# ON FEW-DISTANCE SETS IN THE PLANE

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ABSTRACT. Let $g(k)$ be the maximum size of a planar set that determines at most $k$ distances. We prove

$$\frac{\pi}{3\,\mathcal{C}(\Lambda_{\mathrm{hex}})}\,k\sqrt{\log k}\,(1+o(1)) \;\le\; g(k) \;\le\; C\,k\log k,$$

so $g(k) \asymp k\sqrt{\log k}$ with an explicit hexagonal constant. For any arithmetic lattice $\Lambda$,

$$g_\Lambda(k) \;\ge\; \frac{\pi}{4}\,\mathsf{S}^*(\Lambda)\,k\sqrt{\log k}\,(1+o(1)).$$

We also give quantitative stability: unless $X$ is line–heavy or has two popular nonparallel shifts, either almost all ordered pairs lie below a high quantile of the distance multiset (near–center localization), or a constant fraction of $X\cap W$ lies in one residue class modulo $2\Lambda$.

## 1. Introduction

Let $m(n)$ be the minimum number of distinct distances determined by $n$ planar points. Via the Elekes–Sharir reduction and incidence geometry, Guth–Katz proved $m(n) \gtrsim n/\log n$ [2, 3]. We study the inverse problem

$$g(k) := \max\{\,|X| :\ X \subset \mathbb{R}^2,\ |D(X)| \le k\,\}.$$

The Guth–Katz bound gives $g(k) \lesssim k\log k$, while lattice windows already yield $g(k) \gtrsim k\sqrt{\log k}$ through Bernays–Landau asymptotics for represented norms.

For small $k$, Erdős–Fishburn determined $g(k)$ up to $k \le 5$ and conjectured that extremizers for larger $k$ come from triangular lattice subsets [1].

We determine the growth of $g(k)$ up to constants and make the constants explicit in the lattice setting. For each arithmetic lattice $\Lambda$ we obtain a lower bound with the sharp $k\sqrt{\log k}$ scale and an explicit constant depending on covolume and the Bernays constant of the associated form; universally we retain the $g(k) \lesssim k\log k$ upper bound. We also prove a quantitative stability theorem: unless $X$ is line-heavy or has two popular nonparallel shifts, either almost all ordered pairs lie below a high quantile of the distance multiset or a constant fraction of $X\cap W$ concentrates in a single residue class modulo $2\Lambda$.

## 2. Preliminaries and Notation

**Definition 2.1.** Let $D(X) = \{|x-y| :\ x \ne y\}$. For each realized radius $t \in D(X)$ let

$$m_t \;:=\; \#\{(p,q) \in X^2 :\ p \ne q,\ |p-q| = t.\}$$

Define

$$Q_{\mathrm{ord}}(X) \;:=\; \sum_{t\in D(X)} m_t^2.$$

By Cauchy–Schwarz, writing $n := |X|$ and $k := |D(X)|$,

$$Q_{\mathrm{ord}}(X) \;\ge\; \frac{\big(\sum_{t\in D(X)} m_t\big)^2}{k} \;=\; \frac{n^2(n-1)^2}{k}. \tag{2.1}$$

**Definition 2.2.** Let $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ be a rank–2 lattice. A $\Lambda$–rectangle is

$$W = \{\, a_0 + iv_1 + jv_2 : \ 0 \le i < L_1,\ 0 \le j < L_2 \,\} \qquad (L_1, L_2 \in \mathbb{N}).$$

It is *proper* if $L_1, L_2 \ge 2$.

**Definition 2.3.** Let $\Lambda \subset \mathbb{R}^2$ be a rank–2 lattice with a fixed $\mathbb{Z}$–basis $(v_1, v_2)$. Identify $\Lambda \cong \mathbb{Z}^2$ by

$$u = (u_1, u_2) \in \mathbb{Z}^2 \ \longleftrightarrow\ \lambda(u) := u_1v_1 + u_2v_2 \in \Lambda.$$

Define the quadratic form $Q_\Lambda : \mathbb{Z}^2 \to \mathbb{R}_{\ge 0}$ by $Q_\Lambda(u) := \big|\lambda(u)\big|^2$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$.

We say that $\Lambda$ is arithmetic if $Q_\Lambda$ is proportional to a rational positive–definite binary quadratic form. Equivalently, $\Lambda$ is commensurable with $\mathbb{Z}^2$. In this case, after scaling by a positive real and an $\mathrm{SL}_2(\mathbb{Z})$ change of variables on $u$, we associate to $\Lambda$ a primitive integral positive–definite binary quadratic form $F_\Lambda$ and write $\mathcal{C}(\Lambda) := C(F_\Lambda)$ for its Bernays constant (Appendix A). We also define $s(\Lambda) > 0$ by

$$\forall\, u \in \mathbb{Z}^2 \quad Q_\Lambda(u) = s(\Lambda)\, F_\Lambda(u),$$

so $s(\Lambda)$ records the fixed proportionality between Euclidean squared norms on $\Lambda$ and the integral model. For the unimodular hexagonal lattice one has $s(\Lambda_{\text{hex}}) = 2/\sqrt{3}$.

**Definition 2.4.** For an arithmetic lattice $\Lambda$, write $Q_\Lambda(\lambda) = s(\Lambda)\, F_\Lambda(u)$ with $F_\Lambda$ primitive integral positive definite and $s(\Lambda) > 0$. Let $\mathcal{A}(\Lambda)$ be the covolume and $\mathcal{C}(\Lambda)$ the Bernays constant of $F_\Lambda$. Define

$$\mathsf{S}^*(\Lambda) \ :=\ \frac{s(\Lambda)}{\mathcal{A}(\Lambda)\,\mathcal{C}(\Lambda)}.$$

**Definition 2.5.** Let $\mathcal{L}(X)$ be the finite set of lines determined by unordered pairs of points of $X$. For $\ell \in \mathcal{L}(X)$ set $s_\ell(X) := |X \cap \ell|$. Then

$$\sum_{\ell \in \mathcal{L}(X)} \binom{s_\ell(X)}{2} = \binom{|X|}{2}.$$

**Definition 2.6.** For $\alpha \in (0,1]$, a finite set $X \subset \mathbb{R}^2$ is *$\alpha$–line–heavy* if

$$\max_{\ell \in \mathcal{L}(X)} s_\ell(X) \ \ge\ \alpha\,|X|.$$

When we say line–heavy without specifying $\alpha$, we mean $\alpha$ is an absolute fixed constant.

**Definition 2.7.** For $x \in \mathbb{R}^2$ and $R > 0$, set $B(x,R) := \{y \in \mathbb{R}^2 : \ |y - x| \le R\}$. For a lattice $\Lambda \subset \mathbb{R}^2$, the covering radius $\mu(\Lambda)$ is the least $\rho > 0$ with $B(0,\rho) + \Lambda = \mathbb{R}^2$. We write $\mathcal{A}(\Lambda)$ for the covolume.

**Definition 2.8.** For a rank–2 lattice $\Lambda \subset \mathbb{R}^2$, define the shortest vector length

$$\lambda_1(\Lambda) \ :=\ \min\{\, |\lambda| : \ \lambda \in \Lambda \setminus \{0\} \,\}.$$

**Definition 2.9.** For finite $A, B \subset \mathbb{R}^2$ put $r_{A\to B}(v) := \#\{(a,b) \in A \times B : \ b - a = v\}$ and $r_A(v) := r_{A\to A}(v)$. The additive energy is $E_+(A) := \sum_v r_A(v)^2$.

**Definition 2.10.** Let $\Lambda$ be a rank–2 lattice and fix an aspect–ratio bound $\mathcal{A}_0 \ge 1$. A set $W \subset z + \Lambda$ is *inner–regular with parameters* $(c, R; \mathcal{A}_0)$ if

$$B\big(z, (1-c)R\big) \cap (z + \Lambda) \ \subseteq\ W \ \subseteq\ B(z, R) \cap (z + \Lambda),$$

and every minimal $\Lambda$–aligned rectangle containing $W$ has side lengths within a factor $\mathcal{A}_0$. Implicit constants may depend on $\Lambda$ and $\mathcal{A}_0$.

**Lemma 2.11.** *Let $\Lambda \subset \mathbb{R}^2$ be a lattice with Euclidean covering radius $\mu(\Lambda)$, and fix $\tau \in \mathbb{R}^2$. For every $R > \mu(\Lambda)$,*

$$\{\lambda \in \Lambda : \ |\lambda| \le 2R - 2\mu(\Lambda)\} \ \subseteq \ \{x - y : \ x, y \in (\tau + \Lambda) \cap B(0, R)\}.$$

*Proof.* Let $\lambda \in \Lambda$ with $|\lambda| \le 2R - 2\mu(\Lambda)$. Then the two Euclidean disks $B(0, R - \mu(\Lambda))$ and $B(-\lambda, R - \mu(\Lambda))$ intersect, since

$$\operatorname{dist}(0, -\lambda) = |\lambda| \ \le \ 2(R - \mu(\Lambda)).$$

Pick any $t$ in the intersection, so $|t| \le R - \mu(\Lambda)$ and $|t + \lambda| \le R - \mu(\Lambda)$. By the definition of the covering radius there exists $z \in \tau + \Lambda$ with $|z - t| \le \mu(\Lambda)$. Set $y := z$ and $x := z + \lambda$. Then $x, y \in \tau + \Lambda$, $x - y = \lambda$, and

$$|y| \le |z - t| + |t| \le \mu(\Lambda) + \big(R - \mu(\Lambda)\big) = R,$$

$$|x| = |y + \lambda| \le |(z - t) + (t + \lambda)| \le \mu(\Lambda) + \big(R - \mu(\Lambda)\big) = R.$$

Thus $x, y \in (\tau + \Lambda) \cap B(0, R)$ and $x - y = \lambda$, proving the inclusion. □

## 3. Incidence bounds

**Definition 3.1.** Fix a finite set $X \subset \mathbb{R}^2$. For $g \in \mathrm{SE}(2)$ write $r_g := |\{x \in X : \ g(x) \in X\}|$. Let

$$\mathcal{G} = \mathcal{G}(X) \ := \ \{\, g \in \mathrm{SE}(2) : \ r_g \ge 2 \,\}$$

be the finite set of direct isometries that map at least two points of $X$ into $X$. Write $\mathsf{T} := \{g \in \mathcal{G} : \ g \text{ is a translation}\}$ and $\mathsf{N} := \mathcal{G} \setminus \mathsf{T}$.

**Lemma 3.2.** *Let $X \subset \mathbb{R}^2$ with $|X| = n$ and let $T \subset D(X)$ with $|T| = L$. Then*

$$\sum_{t \in T} m_t \ \le \ \sqrt{Q_{\mathrm{ord}}(X)}\,\sqrt{L}.$$

*Moreover,*

$$Q_{\mathrm{ord}}(X) \ = \ \sum_{t \in D(X)} m_t^2 \ \le \ E_+(X) \ + \ C\, n^3 \log n \ + \ O(n^2).$$

*Proof.* For the first inequality, apply Cauchy–Schwarz to $\sum_{t \in T} m_t$.

For the second, by Lemma 4.1,

$$\sum_{g \in \mathcal{G}} r_g^2 \ = \ Q_{\mathrm{ord}}(X) \ + \ O(n^2),$$

where $\mathcal{G} = \{g \in \mathrm{SE}(2) : r_g \ge 2\}$. Decompose $\mathcal{G}$ into translations $\mathsf{T}$ and non-translations $\mathsf{N}$. For translations,

$$\sum_{g \in \mathsf{T}} r_g^2 \ = \ E_+(X) \ + \ O(n^2).$$

For nontranslations, by the Elekes–Sharir incidence bound and Guth–Katz,

$$\sum_{g \in \mathsf{N}} r_g^2 \ \ll \ n^3 \log n,$$

see Elekes–Sharir [2] and Guth–Katz [3]. Combine to get

$$Q_{\mathrm{ord}}(X) \ \le \ E_+(X) \ + \ C\, n^3 \log n \ + \ O(n^2).$$

□

**Proposition 3.3.** *Let $X \subset \mathbb{R}^2$ with $|X| = n$ and $|D(X)| = k$. List the distinct radii as $t_1 < \cdots < t_k$. For $\theta \in (0,1)$ set $t_\star := t_{\lfloor (1-\theta)k \rfloor}$. If*

$$\sum_{\substack{t \in D(X) \\ t \le t_\star}} m_t \;\ge\; (1-\eta)\, n(n-1) \qquad \text{for some } \eta \in (0, 1/2),$$

*then there exists $z \in X$ such that*

$$\big|X \cap B(z, t_\star)\big| \;\ge\; (1-\eta)\, n.$$

*Proof.* Form the directed graph $\vec{G}$ on vertex set $X$ by placing an arc $p \to q$ between distinct $p, q \in X$ if and only if $|p - q| \le t_\star$. The hypothesis states that the number of ordered edges (arcs) in $\vec{G}$ is at least $(1-\eta)n(n-1)$. Hence the average out–degree satisfies

$$\bar{d} \;=\; \frac{1}{n} \cdot \big[\text{ordered edges}\big] \;\ge\; (1-\eta)(n-1).$$

Choose $z \in X$ with $\deg^+_{\vec{G}}(z) \ge \bar{d}$. Then the number of points of $X$ at distance $\le t_\star$ from $z$ equals $1 + \deg^+_{\vec{G}}(z)$, so

$$|X \cap B(z, t_\star)| \;=\; 1 + \deg^+_{\vec{G}}(z) \;\ge\; 1 + (1-\eta)(n-1) \;\ge\; (1-\eta)\, n.$$

as required. □

**Theorem 3.4.** *Let $\Lambda$ be an arithmetic rank–2 lattice, normalized by similarity so that $\lambda_1(\Lambda) = 1$. There exists $k_0(\Lambda) \in \mathbb{N}$ such that for all $k \ge k_0(\Lambda)$,*

$$\frac{\pi}{4}\, \mathsf{S}^*(\Lambda)\, k\sqrt{\log k}\,(1 + o_\Lambda(1)) \;\le\; g_\Lambda(k) \;\le\; C\, k \log k,$$

*where $C > 0$ is an absolute constant.*

*Proof.* For the lower bound, let $R > \mu(\Lambda)$ and consider disk windows $W_R = (\tau + \Lambda) \cap B(z, R)$ for some $z \in \mathbb{R}^2$. By Proposition 5.1 and Theorem A.1,

$$|D(W_R)| \;=\; \frac{\mathcal{C}(\Lambda)}{s(\Lambda)}\, \frac{4R^2}{\sqrt{\log\big(\frac{4R^2}{s(\Lambda)}\big)}}\,(1 + o_\Lambda(1)).$$

Put $T := \frac{4R^2}{s(\Lambda)}$. Then

$$k = \mathcal{C}(\Lambda)\, \frac{T}{\sqrt{\log T}}\,(1 + o_\Lambda(1)) \qquad (T \to \infty).$$

We invert this asymptotically. Rearranging gives

$$T = \frac{k}{\mathcal{C}(\Lambda)}\, \sqrt{\log T}\,(1 + o_\Lambda(1)).$$

Let $U := \log T$. Taking logs yields

$$U = \log T = \log k - \log \mathcal{C}(\Lambda) + \tfrac{1}{2}\log U + o_\Lambda(1).$$

Since $U \to \infty$, this implies $U = \log k + O(\log\log k)$, hence

$$\sqrt{\log T} = \sqrt{U} = \sqrt{\log k}\,(1 + o_\Lambda(1)).$$

Substituting back gives

$$T = \frac{k}{\mathcal{C}(\Lambda)}\, \sqrt{\log k}\,(1 + o_\Lambda(1)),$$

and therefore

$$R^2 = \frac{s(\Lambda)}{4}\, T = \frac{s(\Lambda)}{4\,\mathcal{C}(\Lambda)}\, k\,\sqrt{\log k}\,(1 + o_\Lambda(1)).$$

Hence

$$\begin{aligned}|W_R| &= \frac{\pi}{\mathcal{A}(\Lambda)}\, R^2 + O_\Lambda(R)\\ &= \frac{\pi}{\mathcal{A}(\Lambda)} \cdot \frac{s(\Lambda)}{4\,\mathcal{C}(\Lambda)}\, k\,\sqrt{\log k}\,(1 + o_\Lambda(1))\\ &= \frac{\pi}{4}\,\mathsf{S}^*(\Lambda)\, k\sqrt{\log k}\,(1 + o_\Lambda(1)),\end{aligned}$$

and so $g_\Lambda(k) \geq |W_R|$ gives the claimed lower bound.

For the upper bound, since $g_\Lambda(k) \leq g(k)$ for every fixed $\Lambda$ and $g(k) \ll k \log k$ by Guth–Katz [3], we obtain $g_\Lambda(k) \ll k \log k$ with an absolute implied constant. □

## 4. Additive structure at positive energy

**Lemma 4.1.** *Let $X \subset \mathbb{R}^2$, $|X| = n$. Put*

$$Q^* = \#\{(p, q, p', q') \in X^4 : \; p \neq q, \; p' \neq q', \; |p - q| = |p' - q'| > 0\}\,.$$

*Then*

$$Q^* \;=\; \sum_{t \in D(X)} m_t^2 \;=\; Q_{\mathrm{ord}}(X).$$

*Let $\mathcal{G} := \{g \in \mathrm{SE}(2) : r_g \geq 2\}$. Then*

$$Q^* \;=\; \sum_{g \in \mathcal{G}} r_g\,(r_g - 1), \qquad \sum_{g \in \mathcal{G}} r_g^2 \;=\; Q^* \;+\; O(n^2).$$

*In particular, writing $\mathsf{T}$ for the translations in $\mathcal{G}$ and $\mathsf{N} := \mathcal{G} \setminus \mathsf{T}$,*

$$\sum_{g \in \mathsf{T}} r_g^2 \;=\; E_+(X) \;+\; O(n^2), \qquad \sum_{g \in \mathsf{N}} r_g^2 \;=\; Q^* \;-\; E_+(X) \;+\; O(n^2).$$

*Proof deferred to Appendix B.* □

### 4.1. From large additive energy to inner–regular lattice windows.

**Lemma 4.2.** *Let $W = \{a_0 + iv_1 + jv_2 : \; 0 \leq i < L_1, \; 0 \leq j < L_2\}$ with $L_1, L_2 \geq 2$.*

*(i) $E_+(W) = \Theta(|W|^3)$. Moreover, if $A \subseteq W$ and $|A| \geq (1 - \varepsilon)|W|$, then*

$$E_+(A) \;\geq\; E_+(W) \;-\; 4\varepsilon\,|W|^3.$$

*(ii) If $A \subset W$ has density $\beta = |A|/|W|$, there exist $s \in \{1, \dots, L_1 - 1\}$, $t \in \{1, \dots, L_2 - 1\}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that*

$$|A \cap (A + \varepsilon_1 s v_1)| \;\geq\; \max\Big\{0,\; \frac{\beta L_1 - 1}{2(L_1 - 1)}\Big\}\,|A|, \quad |A \cap (A + \varepsilon_2 t v_2)| \;\geq\; \max\Big\{0,\; \frac{\beta L_2 - 1}{2(L_2 - 1)}\Big\}\,|A|.$$

*(iii) If $P$ is a proper $\Lambda$–rectangle and $T \subset \mathbb{Z}v_1 + \mathbb{Z}v_2$ is finite, there is a proper GAP $P^\star$ containing $\bigcup_{t \in T}(t + P)$ with side lengths enlarged by the spans of the $T$–coefficients and*

$$|P^\star| = |P| + \Delta_\alpha L_2 + \Delta_\gamma L_1 + \Delta_\alpha \Delta_\gamma.$$

*Proof.* (i) Write each difference as $u = u_1v_1 + u_2v_2$ with $u_i \in \mathbb{Z}$. A pair $(x, y) \in W^2$ contributes to $r_W(u)$ iff the coordinates differ by $u_1, u_2$, hence $r_W(u) = \max\{0, L_1 - |u_1|\} \cdot \max\{0, L_2 - |u_2|\}$ and

$$E_+(W) = \Big( \sum_{d=-(L_1-1)}^{L_1-1} (L_1 - |d|)^2 \Big) \Big( \sum_{e=-(L_2-1)}^{L_2-1} (L_2 - |e|)^2 \Big).$$

For a single side, $\sum_{d=-(L-1)}^{L-1} (L - |d|)^2 = \frac{2}{3}L^3 + \frac{1}{3}L$, giving $E_+(W) = \Theta(|W|^3)$. For the deletion bound, removing one point from $W$ destroys at most two ordered pairs for any fixed $u$, hence

$$r_A(u) \;\ge\; r_W(u) \;-\; 2\,|W \setminus A| \;=\; r_W(u) \;-\; 2\varepsilon|W|.$$

Therefore

$$E_+(A) = \sum_u r_A(u)^2 \;\ge\; \sum_u r_W(u)^2 \;-\; 4\,|W \setminus A| \sum_u r_W(u).$$

Since $\sum_u r_W(u) = |W|^2$ and $|W \setminus A| = \varepsilon|W|$, we get $E_+(A) \;\ge\; E_+(W) \;-\; 4\varepsilon\,|W|^3$. (ii) Index $W$ by $(i, j)$ with $0 \le i < L_1$, $0 \le j < L_2$. For a fixed column $j$ let $b_j := |\{i : \; (i, j) \in A\}|$. For $s \in \{1, \dots, L_1 - 1\}$ put $N_j(s) := \sum_{i=0}^{L_1-1-s} \mathbf{1}_A(i, j)\, \mathbf{1}_A(i + s, j)$. Then $\sum_{s=1}^{L_1-1} N_j(s) = \binom{b_j}{2}$. Averaging over $s$ and summing over $j$ gives

$$\frac{1}{L_1 - 1} \sum_{s=1}^{L_1-1} |A \cap (A + sv_1)| = \frac{1}{L_1 - 1} \sum_{j=0}^{L_2-1} \binom{b_j}{2}.$$

By Cauchy–Schwarz, $\sum_{j=0}^{L_2-1} \binom{b_j}{2} \ge \frac{|A|^2}{2L_2} - \frac{|A|}{2}$, so for some $s$

$$|A \cap (A + sv_1)| \;\ge\; \Big( \frac{\beta L_1 - 1}{2(L_1 - 1)} \Big) |A|.$$

If the RHS is negative, use the trivial 0 bound. Replacing $s$ by $-s$ if needed gives $\varepsilon_1$. The $v_2$ case is identical with rows/columns swapped, giving $t$ and $\varepsilon_2$.

(iii) Write each $t \in T$ as $t = \alpha_t v_1 + \gamma_t v_2$ with $\alpha_t, \gamma_t \in \mathbb{Z}$ and set

$$\alpha_{\min} = \min_t \alpha_t, \;\; \gamma_{\min} = \min_t \gamma_t, \;\; \Delta_\alpha = \max_t \alpha_t - \alpha_{\min}, \;\; \Delta_\gamma = \max_t \gamma_t - \gamma_{\min}.$$

Let $a_0^\star := a_0 + \alpha_{\min} v_1 + \gamma_{\min} v_2$ and $P^\star := \{a_0^\star + iv_1 + jv_2 : \; 0 \le i < L_1 + \Delta_\alpha, \; 0 \le j < L_2 + \Delta_\gamma\}$. Then $\bigcup_{t \in T}(t + P) \subset P^\star$ and $|P^\star| = (L_1 + \Delta_\alpha)(L_2 + \Delta_\gamma) = |P| + \Delta_\alpha L_2 + \Delta_\gamma L_1 + \Delta_\alpha \Delta_\gamma$. ☐

**Proposition 4.3.** *Let $v_1, v_2$ be nonparallel, $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, and*

$$P = \{a_0 + iv_1 + jv_2 : \; 0 \le i < L_1, \; 0 \le j < L_2\}$$

*with $L_1 \ge L_2 \ge 2$. Let $A \subseteq P$ and write $\beta := |A|/|P| \in [0, 1]$. Then there exists a $\Lambda$–rectangle $W \subseteq P$ of side lengths $L_2 \times L_2$ with exactly $4L_2 - 4$ lattice boundary points (hence $\asymp |W|^{1/2}$) such that*

$$\frac{|A \cap W|}{|W|} \;\ge\; \frac{\beta}{2}.$$

*Proof.* Write the column sums $b_i := |\{\, j \in \{0, \dots, L_2 - 1\} : \; (i, j) \in A \,\}|$ for $i = 0, \dots, L_1 - 1$, so that $\sum_{i=0}^{L_1-1} b_i = |A| = \beta L_1 L_2$.

Consider the $L_1$ cyclic length–$L_2$ column windows

$$\mathcal{W}_s^{\mathrm{cyc}} \;:=\; \{\, s, s + 1, \dots, s + L_2 - 1 \,\} \pmod{L_1} \qquad (s = 0, 1, \dots, L_1 - 1).$$

Each point of $A$ lies in exactly $L_2$ of these cyclic windows, hence

$$\frac{1}{L_1} \sum_{s=0}^{L_1-1} \sum_{i \in \mathcal{W}_s^{\mathrm{cyc}}} b_i \;=\; \frac{L_2}{L_1} \sum_{i=0}^{L_1-1} b_i \;=\; \frac{L_2}{L_1}\, |A|.$$

Therefore there exists $s^*$ with

$$\sum_{i\in\mathcal{W}_{s^*}^{\mathrm{cyc}}} b_i \;\ge\; \frac{L_2}{L_1}\,|A|.$$

If the window $\mathcal{W}_{s^*}^{\mathrm{cyc}}$ is nonwrapping (i.e. $s^*\le L_1-L_2$), put

$$W \;:=\; \{a_0+iv_1+jv_2:\; s^*\le i\le s^*+L_2-1,\; 0\le j<L_2\}\;\subset P.$$

Then $|A\cap W|=\sum_{i\in\mathcal{W}_{s^*}^{\mathrm{cyc}}} b_i\ge (L_2/L_1)|A|$, hence

$$\frac{|A\cap W|}{|W|}\;\ge\;\frac{(L_2/L_1)|A|}{L_2^2}\;=\;\frac{|A|}{L_1L_2}\;=\;\beta\;\ge\;\frac{\beta}{2}.$$

If $\mathcal{W}_{s^*}^{\mathrm{cyc}}$ wraps (so $s^*>L_1-L_2$), it decomposes as a disjoint union of two contiguous nonwrapping parts

$$J_1=[\,s^*,\,L_1-1\,],\qquad J_2=[\,0,\,s^*+L_2-1-L_1\,],$$

with $|J_1|+|J_2|=L_2$, hence $\max\{|J_1|,|J_2|\}\ge\lceil L_2/2\rceil$. One of these parts, call it $J$, satisfies

$$\sum_{i\in J} b_i\;\ge\;\tfrac12\sum_{i\in\mathcal{W}_{s^*}^{\mathrm{cyc}}} b_i\;\ge\;\frac12\cdot\frac{L_2}{L_1}\,|A|.$$

Since $J$ is a prefix (resp. suffix) of $[0,L_1-1]$ and $|J|\ge\lceil L_2/2\rceil$, the block $[0,L_2-1]$ (resp. $[L_1-L_2,L_1-1]$) is a contiguous nonwrapping interval of length $L_2$ that contains $J$. We take that block as our window.

Let $W$ be the nonwrapping $L_2$–column block in $P$ obtained by extending $J$ on one side to length $L_2$ (this is always possible since $J$ is a prefix or suffix of $[0,L_1-1]$). Then $W\subset P$ and

$$|A\cap W|\;\ge\;\sum_{i\in J} b_i\;\ge\;\frac12\cdot\frac{L_2}{L_1}\,|A|.$$

Dividing by $|W|=L_2^2$ gives

$$\frac{|A\cap W|}{|W|}\;\ge\;\frac12\cdot\frac{L_2}{L_1}\cdot\frac{|A|}{L_2^2}\;=\;\frac12\cdot\beta.$$

In both cases $W$ is a $\Lambda$–rectangle of side lengths $L_2\times L_2$, and $W$ has exactly $4L_2-4$ boundary points, as claimed.

□

**Lemma 4.4.** *Let $\Lambda=\mathbb{Z}v_1\oplus\mathbb{Z}v_2$ and let $W\subset z+\Lambda$ be inner–regular with parameter $c\in[0,1)$ and radius $R$, with $(1-c)R>\mu(\Lambda)$ and bounded aspect ratio.*

*(i) We have*

$$B(z,(1-c)R)\cap(z+\Lambda)\;\subseteq\;W\;\subseteq\;B(z,R)\cap(z+\Lambda),$$

*and $R\asymp_\Lambda |W|^{1/2}$. There exists a subset $W_{\mathrm{in}}\subset W$ with*

$$|W\setminus W_{\mathrm{in}}|\;\ll_\Lambda\;|W|^{1/2}$$

*such that $W_{\mathrm{in}}+t\subset W$ for every $t\in\Lambda$ whose $(v_1,v_2)$–coordinates lie in $\{0,1\}^2$.*

*(ii) Fix $\varepsilon\in(0,1)$ and $\delta\in(0,1]$. Put*

$$\rho_\varepsilon\;:=\;(1-c-\varepsilon)R-\mu(\Lambda)\;>\;0.$$

*If $\lambda\in\Lambda$ satisfies $|\lambda|\le(2-\delta)\rho_\varepsilon$, then*

$$r_W(\lambda)\;\ge\;c_0(\Lambda,c,\varepsilon,\delta)\,R^2.$$

*Consequently, if $X \subset W$ with $|W \setminus X| = o_\Lambda(R^2)$, then for all sufficiently large $R$ (depending on $\Lambda, c, \varepsilon, \delta$) every such $\lambda$ lies in $D(X)$. Moreover, all differences $x - y$ with $x, y \in W$ satisfy $|x - y| \le 2R$, and for every $\lambda \in \Lambda$ with*

$$|\lambda| \;\le\; 2(1-c)R - 2\mu(\Lambda)$$

*there exist $x, y \in W$ with $x - y = \lambda$.*

*Proof.* (i) The containment and the estimate $R \asymp_\Lambda |W|^{1/2}$ follow from bounded aspect ratio and norm equivalence on $\Lambda$. For the residue-stable core $W_{\mathrm{in}}$ take

$$W_{\mathrm{in}} := B(z, (1-c)R - \Delta) \cap (z + \Lambda), \qquad \Delta := \max\{|v_1|, |v_2|, |v_1 + v_2|\}.$$

Then $W_{\mathrm{in}} + t \subset B(z, (1-c)R) \subset W$ for $t \in \{0, v_1, v_2, v_1 + v_2\}$, and the removal bound $|W \setminus W_{\mathrm{in}}| \ll_\Lambda R \asymp_\Lambda |W|^{1/2}$ follows from lattice–point counting in a belt of fixed thickness (via Lemma B.1).

(ii) Put $\rho := (1-c)R - \mu(\Lambda)$ and fix $\varepsilon \in (0,1)$. Let $\rho_\varepsilon := (1 - c - \varepsilon)R - \mu(\Lambda) > 0$. Fix any $\delta \in (0,1]$. If $|\lambda| \le (2-\delta)\rho_\varepsilon$, then the two disks $B(z, \rho_\varepsilon)$ and $B(z, \rho_\varepsilon) - \lambda$ have a lens $L$ whose area satisfies $\operatorname{area}(L) \gg_\delta \rho_\varepsilon^2 \asymp R^2$. By Corollary B.2 the lens contains $\gg_{\Lambda,\varepsilon,\delta} R^2$ points of the translate $z + \Lambda$, each giving an ordered pair $(x, y) \in W \times W$ with $y - x = \lambda$.

If $X \subset W$ with $|W \setminus X| = \Delta$, then for every fixed $\lambda$,

$$r_X(\lambda) \;\ge\; r_W(\lambda) \;-\; 2\Delta,$$

since removing a single point deletes at most two ordered $\lambda$-pairs.

Hence

$$r_W(\lambda) \;\gg_{\Lambda,c,\varepsilon,\delta}\; R^2 \qquad \text{for all } |\lambda| \le (2-\delta)\rho_\varepsilon.$$

Consequently, if $X \subset W$ with $|W \setminus X| = o_\Lambda(R^2)$, then for every fixed $\delta \in (0,1]$ and all sufficiently large $R$, every $\lambda$ with $|\lambda| \le (2-\delta)\rho_\varepsilon$ lies in $X - X$, hence $|\lambda| \in D(X)$.

All differences $x - y$ with $x, y \in W$ satisfy $|x - y| \le 2R$. Moreover, for every $\lambda \in \Lambda$ with $|\lambda| \le 2(1-c)R - 2\mu(\Lambda)$ there exist $x, y \in W$ with $x - y = \lambda$, by Lemma 2.11 applied in $z + \Lambda$ with radius $(1-c)R$. □

## 5. Counting realized distances in lattice windows

**Proposition 5.1.** *Let $\Lambda$ be a rank–2 arithmetic lattice and let $W_R$ be inner–regular:*

$$B(z, (1-c)R) \cap (z + \Lambda) \;\subseteq\; W_R \;\subseteq\; B(z, R) \cap (z + \Lambda)$$

*for some fixed $c \in [0,1)$ with $(1-c)R > \mu(\Lambda)$. Then, as $R \to \infty$,*

$$\frac{\mathcal{C}(\Lambda)}{s(\Lambda)} \frac{4(1-c)^2 R^2}{\sqrt{\log\left(\frac{4R^2}{s(\Lambda)}\right)}} (1 + o_{\Lambda,c}(1)) \;\le\; |D(W_R)| \;\le\; \frac{\mathcal{C}(\Lambda)}{s(\Lambda)} \frac{4R^2}{\sqrt{\log\left(\frac{4R^2}{s(\Lambda)}\right)}} (1 + o_\Lambda(1)).$$

*Here $o_{\Lambda,c}(1)$ and $o_\Lambda(1)$ are uniform for fixed $\Lambda$ (and fixed $c$ in the lower bound).*

*Proof.* By Lemma 2.11 applied in $z + \Lambda$ with radius $(1-c)R$, every $\lambda \in \Lambda$ with $|\lambda| \le 2(1-c)R - 2\mu(\Lambda)$ occurs as a difference $x - y$ with $x, y \in W_R$, while trivially all differences satisfy $|x - y| \le 2R$. Since $Q_\Lambda(\lambda) = s(\Lambda)\, F_\Lambda(u)$ for a primitive integral positive–definite binary quadratic form $F_\Lambda$,

$$\mathcal{R}_{F_\Lambda}\Big(\tfrac{(2(1-c)R - 2\mu(\Lambda))^2}{s(\Lambda)}\Big) \;\le\; |D(W_R)| \;\le\; \mathcal{R}_{F_\Lambda}\Big(\tfrac{(2R)^2}{s(\Lambda)}\Big).$$

Bernays–Landau for fixed $F_\Lambda$ gives $\mathcal{R}_{F_\Lambda}(U) = \mathcal{C}(\Lambda)\, U/\sqrt{\log U}\,(1 + o(1))$. Since

$$\tfrac{(2(1-c)R - 2\mu(\Lambda))^2}{s(\Lambda)} = \tfrac{4(1-c)^2 R^2}{s(\Lambda)} (1 + O_\Lambda(R^{-1})),$$

the lower main term carries $(1-c)^2$; the $\mu(\Lambda)$ correction is absorbed by $o(1)$. Replacing $U$ by a fixed multiplicative constant changes $\sqrt{\log U}$ by $1+o(1)$, so both denominators may be written as $\sqrt{\log(\frac{4R^2}{s(\Lambda)})}(1+o(1))$. ☐

**Definition 5.2.** For a rank–2 lattice $\Lambda$ and $k \in \mathbb{N}$ put

$$g_\Lambda(k) \;:=\; \max\{\,|X| : \;\exists\,\tau \in \mathbb{R}^2 \;\text{ with } X \subset \tau+\Lambda,\; |D(X)| \le k\,\}.$$

## 6. Residue classes and concentration

### 6.1. **Rigidity of near–optimizers.**

**Proposition 6.1.** *Let $A_0 \subset \mathbb{R}^2$ be finite. Suppose there exists $\mathcal{U} \subset A_0 - A_0$ with $|\mathcal{U}| \ge \beta\,|A_0|$ and*

$$|A_0 \cap (A_0+u)| \;\ge\; \rho\,|A_0| \qquad (\forall\, u \in \mathcal{U}),$$

*for some $\beta, \rho \in (0,1]$. Then there exist nonparallel vectors $u_1, u_2$, a full-rank lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, a $\Lambda$–rectangle $W$, and a set $A \subseteq A_0 \cap W$ such that*

$$|W| \;\ge\; c(\beta,\rho)\,|A_0|, \qquad |A| \;\ge\; c(\beta,\rho)\,|A_0|, \qquad |A \cap (A+u_i)| \;\ge\; c(\beta,\rho)\,|A| \quad (i=1,2).$$

*Proof.* By Proposition 6.2, $E_+(A_0) \ge \beta\rho^2\,|A_0|^3$. Apply Proposition B.3(i) to obtain $A' \subseteq A_0$ with $|A'| \ge c_1(\beta,\rho)|A_0|$ and $|A'-A'| \le K|A'|$ where $K \le C_1(\beta,\rho)$. By Proposition B.3(ii), $A'$ lies in a proper rank–2 GAP $P$ with $|P| \le C_2(\beta,\rho)|A'|$. The two GAP steps give nonparallel $v_1, v_2$ and the lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$. Apply Proposition 4.3 to $A' \subset P$ to obtain a $\Lambda$–rectangle $W \subset P$ with $|W| \ge c_2(\beta,\rho)|A'|$ and $|A' \cap W| \ge c_3(\beta,\rho)|W|$. Set $A := A' \cap W$. Then $|A| \ge c(\beta,\rho)\,|W|$, so $\beta_W := |A|/|W| \ge c(\beta,\rho)$. Apply Lemma 4.2(ii) to $A \subset W$: there exist $s, t \in \{1,\dots,L_2-1\}$ and signs $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that

$$|A \cap (A+\varepsilon_1 s v_1)| \;\ge\; c'(\beta,\rho)\,|A|, \qquad |A \cap (A+\varepsilon_2 t v_2)| \;\ge\; c'(\beta,\rho)\,|A|.$$

Since $s, t \le L_2 - 1$, these overlaps occur entirely inside $W$, so no boundary loss arises. This gives the desired two nonparallel heavy shifts inside a single $\Lambda$–rectangle $W$, with $|W| \ge c(\beta,\rho)|A_0|$ and $|A| \ge c(\beta,\rho)|A_0|$.

☐

**Proposition 6.2.** *Let $A_0 \subset \mathbb{R}^2$ and suppose there exists $\mathcal{U} \subset A_0 - A_0$ with $|\mathcal{U}| \ge \beta\,|A_0|$ and $|A_0 \cap (A_0+u)| \ge \rho\,|A_0|$ for all $u \in \mathcal{U}$. Then*

$$E_+(A_0) \;\ge\; \sum_{u \in \mathcal{U}} r_{A_0}(u)^2 \;\ge\; \beta\,\rho^2\,|A_0|^3.$$

*Consequently, by Proposition B.3, there exists $A' \subseteq A_0$ with $|A'| \ge c(\beta,\rho)|A_0|$ and $|A'-A'| \le K(\beta,\rho)\,|A'|$, and $A'$ lies in a proper rank–2 GAP $P$ with $|P| \le C\,K(\beta,\rho)^C\,|A'|$.*

*Proof.* Immediate from $E_+(A_0) = \sum_v r_{A_0}(v)^2$, the hypothesis, and the standard BSG and Freiman statements (Prop. B.3). ☐

**Lemma 6.3.** *Let $A \subset \mathbb{R}^2$ with $|A-A| \le K|A|$. Then*

$$\Big|\Big\{u \in A-A : \; r_A(u) \;\ge\; \frac{|A|}{2K}\Big\}\Big| \;\ge\; \frac{|A|}{2}.$$

*Proof.* Write $D := |A-A|$ and $M := |\{u : \; r_A(u) \ge T\}|$. For any $T \in (0,|A|)$,

$$|A|^2 = \sum_u r_A(u) \;\le\; M|A| + (D-M)T.$$

With $T = |A|/(2K)$ and $D \le K|A|$ this gives

$$|A|^2 \ \le\ \frac{|A|^2}{2} \ +\ M|A|\Big(1 - \frac{1}{2K}\Big),$$

hence $M \ge |A|/(2 - 1/K) \ge |A|/2$. □

**Lemma 6.4.** *Let $A \subset \mathbb{R}^2$ be finite and fix a direction $u$. For each line $L \parallel u$, write $s_L := |A \cap L|$. Then*

$$\sum_{\substack{v \parallel u \\ v \ne 0}} r_A(v) \ =\ \sum_{L \parallel u} s_L\,(s_L - 1).$$

*Proof.* Partition ordered pairs $(x, y) \in A^2$ with $x \ne y$ by the line $L$ parallel to $u$ that contains them. Pairs from $L$ contribute exactly $s_L(s_L - 1)$, and each such pair has difference $y - x$ parallel to $u$. Summing over $L$ gives the identity. □

## 6.2. **Period lattices and residue decomposition.**

**Proposition 6.5.** *Let $L \subset \mathbb{R}^2$ be a rank–2 lattice and let $X' \subset \mathbb{R}^2$ be finite. Write its canonical decomposition modulo $2L$ as $X' = \bigsqcup_{j=1}^m X_j$ with $1 \le m \le 4$ and $X_j \subset c_j + 2L$. Put $N := |X'|$ and $m_j := |X_j|$. Then*

$$E_+(X') \ \le\ 4\,N^2 \max_{1 \le j \le m} m_j.$$

*In particular, if $\delta := 1 - \max_j(m_j)/N \in [0, 3/4]$, then $E_+(X') \le 4(1 - \delta)N^3$.*

*Proof.* Write $r_{X_i \to X_j}(v) := |\{(x, y) \in X_i \times X_j :\ y - x = v\}|$. Then $r_{X'}(v) = \sum_{j=1}^m r_{X' \to X_j}(v)$ and by Cauchy–Schwarz,

$$E_+(X') = \sum_v r_{X'}(v)^2 \ \le\ 4 \sum_{j=1}^m \sum_v r_{X' \to X_j}(v)^2.$$

For fixed $j$, $\sum_v r_{X' \to X_j}(v) = N m_j$ and $r_{X' \to X_j}(v) \le m_j$, hence $\sum_v r_{X' \to X_j}(v)^2 \le N m_j^2$. Summing over $j$ gives $E_+(X') \le 4N \sum_j m_j^2 \le 4N(\max_j m_j) \sum_j m_j = 4N^2 \max_j m_j$. The final inequality follows by substituting $m_{\max} = (1 - \delta)N$. □

**Corollary 6.6.** *Let $X' \subset P$ decompose as $X' = \bigsqcup_{j=1}^m X_j$ into residue classes modulo $2L$, with $m \le 4$ and $N := |X'|$. If $E_+(X') \ge \alpha\,N^3$ for some $\alpha \in (0, 1]$, then*

$$\max_{1 \le j \le m} |X_j| \ \ge\ \frac{\alpha}{4}\,N.$$

*Proof.* By Proposition 6.5, $E_+(X') \le 4N^2 \max_j |X_j|$. Thus $4N^2 \max_j |X_j| \ge \alpha N^3$, so $\max_j |X_j| \ge (\alpha/4)N$. □

# 7. Hex construction and global bounds

## 7.1. **Arithmetic reduction.**

**Theorem 7.1.** *As $k \to \infty$,*

$$\frac{\pi}{3\,\mathcal{C}(\Lambda_{\rm hex})}\,k\sqrt{\log k}\,(1 + o(1)) \ \le\ g(k) \ \le\ C\,k \log k.$$

*for some absolute constant $C > 0$.*

*Proof.* The lower bound follows from Theorem 3.4 applied to $\Lambda_{\text{hex}}$. For the upper bound, by Guth–Katz [3], any $n$-point planar set determines at least $c\,n/\log n$ distinct distances. Hence if $|D(X)| \le k$ then

$$n \;\le\; C_1\,k\,\log n.$$

Define $f(x) := x - C_1 k \log x$. Then $f'(x) = 1 - \frac{C_1 k}{x}$, so $f'$ is $\ge \frac{1}{2}$ on $[2C_1 k, \infty)$. Set $M := 2C_1 k \log k$. For all sufficiently large $k$ we have $M \ge 2C_1 k$ and

$$C_1 k \log M \;<\; M.$$

For $k$ sufficiently large we have $\log\log k \le \frac{1}{2}\log k$, hence $\log M = \log\big(2C_1 k \log k\big) \le \log k + \log(2C_1) + \log\log k \le \frac{3}{2}\log k + O(1)$, which yields $C_1 k \log M < M$.

If $n \ge M$, then $f(n) \ge f(M) > 0$, contradicting $n \le C_1 k \log n$. Hence $n < M = 2C_1 k \log k$, and taking the supremum yields $g(k) \ll k \log k$. □

7.2. **Quantitative stability.**

**Lemma 7.2.** *Let $X \subset \mathbb{R}^2$ with $|X| = n$ and $|D(X)| \le k$, and assume $k \le C\,n/\log n$. Fix $\sigma \in (0, 1/4]$ and set $\theta_k = (\log k)^{-1/2-\sigma}$. Let $t_\star$ be the $(1-\theta_k)$–quantile of $D(X)$. Then for all sufficiently large $k$, at least one holds:*

*(i) some line $\ell$ satisfies $|X \cap \ell| \ge c\,n$;*
*(ii) there exist two nonparallel vectors $v_1, v_2$ with $r_X(v_i) \ge c\,n$ $(i = 1, 2)$;*
*(iii) there exists $z \in X$ with $\big|X \cap B(z, t_\star)\big| \ge (1 - o(1))\,n$.*

*Here $c, C > 0$ are absolute constants.*

*Proof.* By (2.1) and $k \le Cn/\log n$, $Q_{\text{ord}}(X) \ge c_0 n^3 \log n$. By Lemma 3.2, $Q_{\text{ord}}(X) \le E_+(X) + C_1 n^3 \log n + O(n^2)$. If $E_+(X) \ge c_2 n^3$, apply Proposition B.3(i) to get $A \subseteq X$ with $|A| \ge cn$ and $|A - A| \le K|A|$ where $K \le C$. By Lemma 6.3, either two nonparallel $u$ in $A - A$ satisfy $r_A(u) \gg n$ (yielding (ii) for $X$), or all popular $u$ are parallel. Then Lemma 6.4 gives

$$\sum_{L \parallel u} s_L(s_L - 1) \;=\; \sum_{v \parallel u} r_A(v) \;\ge\; \frac{|A|}{2} \cdot \frac{|A|}{2K} \;\gg\; |A|^2,$$

so $\max_{L\parallel u} s_L \gg |A| \gg n$, yielding (i) with an absolute constant. If $E_+(X) < c_2 n^3$, then

$$\sum_{\text{top } L} m_t \;\le\; \sqrt{Q_{\text{ord}}(X)}\,\sqrt{L} \;\le\; C_3\, n^2\, \theta_k^{1/2}$$

with $L = \lfloor \theta_k k \rfloor$. Since $\theta_k^{1/2} = o(1)$, the bottom $(1-\theta_k)k$ radii support $(1 - o(1))n(n-1)$ ordered pairs. Proposition 3.3 yields (iii). □

**Theorem 7.3.** *There exist absolute constants $c, C > 0$ such that the following holds. Let $X \subset \mathbb{R}^2$ with $|X| = n$ and $|D(X)| \le k$, and assume $k \le C\,n/\log n$. Then either*

*(i) some line $\ell$ contains at least $c\,n$ points of $X$; or*
*(ii) there exist nonparallel vectors $v_1, v_2$, a rank–2 lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, a $\Lambda$–rectangle $W$, and a set $A \subseteq X \cap W$ with $|W| \ge c\,n$ and $|A| \ge c\,n$ such that $|A \cap (A + v_i)| \ge c\,|A|$ for $i = 1, 2$. Moreover, with $N := |X \cap W|$ and the residue decomposition modulo $2\Lambda$,*

$$E_+(X \cap W) \;\le\; 4\,N^2 \max_j |X_j|, \quad \textit{so if } E_+(X \cap W) \ge \alpha N^3 \textit{ then } \max_j |X_j| \ge (\alpha/4)N;$$

*(iii) There exists $z \in X$ such that $\big|X \cap B(z, t_\star(X, \theta_k))\big| \ge (1 - o(1))\,n$.*

*Proof.* As above, $Q_{\mathrm{ord}}(X) \gg n^3 \log n$ and $Q_{\mathrm{ord}}(X) \le E_+(X) + Cn^3 \log n + O(n^2)$. If $E_+(X) \gg n^3$, apply Proposition 6.1 to obtain (ii) and then Proposition 6.5 for the residue estimate. Otherwise use Lemma 7.2 to obtain (i) or (iii). $\square$

## Appendix A. Bernays–Landau asymptotic

**Theorem A.1** (Bernays–Landau). *Let $F$ be a primitive positive definite integral binary quadratic form and $\mathcal{R}_F(T) := \#\{n \le T : n \text{ is represented by } F\}$. Then*

$$\mathcal{R}_F(T) = C(F)\,\frac{T}{\sqrt{\log T}}\,(1 + o(1)) \qquad (T \to \infty),$$

*where $C(F) > 0$ depends only on $F$.*

*Proof.* Bernays [5]; see also Tenenbaum [4, Ch. III.4]. $\square$

## Appendix B. Combinatorial and geometric facts

*Proof of Lemma 4.1.* The first identity $Q^* = \sum_t m_t^2$ is immediate from the definition of $m_t$, which counts ordered pairs $(p, q)$ with $p \ne q$ and $|p - q| = t$.

For the isometry identity, each ordered pair of ordered pairs $((p, q), (p', q'))$ with $p \ne q$, $p' \ne q'$, $|p-q| = |p'-q'| > 0$ determines a unique direct isometry $g \in \mathrm{SE}(2)$ with $g(p) = p'$ and $g(q) = q'$. Moreover such a $g$ satisfies $r_g \ge 2$, so $g \in \mathcal{G}$. Conversely, any $g \in \mathcal{G}$ and any ordered distinct pair $(x, y)$ with $g(x), g(y) \in X$ produce one such quadruple. Thus $Q^* = \sum_{g\in\mathcal{G}} r_g(r_g - 1)$.

Summing gives $\sum_{g\in\mathcal{G}} r_g^2 = Q^* + \sum_{g\in\mathcal{G}} r_g$. Each ordered pair $(x, y) \in X^2$ with $x \ne y$ contributes to at most $O(1)$ isometries in $\mathcal{G}$ (those determined by $(x, y)$ together with a second matched ordered pair), hence $\sum_{g\in\mathcal{G}} r_g = O(n^2)$ and $\sum_{g\in\mathcal{G}} r_g^2 = Q^* + O(n^2)$.

For translations, $\sum_{g\in\mathsf{T}} r_g^2 = \sum_v r_X(v)^2 = E_+(X)$, with the $O(n^2)$ adjustment if one includes $r_g = 1$ maps, which we do not since $g \in \mathcal{G}$. Subtracting yields the non-translation identity. $\square$

**Lemma B.1.** *Let $\Lambda \subset \mathbb{R}^2$ be a lattice with covolume $\mathcal{A}(\Lambda)$. There exist constants $r_\Lambda, R_\Lambda \asymp_\Lambda 1$ and $C_\Lambda > 0$ such that for every translate $\tau + \Lambda$ and every bounded convex set $K \subset \mathbb{R}^2$ with piecewise $C^1$ boundary,*

$$\big|(\tau + \Lambda) \cap K\big| \;=\; \frac{\mathrm{area}(K)}{\mathcal{A}(\Lambda)} \;+\; O_\Lambda(1 + \mathrm{perim}(K)),$$

*uniformly in $\tau$.*

*Proof.* Fix a fundamental domain $\mathcal{F}$ for $\Lambda$ with $B(0, r_\Lambda) \subset \mathcal{F} \subset B(0, R_\Lambda)$. Let $\mathcal{F}_\lambda := \lambda + \mathcal{F}$ for $\lambda \in \tau + \Lambda$. Cells with $\mathcal{F}_\lambda \subset K$ contribute exactly $\mathrm{area}(K)/\mathcal{A}(\Lambda)$ up to an error bounded by the number of boundary cells. A cell intersects $\partial K$ only if $\lambda \in \partial K + B(0, R_\Lambda)$. Hence the number of boundary cells is at most

$$\frac{\mathrm{area}(\partial K + B(0, R_\Lambda))}{\mathrm{area}(\mathcal{F})} \;\ll_\Lambda\; 1 + \mathrm{perim}(K),$$

since $\mathrm{area}(\partial K + B(0, R_\Lambda)) \le c_\Lambda\,(R_\Lambda^2 + R_\Lambda\,\mathrm{perim}(K))$ for convex $K$ with piecewise $C^1$ boundary. Each boundary cell changes the count by at most 1, giving the stated error uniformly in $\tau$. $\square$

**Corollary B.2.** *Let $\rho > 0$ and $u \in \mathbb{R}^2$ with $|u| \le 2\rho$. For any $z \in \mathbb{R}^2$, the lens $L := B(z, \rho) \cap (B(z, \rho) - u)$ satisfies*

$$\big|(\tau + \Lambda) \cap L\big| \;=\; \frac{\mathrm{area}(L)}{\mathcal{A}(\Lambda)} \;+\; O_\Lambda(\rho),$$

*uniformly in* $\tau, z, u$*. The perimeter obeys* $\mathrm{perim}(L) \le 4\pi\rho$*. In addition,*

$$\mathrm{area}(L) \;=\; 2\rho^2 \arccos\Big(\frac{|u|}{2\rho}\Big) \;-\; \frac{|u|}{2}\sqrt{4\rho^2 - |u|^2},$$

*so if* $|u| \le (2-\delta)\rho$ *with fixed* $\delta \in (0,1]$ *then* $\mathrm{area}(L) \gg_\delta \rho^2$*. Hence* $\big|(\tau + \Lambda) \cap L\big| \gg_{\Lambda,\delta} \rho^2$*.*

*Proof.* Apply Lemma B.1 with $K = L$. The boundary of $L$ consists of two circular arcs of radius $\rho$, so $\mathrm{perim}(L) \le 4\pi\rho$. The explicit area formula is standard and yields the stated lower bound when $|u| \le (2-\delta)\rho$. □

**Proposition B.3.** *There exist absolute constants* $C, c > 0$ *such that:*

*(i) (BSG) If* $A \subset \mathbb{R}^2$ *satisfies* $E_+(A) \ge \kappa\,|A|^3$ *with* $0 < \kappa \le 1$*, then there exists* $A' \subseteq A$ *with*

$$|A'| \;\ge\; c\,\kappa^C\,|A| \qquad \textit{and} \qquad |A' - A'| \;\le\; C\,\kappa^{-C}\,|A'|.$$

*(ii) (Freiman in* $\mathbb{R}^2$*) If* $A' \subset \mathbb{R}^2$ *has* $|A' - A'| \le K\,|A'|$*, then* $A'$ *lies in a proper rank–2 GAP* $P$ *with* $|P| \le C\,K^C\,|A'|$*.*

*Proof.* (i) This is the Balog–Szemerédi–Gowers theorem. See, e.g., [6] or TaoVu. Applied to the additive energy $E_+(A) \ge \kappa|A|^3$ it yields $A' \subseteq A$ with $|A'| \ge c\,\kappa^C|A|$ and $|A' - A'| \le C\,\kappa^{-C}|A'|$, for absolute constants $c, C > 0$.

(ii) This is Freiman's theorem in $\mathbb{R}^2$. If $|A' - A'| \le K|A'|$, then $A'$ is contained in a proper rank-2 generalized arithmetic progression $P$ with $|P| \le C\,K^C|A'|$. See [7]. □

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