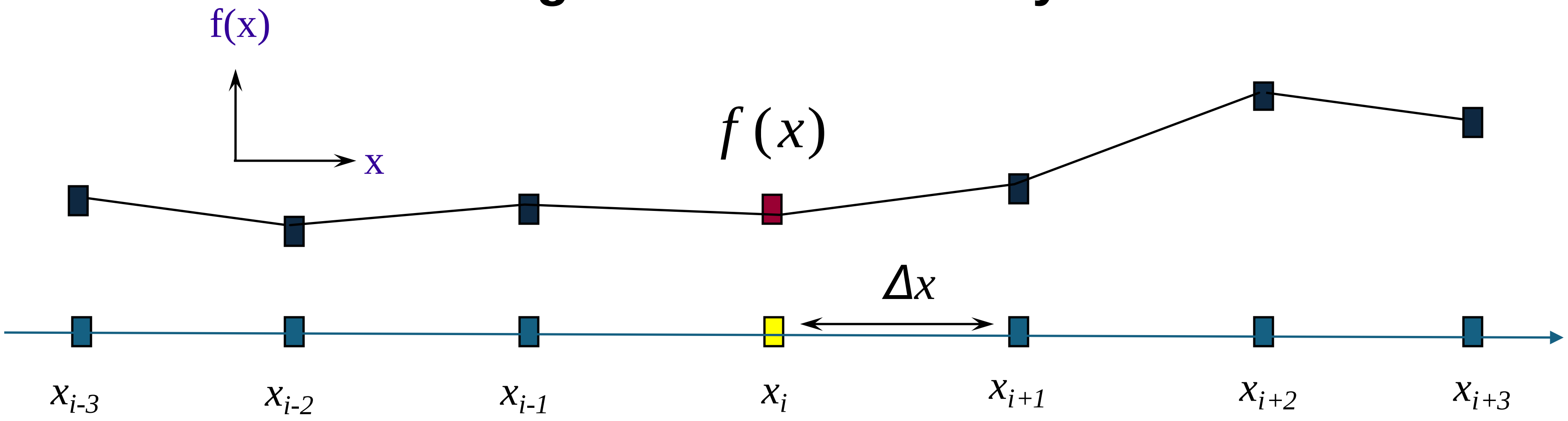
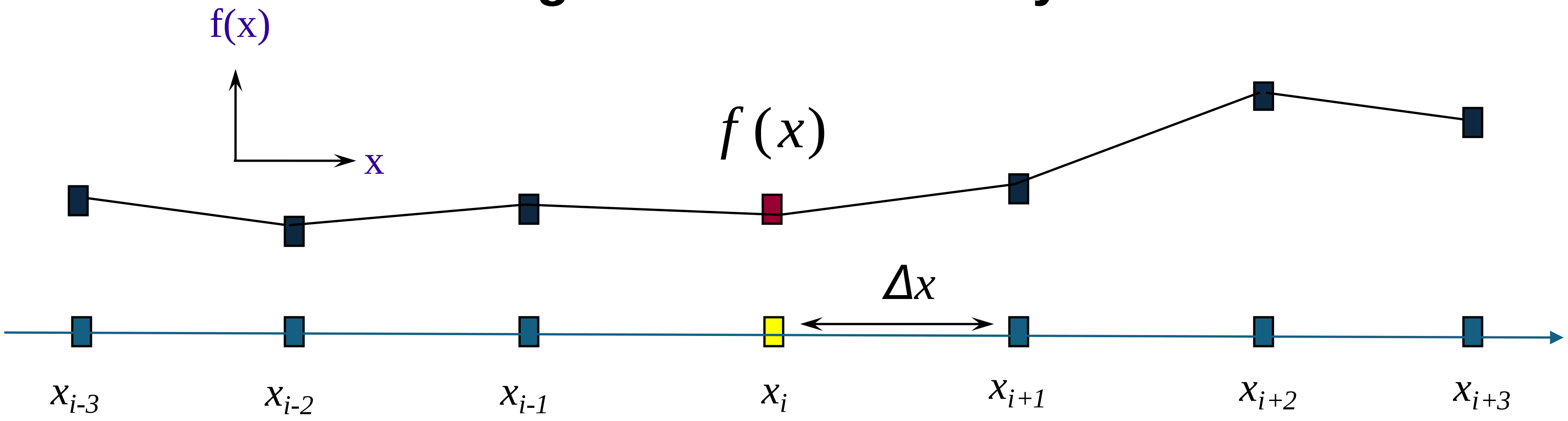
Higher-order accuracy



What is the (approximate) value of the first derivative at the desired location?

Higher-order accuracy



What is the (approximate) value of the first derivative at the desired location?

The first derivative, to 2nd order accuracy is

$$f_i' = \frac{-f_{i-1} + f_{i+1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

to 4th order accuracy, it is

$$f_i' = \frac{f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}}{12\Delta x} + \mathcal{O}(\Delta x^4)$$

to 6th order accuracy, it is

$$f_i' = \frac{-f_{i-3} + 9f_{i-2} - 45f_{i-1} + 45f_{i+1} - 9f_{i+2} + f_{i+3}}{60\Delta x} + \mathcal{O}(\Delta x^6)$$

Finite Difference Coefficients

These coefficients come from Taylor expansion. Suppose that we want to compute df/dx to 2nd order. Using a 3-point stencil, we have

$$\frac{df}{dx} = \frac{1}{h}[Af(x-h) + Bf(x) + Cf(x+h)]$$

where $h = \Delta x$. According to the table above, we expect to find $A = -1/2$, $B = 0$ and $C = 1/2$. Let us prove this.

If we Taylor expand around x,

$$\begin{aligned} Af(x-h) &= Af(x) + Af'(x)(-h) + Af''(x)\frac{h^2}{2} \\ Bf(x) &= Bf(x) \\ Cf(x+h) &= Cf(x) + Cf'(x)(h) + Cf''(x)\frac{h^2}{2} \end{aligned}$$

Summing them all

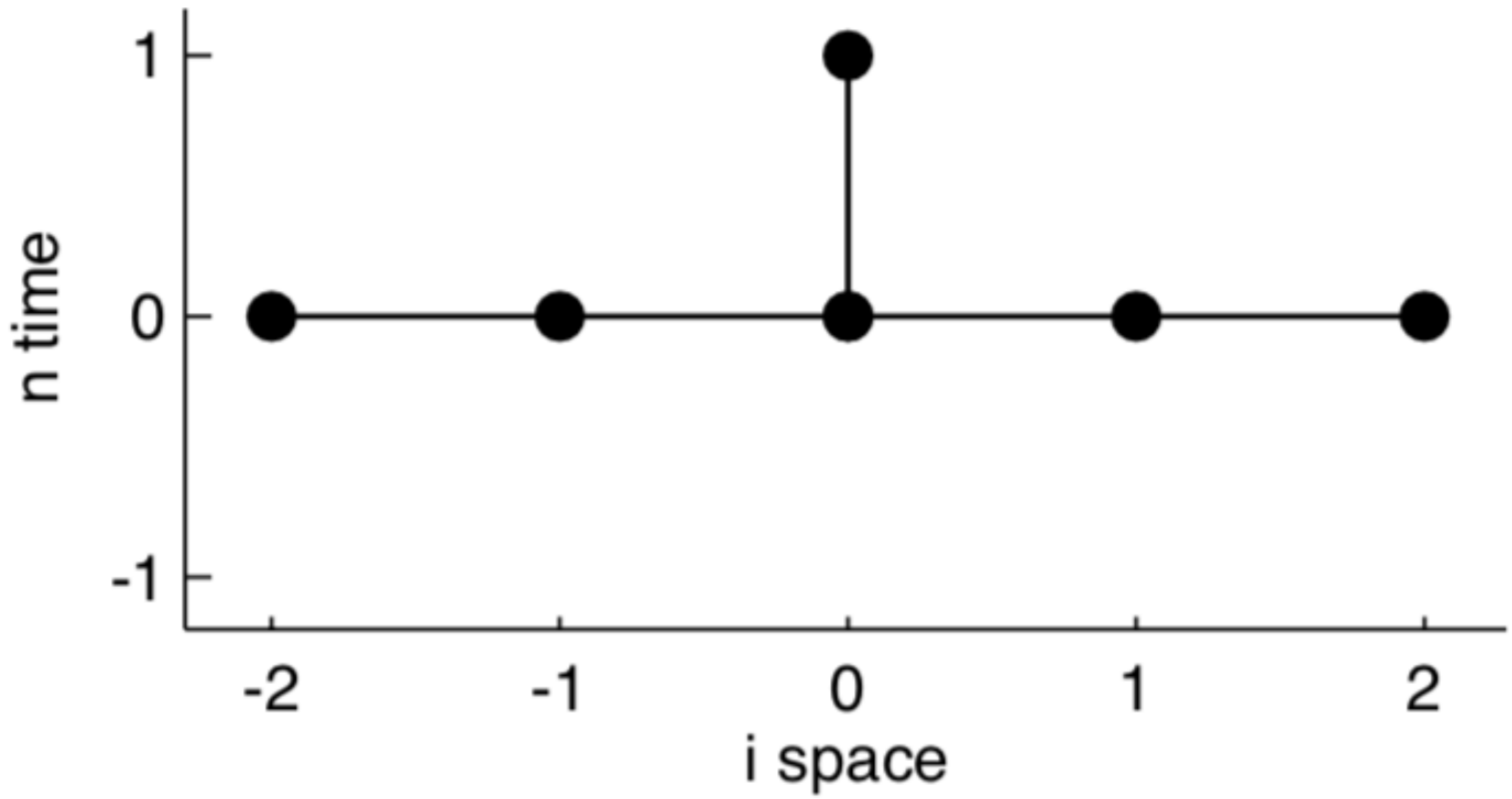
$$h\frac{df}{dx} = f(x)(A+B+C) + f'(x)(A-C)h + f''(x)\frac{h^2}{2}(A+C)$$

Since only the first derivative should survive in the RHS, this leads to the conditions

$$\begin{aligned} A+B+C &= 0 \\ A-C &= 1 \\ A+C &= 0 \end{aligned}$$

Leading to $A = -1/2$, $C = 1/2$, $B = 0$, as expected.

Stencil for 4th order accuracy

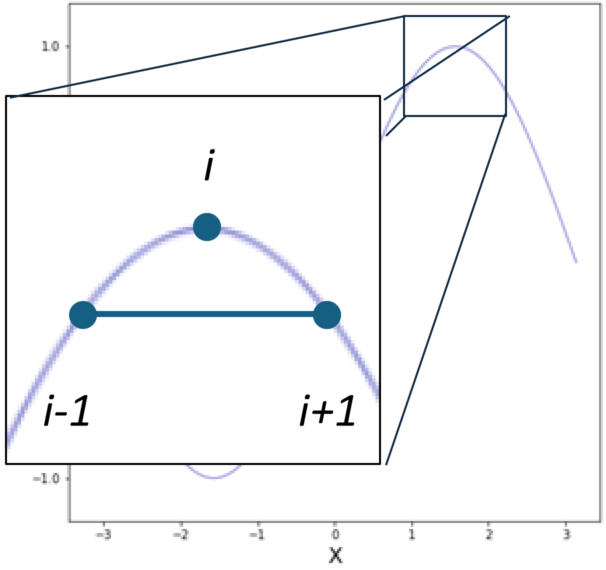


- Higher orders naturally require a larger stencil
- More points are needed to reach the higher order terms in the Taylor series expansion

Geometrical Interpretation

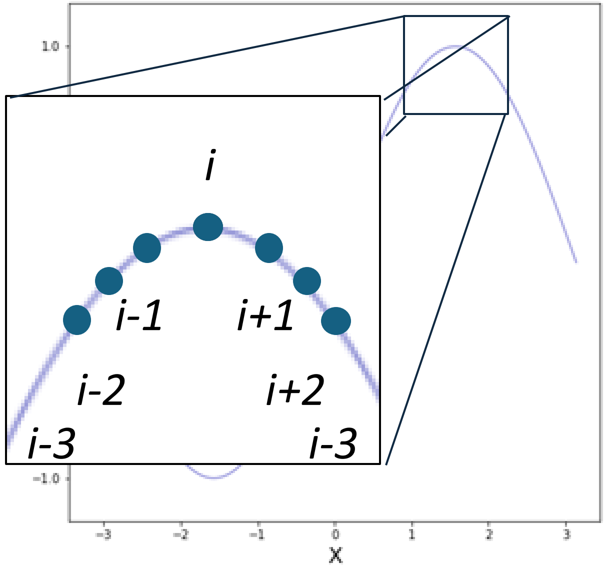
2nd order accuracy

$$f_i' = \frac{-f_{i-1} + f_{i+1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$



6th order accuracy

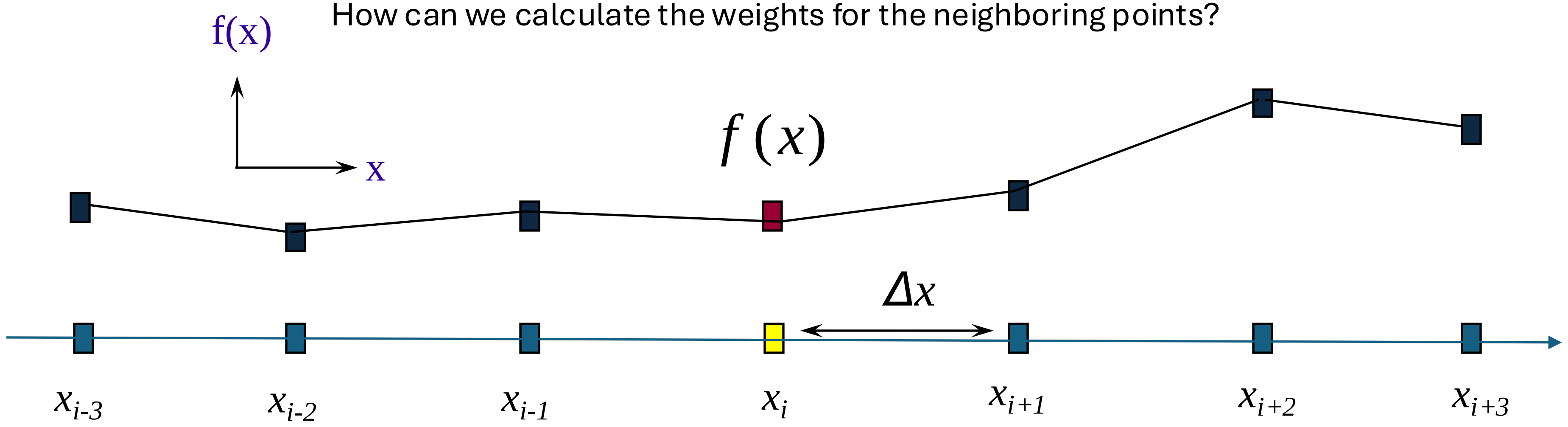
$$f_i' = \frac{-f_{i-3} + 9f_{i-2} - 45f_{i-1} + 45f_{i+1} - 9f_{i+2} + f_{i+3}}{60\Delta x} + \mathcal{O}(\Delta x^6)$$



With more points, the function is better sampled!

Finite Difference Coefficients

How can we calculate the weights for the neighboring points?



General formula!

$$\boxed{\sum_{i=0}^{N-1} s_i^n c_i = d!\,\delta(n-d) \quad \text{for } 0 < n < N-1}$$

s = (-3,-2,-1,0,1,2,3) stencil positions
N = size of stencil
d = order of the derivative

en.wikipedia.org/wiki/Finite_difference_coefficient

$$\sum_{i=0}^{N-1} s_i^n c_i = d!\,\delta(n-d) \quad \text{for } 0 < n < N-1$$

s = (-3,-2,-1,0,1,2,3) stencil positions
N = size of stencil
d = order of the derivative

Finite difference coefficient

7 languages

Article Talk Read Edit View history Tools

From Wikipedia, the free encyclopedia

In mathematics, to approximate a derivative to an arbitrary order of accuracy, it is possible to use the finite difference. A finite difference can be **central**, **forward** or **backward**.

Central finite difference [edit]

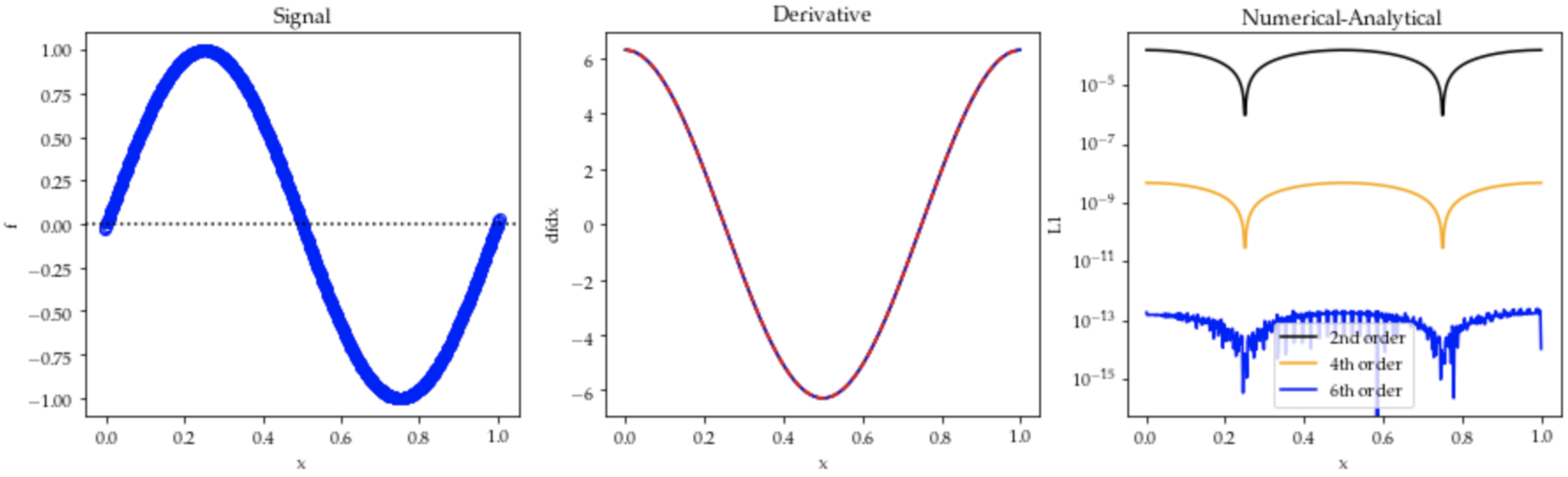
This table contains the coefficients of the **central** differences, for several orders of accuracy and with uniform grid spacing:[1]

Derivative	Accuracy	−5	−4	−3	−2	−1	0	1	2	3	4	5
1	2					−1/2	0	1/2				
	4				1/12	−2/3	0	2/3	−1/12			
	6			−1/60	3/20	−3/4	0	3/4	−3/20	1/60		
	8		1/280	−4/105	1/5	−4/5	0	4/5	−1/5	4/105	−1/280	
2	2					1	−2	1				
	4				−1/12	4/3	−5/2	4/3	−1/12			
	6			1/90	−3/20	3/2	−49/18	3/2	−3/20	1/90		
	8		−1/560	8/315	−1/5	8/5	−205/72	8/5	−1/5	8/315	−1/560	
3	2				−1/2	1	0	−1	1/2			
	4			1/8	−1	13/8	0	−13/8	1	−1/8		
	6		−7/240	3/10	−169/120	61/30	0	−61/30	169/120	−3/10	7/240	
4	2				1	−4	6	−4	1			
	4			−1/6	2	−13/2	28/3	−13/2	2	−1/6		
	6		7/240	−2/5	169/60	−122/15	91/8	−122/15	169/60	−2/5	7/240	
5	2			−1/2	2	−5/2	0	5/2	−2	1/2		
	4		1/6	−3/2	13/3	−29/6	0	29/6	−13/3	3/2	−1/6	
	6	−13/288	19/36	−87/32	13/2	−323/48	0	323/48	−13/2	87/32	−19/36	13/288
6	2			1	−6	15	−20	15	−6	1		
	4		−1/4	3	−13	29	−75/2	29	−13	3	−1/4	
	6	13/240	−19/24	87/16	−39/2	323/8	−1023/20	323/8	−39/2	87/16	−19/24	13/240

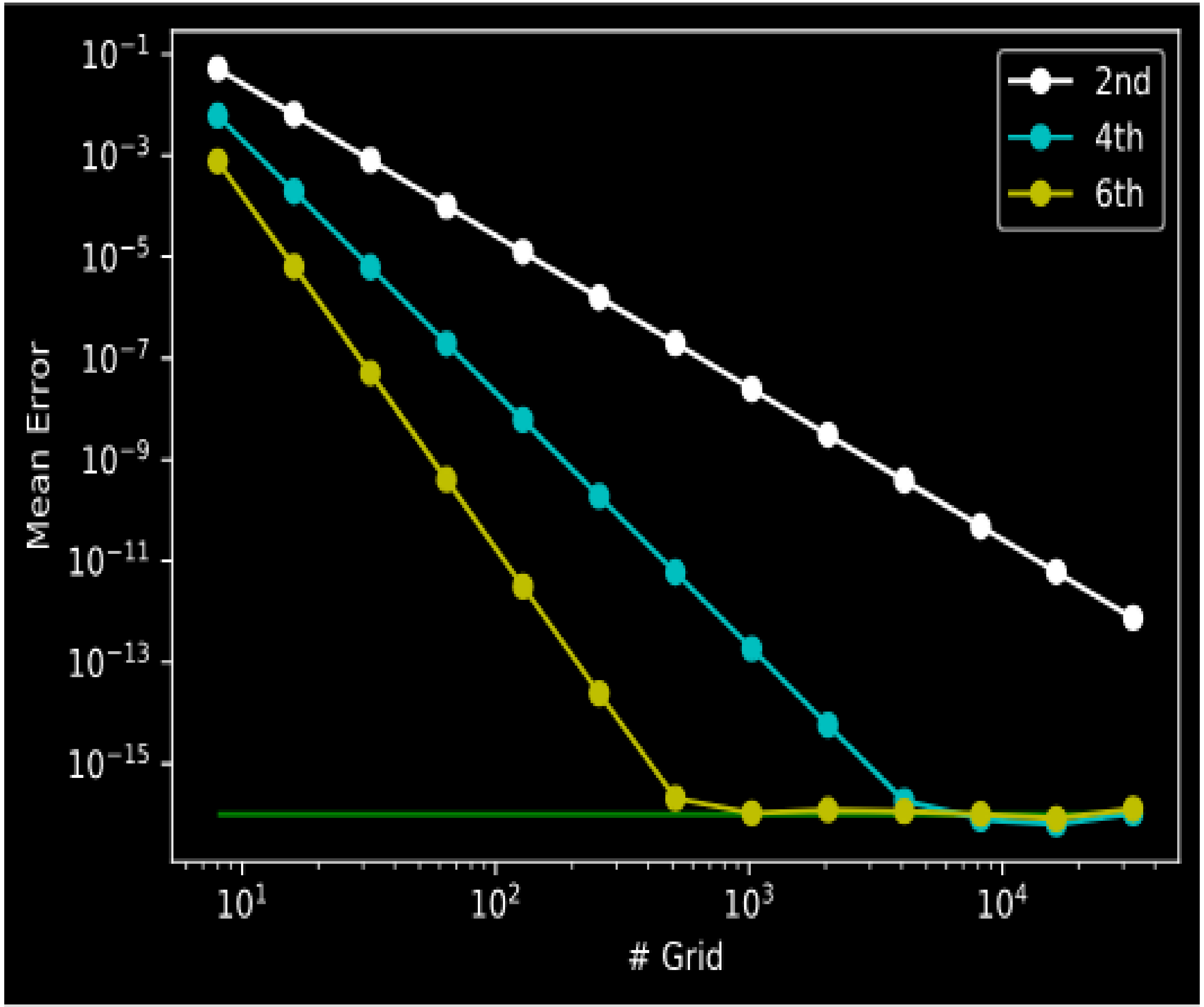
High order derivatives

Class exercise.

- Define a signal f(x)=sin(kx)
  - k=2π,
  - grid from 0 to 1
- The analytical derivative is k*cos(kx)
- Take the numerical derivative in 2^{nd}, 4^{th}, and 6^{th} order
- Compare the norm |analytical-numerical|



Higher-order accuracy



- Higher order derivative = smaller truncation error.
- Higher orders naturally require a larger stencil.
- Higher order derivatives approach machine precision faster.

Finite Differences - Summary

- Conceptually the simplest of the numerical methods and can be learned quite quickly
- Depending on the physical problem FD methods are conditionally stable (relation between time and space increment)
- High-order FD methods have difficulties concerning damping at the grid scale
- FD methods are usually explicit and therefore very easy to implement and efficient on parallel computers
- FD methods work best on regular, rectangular grids