1|open) \; = \; 0.6$

$P(z_1|closed) \; = \; 0.3$

$P(open) \; = \; 0.5$

$P(closed) \; = \; 0.5$

<u>Updated Information:</u>

$P(open|z_1) = 0.67$

$P(closed|z_1) = .33$

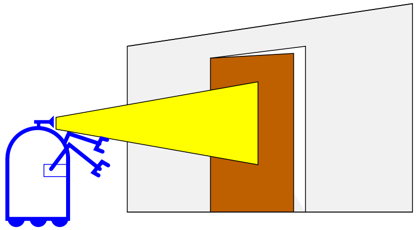
Timestep 2

$$P(open|z_{1:2}) = \eta\, P(z_2|open)P(open|z_1)$$

?

Let's assume for this example that measurement z_2 comes from the same sensor as z_1, so we can use the same sensor model (given). If we had a second sensor with different properties, we could use those here instead.

Door Measurement Example #2



$$P(x|z_{1:n}) = \eta\ P(z_n|x)P(x|z_{1:n-1})$$

Timestep 1

Originally Given information:
$P(z_1|open) = 0.6$
$P(z_1|closed) = 0.3$
$P(open) = 0.5$
$P(closed) = 0.5$

Updated Information:
$P(open|z_1) = 0.67$
$P(closed|z_1) = .33$

Timestep 2

$$P(open|z_{1,2}) = \eta\ P(z_2|open)P(open|z_1)$$

$$P(open|z_{1,2}) = \eta\ 0.6 \ * 0.67 = \overset{\eta}{\cancel{\alpha}}\ 0.4$$

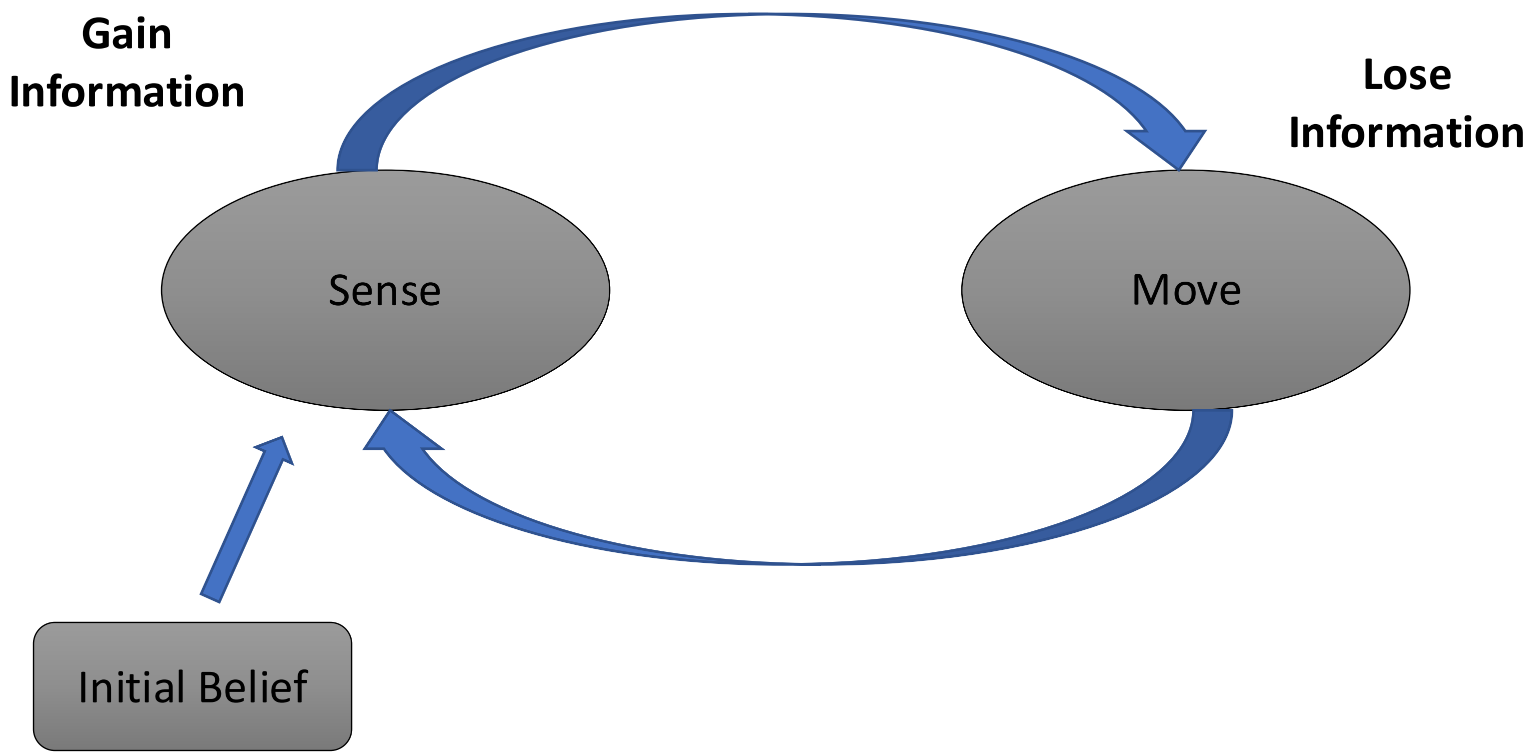
$$P(closed|z_{1,2}) = \eta\ P(z_2|closed)P(closed|z_1)$$

$$P(closed|z_{1,2}) = \eta\ 0.3 \ * 0.33 = \overset{\eta}{\cancel{\alpha}}\ 0.1$$

$$\eta = (0.4 + 0.1)^{-1} = 2$$

$$\boldsymbol{P(open|z_{1,2}) = 0.8}$$

More confident!


Gain Information
Lose Information
Sense
Move
Initial Belief

Actions

- Actions are never carried out with absolute certainty.
- In contrast to measurements, actions generally increase the uncertainty.

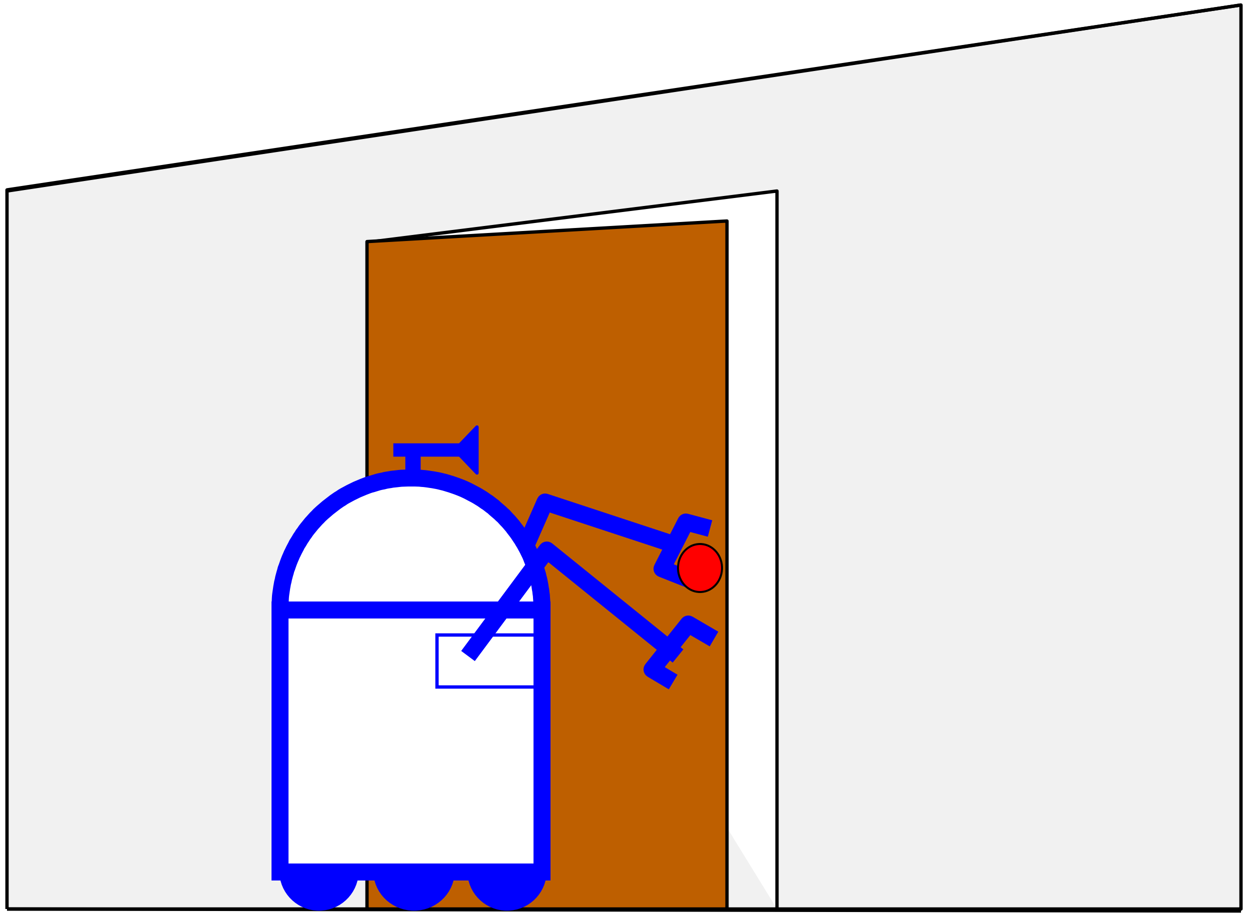
Modeling Actions

To incorporate the outcome of an action *u* into the current “belief”, we use the conditional probability:

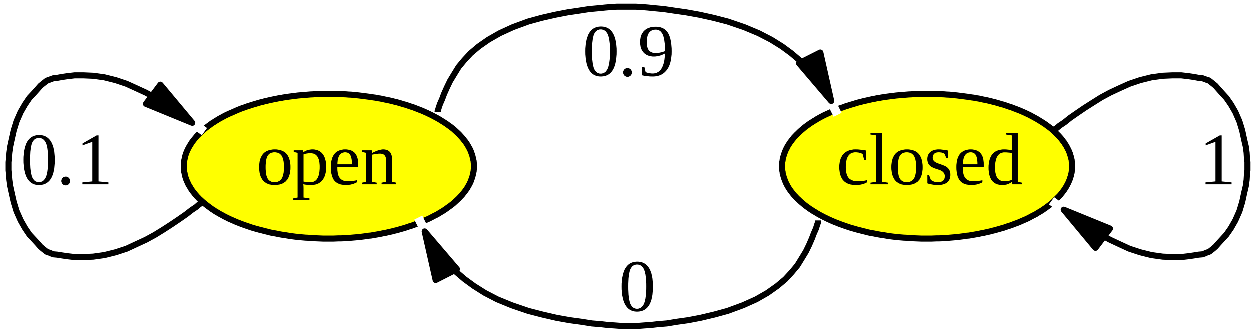
$$P(x_n | u_n, x_{n-1})$$

This term specifies that executing control input $\mathrm{u_n}$ in state $\mathrm{x_{n-1}}$ changes the state to $\mathrm{x_n}$ with the given probability.

Example: Pushing to close the door



$P(x_n | u_n, x_{n-1})$ for $u_n = $ "$push$":



If the door is open, the action "push" successfully closes the door in 90% of all cases.

Integrating the Outcome of Actions

Continuous case:

$$P(x_n|u_n) = \int P(x_n|u_n, x_{n-1})P(x_{n-1})\, dx_{n-1}$$

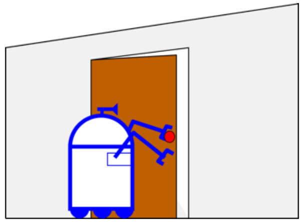
Discrete case:

$$P(x_n|u_n) = \sum_{x_{n-1}} P(x_n|u_n, x_{n-1})P(x_{n-1})$$

Example:

$$P(closed|u) = P(closed|u, open)P(open) + P(closed|u, closed)P(closed)$$

Door Example #3



$$P(x_3|z_1, z_2, u_3) = \sum_{x_2} P(x_3|u_3, x_2)P(x_2|z_1, z_2)$$

From before:

Originally Given information:

$P(z_1|open) = 0.6$

$P(z_1|closed) = 0.3$

$P(open) = 0.5$

$P(closed) = 0.5$

Newly Obtained Information:

$P(open|z_1) = 0.67$

$P(closed|z_1) = .33$

$P(open|z_{1,2}) = 0.8$

Rewritten in more consistent notation:

$P(door_{open}|z_{1,2}) = 0.8$

Action Information:

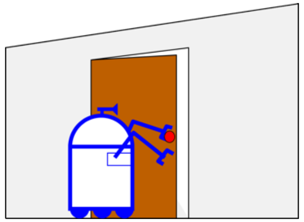
$P(door_{closed}|door_{open}, push) = 0.9$

$P(door_{closed}|door_{closed}, push) = 1$

$P(door_{open}|door_{open}, push) = 0.1$

$P(door_{open}|door_{closed}, push) = 0$

Door Example #3



$$P(x_3|z_1,z_2,u_3) = \sum_{x_2} P(x_3|u_3,x_2)P(x_2|z_1,z_2)$$

$$P(door_{closed}|door_{open},push) = 0.9$$
$$P(door_{closed}|door_{closed},push) = 1$$
$$P(door_{open}|door_{open},push) = 0.1$$
$$P(door_{open}|door_{closed},push) = 0$$

$$P(door_{open}|z_{1,2}) = 0.8$$

$$P(x_3|z_1,z_2,u_3) = \sum_{x_2} P(x_3|u_3,x_2)P(x_2|z_1,z_2)$$

$$P(door_{open}|z_1,z_2,push) = P(door_{open}|push,door_{open})P(door_{open}|z_1,z_2) + P(door_{open}|push,door_{closed})P(door_{closed}|z_1,z_2)$$

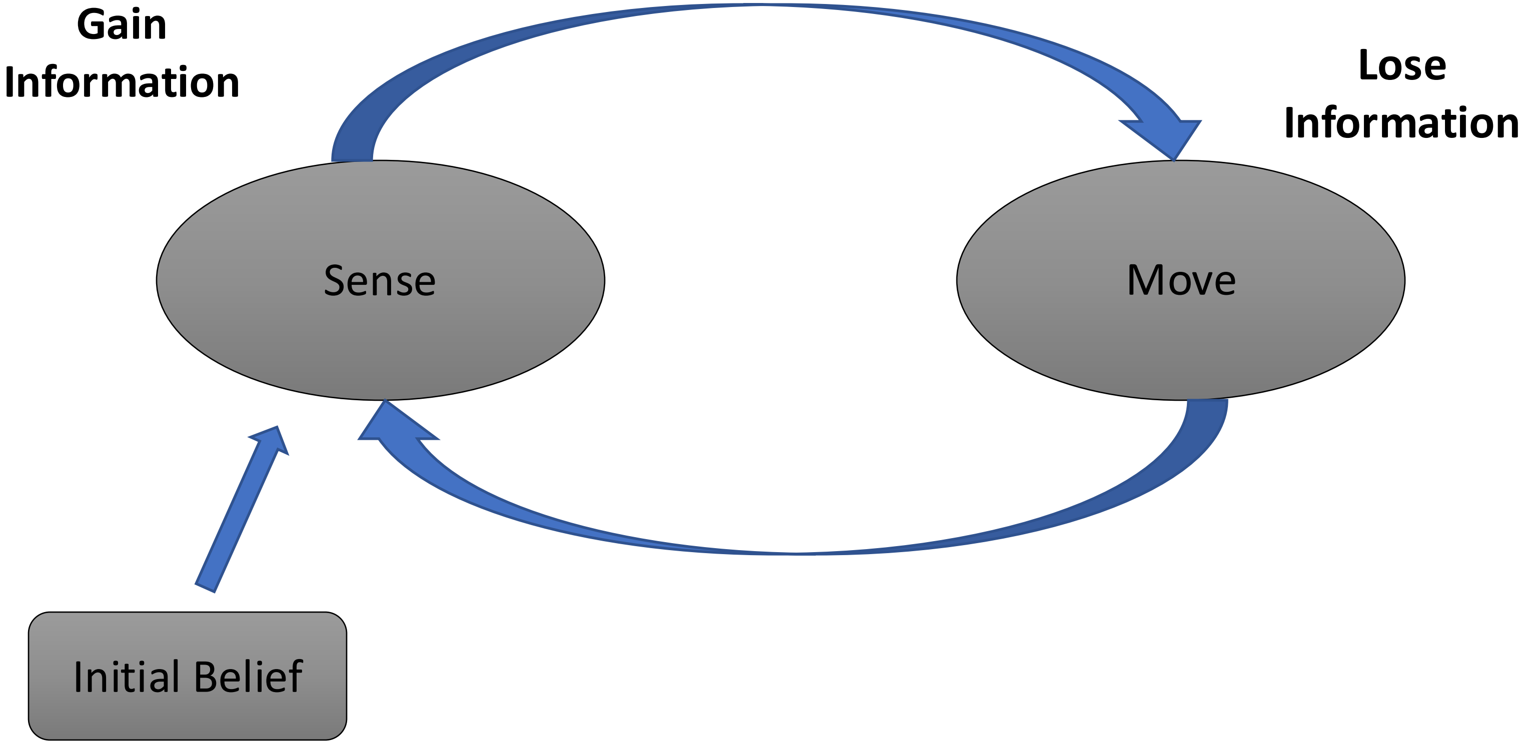
$$= 0.1 * 0.8 + 0 * 0.2$$

$$= \mathbf{0.08}$$

Localization

$P(x|z_{1:n}) = \eta \, P(z_n|x)P(x|z_{1:n-1})$

$P(x_n|u_n) = \sum P(x_n|u_n, x_{n-1})P(x_{n-1})$

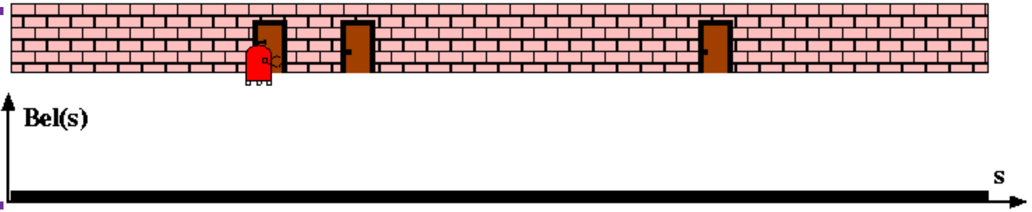


Common mistake...

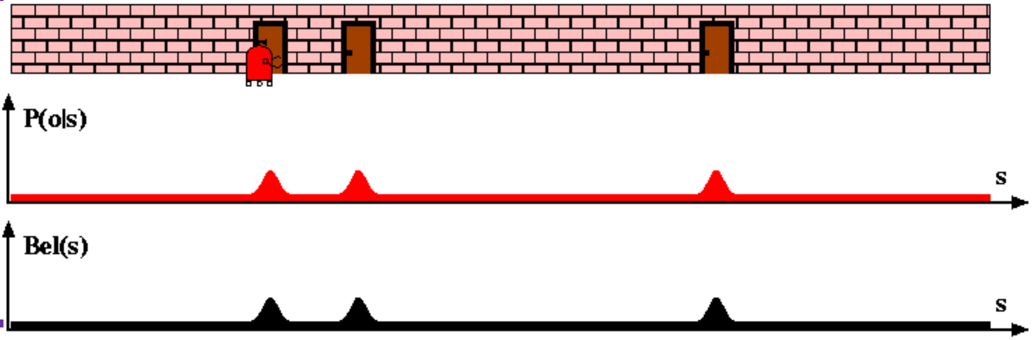
✓ $$P(x \mid y) = 1 - P(\neg x \mid y)$$

✗ $$P(x \mid y) \neq 1 - P(x \mid \neg y)$$

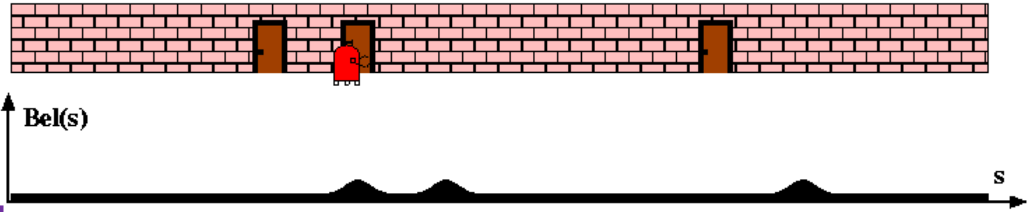
Initial Guess: Could be anywhere…



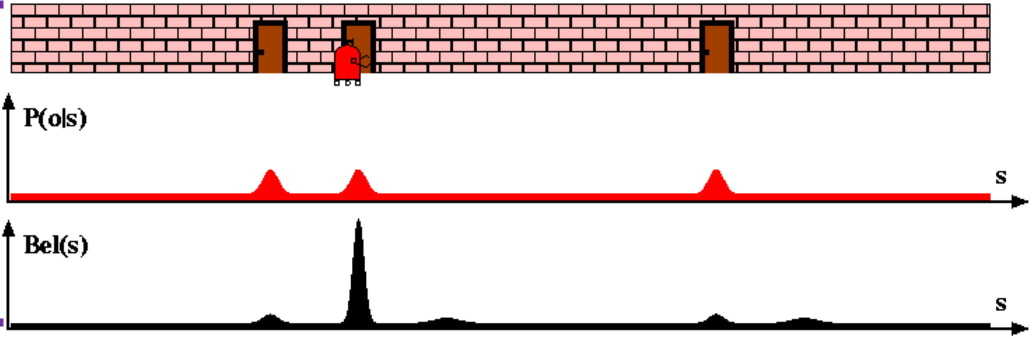
Take a measurement: we’re probably in front of a door…



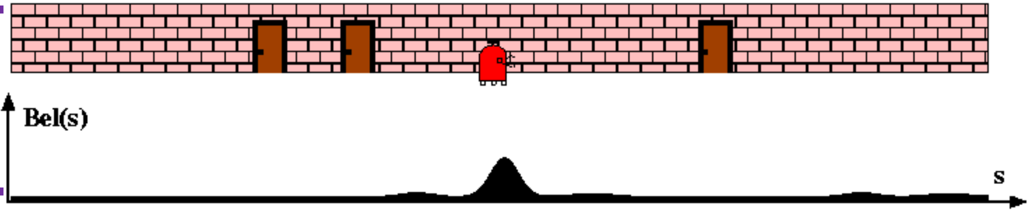
Execute an action – move to the right by about a meter… probability mass “spreads out”



Take another measurement. It seems we’re in front of a door again (red). Given what we believed before about position, the most likely place now is the second door.



Execute an action – move to the right by about a meter… probability mass “spreads out”



Reference

- **Probabilistic Robotics** by Thrun, Burgard and Fox.. Chapter 2 (available on Canvas)