Bulk Viscosity

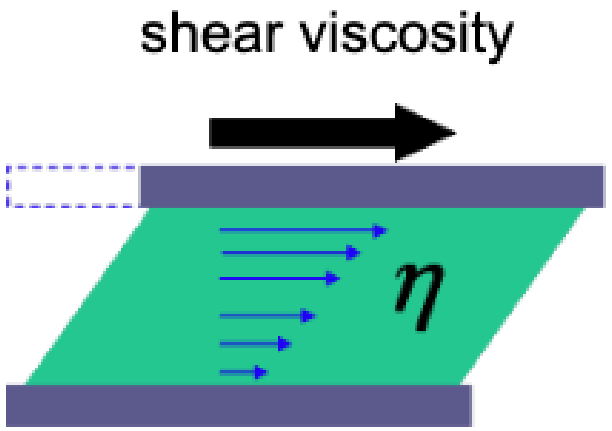
Resistance to Expansion/Compression

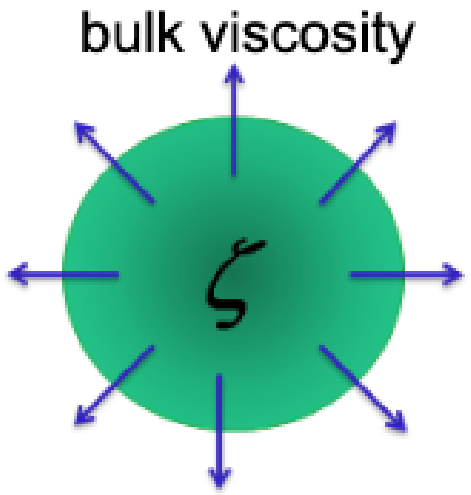
To deal with discontinuities, we borrow a concept from non-ideal gases, which is the notion of *bulk viscosity* (also referred to as *volume viscosity*, or *second viscosity*).

Bulk viscosity represents the fluid's ***resistance to compression***, which microphysically stems from the finite time it takes for energy given to the fluid to be distributed to the rovibrational molecular modes.

Monoatomic gases and ideal gases do not have bulk viscosity.

Another way to think of second viscosity is the dissipation that happens while the gas is out of equilibrium following an expansion or compression event. Usually the restoration of equilibrium is fast, but if the relaxation time is long, dissipation will occur.



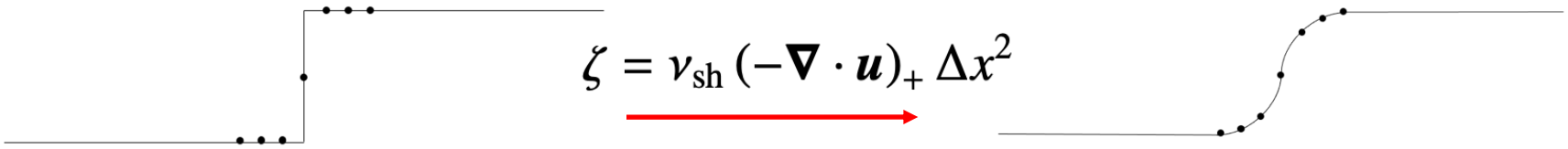


Shock Viscosity

Construct an artificial bulk viscosity that is activated only when the flow is undergoing strong compression.

This bulk viscosity will smooth discontinuities so that the stencil can resolve them.

We refer to this as a *shock viscosity*.



Shock viscosity takes the form of a bulk viscosity.
The coefficient is proportional to the positive part of the convergence
(the normalization by Δx^2 is only to keep the coefficient nu_{sh} close to unity).

$$\boldsymbol{\tau}_{ij} = 2\rho\nu \mathsf{S}_{ij} + \rho\zeta_{\rm shock}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}.$$

Viscous force

$$\rho^{-1}\boldsymbol{F}_{\rm visc} = \nu\left(\nabla^2\boldsymbol{u} + \frac{1}{3}\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u} + 2\mathbf{S}\cdot\boldsymbol{\nabla}\ln\rho\right) + \zeta_{\rm shock}\left[\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \left(\boldsymbol{\nabla}\ln\rho + \boldsymbol{\nabla}\ln\zeta_{\rm shock}\right)\boldsymbol{\nabla}\cdot\boldsymbol{u}\right].$$

Viscous heating

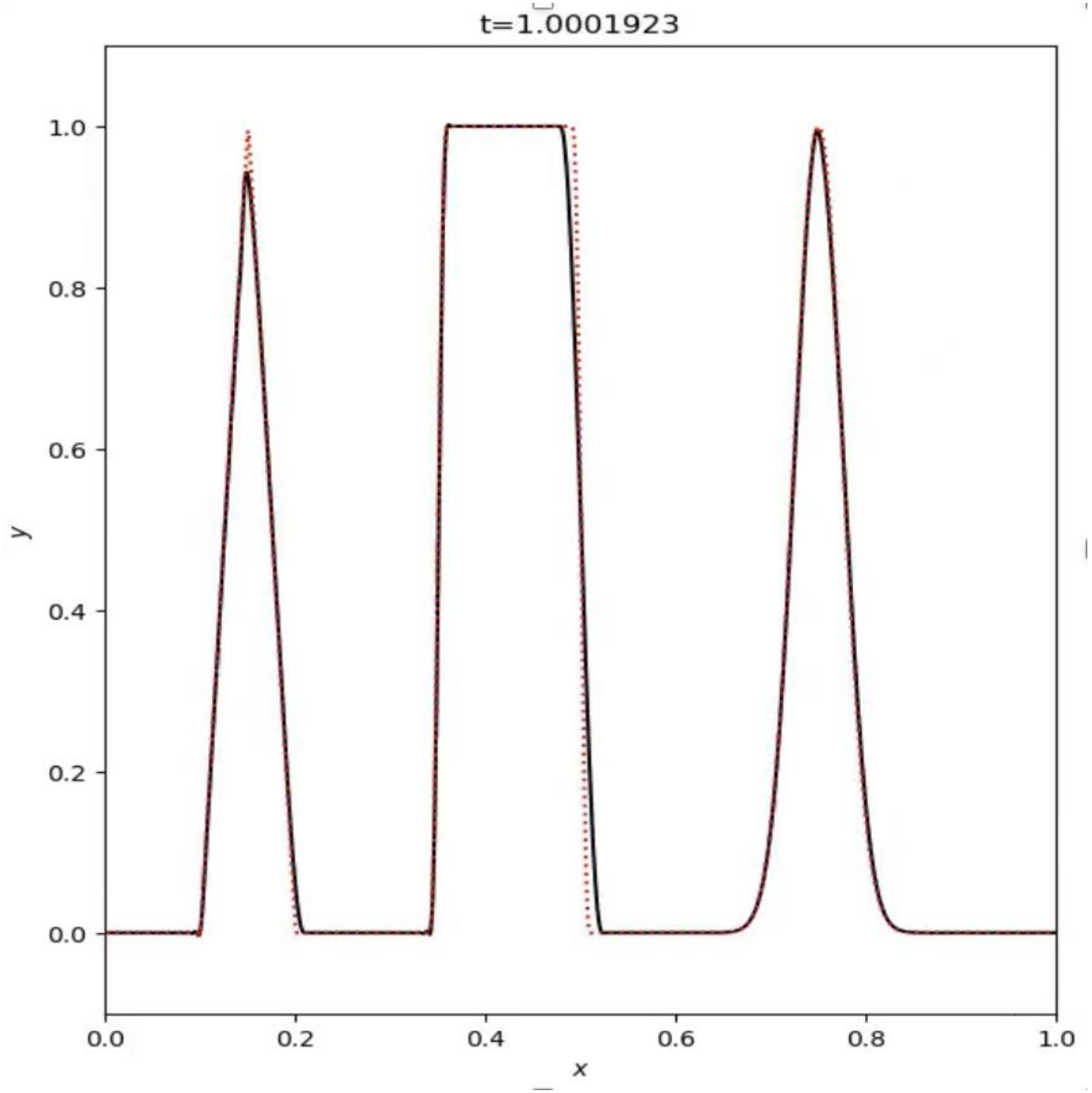
$$\rho^{-1}\Gamma_{\rm visc} = 2\nu\mathbf{S}^2 + \zeta_{\rm shock}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2.$$

Shock nu=1.0

High order scheme (6th space, 4th time)
Hyper viscosity Re_{Ny} =1
Resolution 1024

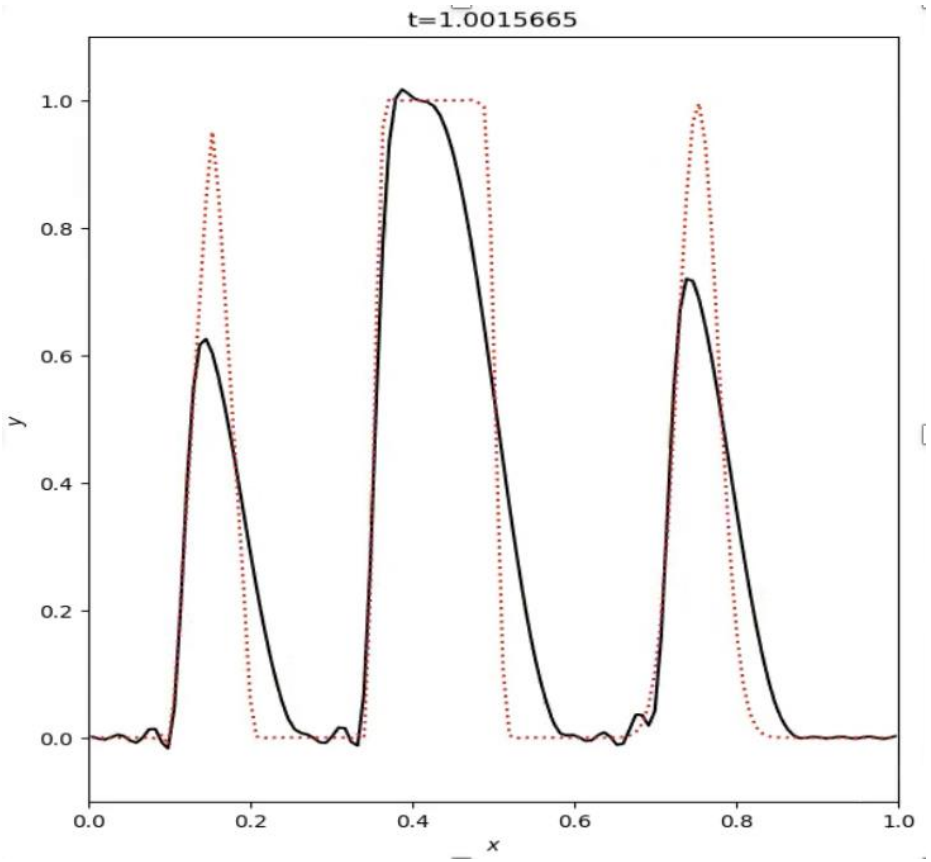
Shock viscosity ζ=1.0

A few problems with the scheme are: the viscosity affects the pre-shock region and the shock is therefore not as sharp anymore as we had before. Also, it is not sufficient to completely eliminate post-shock oscillations.



Shock viscosity

Low Resolution



High order scheme (6th space, 4th time)
Hyper viscosity Re_{Ny} =1
Resolution 128

Shock viscosity ζ=1.0

Too much diffusion at low resolution

High order is not an advantage if the function has discontinuities

$$f(x \pm dx) = f(x) \pm \mathrm{d}x f'(x) + \frac{dx^2}{2!} f''(x) \pm \frac{dx^3}{3!} f'''(x) + \frac{dx^4}{4!} f''''(x) \pm \ldots$$

A high order scheme presupposes that each term is successively smaller.
The assumption is violated at discontinuities.
At discontinuities, low order schemes will perform better.

Finite Volume Method

The idea of the finite volume method is to create cells out of the grid, instead of just sampling grid points.

The grid points now become cell centers located at x_i , and we now have cell interfaces or cell walls located at the half-points.

In 3D the cells can be regarded as control volumes containing the conserved quantities q.

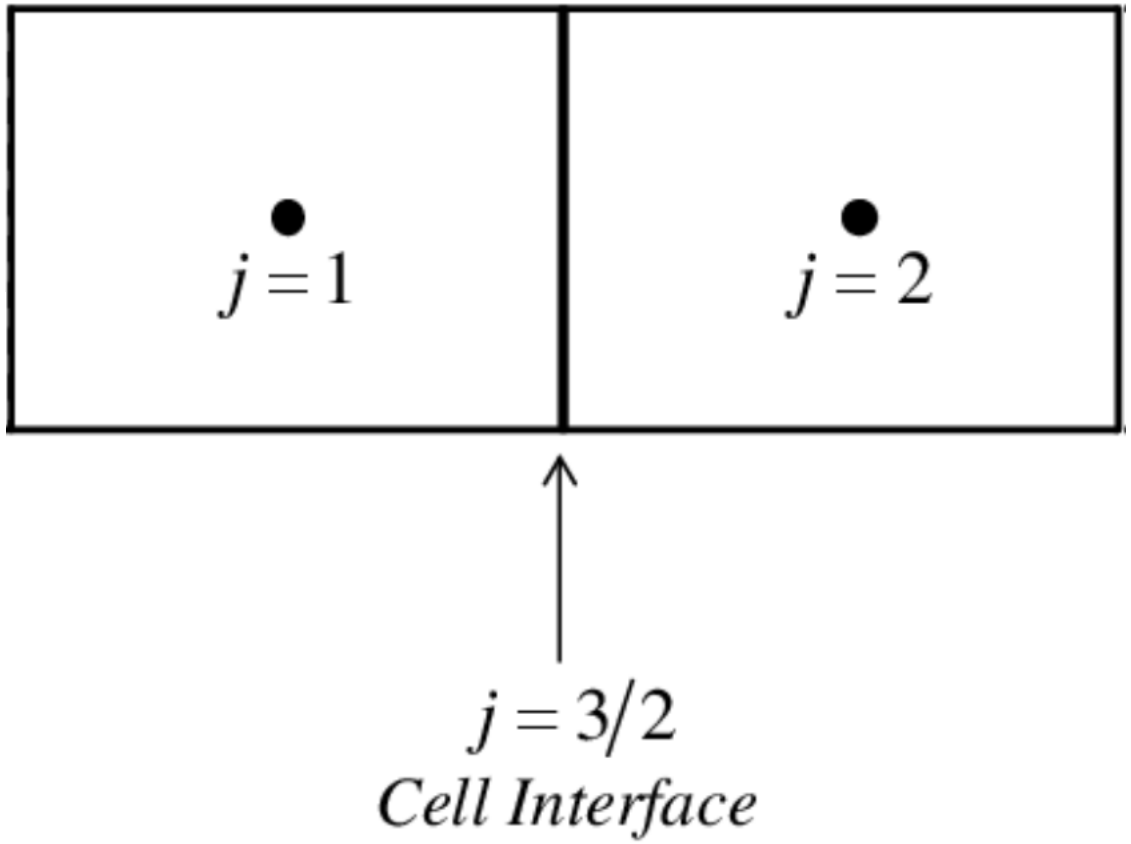
The conservation equation is then invoked

$$\frac{\partial q}{\partial t} + \nabla \cdot f(q) = 0$$

Leading to the following recurrence relation (see Lecture notes)

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}\left[f_{i+1/2}^{n+1/2} - f_{i-1/2}^{n+1/2}\right]$$

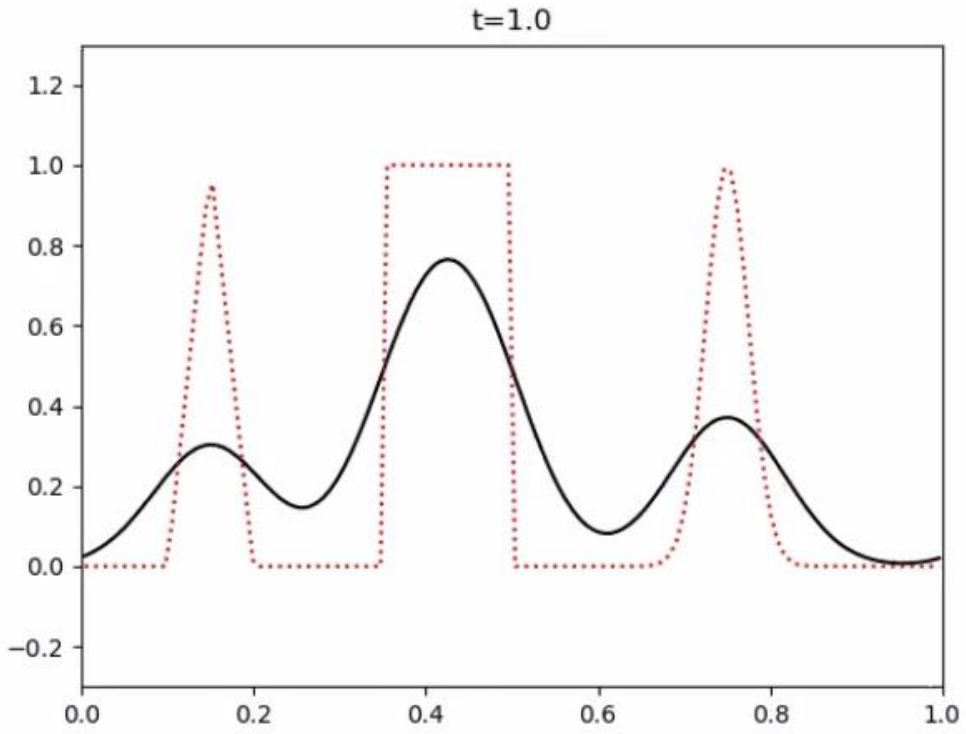
The different schemes vary in how they approximate the flux at cell interfaces.



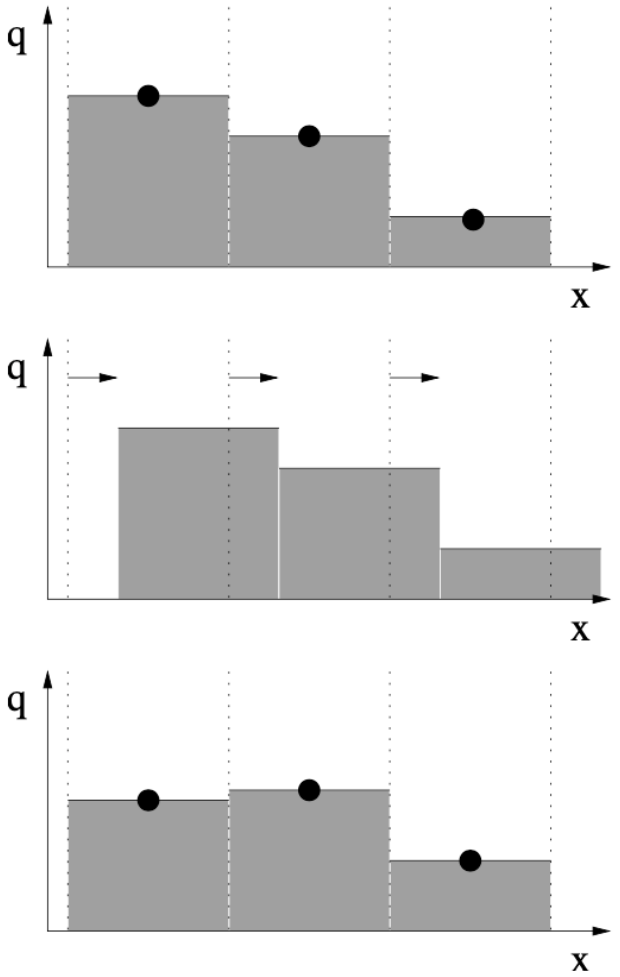
Lower order schemes: Donor cell

The simplest flux conserving scheme is the donor-cell scheme. In this scheme the “average interface state” is simply the value of the upwind cell.

$$f_{i+1/2}^{n+1/2} \approx q_i^n u_{i+1/2}$$



The Donor-Cell algorithm, as the upstream differencing method, is very diffusive. But it is the basis of better methods.



The physical interpretation of this method is the following. One assumes that the density is constant within each cell. We then let the material flow through the cell interfaces, from left to right.

Since the density is constant, and since the CFL condition makes sure that the flow is no further than 1 grid cell spacing at maximum, the flux through the cell interface is constant.

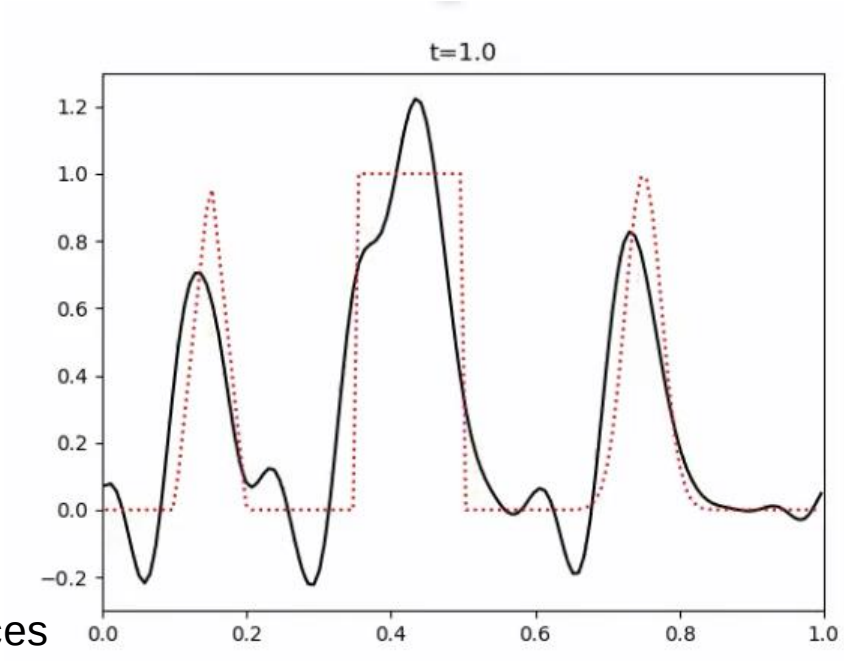
Once the time step if finished, the state in each cell has the form of a step function. To get back to the original sub-grid model we need to average the quantity q(x) out over each cell

Slope of donor cell

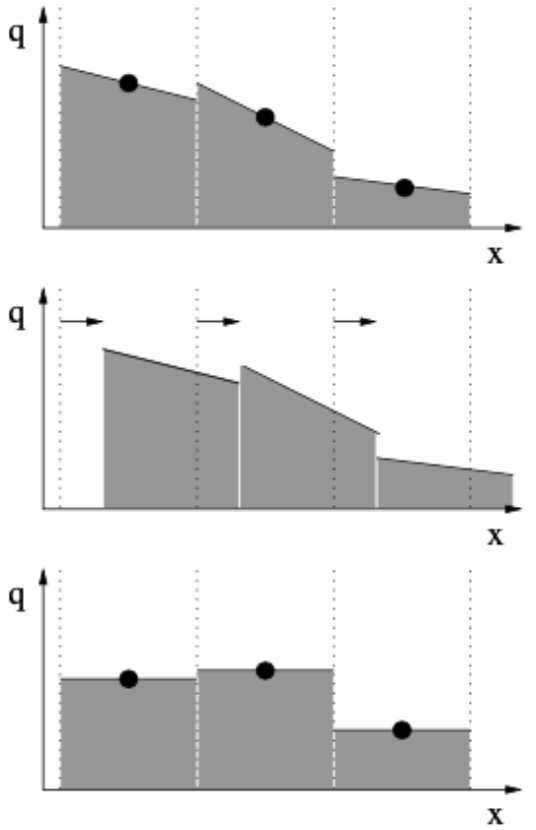
$$f_{i+1/2} = v\left[q_i + \frac{1}{2}\delta q_i(1 - C)\right]$$

Lax-Wendroff: forward slope

$$\delta q_i = \delta q_{\text{forward}} = q_{i+1} - q_i$$



The scheme introduces ringing again at the discontinuity.



The donor-cell algorithm is a simple algorithm based on the idea that at the beginning of each time step the state within each cell is constant throughout the cell.

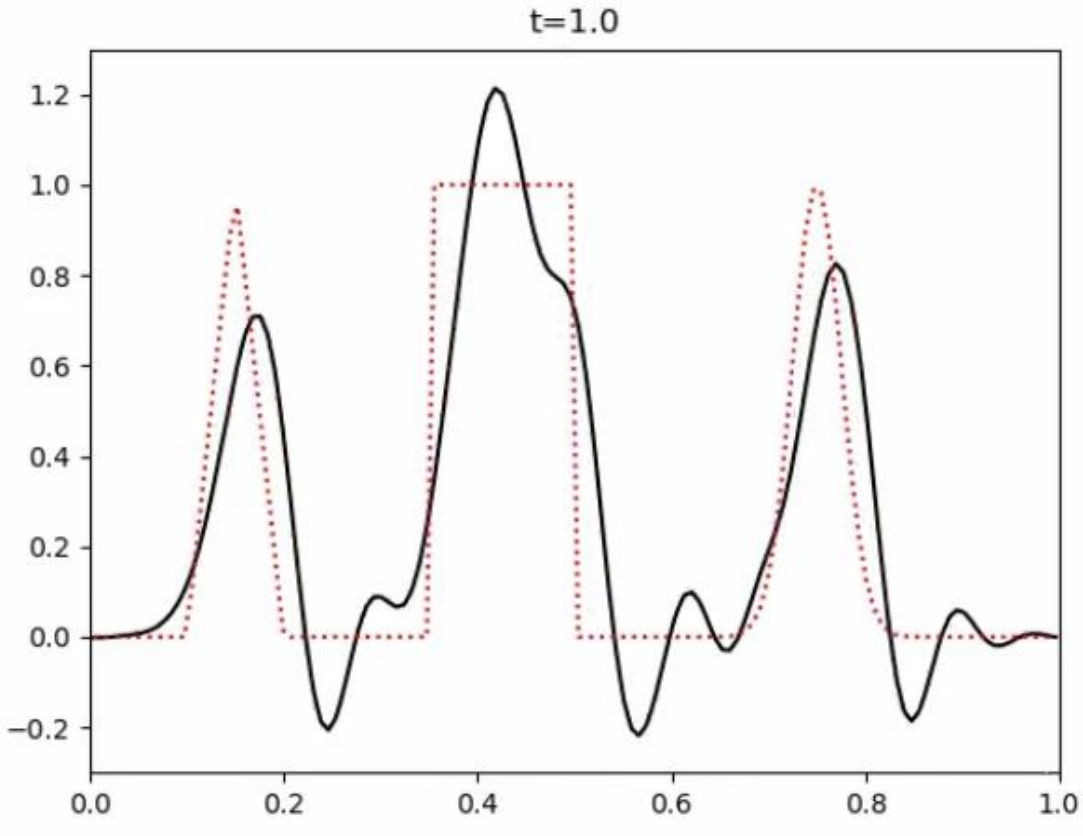
But one could also make another assumption: that the function is varying at scales smaller than the grid cell. We can use the information on the slope of the function.

Slope of donor cell

$$f_{i+1/2} = v\left[q_i + \frac{1}{2}\delta q_i(1 - C)\right]$$

Beam-Warming: backward slope

$$\delta q_i = \delta q_{\text{backward}} = q_i - q_{i-1}$$



At each grid cell i, one can choose between two ”natural” slopes to use.

These slopes are determined by the difference between the points

- i and i + 1 (forward slope, the Lax-Wendroff scheme)
- i−1 and i (backward slope, the Beam-Warming scheme).

The LW scheme rings downstream.
The BM scheme rings upstream.

Slope of donor cell

$$f_{i+1/2} = v\left[q_i + \frac{1}{2}\delta q_i(1 - C)\right]$$

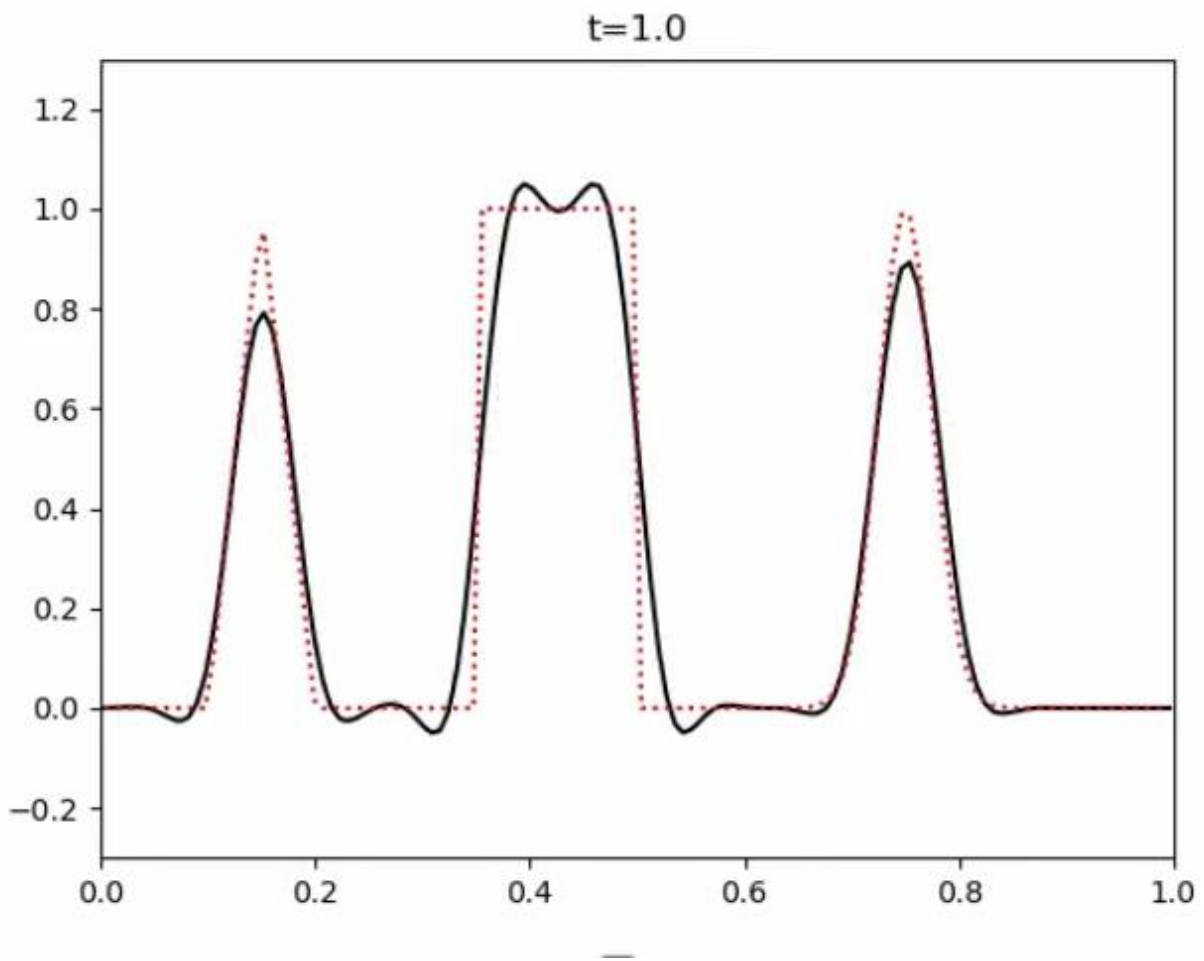
Fromm: centered

$$\delta q_i = \frac{1}{2}\left(\delta q_{\text{forward}} + \delta q_{\text{backward}}\right)$$

The Fromm scheme is the average of forward and backward, equivalent to central differences.

The result is smooth with some amount of overshoot.

The initial shape of the functions is recognizable.



Slope of donor cell

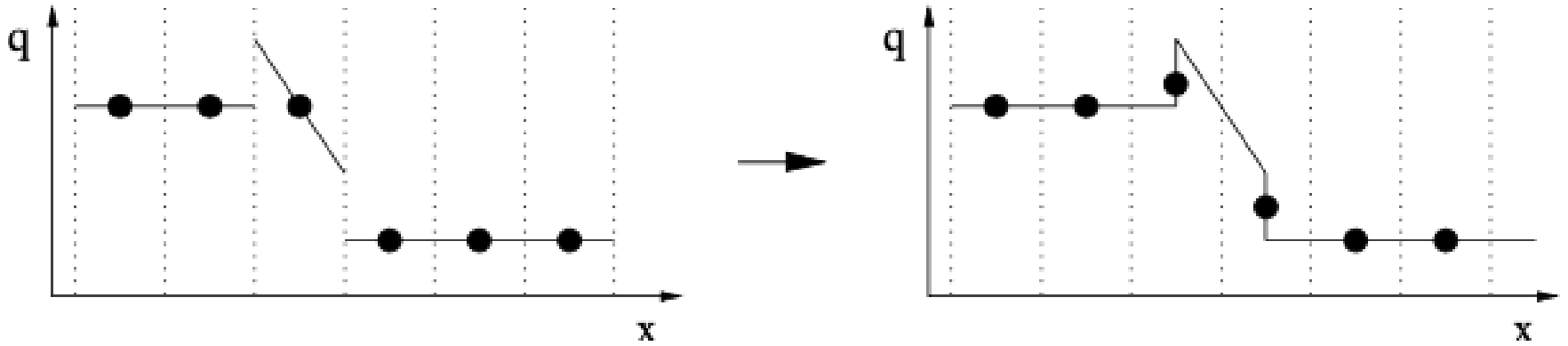
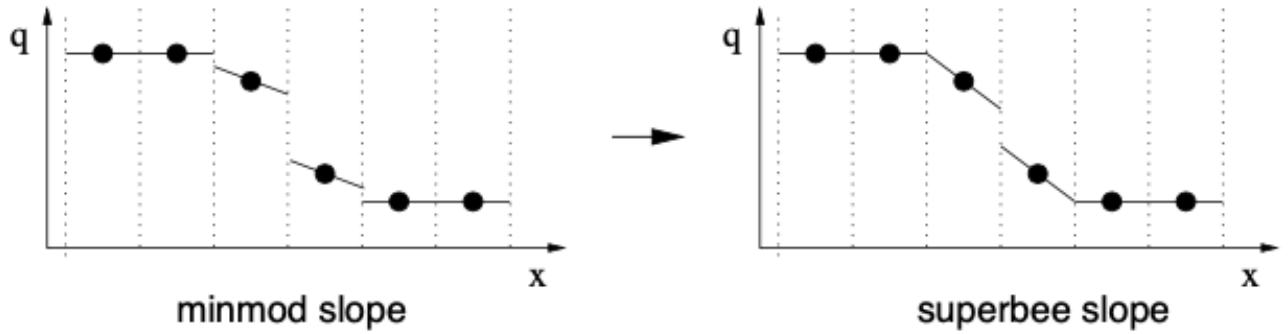


Illustration of how the LW scheme leads to ringing at discontinuities.

Slope limiters



A successful method to prevent such overshoots is the use of slope limiters. These are non-linear conditions that modify the slope if, and only if, this is necessary to prevent over- shoots.

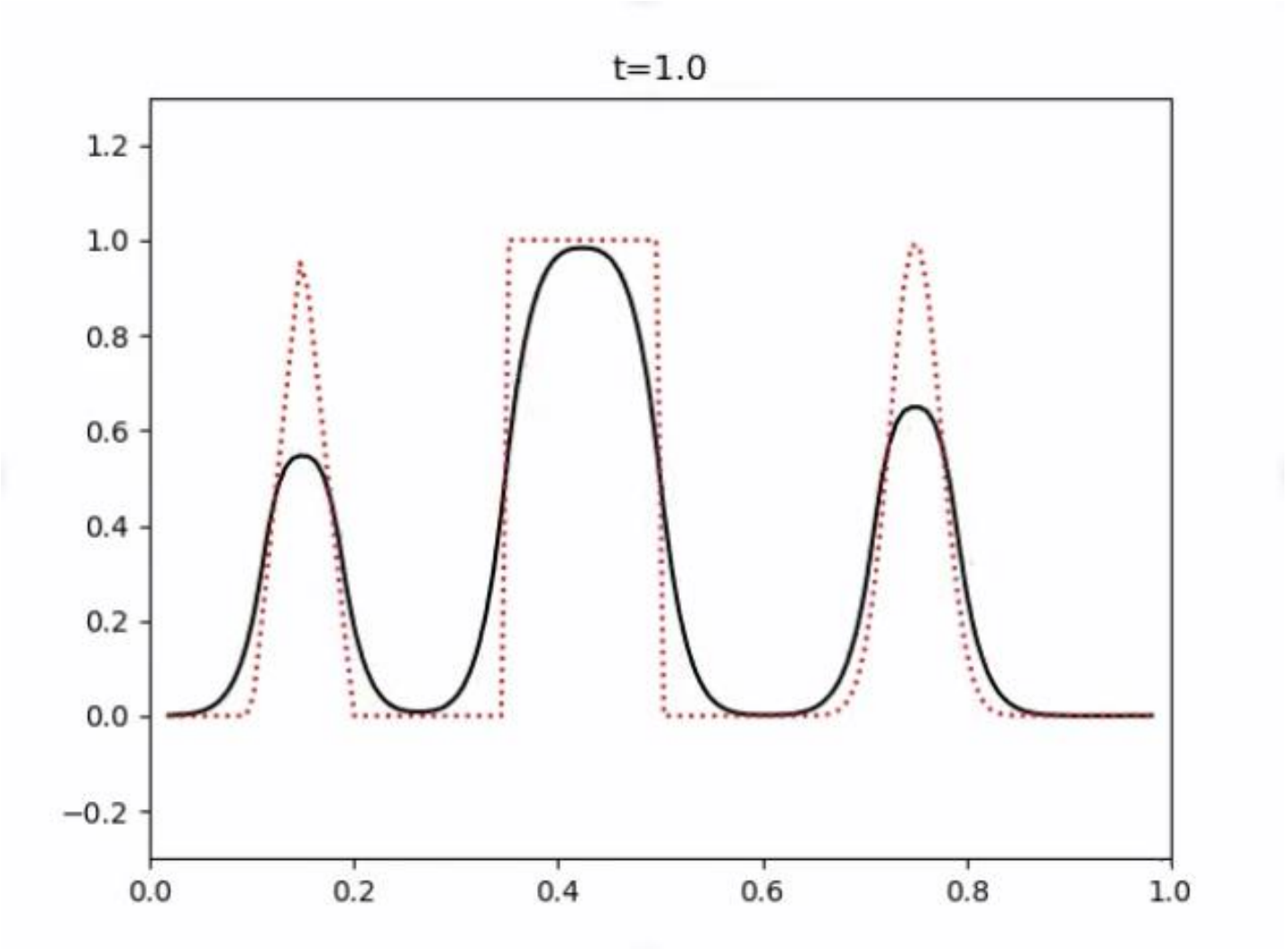
In these schemes, nonlinear corrections are inserted with the purpose of hindering the oscillations from going unstable. The way these schemes achieve this result is by limiting the slope to well-behaved values.

Slope limiter: Minmod

$$f_{i+1/2} = v\left[q_i + \frac{1}{2}\delta q_i(1 - C)\right]$$

MinMod

$$\begin{aligned}\delta q_i &= \min[\max(\delta q_f, 0), \max(\delta q_b, 0)] \\ &+ \max[\min(\delta q_f, 0), \min(\delta q_b, 0)]\end{aligned}$$

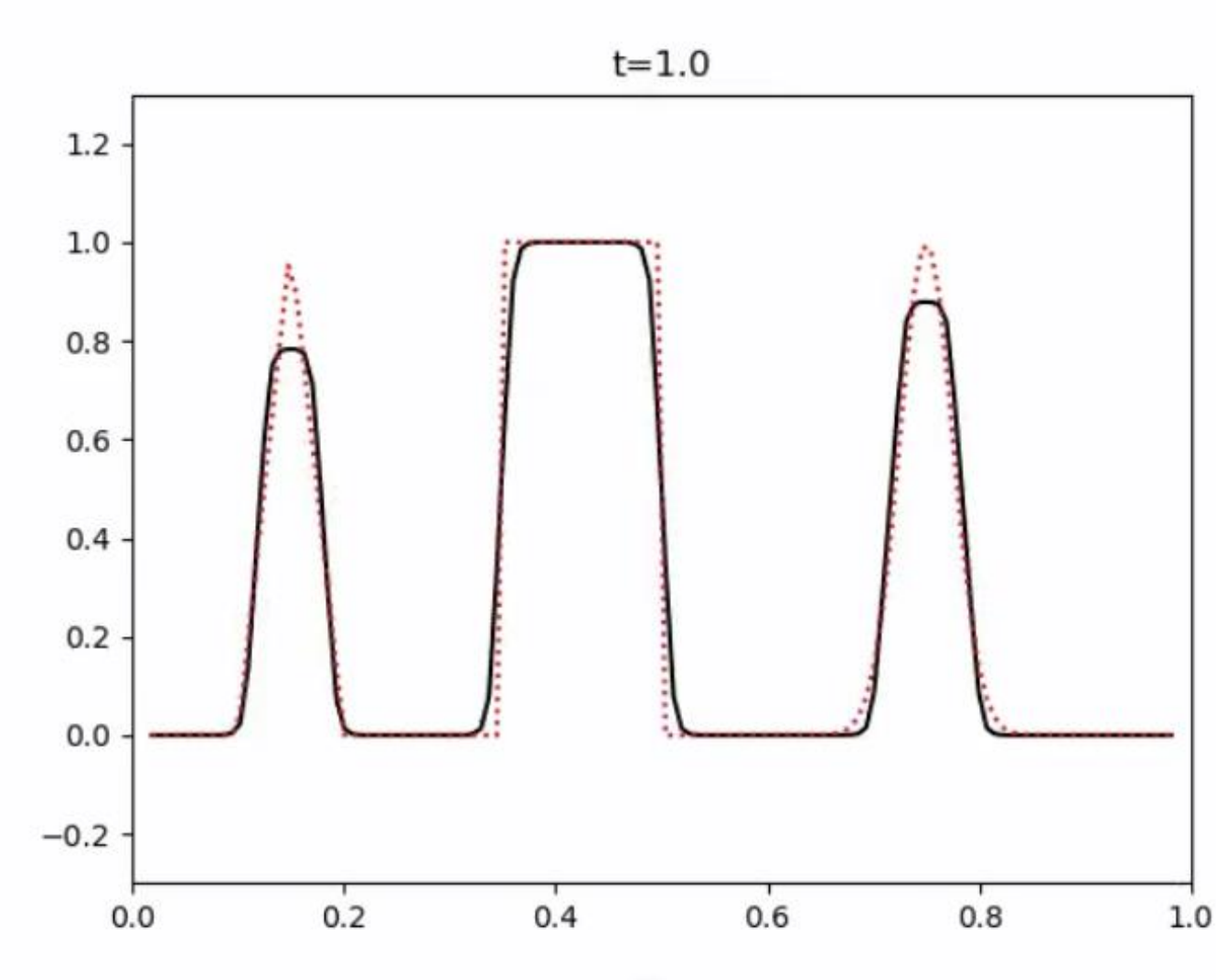


Slope limiter: Superbee

$$f_{i+1/2} = v\left[q_i + \frac{1}{2}\delta q_i(1 - C)\right]$$

Superbee

$$\delta q_i = \left[\text{sign}(\delta q_f) + \text{sign}(\delta q_b)\right] \times \min\left\{\text{abs}(\delta q_f), \text{abs}(\delta q_b), \frac{1}{2}\max\left[\text{abs}(\delta q_f), \text{abs}\,(\delta q_b)\right]\right\}$$

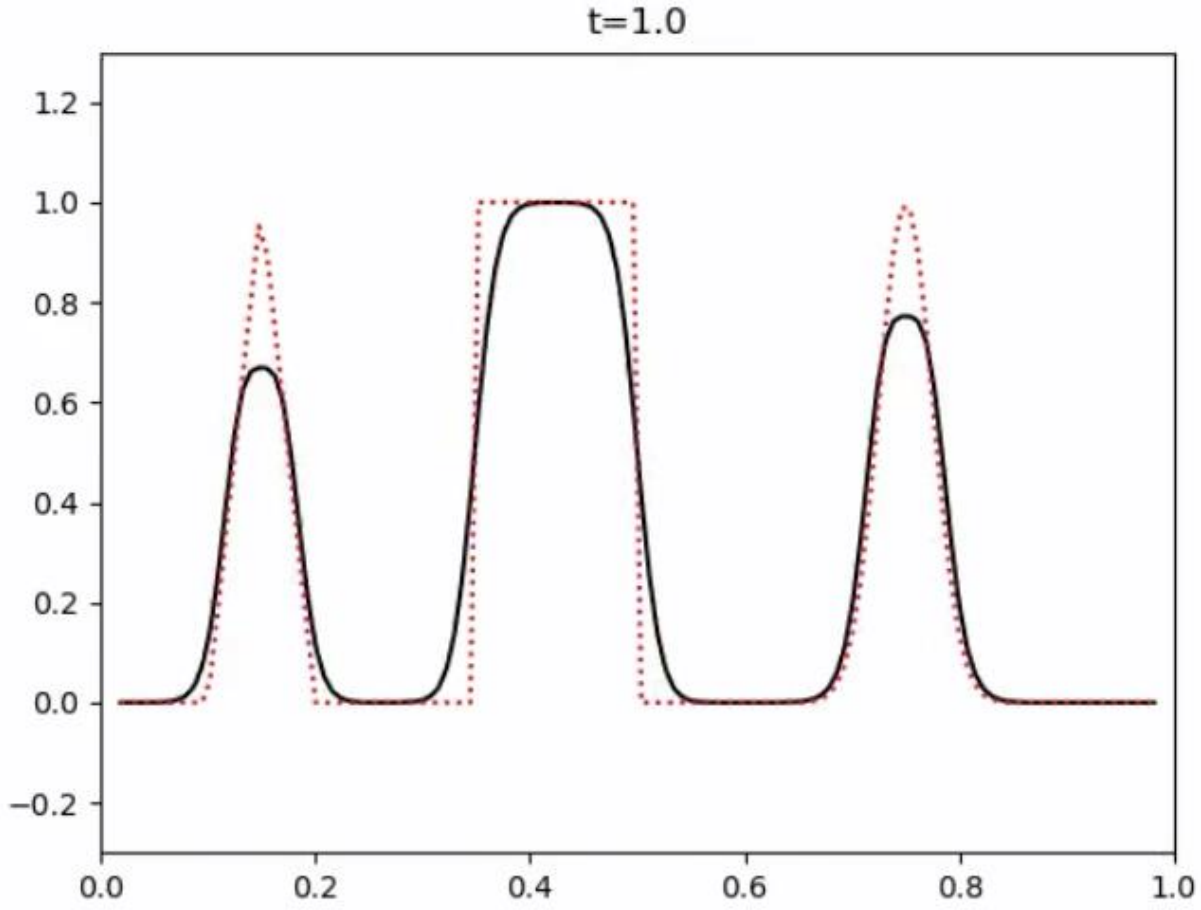


Slope limiter: Van Leer

$$f_{i+1/2} = v\left[q_i + \frac{1}{2}\delta q_i(1 - C)\right]$$

Van Leer

$$\delta q_i = \begin{cases} \frac{2}{\delta q_f^{-1} + \delta q_b^{-1}} & \text{if} \quad \delta q_f \delta q_b > 0 \\ 0 & \text{if} \quad \delta q_f \delta q_b \leq 0 \end{cases}$$

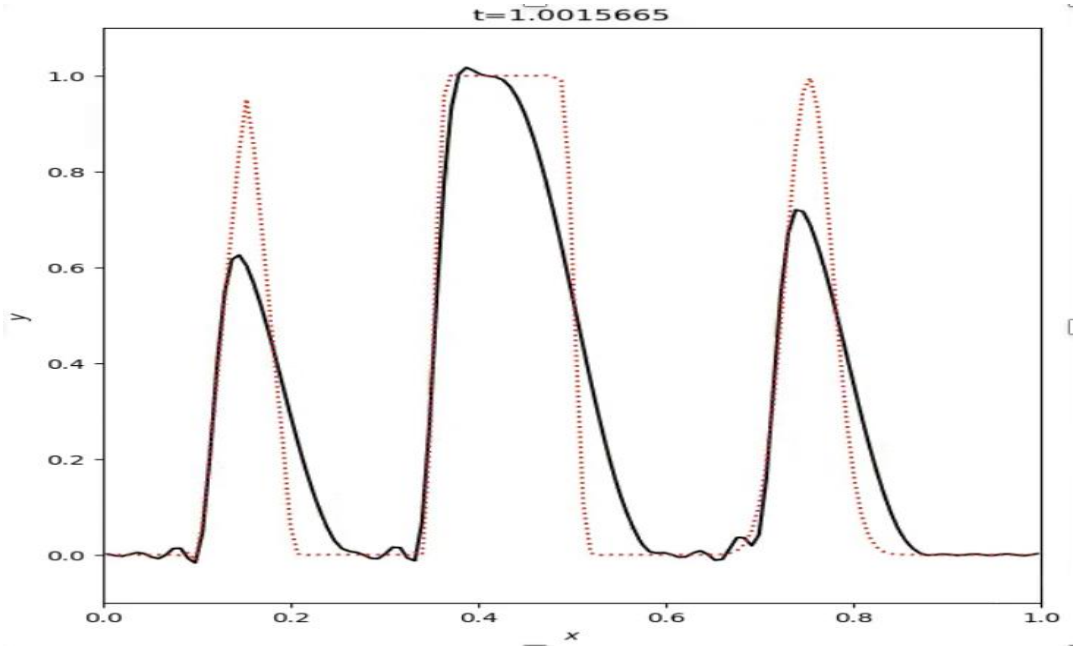


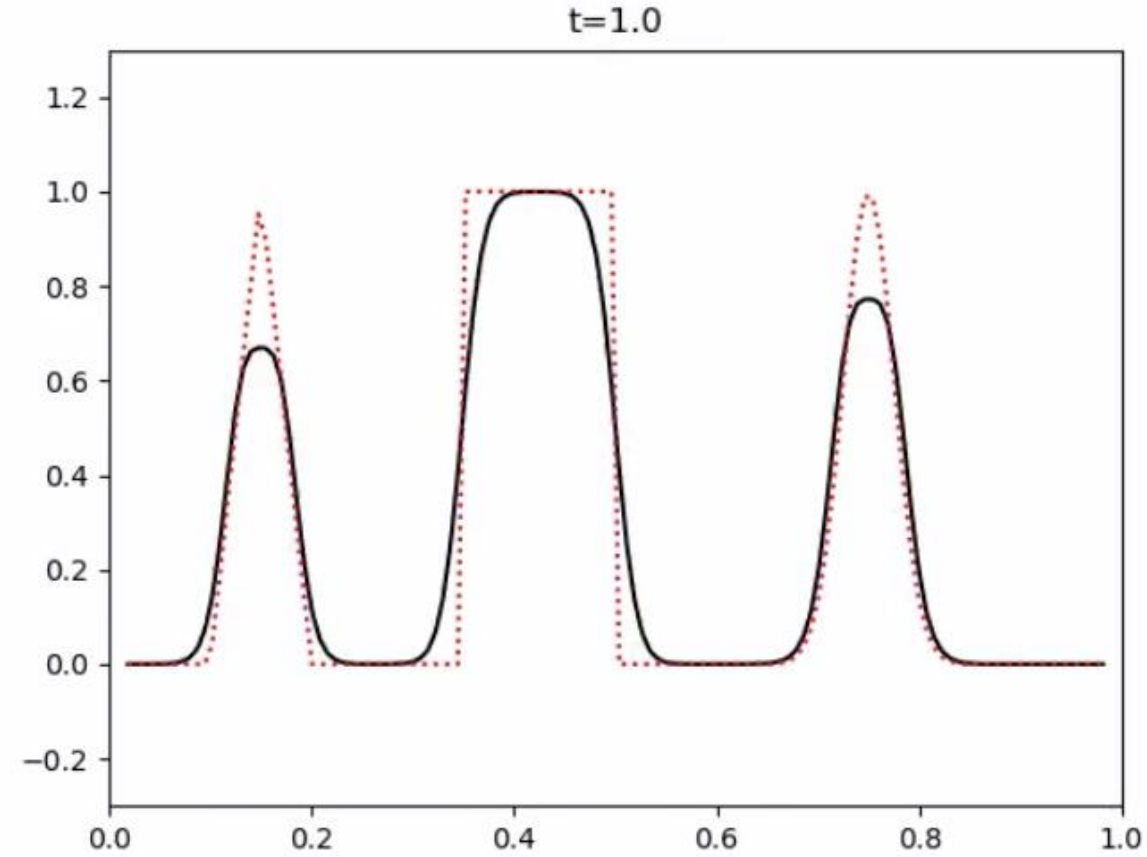
Finite Difference vs Finite Volume at same resolution

At resolution Nx=128, it is seen that the shock diffusion scheme (left) smoothes the shock significantly at the pre-shock Front, and does not completely prevent ringing at the post-shock region.

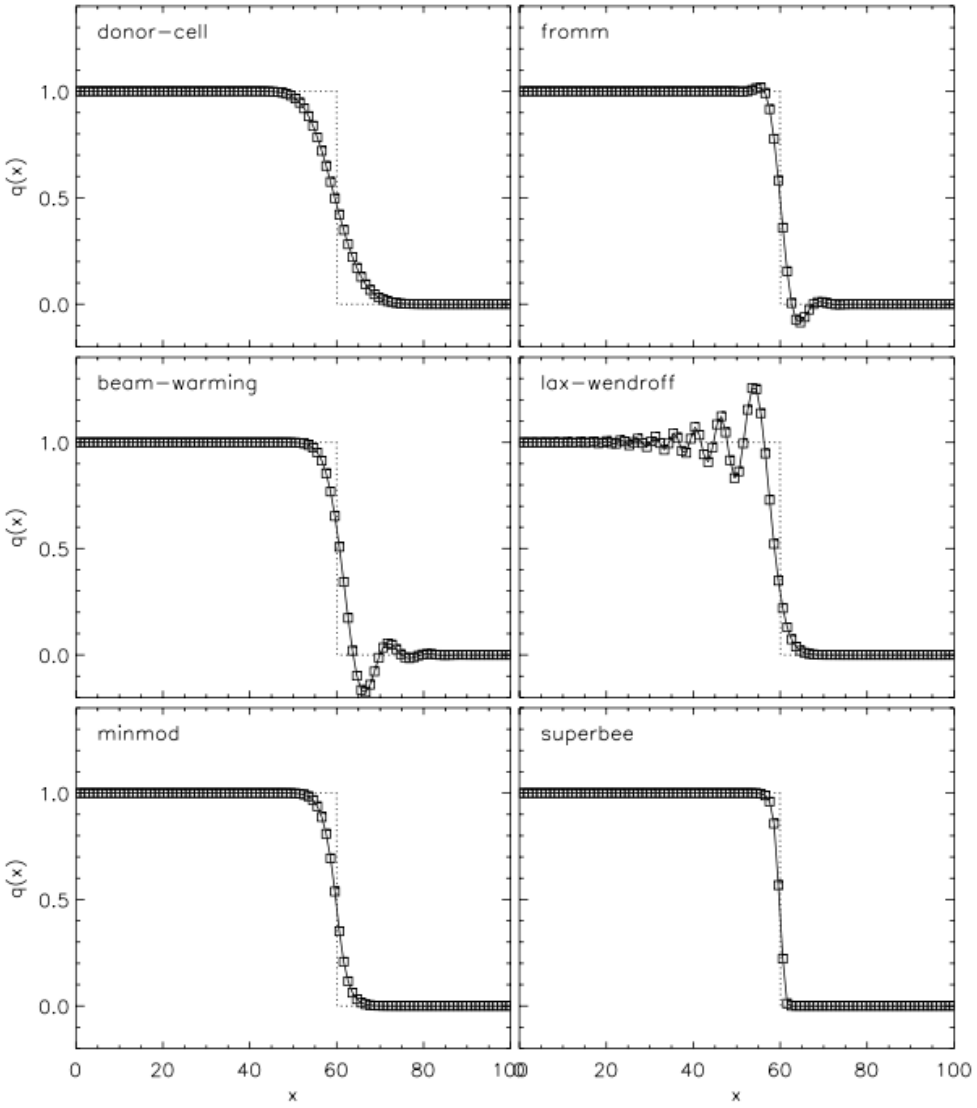
The Finite Volume scheme with Van Leer reconstruction, OTOH, is smooth.

The advantage of shock diffusion is that the amount of diffusion is controlled. But for flows where shocks abound, a finite volume scheme is preferred.



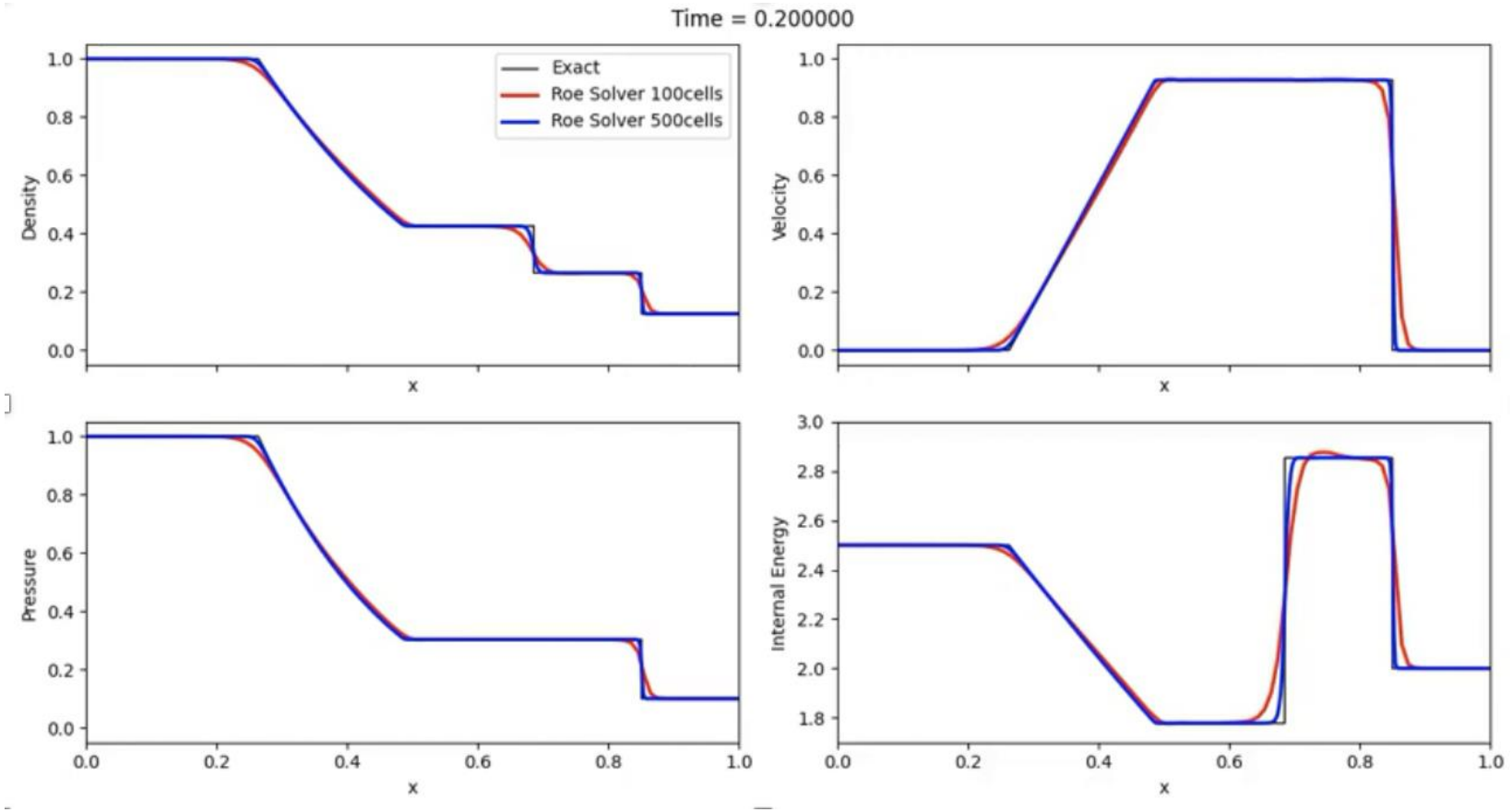


Slopes of donor cell: Summary



Sod Shock

Sod shock test tube with finite volume method.



Sod Shock

Sod shock test tube with finite difference method.