4th order Runge-Kutta (RK4)

Take the initial value problem $\frac{dy}{dt} = f(t, y)$

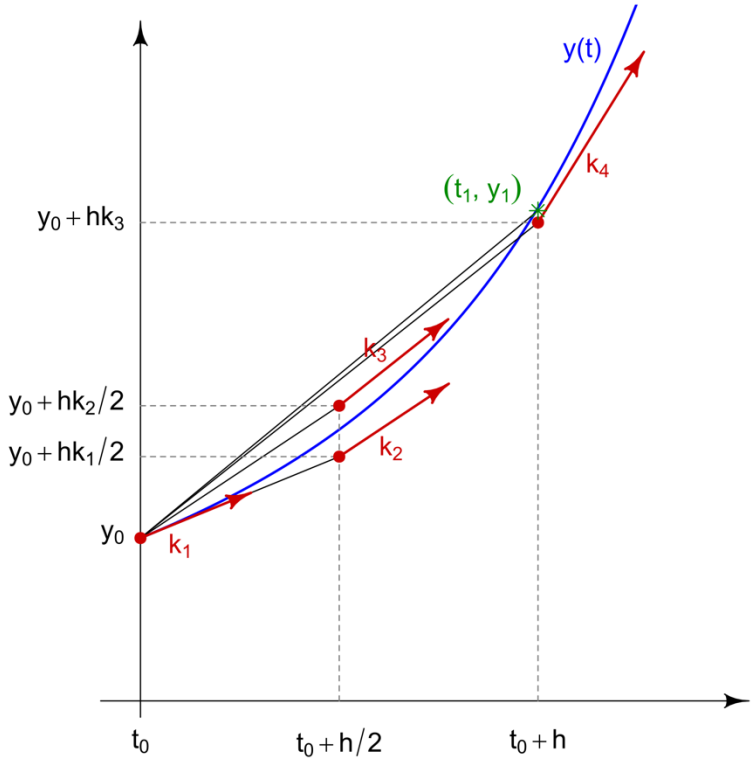
where *f* and t_0,y_0 are given.

Then apply recursively the following formula to approximate the unknown function *y*

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$t_{n+1} = t_n + h$$

Define the slopes

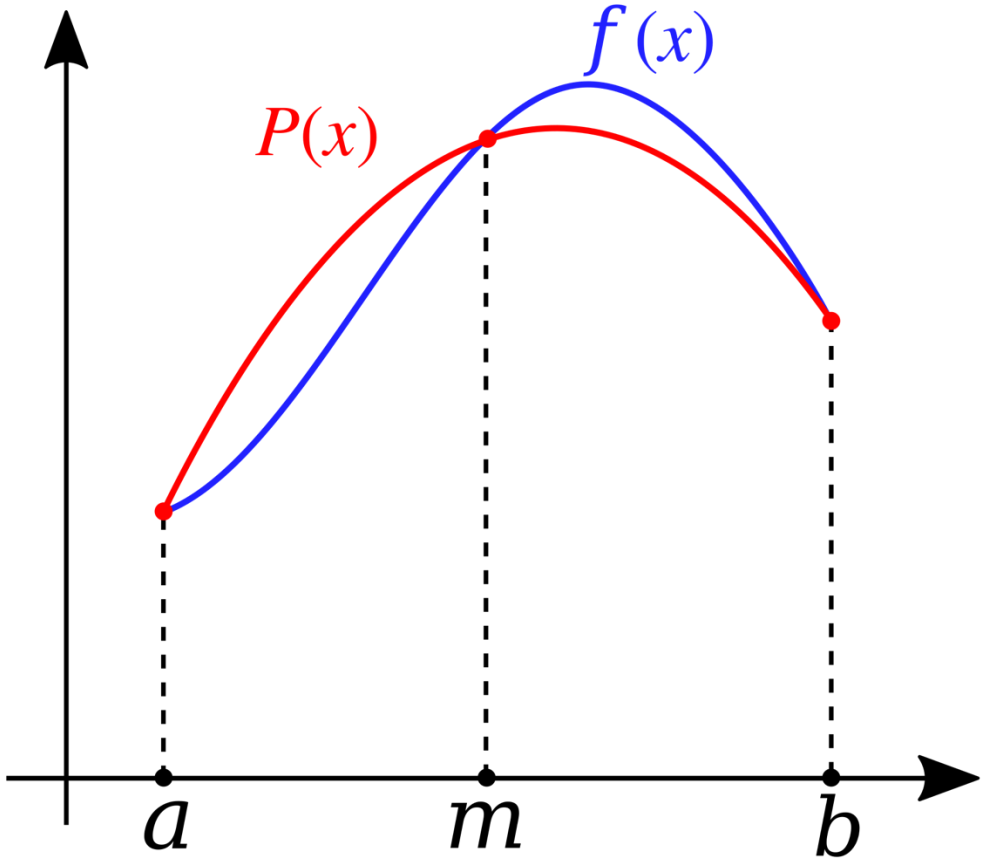
$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right), \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right), \\ k_4 &= f(t_n + h, y_n + hk_3)\,. \end{aligned}$$



Runge-Kutta and Simpson Rule

If the function depends only on one variable, then the two midpoints are equal, k2=k3, and we find Simpson's integration rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

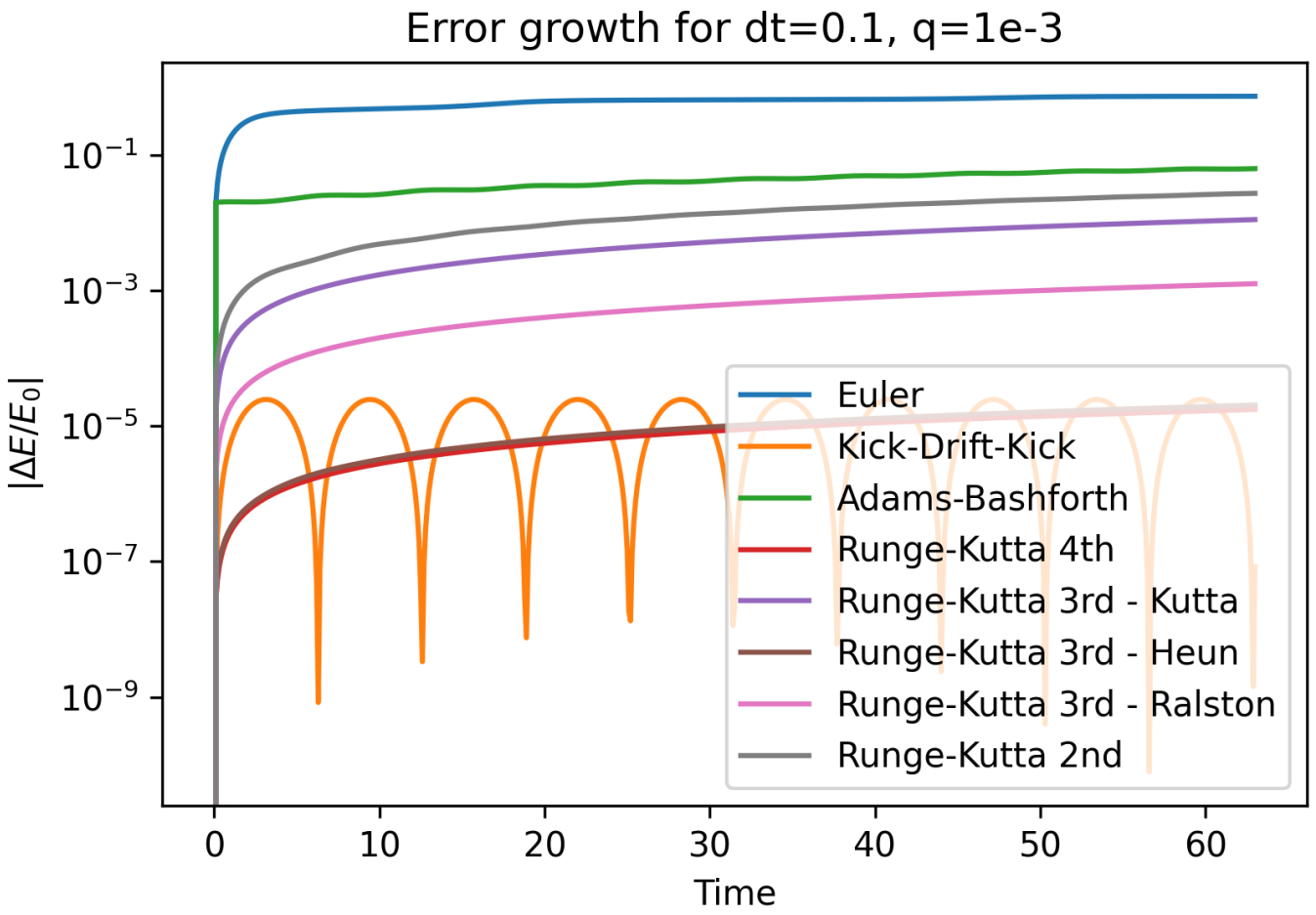


Runge-Kutta Energy Error

Runge-Kutta is a family of algorithms, with varying order (see Notes).

Euler integration is a 1st order Runge-Kutta.

Even though RK4 has an unbounded error in energy, the error can be comparable in magnitude to that achieved by KDK.

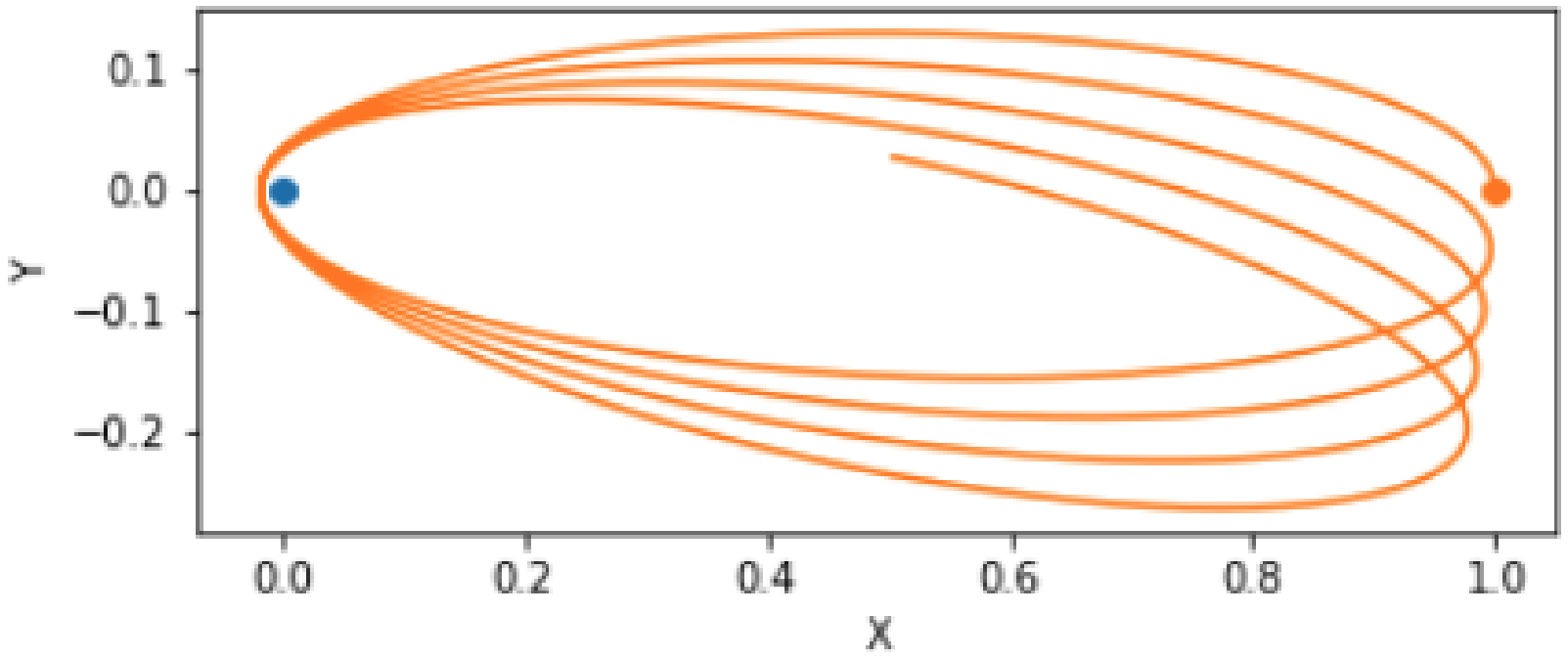


For fixed timesteps, either algorithm can fail, because they may be taking too large timesteps compared to what the system needs.

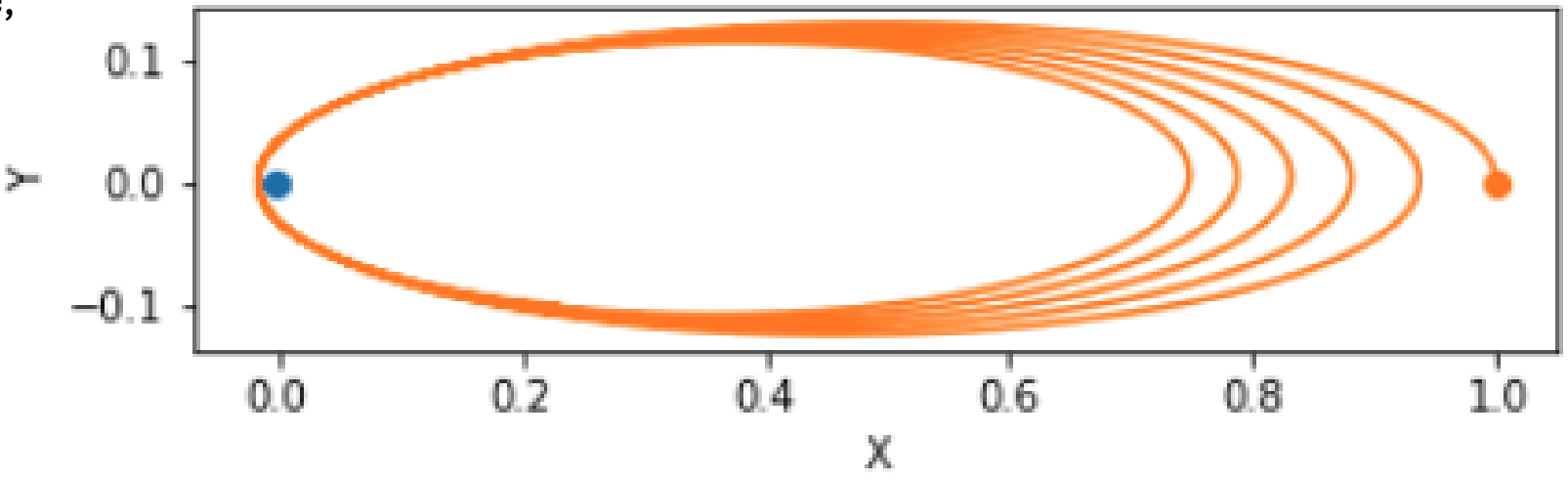
Here, one sees that even though KDK conserves energy, it will have phase errors when the timestep is too large, leading to precession.

Runge-Kutta keeps the phase accurate, but will dissipate energy (decrease in semimajor axis).

KDK Leapfrog – Fixed timestep 10^{-3}

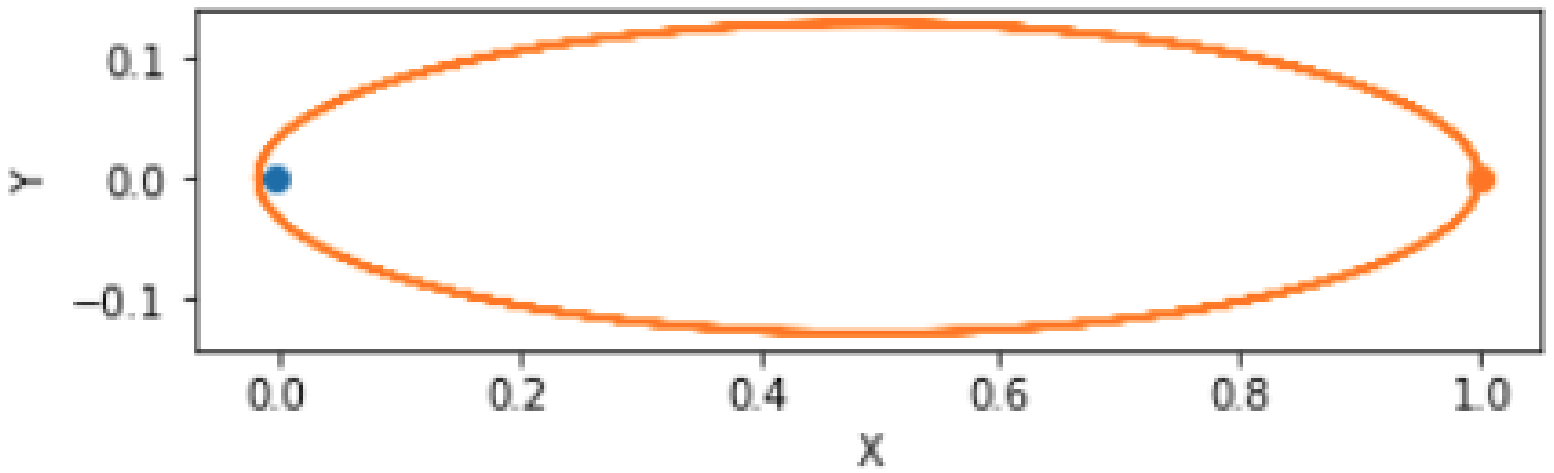


RK4 – Fixed timestep 10^{-3}



KDK/RK4 – Adaptive timestep

The usual algorithm used is *adaptive* timestep.

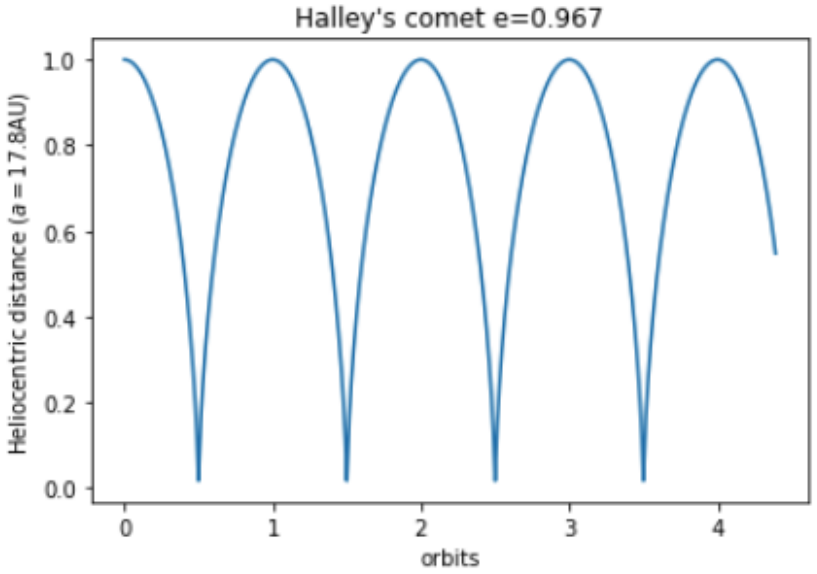


Define a tolerance, integrate with two algorithms, decreasing the timestep until the two algorithms yield error under the tolerance. When that happens (the algorithms converge), we accept the solution.

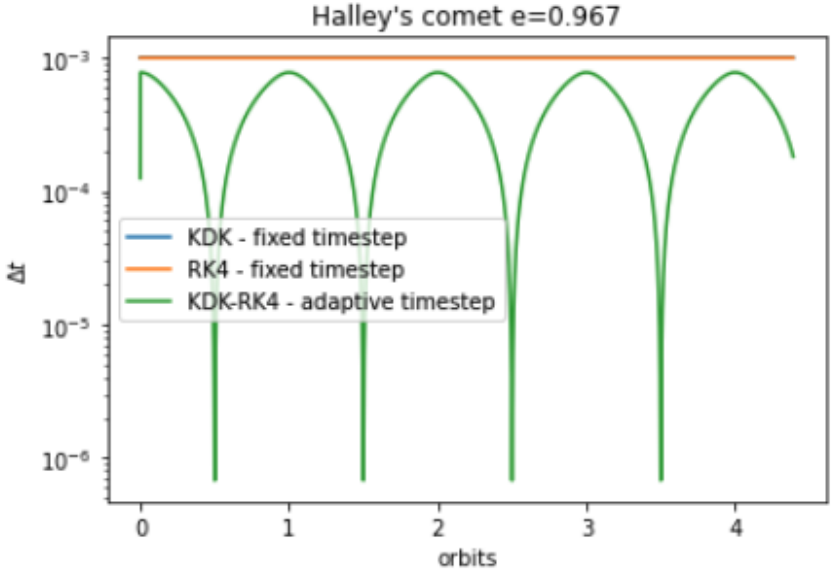
In the example above, a KDK and a RK4 solution were computed simultaneously.

KDK/RK4 – Adaptive timestep

Heliocentric Distance



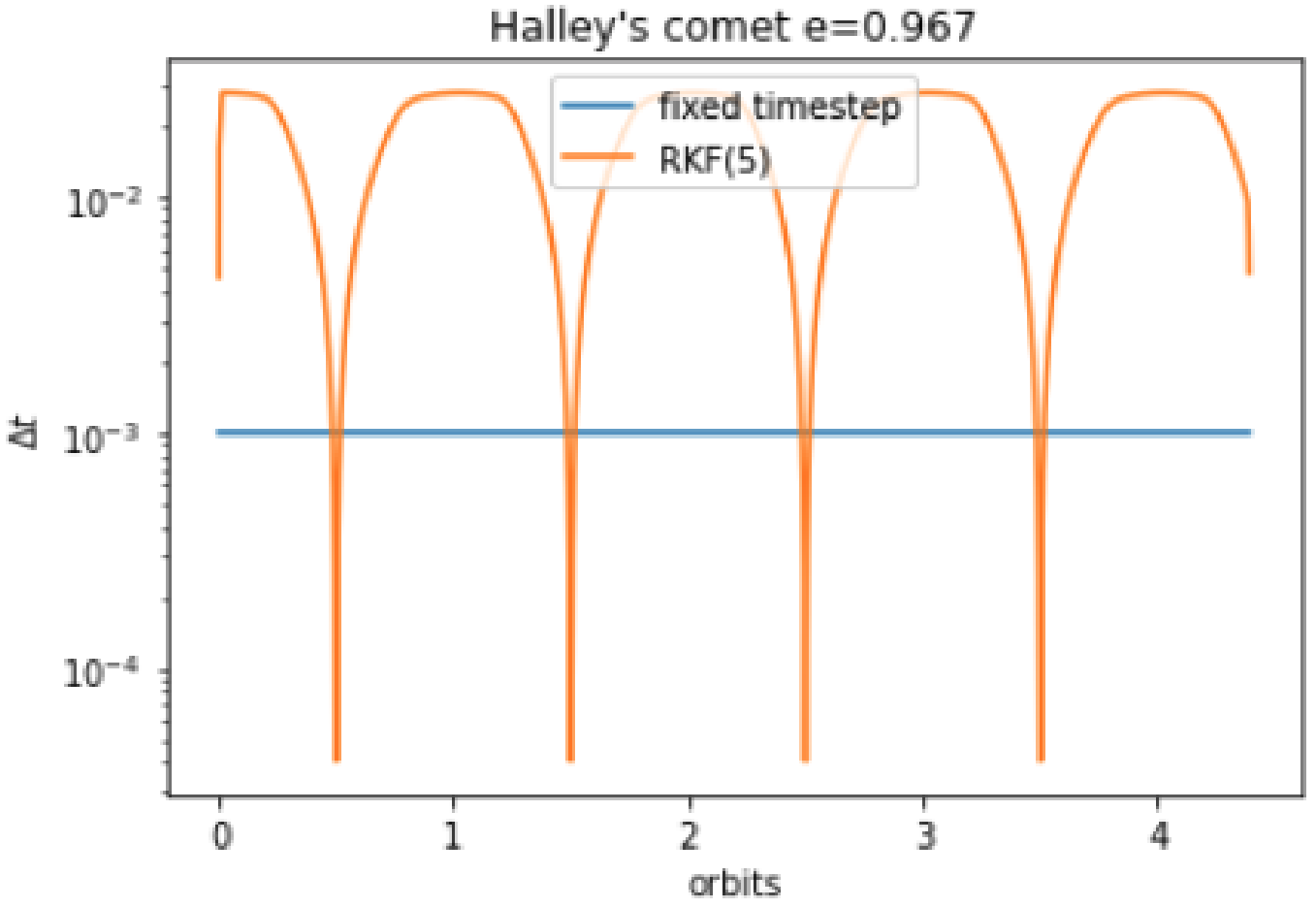
Timestep



As the figures above show, the timestep follows the heliocentric distance. At closest approach the comet is fastest, according to Kepler's 2nd law. That naturally necessitates shorter timesteps to accurate capture the orbital evolution.

RKF(5) Adaptive timestep

A popular adaptive integrator is the RKF(5), which is a 4th order Runge Kutta, compared to a 5th order. As seen here, it can do with very large timesteps at aphelion.



Back to hydro: 4rd order RK in time, 6th order space

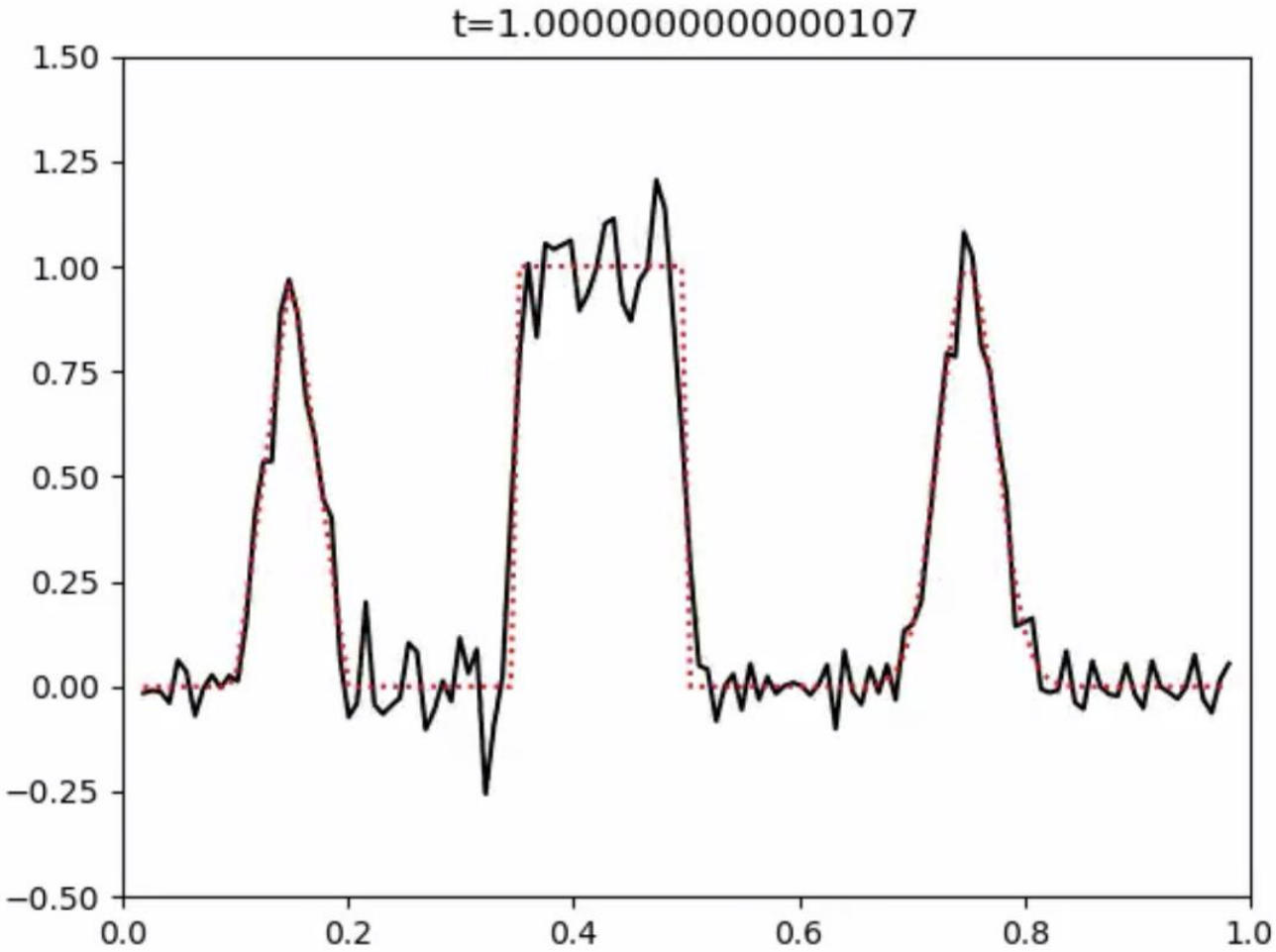
Now that we understand time integrators, let us return to hydrodynamics.

Let's test the advection of the function shown here, a hat function, a block function, and a Gaussian.

$$\frac{\partial f}{\partial t} = -u\frac{\partial f}{\partial x}$$

The test shows the advection using RK4 for time integration of the LHS, and 6th order derivatives in space for the RHS.

The scheme develops ringing at the discontinuities.



Gibbs Phenomena

A common criticism of high order schemes is their tendency to produce Gibbs phenomena (ripples) near discontinuities. Consequently one needs a small amount of diffusion to damp out the modes near the Nyquist frequency. Thus, one needs to replace eq. (50) by the equation

$$\dot{f} = -uf' + \nu f''. \tag{56}$$

The question is now how much diffusion is necessary, and how this depends on the spatial order of the scheme.

In figure 2 we plot the result of advecting the periodic step-like function, $f(kx)$, over 5 wavelengths, corresponding to a time $T = L/u$. The goal is to find the minimum diffusion coefficient ν necessary to avoid wiggles in the solution. In the first two panels one sees that for a 6th order scheme the diffusion coefficient has to be approximately $\nu = 0.01\,u\delta x$. For $\nu = 0.005\,u\delta x$ there are still wiggles. For a 10th order scheme one can still use $\nu = 0.005\,u\delta x$ without producing wiggles, while for a spectral scheme of nearly infinite order one can go down to $\nu = 0.002\,u\delta x$ without any problems.

We may thus conclude that all these schemes need some diffusion, but that the diffusion coefficient can be much reduced when the spatial order of the scheme is high. In that sense it is therefore not true that high order schemes are particularly vulnerable to Gibbs phenomena, but rather the contrary!

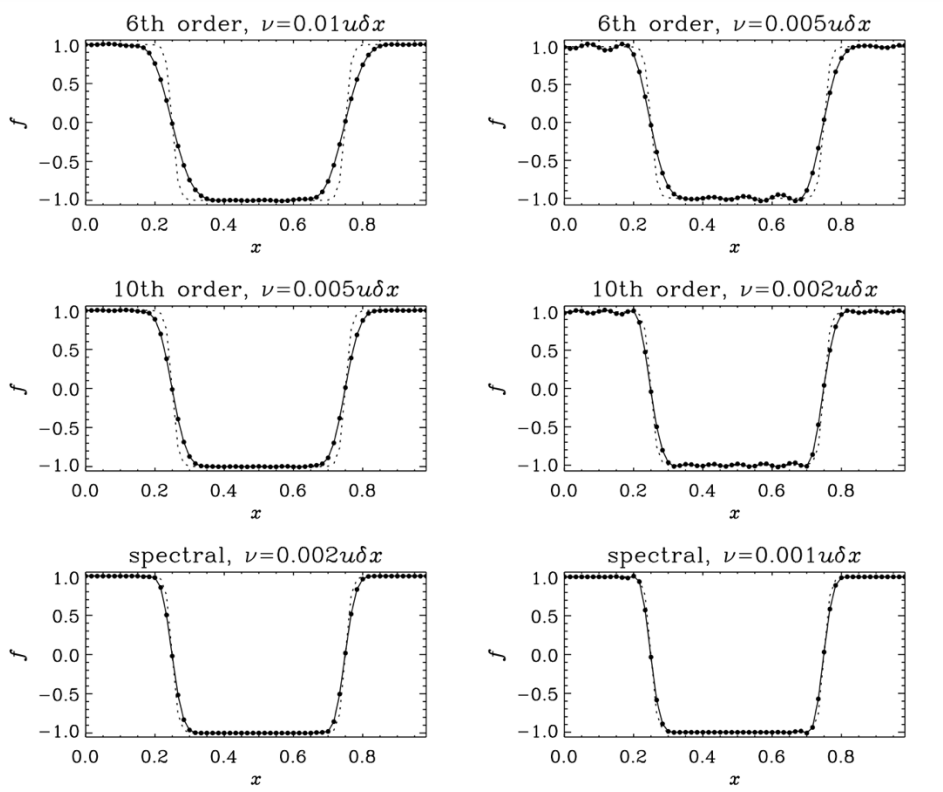


Figure 2: Resulting profile after advecting a step-like function 5 times through the periodic mesh. The dots on the solid line give the location of the function values at the computed meshpoints and the dotted line gives the original profile. For the panels on the right hand side the diffusion coefficient is too small and the profile shows noticeable wiggles. $\delta x = 1/60$.

Brandenburg & Dobler (2002)

Viscosity

$$Re_{Ny} = u\,\Delta x / nu = 1$$

Even though astrophysical flows are usually high Reynolds numbers so viscosity is negligible, numerical schemes usually require artificial viscosity for stability.

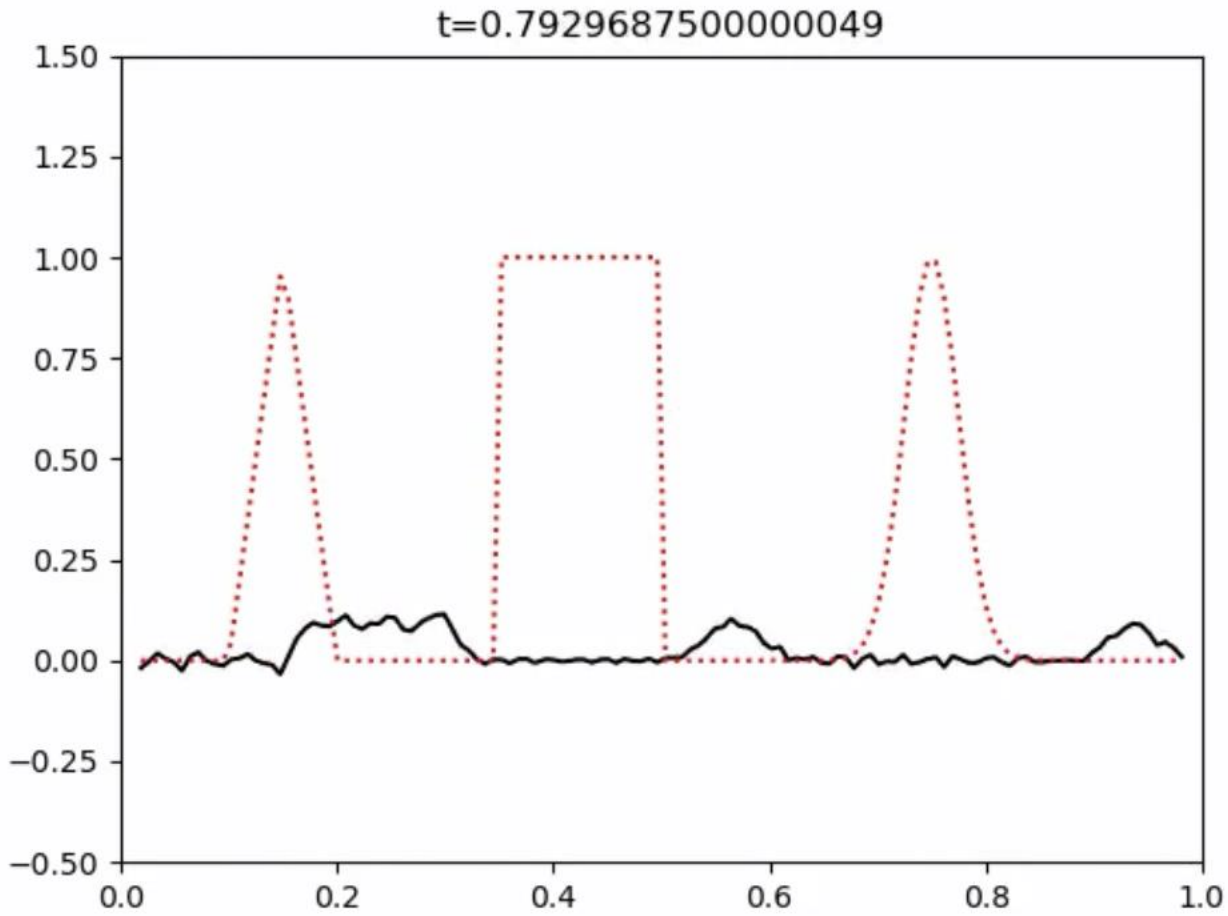
Viscosity is a Laplacian operator, so it operates in Fourier space as k^2: it is stronger for the smallest wavelengths.

We can use this property and set an artificial viscosity that has Reynolds number 1 for the smallest wavelength of the grid, $\lambda=2\Delta x$.

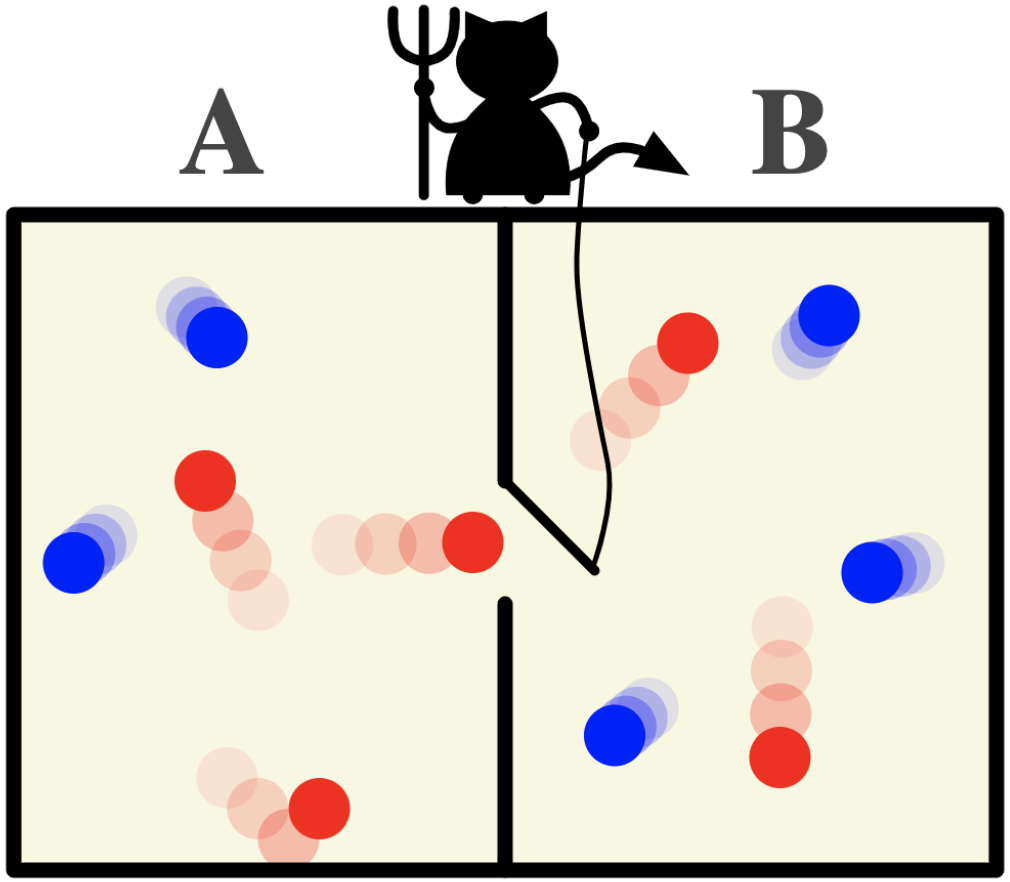
This wavelength is the *Nyquist* wavelength.

The viscosity associated is $nu = u\,\Delta x/Re$.

It works at damping the signal, but it will also damp everything else…



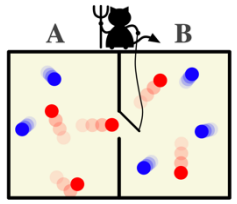
Let there be demons…



Demons in physics are hypothetical beings that violate the laws of nature in some particular way.

Numerical simulations need demons to deal with the breakdown
of continuum physics that happens at the grid scale

Hyperdiffusion

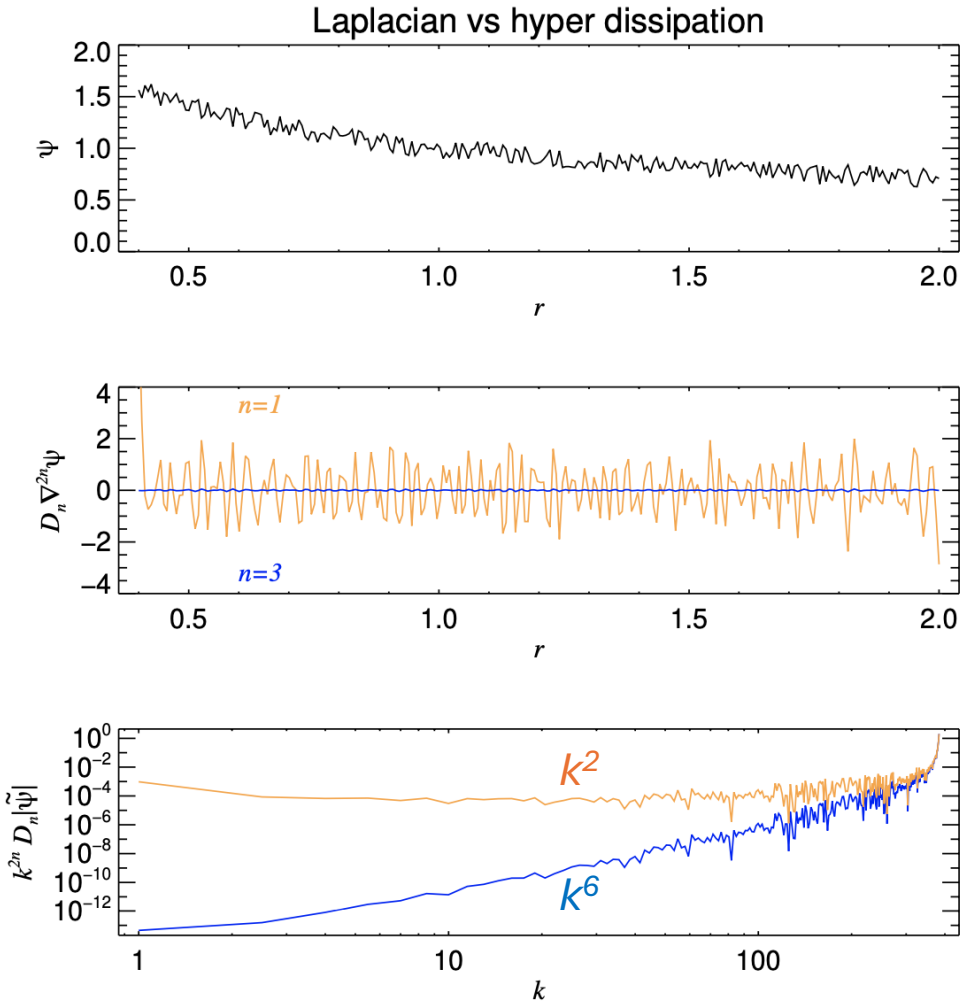


Hyper-dissipation allows **low Reynolds number** at the **grid scale** while maintaining **high Reynolds number** at the **inertial range**.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathcal{J} = 0.$$

Laplacian dissipation	Hyper dissipation
$\mathcal{J} = -D\nabla\rho$	$\mathcal{J}^{(n)} = (-1)^n D^{(n)} \nabla^{2n-1}\rho.$
$\frac{\partial \rho}{\partial t} = D\nabla^2\rho$	$\frac{\partial \rho}{\partial t} = D^{(3)}\nabla^6\rho,$
(n=1)	(n=3)
$\propto k^2$	$\propto k^{2n}$

$$\mathrm{Re}_{\mathrm{grid}} = u_{\mathrm{rms}} / (\nu_n k_{\mathrm{Ny}}^{2n-1})$$



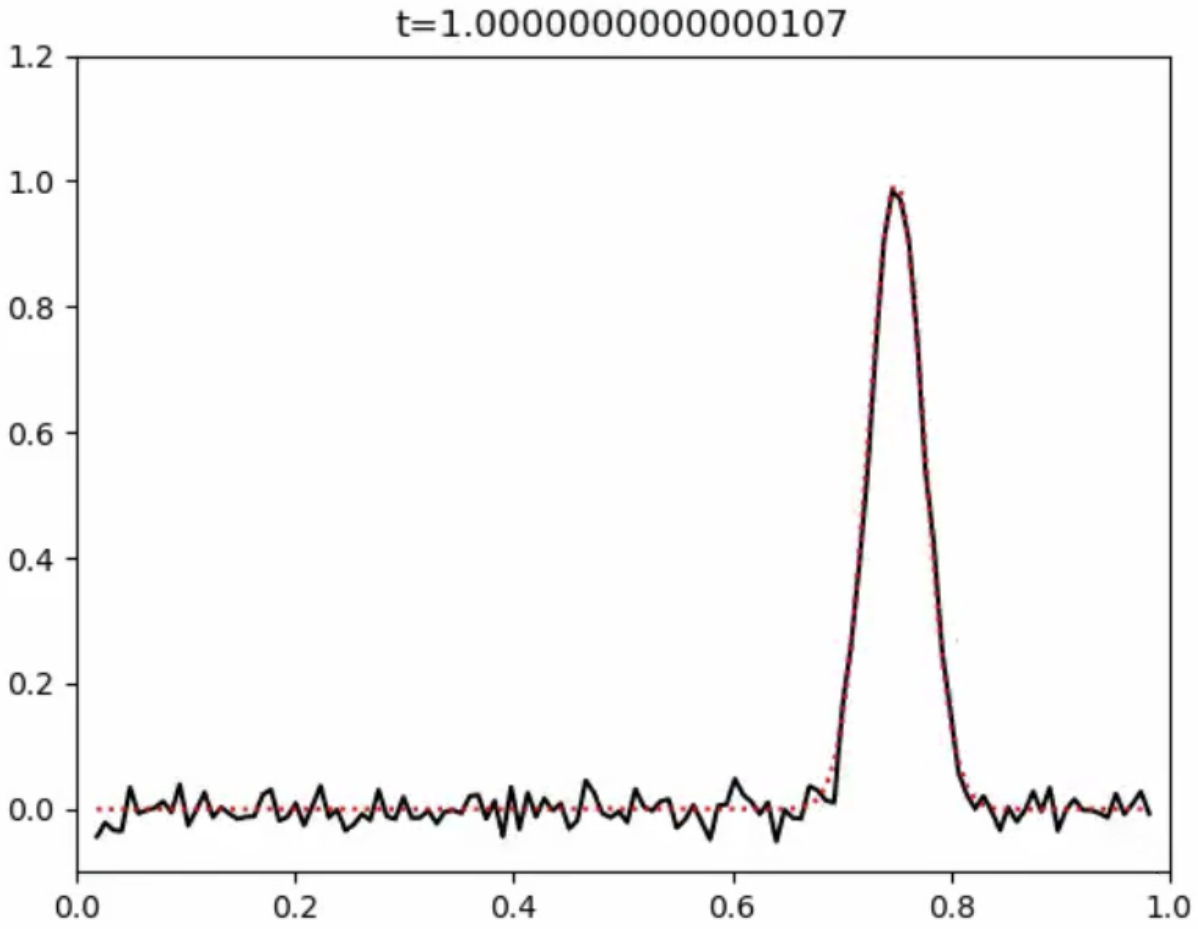
With a k^6 dependency in Fourier space, hyperviscosity allows keeping high viscous power at the Nyquist scale, while maintaining the integral scales of the flow quiescent.

$Re_{Ny} = u\,\Delta x^5 / nu_{hyper} = 1$

With a k^6 dependency in Fourier space, hyperviscosity allows keeping high viscous power at the Nyquist scale, while maintaining the integral scales of the flow quiescent.

Hyperviscosity keeps the viscosity where it is needed: at the grid scale.

Caution though: it is unphysical.



$Re_{Ny} = u\,\Delta x^5 / nu_{hyper} = 1.0$

Still bad at discontinuities….

Next class, *shock viscosity*.