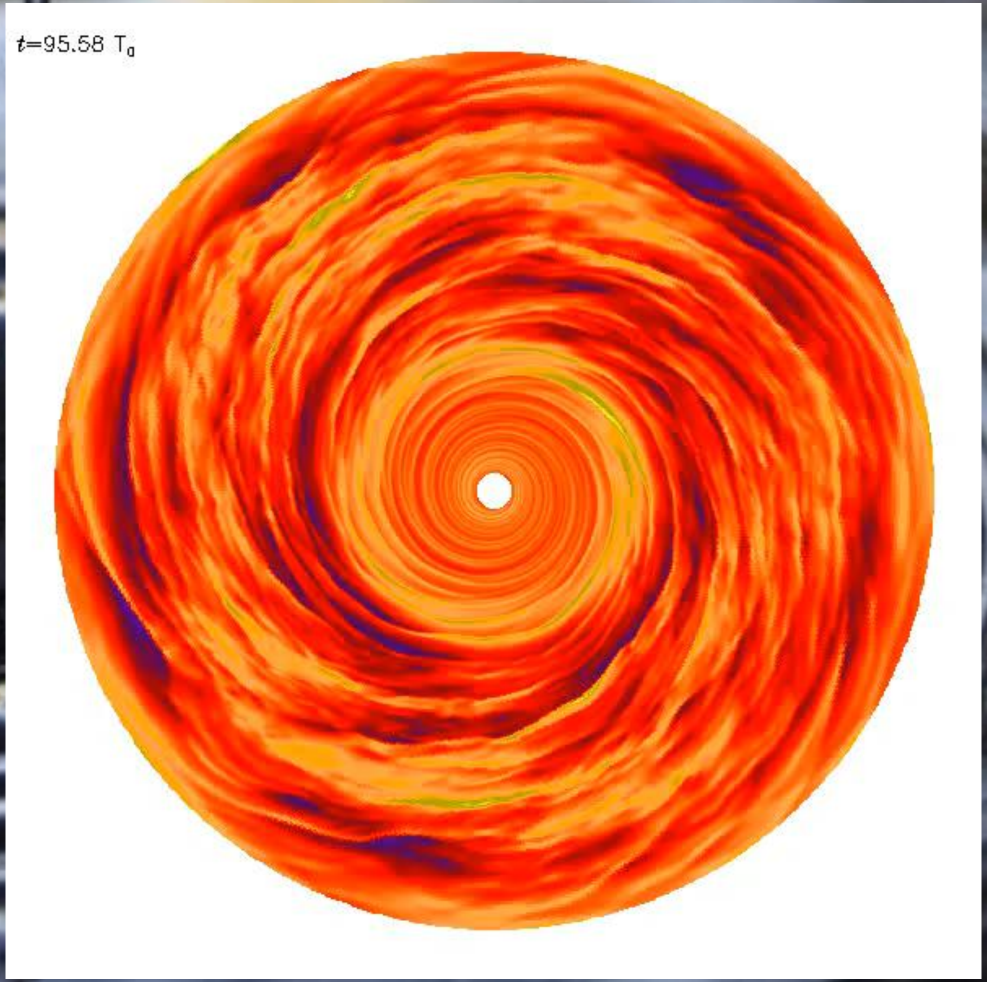
# Dynamics and Hydrodynamics

## Prof Wladimir Lyra

Class hours: Tue/Thu
10:30am - 11:45am

NMSU
ASTRONOMY




Lyra, W., Turner, N. & McNally, C. 2015, A&A, 574, A10
*Rossby wave instability does not require sharp resistivity gradients*

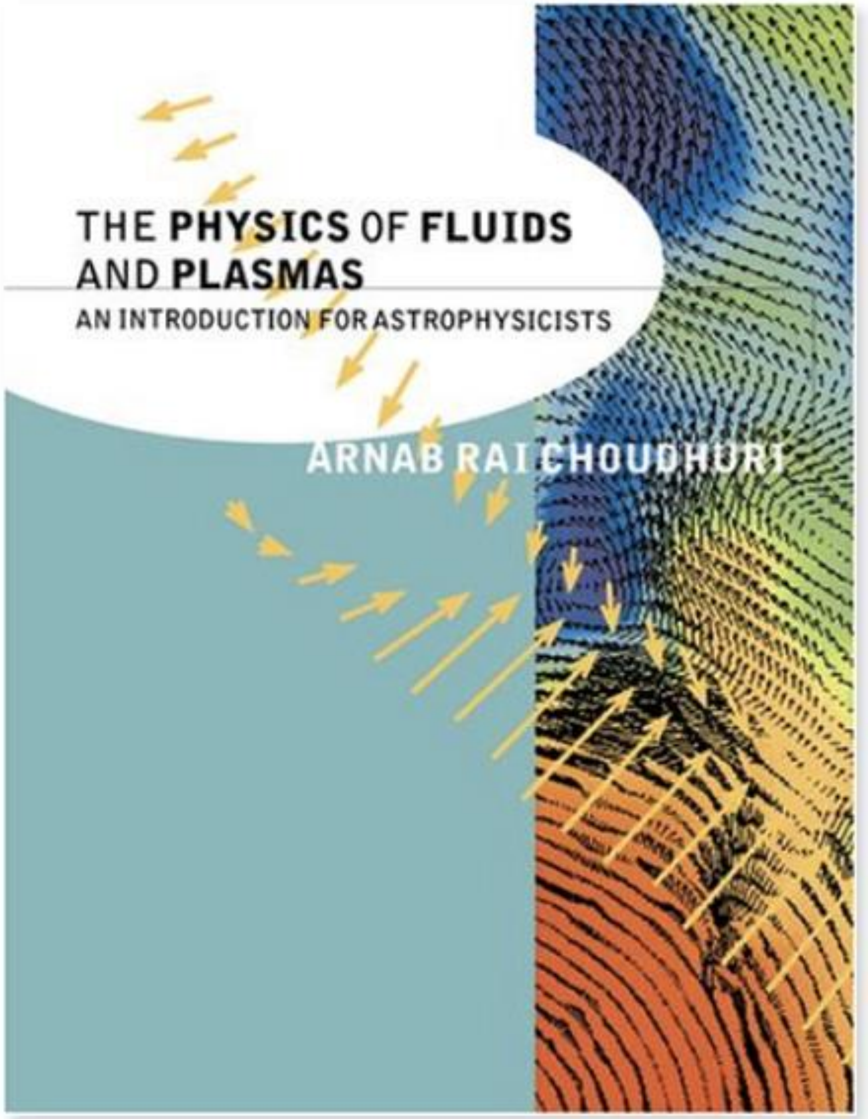
# Course Sequence

| Class Flow | Class # | | Planned |
|---|---|---|---|
| Part 1 - Basic equations | 1 | 16-Jan | Intro to hydro:  Evolution equations. |
| | 2 | 21-Jan | Lagrangian vs Eulerian formulation. |
| | 3 | 23-Jan | Viscous flows; Viscous tensor, Reynolds number. |
| | 4 | 28-Jan | Vorticity Equation, Kelvin's vorticity theorem, baroclinicity. |
| | 5 | 30-Jan | Accretion disks. |
| | 6 | 4-Feb | Shocks, Rankine-Hugoniot jump conditions. |
| | 7 | 6-Feb | Point explosion, Sedov problem, Supernova Remnants. |
| | 8 | 11-Feb | Checkpoint A |
| | 9 | 13-Feb | Travel -- (cancel class, online class?) |
| Part 2 - Waves and instabilities | 10 | 18-Feb | Acoustic waves; Linearization. Dispersion relation. |
| | 11 | 20-Feb | Turbulence, Reynolds stress, Kolmogorov cascade. |
| | 12 | 25-Feb | Rayleigh-Taylor & Kelvin Helmholtz. |
| | 13 | 27-Feb | Jeans instability. |
| | 14 | 4-Mar | Toomre instability. |
| | 15 | 6-Mar | Midterm |
| | Spring break | | |
| Part 3 - Stellar Dynamics | 16 | 18-Mar | Potential theory, Jeans equation. |
| | 17 | 20-Mar | Halo and disk potential. NFW Dark Matter Halo. |
| | 18 | 25-Mar | Travel -- (cancel class? online class?) |
| | 19 | 27-Mar | Travel --  (cancel class? online class?) |
| | 20 | 1-Apr | Potential of ellipsoids, Homoeoids. |
| | 21 | 3-Apr | Stellar Orbits. |
| | 22 | 8-Apr | Two-body relaxation, Dynamical Friction. |
| | 23 | 10-Apr | Checkpoint B |
| Part 4 - Numerics | 24 | 15-Apr | Discretization techniques. |
| | 25 | 17-Apr | Compiled languages (Fortran and C). |
| | 26 | 22-Apr | Spatial discretization - High order finite difference coefficients. |
| | 27 | 24-Apr | Time discretization - Euler and Runge-Kutta. |
| | 28 | 29-Apr | Von Neumann stability analysis and Courant number. |
| | 29 | 1-May | Diffusion, the viscosity operator, treatment of shocks. |
| | | | Final |

# Grading

*Checkpoint A (10%)*
*Midterm (20%)*
*Checkpoint B (10%)*
*Final (20%)*
*Homework Assignments (15%)*
*Project (25%)*

# Book – Part I and II



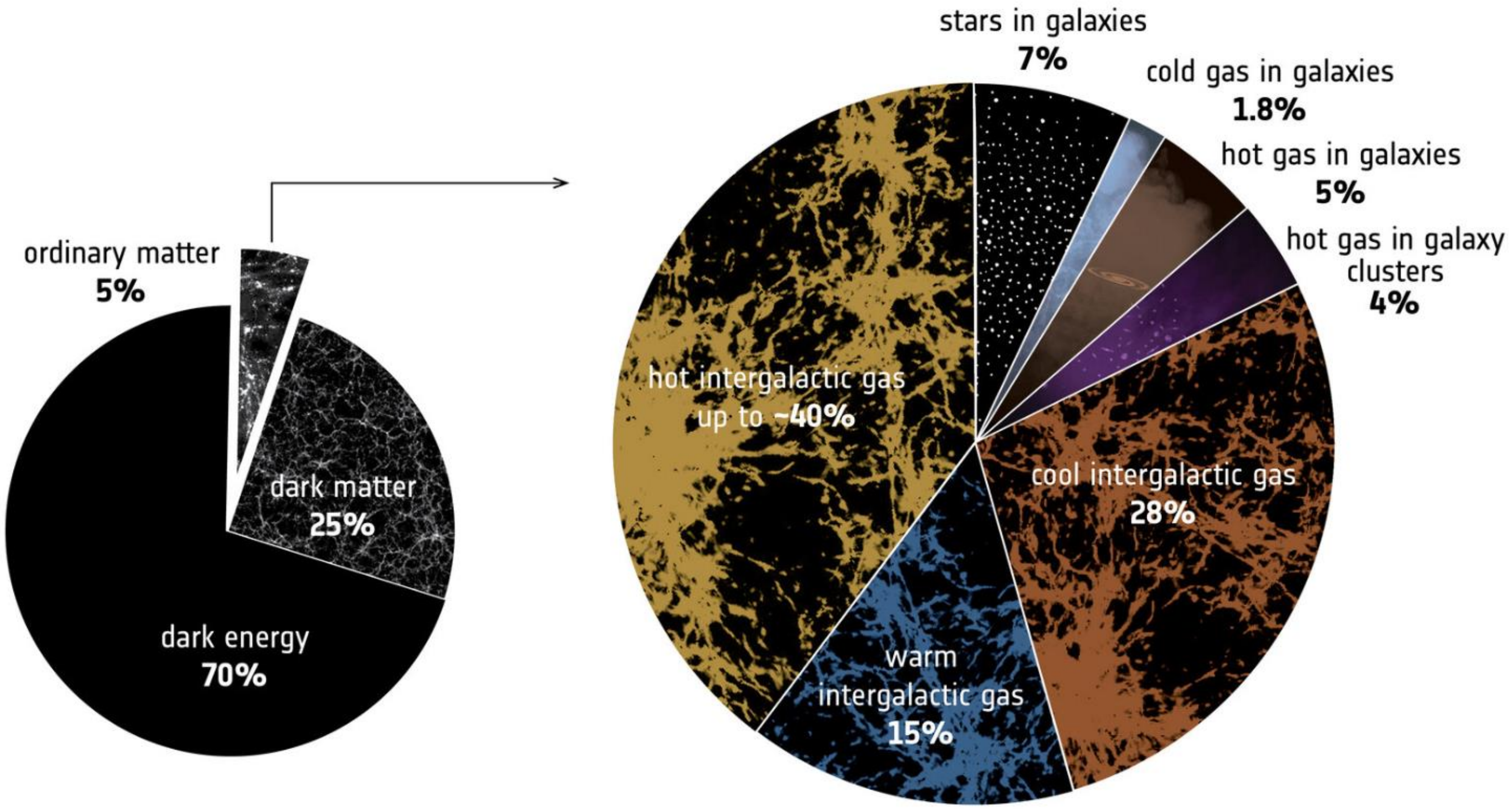
## Lecture 1 - Macroscopic view of Hydrodynamics

Learning Objectives:

When is a macroscopic fluid description valid.

Understand the equations of hydrodynamics from conservation laws.

# Baryonic Matter in the Universe is mostly gas



## The equations of hydrodynamics

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) &= 0 \\
\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) &= -\boldsymbol{\nabla} p + \rho \boldsymbol{f} \\
\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (E \boldsymbol{u}) &= -\boldsymbol{\nabla} \cdot (p \boldsymbol{u}) + \rho \boldsymbol{u} \cdot \boldsymbol{f}
\end{aligned}
$$

Notice that if we define $Q\boldsymbol{u}$ as the flux of quantity $Q$, they all have this form:

$$
\frac{\partial}{\partial t}\,(\text{density of quantity}) + \boldsymbol{\nabla} \cdot\ (\text{flux of quantity}) = \text{sources} - \text{sinks}
$$

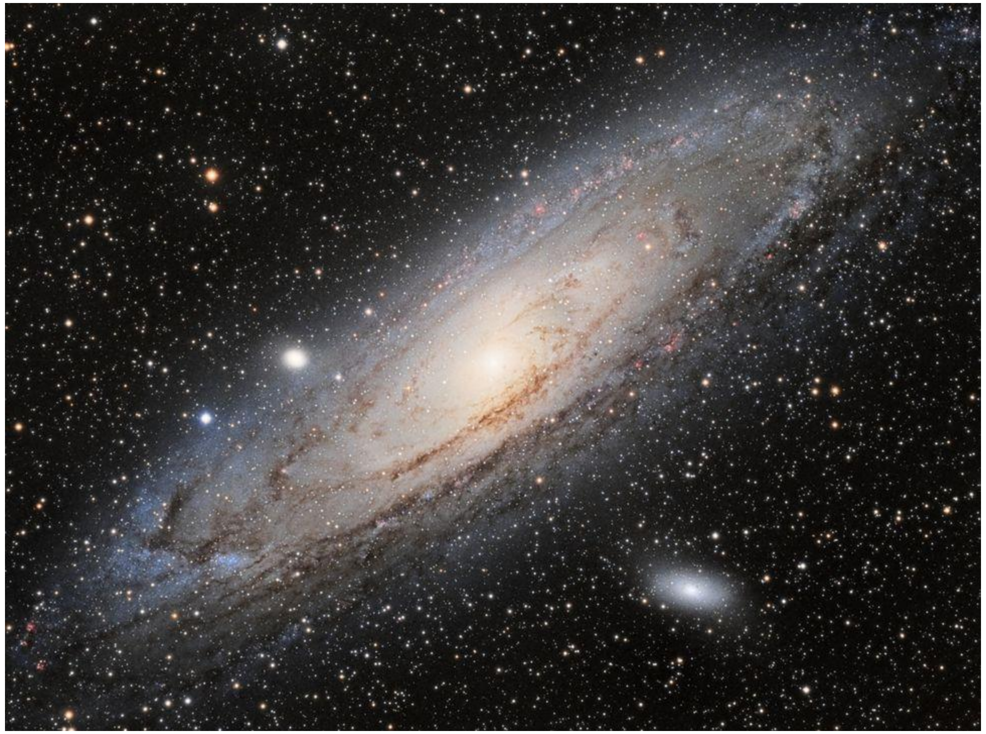
| Level | Description of state | Dynamical equations |
|---|---|---|
| 0: *N* quantum particles | $\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)$ | Schrödinger's equation |
| 1: *N* classical particles | $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N;}\,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_N$ | Newton's laws or Hamilton's equations |
| 2: Distribution function | $f(\boldsymbol{x},\boldsymbol{u},t)$ | Boltzmann's equation |
| 3: Continuum model | $\rho(\boldsymbol{x}),u(\boldsymbol{x}),p(\boldsymbol{x})$ | Hydrodynamics equations |

Level 0 to 1: Separation between particles much larger than de Broglie wavelength

Level 1 to 2: when *N* becomes too large, then it's impractical to solve $N \ll 1$ equations. Define then the distribution function f for the particle number density in phase space. This is only possible when the system is either collisionless or if binary collisions are the only interaction between particles.

Level 2 to 3: Through frequent binary collisions, the system relaxes to the Maxwell-Boltzmann distribution. This occurs if the mean-free path is small compared to the length scale of the system. Otherwise, a distribution function needs to be applied (in this case a continuity equation still can be used, but the "fluid" contains no pressure).

## Interstellar Medium



Number density: 10 $cm^{-3}$

Mean free path: $\lambda = 1/n\sigma$ = $10^{14}$ cm ~ 10AU

Typical scales: 100 pc

*Hydro is applicable!*

# Atmosphere

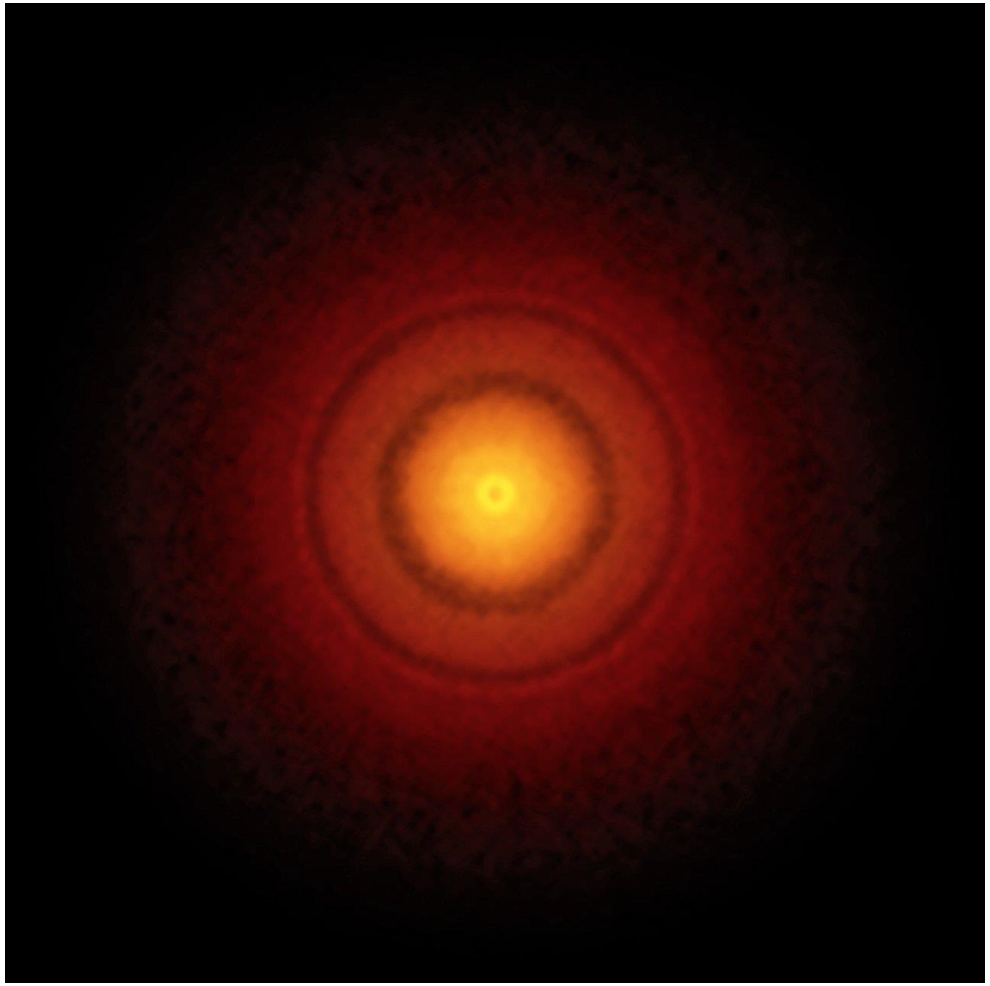
Number density: $10^{19}$ $cm^{-3}$

Mean free path: $\lambda=1/n\sigma = 10^{-4}$ cm

Typical scales: m-km

*Hydro is applicable!*

# Protoplanetary Disk



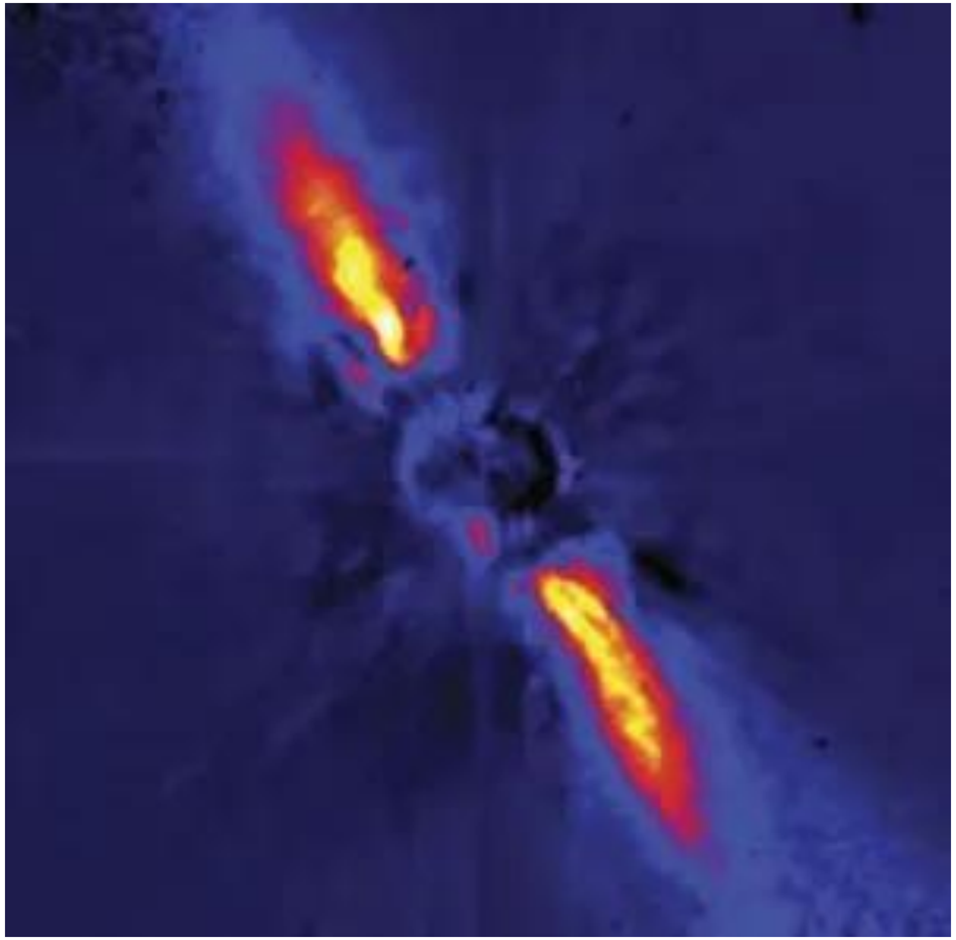
Number density: $10^{15}$ $cm^{-3}$

Mean free path: $\lambda = 1/n\sigma$ = 1 cm

Typical scales: AU

*Hydro is applicable!*

# Gas in Debris Disks



Number density: 100 $cm^{-3}$

Mean free path: $\lambda = 1/n\sigma = 10^{13}$ cm ~ 1AU

Typical scales: 10 AU

Borderline…
Proceed with caution.

# Cometary Tail

Number density: 100 $cm^{-3}$

Mean free path: $\lambda = 1/n\sigma = 10^{13}$ cm ~ 1AU

Typical scales: $10^{9-10}$ cm

Hydrodynamics not applicable

# Consider a typical fluid

The Rio Grande in Las Cruces

# … and define a fluid element

A fluid element is a region over which we can define local variables (density, velocity, temperature, etc).

To be uniform, this patch must be

• Small enough that we can ignore systematic variations across it for any variable $Q$ we are interested in: $l_{\mathrm{patch}} \ll Q / |\nabla Q|$

• large enough to contain sufficient particles to ignore fluctuations due to the finite number of particles $n\, l^{3}_{\mathrm{patch}} \gg 1$

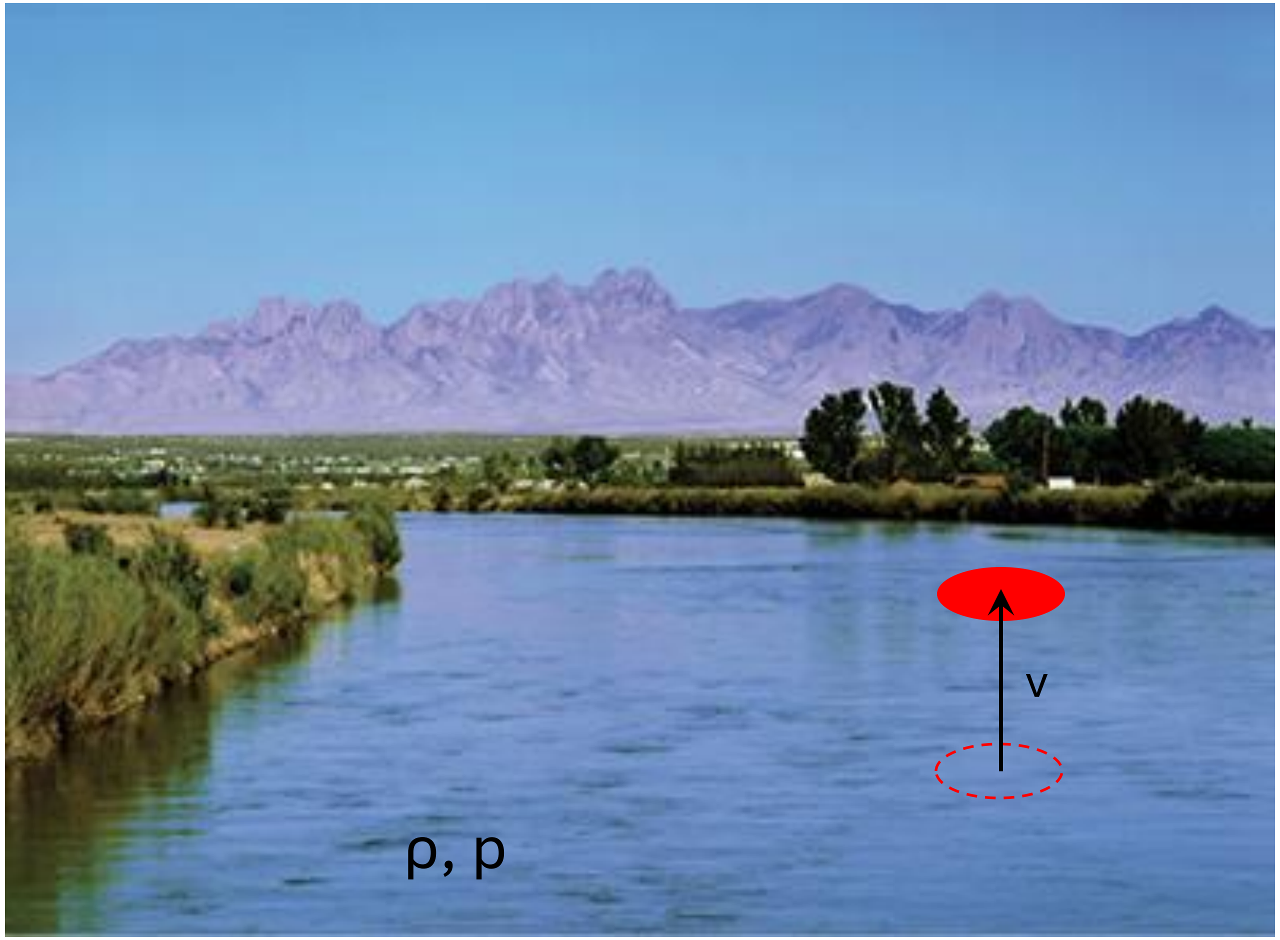
• Much bigger than mean free path: $l_{\mathrm{patch}} \gg l_{\mathrm{mfp}}$

## Coherent motion

- The fluid is described by local macroscopic variables (e.g., density, temperature, bulk velocity).

- The dynamics of the fluid is governed by internal forces (e.g. pressure gradients) and external forces (e.g. gravity).

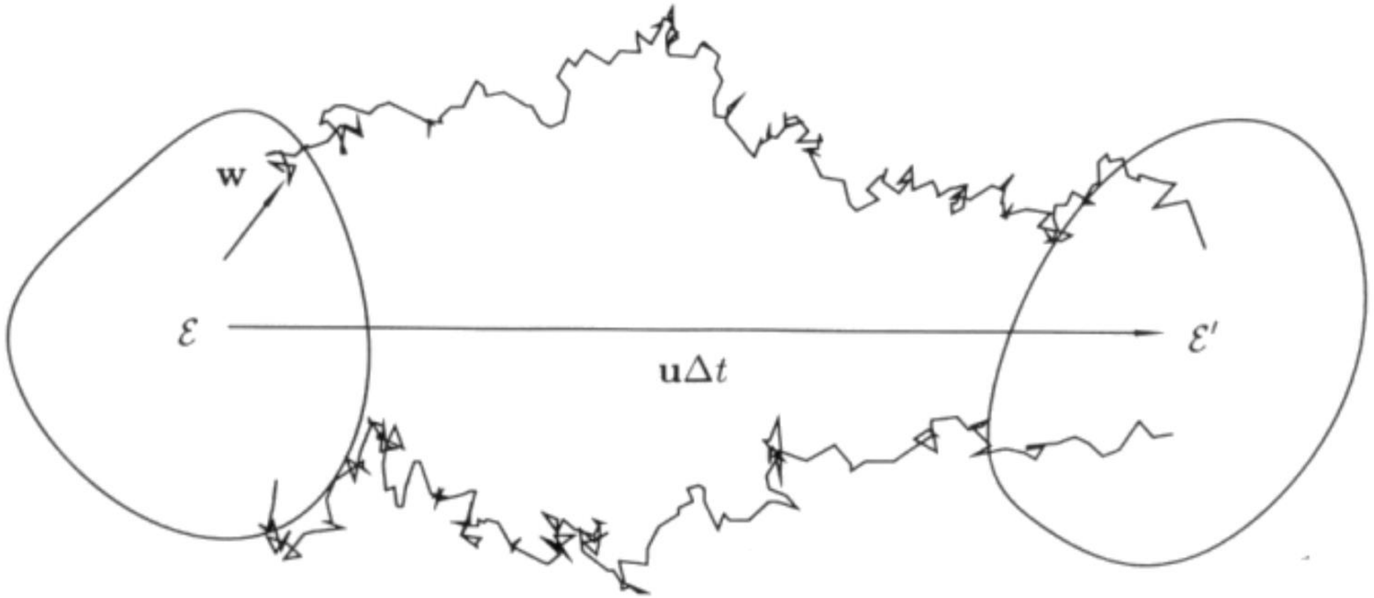
- The changes of macroscopic properties in a fluid element are described by conservation laws.



# Bulk velocity

The fluid element moves with a bulk velocity, the mean flow velocity.

The individual particles move with the mean flow velocity, plus a random velocity component.



# Control Volume

3D generalization of fluid patch.

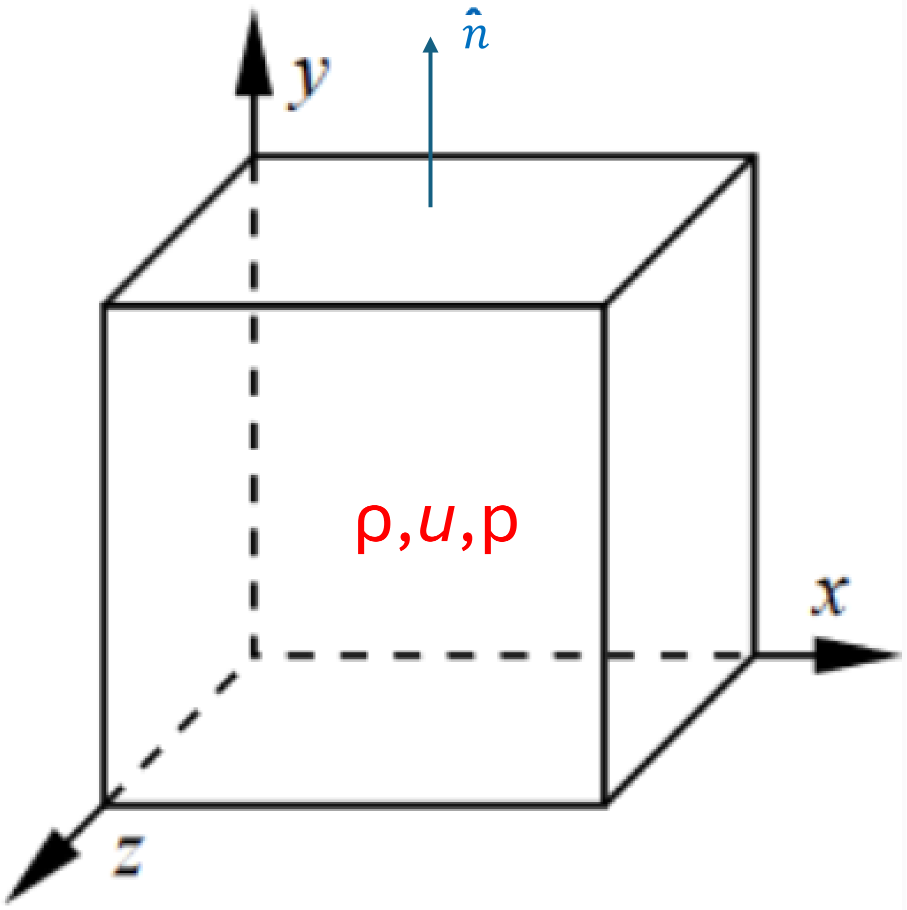
Unit length cube

$l_{\text{mfp}} \ll L \ll Q \,/\, |\nabla Q|$

$n L^3 \gg 1$

The time rate of change of a quantity inside this volume:

- Changes of this quantity (volumetric contributions)
- Surface effects (net transport across the surface)

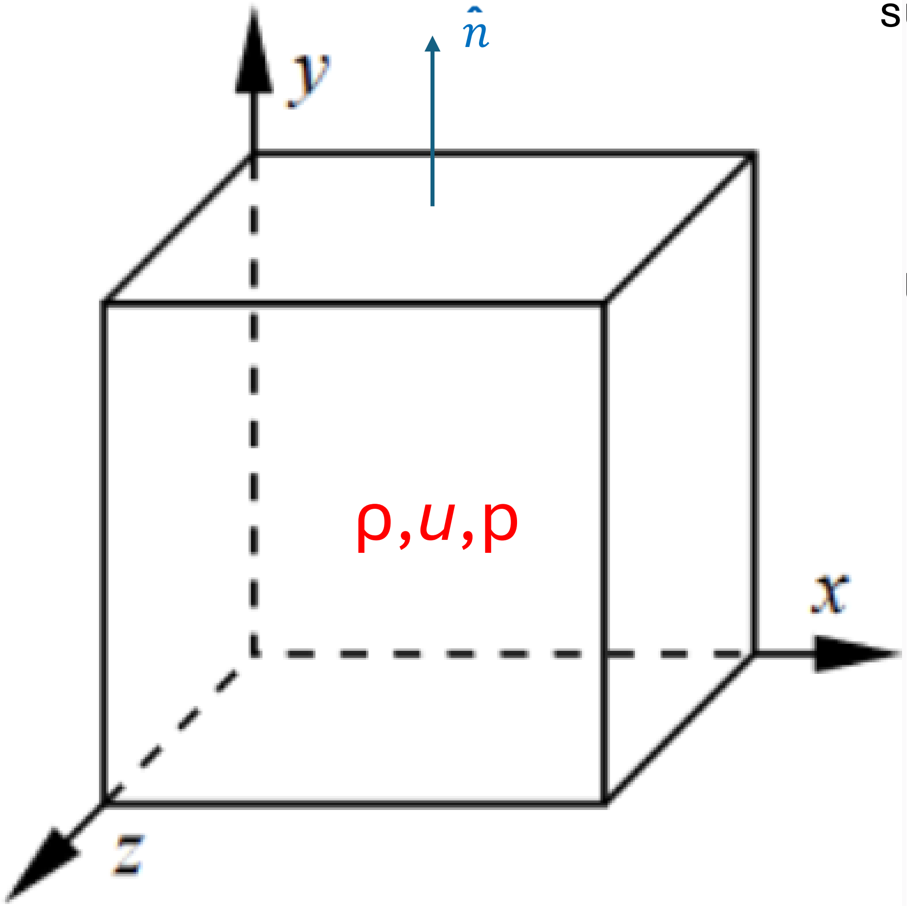



$$\frac{d}{dt}\int_V \rho dV = -\oint_A \rho \boldsymbol{u} \cdot d\boldsymbol{A}$$

$$\frac{d}{dt}\int_V \rho dV = -\int_V \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) dV$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0$$

# Continuity Equation (Mass Conservation)




Apply mass conservation to the cube. There is no mass creation or annihilation. The change in mass is equal to the mass flux across the surfaces.

$$\frac{d}{dt}\int_V \rho dV = -\oint_A \rho \boldsymbol{u} \cdot d\boldsymbol{A}$$

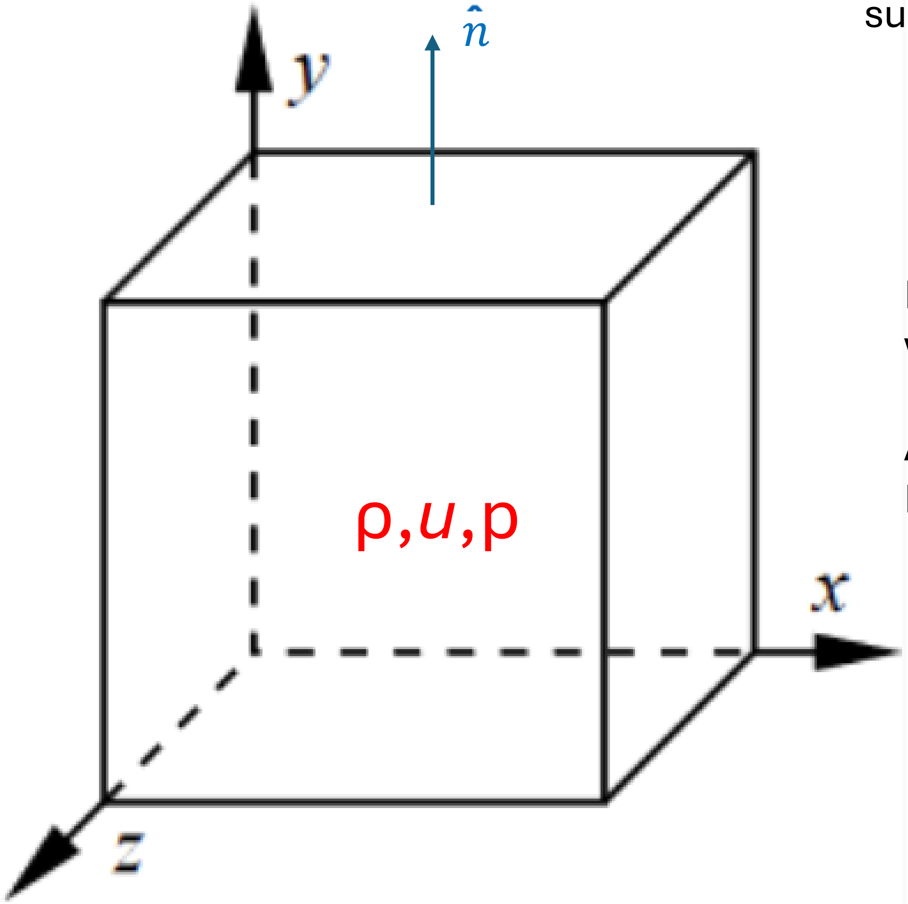
Use Gauss' theorem

$$\frac{d}{dt}\int_V \rho dV = -\int_V \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) dV$$

Use Leibniz' rule for differentiation under the integral, and put all under the same integrand

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0}$$

# Euler Equation (Momentum Conservation)




Apply momentum conservation to the cube. Now there are sources of momentum: the forces. These forces can be body forces (e.g. gravity) and surface forces (pressure).

The change in momentum is equal to the momentum flux across the surfaces, plus surface and body forces.

$$\frac{d}{dt}\int_V (\rho \boldsymbol{u})\, dV = -\oint_A (\rho \boldsymbol{u})\, \boldsymbol{u} \cdot d\boldsymbol{A} - \oint_A \mathbf{P} \cdot d\boldsymbol{A} + \int_V \rho \boldsymbol{f}\, dV$$

Mathematically, the body force has to be expressed by a 2nd rank tensor to dot with the area and produce a vector. For a pure normal force (pressure), $\mathbf{P}_{ij} = p\, \delta_{ij}$.

Applying Gauss theorem, we put all terms under the same volume integral, to find Euler's equation

$$\boxed{\frac{\partial(\rho \boldsymbol{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\boldsymbol{\nabla} p + \rho \boldsymbol{f}}$$