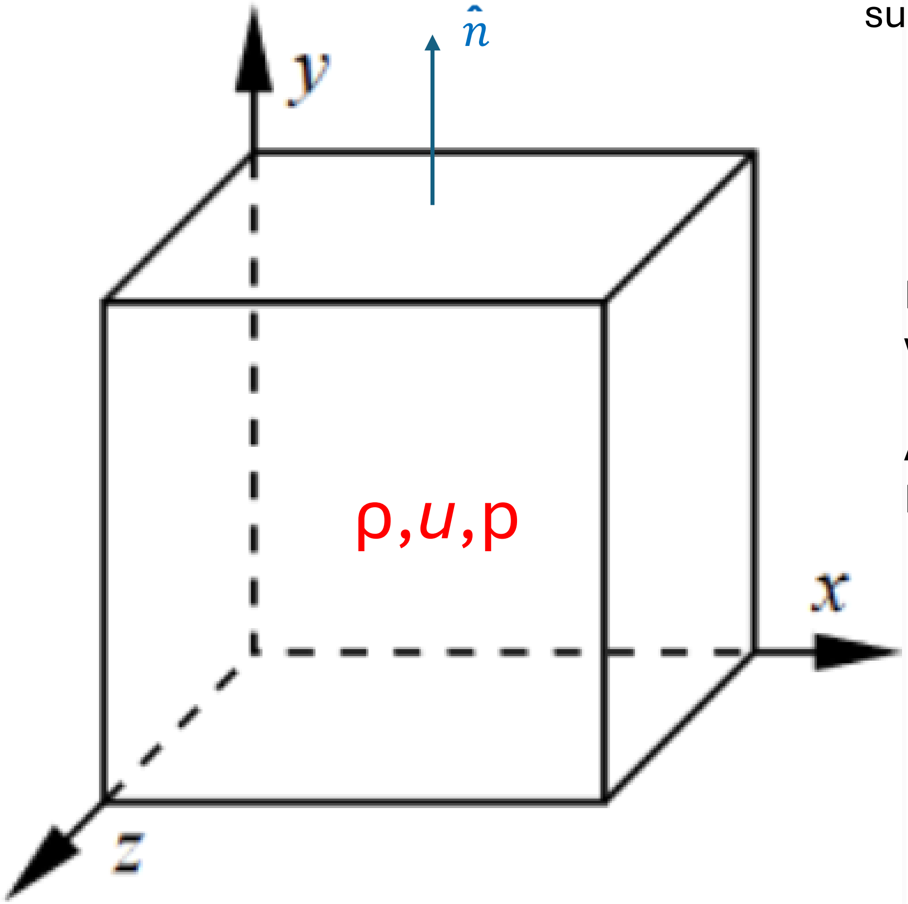
# Module 1, Lecture 2 - Macroscopic view of Hydrodynamics

Learning Objectives:

- Concept of flux.
- Pressure as a tensor.
- Energy Equation

# Euler Equation (Momentum Conservation)




Apply momentum conservation to the cube. Now there are sources of momentum: the forces. These forces can be body forces (e.g. gravity) and surface forces (pressure).

The change in momentum is equal to the momentum flux across the surfaces, plus surface and body forces.

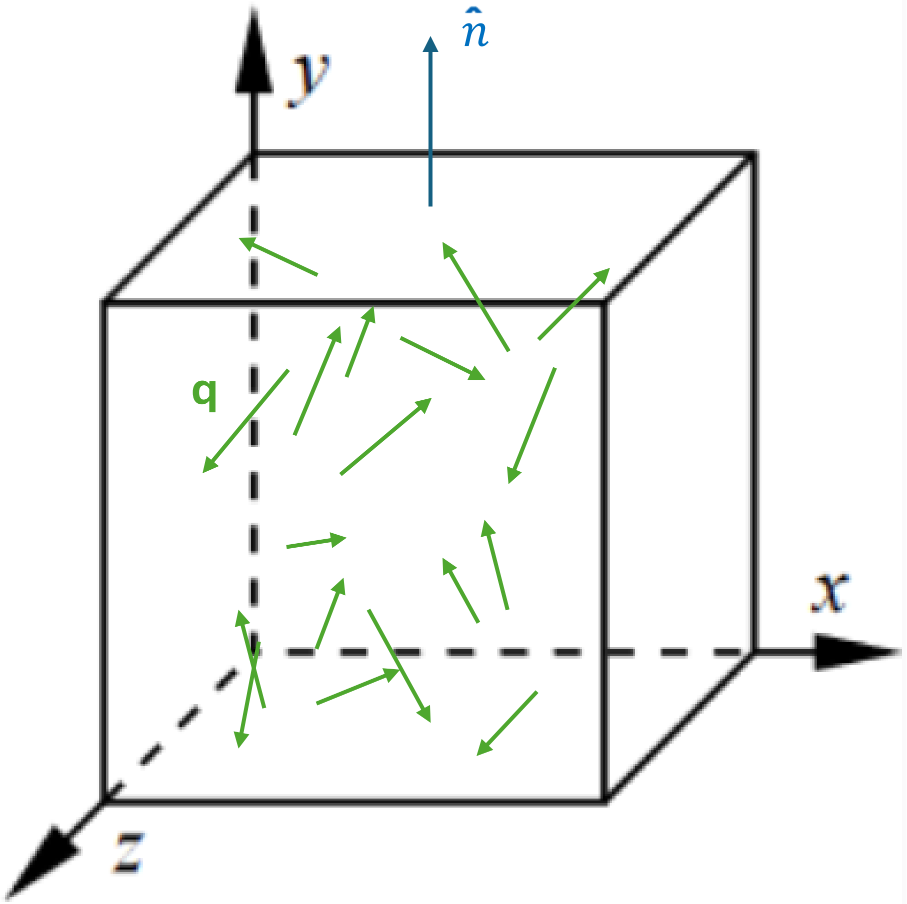
$$\frac{d}{dt}\int_V (\rho \boldsymbol{u})\, dV = -\oint_A (\rho \boldsymbol{u})\, \boldsymbol{u} \cdot d\boldsymbol{A} - \oint_A \mathbf{P} \cdot d\boldsymbol{A} + \int_V \rho \boldsymbol{f}\, dV$$

Mathematically, the body force has to be expressed by a 2nd rank tensor to dot with the area and produce a vector. For a pure normal force (pressure), $\mathbf{P}_{ij} = p\,\delta_{ij}$.

Applying Gauss theorem, we put all terms under the same volume integral, to find Euler's equation

$$\boxed{\frac{\partial(\rho \boldsymbol{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\boldsymbol{\nabla} p + \rho \boldsymbol{f}}$$

# Euler Equation – The momentum flux




The $\rho \boldsymbol{u} \otimes \boldsymbol{u}$ term is momentum flux.

Consider any vector field ***q*** (shown in green inside the cube).

The interpretation of ***qu*** as the flux of ***q*** is intuitive, though in this case ***qu*** has to be a 2nd rank tensor. In index notation, ***qu*** is the tensor **C** with components $C_{ij} = q_i u_j$. It means the component *i* of the vector ***q*** flowing in the direction *j*.

The vector ***q*** flows in all directions. It is only the flux normal to the area that flows in/out of the box.

Substituting ***q***=*ρ**u*** does not change this interpretation. It adds, however, the non-intuitive notion that the *velocity advects itself*. This is a manifestation of Newton's first law: an object in motion remains in motion.

# Euler Equation – Momentum flux as ram pressure

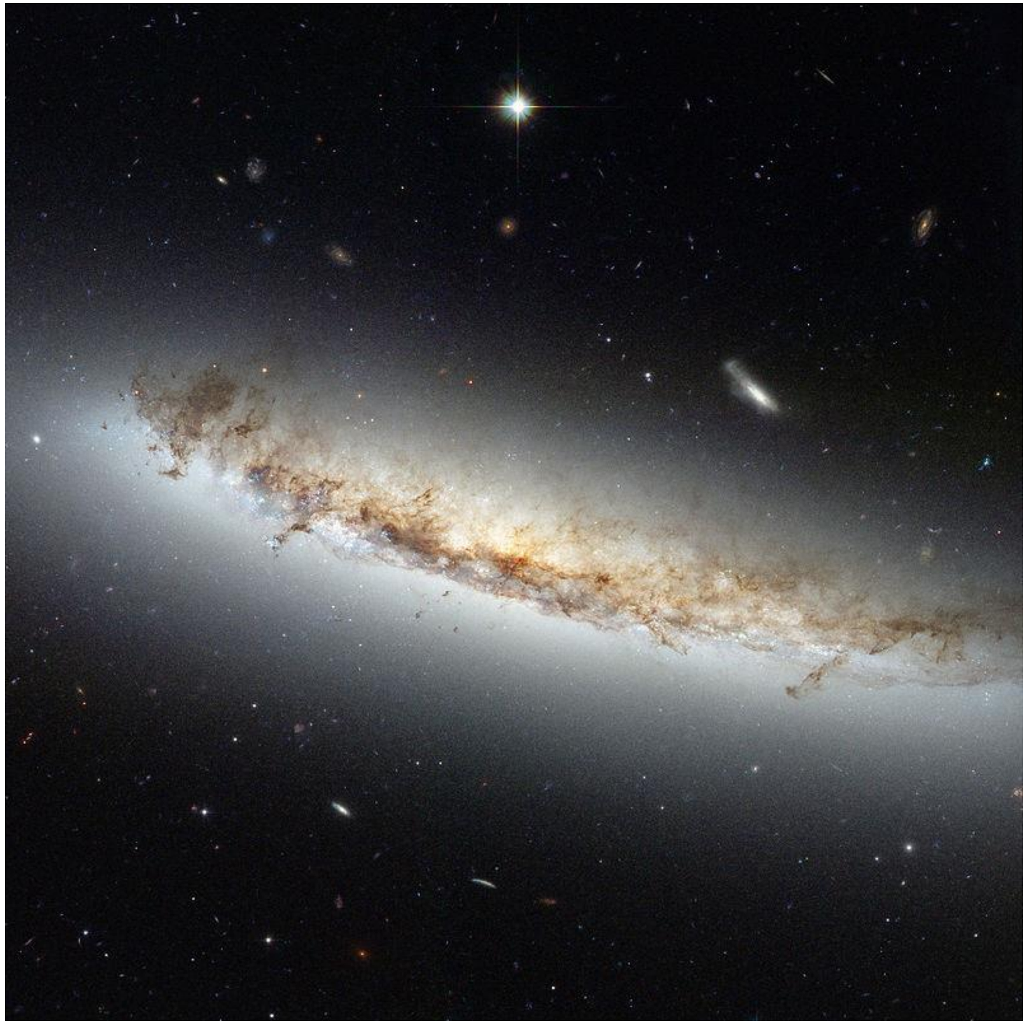


Figure 1.2: NGC 4402. The galaxy is falling into the Virgo cluster, and undergoing ram pressure stripping by the intracluster medium (ICM), which is high pressure (hot and dense).

The body force has to be expressed by a $2^{nd}$ rank tensor to dot with the area and produce a vector.

$$\frac{\partial(\rho \boldsymbol{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} \cdot \mathbf{P} - \rho \boldsymbol{f}$$

So $\rho \boldsymbol{u} \otimes \boldsymbol{u}$ has unit of pressure.

This is the ***ram pressure***.

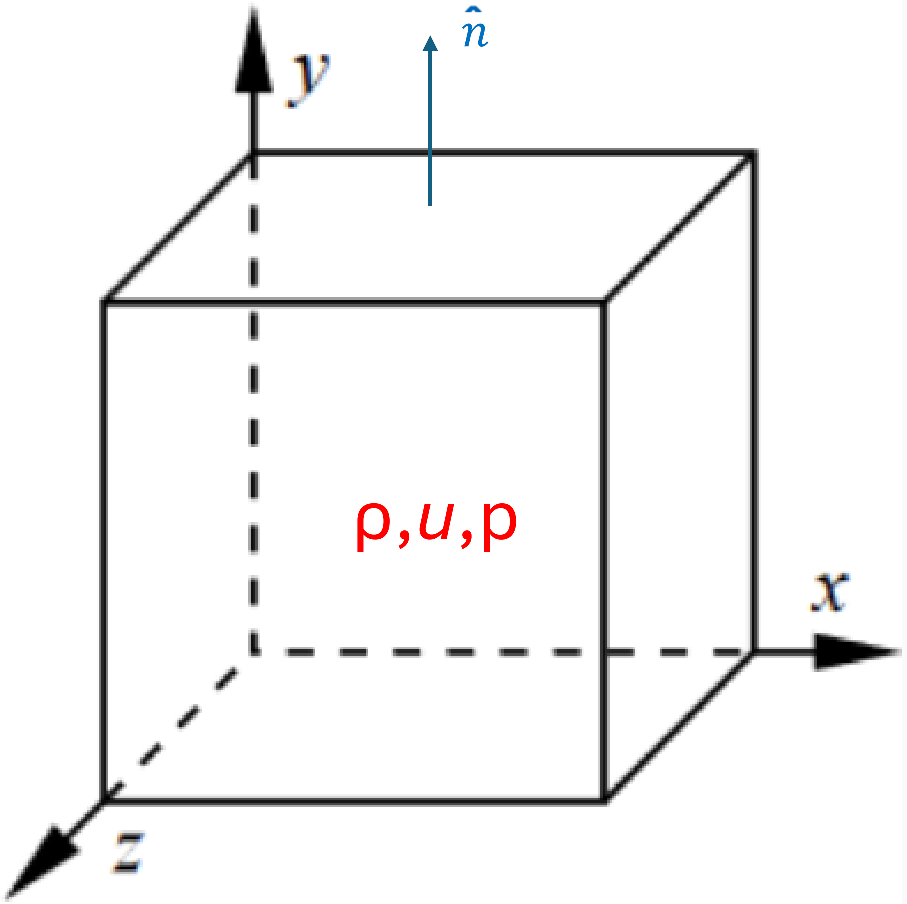
The combined terms $\rho u_i u_j + P_{ij}$ is called the stress tensor.

Notice why they have the same units. Microphysically, ram pressure and thermal pressure are similar. The **thermal pressure** is caused by molecules hitting the surface due to their **random motions** within the fluid. The **ram pressure** is caused by molecules hitting the surface due to the **bulk motion of the fluid**.

## Energy Equation

The conservation of energy is a statement of the first law of thermodynamics: energy change is added heat and work done:

$$du = dq - dw$$




Let us ignore heating and cooling for the moment and set $dq=0$. Applying $du=-dw$ to the control volume implies

*du:* change in total fluid energy (time change of volume integral plus surface integral of the energy flux),

*dw:* work done by the pressure and body forces.

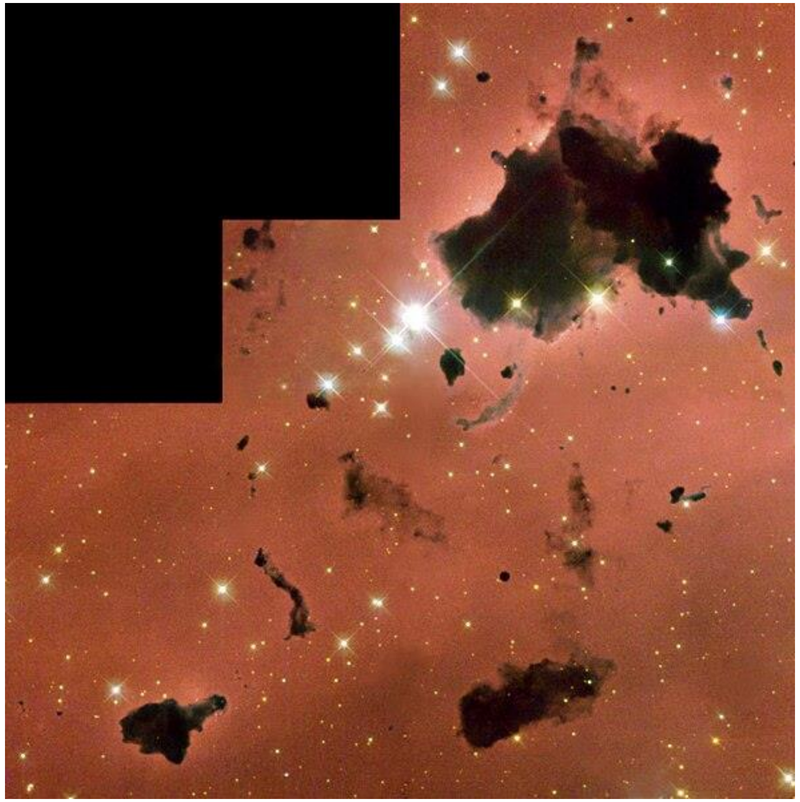
Consider the only energies to be kinetic energy of the bulk flow, and the thermal (internal) energy *e* of the fluid. Then:

$$\underbrace{\frac{d}{dt}\int_V \left(\frac{\rho u^2}{2} + \rho e\right) dV}_{\text{change in the total fluid energy}} = -\underbrace{\oint_A \left(\frac{\rho u^2}{2} + \rho e\right) \boldsymbol{u}\cdot d\boldsymbol{A}}_{\text{energy flux through the surface}} - \underbrace{\oint_A \boldsymbol{u}\cdot(\mathbf{P}\cdot d\boldsymbol{A})}_{\text{work done by pressure}} + \underbrace{\int_V \rho\boldsymbol{u}\cdot\boldsymbol{f}\,dV}_{\text{work done by body forces}}$$

Call $E=\rho u^2/2 + \rho e$ and apply Gauss theorem

$$\boxed{\frac{\partial E}{\partial t} + \nabla\cdot(E\boldsymbol{u}) = -\nabla\cdot(p\boldsymbol{u}) + \rho\boldsymbol{u}\cdot\boldsymbol{f}}$$

# Typical external forces in astrophysics: gravity



$\boldsymbol{f} = \boldsymbol{g}$

Acceleration $\boldsymbol{g}$ given by Poisson's equation

$$\nabla \cdot \boldsymbol{g} = -4\pi G\rho$$

## Typical external forces in astrophysics: radiation pressure

$$\boldsymbol{f}_{\rm rad} = \frac{1}{c}\int \kappa_\nu \boldsymbol{F}_\nu d\nu$$

## External forces

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) &= 0 \\
\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) &= -\boldsymbol{\nabla} p + \rho \boldsymbol{g} + \rho \boldsymbol{f}_{\mathrm{rad}} \\
\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (E \boldsymbol{u}) &= -\boldsymbol{\nabla} \cdot (p \boldsymbol{u}) + \rho \boldsymbol{u} \cdot \boldsymbol{g} + \rho \boldsymbol{u} \cdot \boldsymbol{f}_{\mathrm{rad}} + \mathcal{H} - \mathcal{C}
\end{aligned}
$$

gravity

radiation pressure

heating and cooling by radiation

# Building Intuition

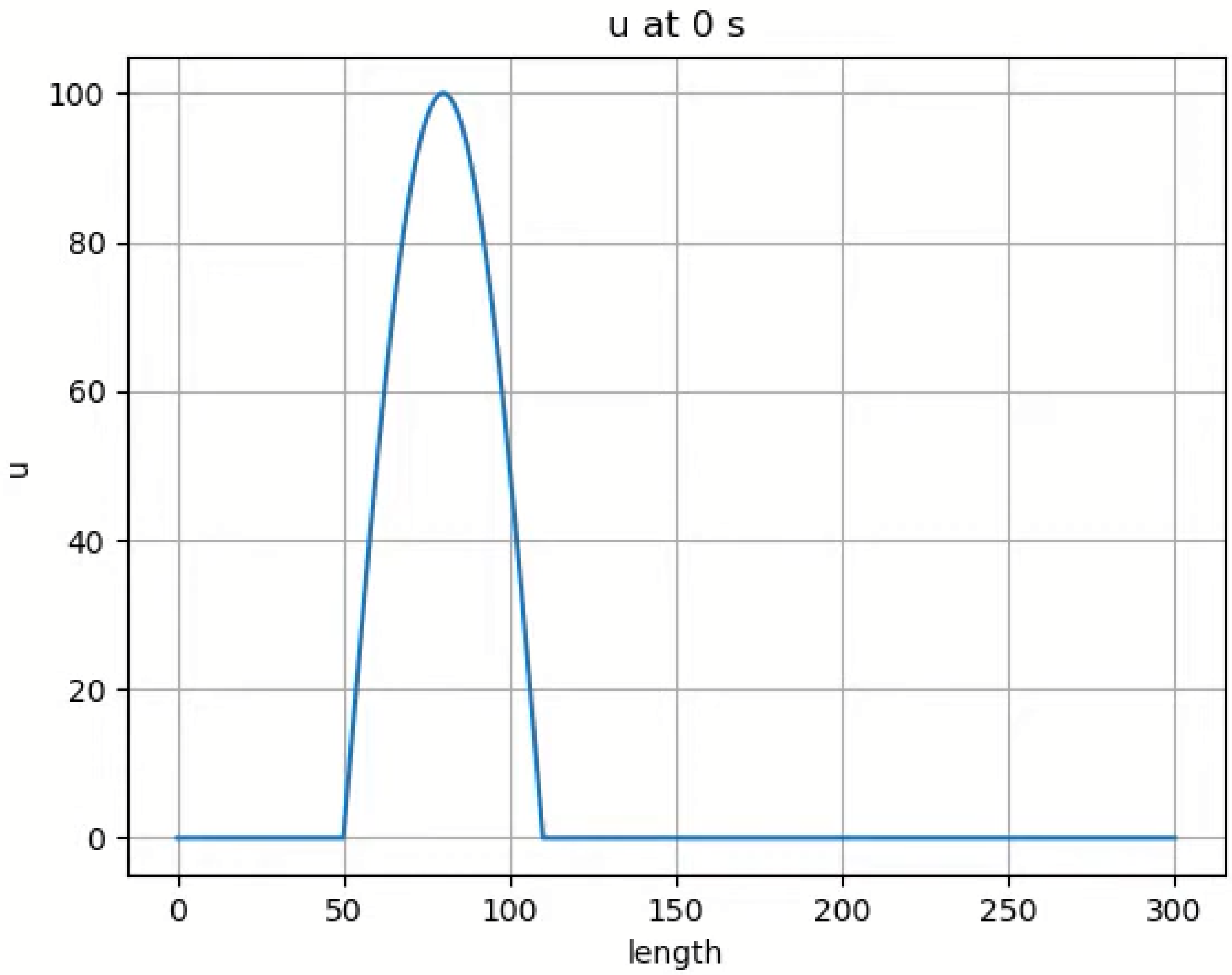
In the following equation, where $\rho$ is the density and $\boldsymbol{u}$ the velocity

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\, \rho = -\rho\, (\nabla \cdot \mathbf{u})$$

the second term in the left-hand side represents

A) Compression
***B) Advection***
C) Pressure
D) Stretching

# Advection equation




In 1D, the advection equation is

$$\frac{\partial f(x,t)}{\partial t} = -u\frac{\partial f(x,t)}{\partial x}$$

General solution:

*f(x,t) = f(x-vt,0)*

# Building Intuition

In the following equation, where $\rho$ is the density and $\boldsymbol{u}$ the velocity

$$\frac{\partial\rho}{\partial t} + (\mathbf{u} \cdot \nabla)\,\rho = -\rho\,(\nabla \cdot \mathbf{u})$$

the term in the right-hand side represents

***A) Compression***
B) Advection
C) Pressure
D) Stretching

# The compression term

$\nabla \cdot \vec{v} < 0$

**Compression**

Negative divergence (convergence)

Density *increases*

$\nabla \cdot \vec{v} > 0$

**Expansion**

Positive divergence

Density *decreases*

# Building Intuition

If the density is constant, the fluid is incompressible. Given the continuity equation

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\, \rho = -\rho\, (\nabla \cdot \mathbf{u})$$

which expression below describes incompressibility?

A) $\nabla . u = 0$

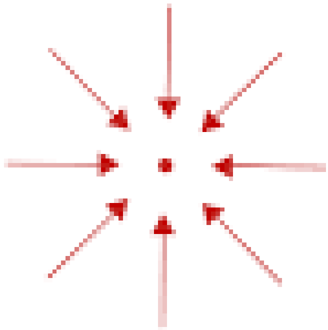
B) $\nabla \times u = 0$

C) $\partial\rho/\partial t=0$

D) $(u.\nabla)\ \rho = 0$

# Compressibility

$$\nabla \cdot \vec{v} < 0$$

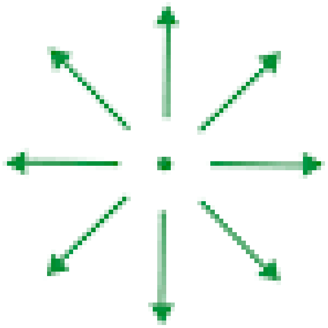


**Compression**

Negative divergence (convergence)

Density *increases*

$$\nabla \cdot \vec{v} > 0$$

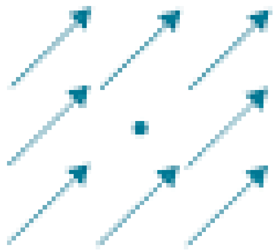


**Expansion**

Positive divergence

Density *decreases*

$$\nabla \cdot \vec{v} = 0$$



**Incompressibility**

Zero divergence

Constant density