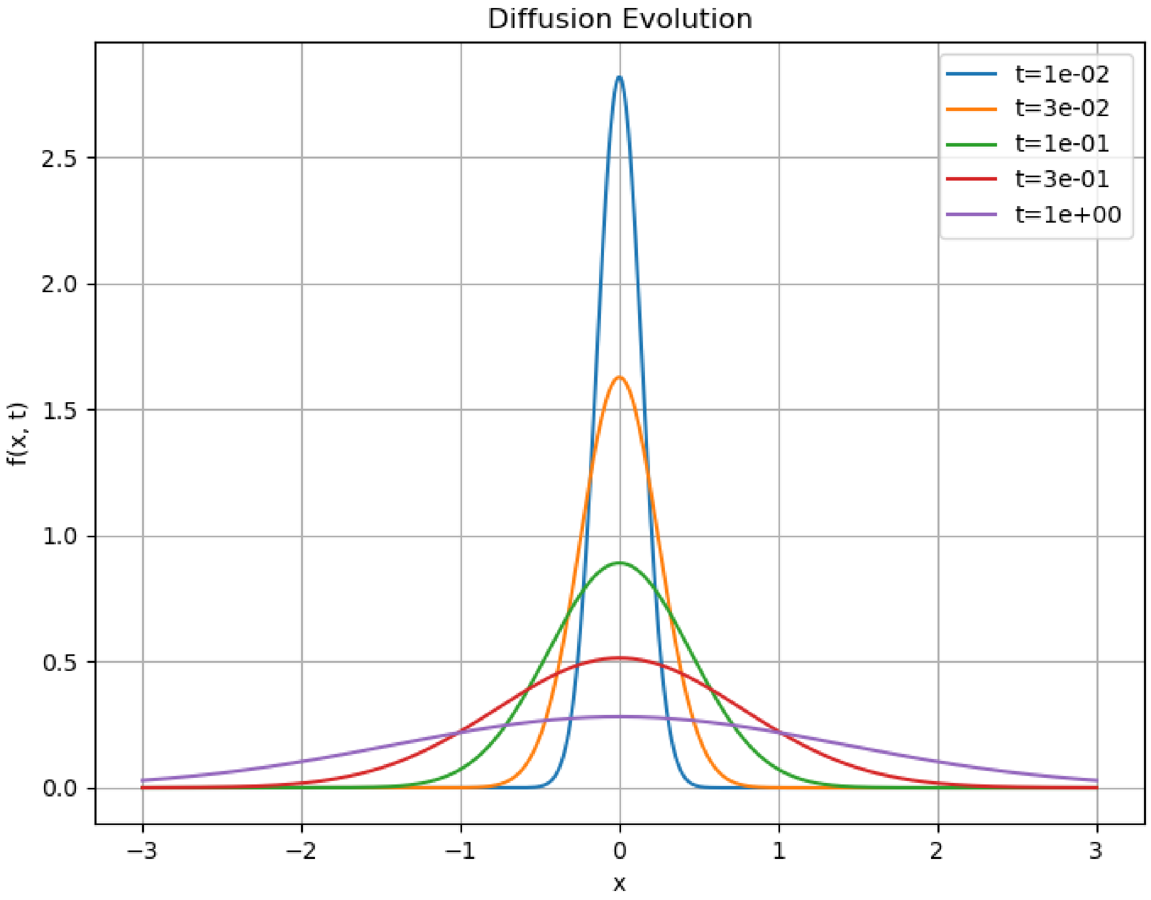
Diffusion equation – General solution

$$\frac{\partial f(x,t)}{\partial t} = K \frac{\partial^2 f(x,t)}{\partial x^2}$$



The general solution to this equation is (show as homework)

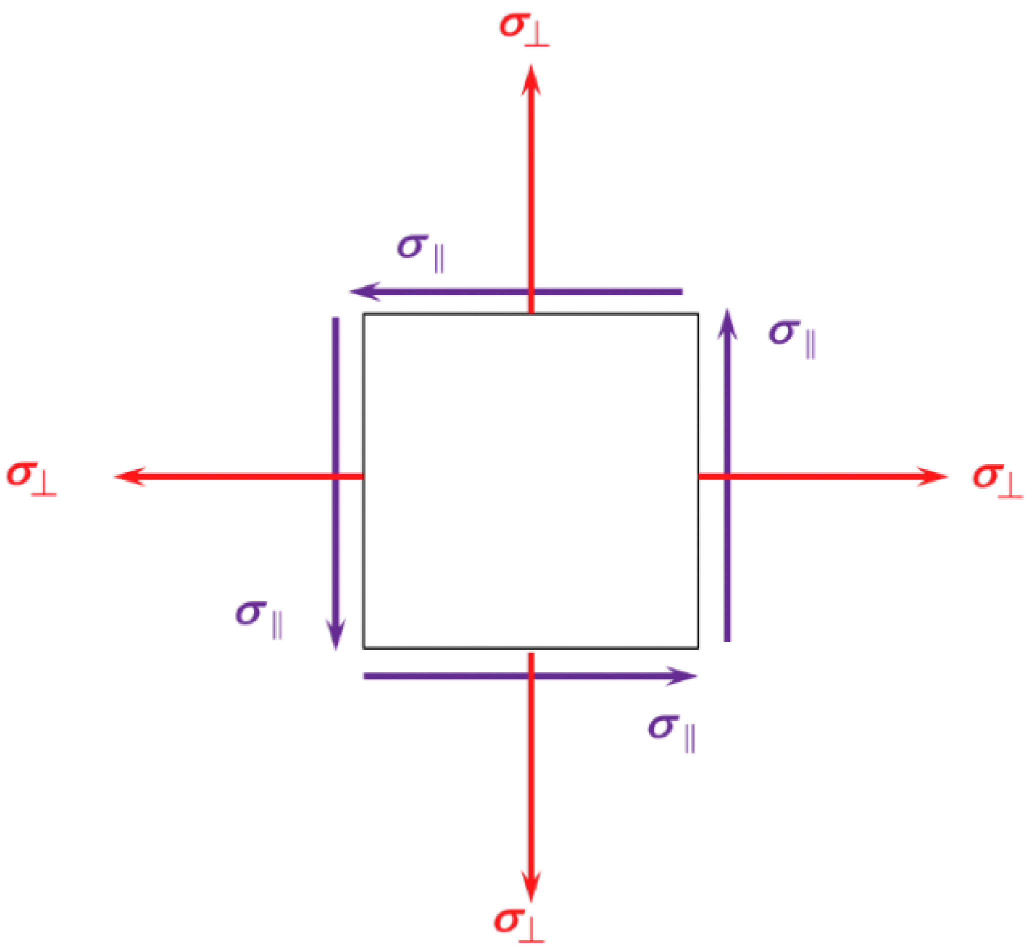
$$f(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-\frac{x^2}{4Dt}}$$

Which is a Gaussian of time-dependent width σ(t)=sqrt(2Dt).

The peak amplitude decreases and the width decreases in time: the bump spreads. Peaks are eroded, valleys are filled. So is the nature of diffusion.

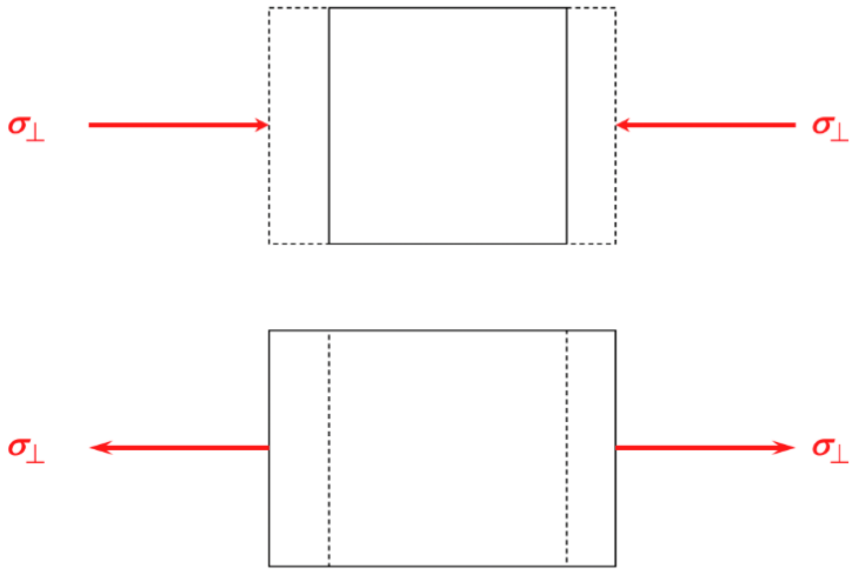
Surface forces

Surface forces can be either normal or parallel to the surface.
This is described by a second-rank tensor, the stress tensor.

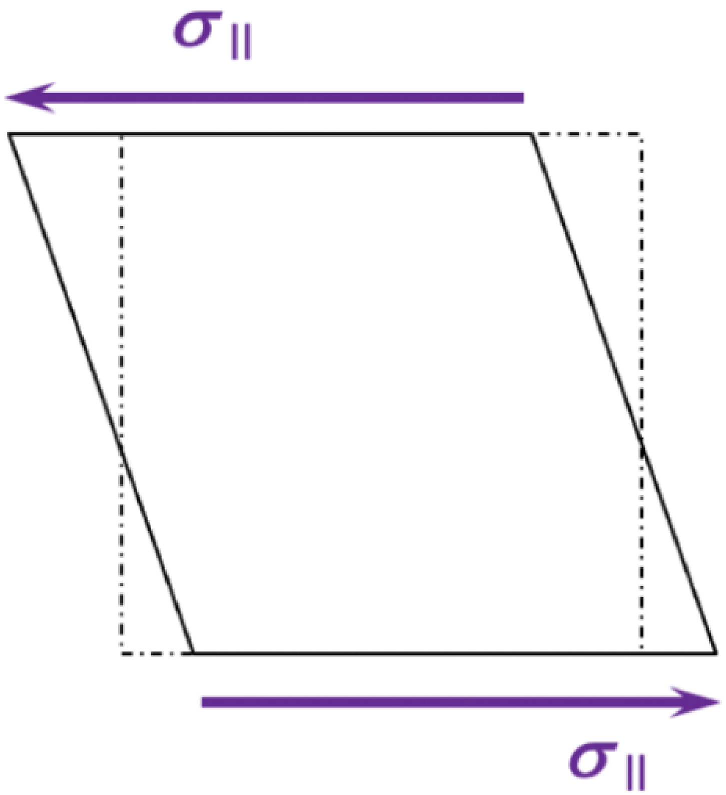


Pressure and Stress

Perpendicular force:
Pressure



Tangential force:
Stress



Normal forces are *pressure.* Parallel forces are *stresses*.
Stress lead to deformation in the fluid. The stress tensor is

$$P_{ij} = p\delta_{ij} + \pi_{ij}$$

π_{ij} is the viscous tensor. It has to be traceless to not contribute a normal force to $\nabla \cdot P$

Viscosity

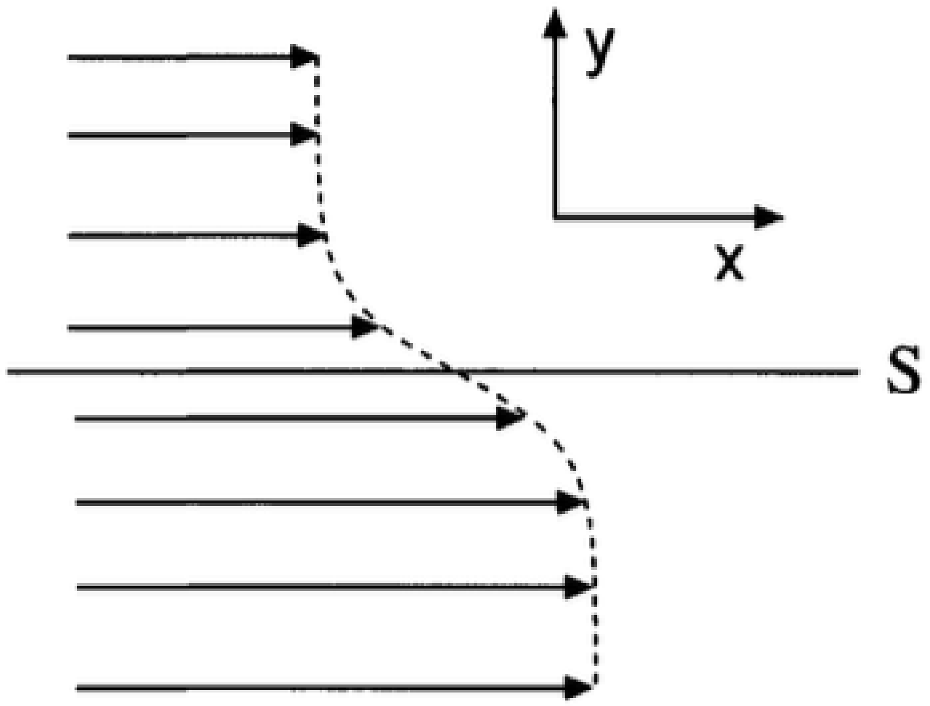


Figure from Choudhuri

If there is friction inside the fluid, faster-moving parcels will accelerate slow-moving parcels at their interface. The situation is shown in the figure.

In the region of shear (velocity gradient), we expect the viscous force to operate. **The viscous force is proportional to the shear.**

Definition of shear: the gradient of the velocity in a different direction:

$$\frac{\partial u_i}{\partial x_j}$$

Viscosity

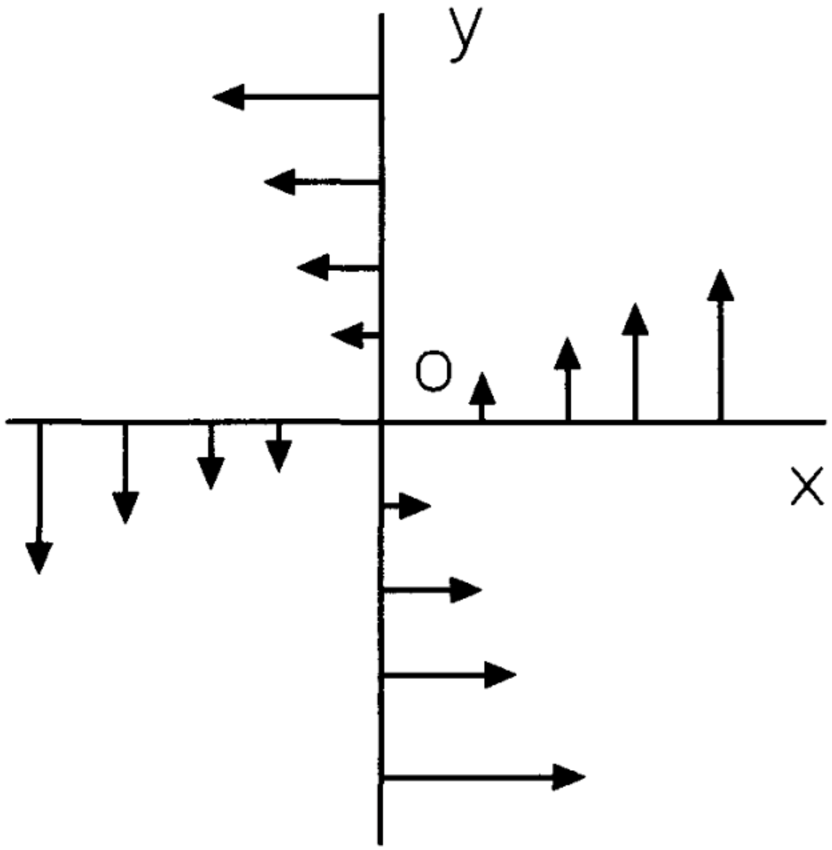


Figure from Choudhuri

Need to construct a viscous tensor that results in a force that

1) Cancels for rigid rotation
2) Results in no normal force (traceless)

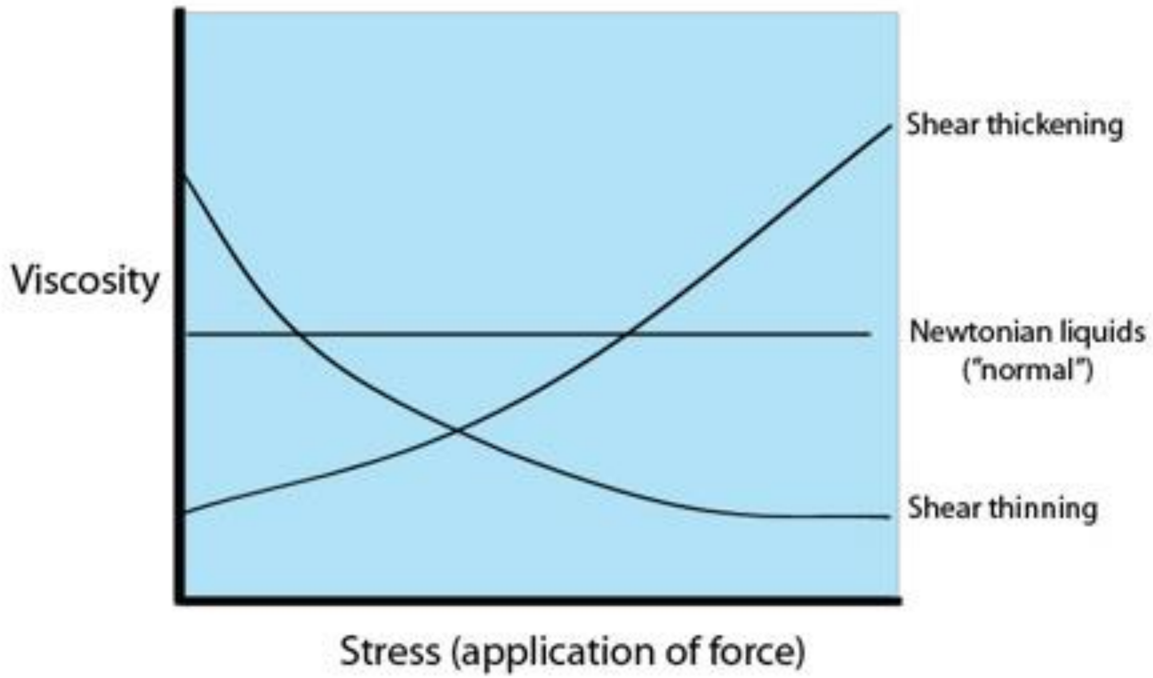
The most general second rank tensor constructed with linear velocity gradients is

$$\pi_{ij} = a\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + b\delta_{ij}\nabla\cdot\boldsymbol{u}$$

The traceless condition requires *b=-2a/3*. Thus for *a=-μ*

$$\pi_{ij} = -\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{u}\right)$$

Non-Newtonian Fluids



Aside:

Flows whose viscosity behave as we derived are called *Newtonian fluids* (as it was Sir Isaac who postulated that the viscous force should be proportional to the shear).

There are fluids that do not obey this law, for which the viscosity depends on the force applied.

These *non-Newtonian fluids* can be either shear-thickening (viscosity increases with force applied), or shear-thinning (viscosity decreases with force applied).

NEWTONIAN LIQUIDS

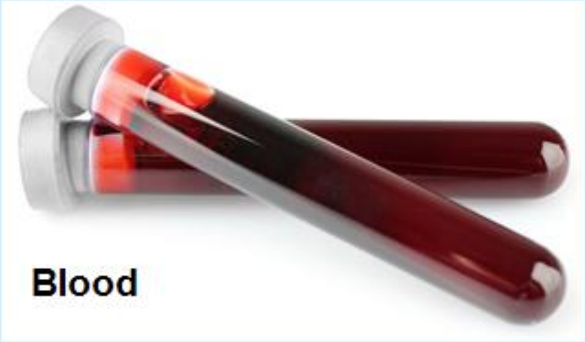
Olive Oil

NON-NEWTONIAN LIQUIDS

Asphalt

Lava

Molten Metal



Blood

Toothpaste

Paint

Glue

Corn flour and water

Whipped Cream

Gypsum paste

Newtonian Fluids

Honey is a Newtonian fluid. Its viscosity does not depend on applied force.

Non-Newtonian Fluids

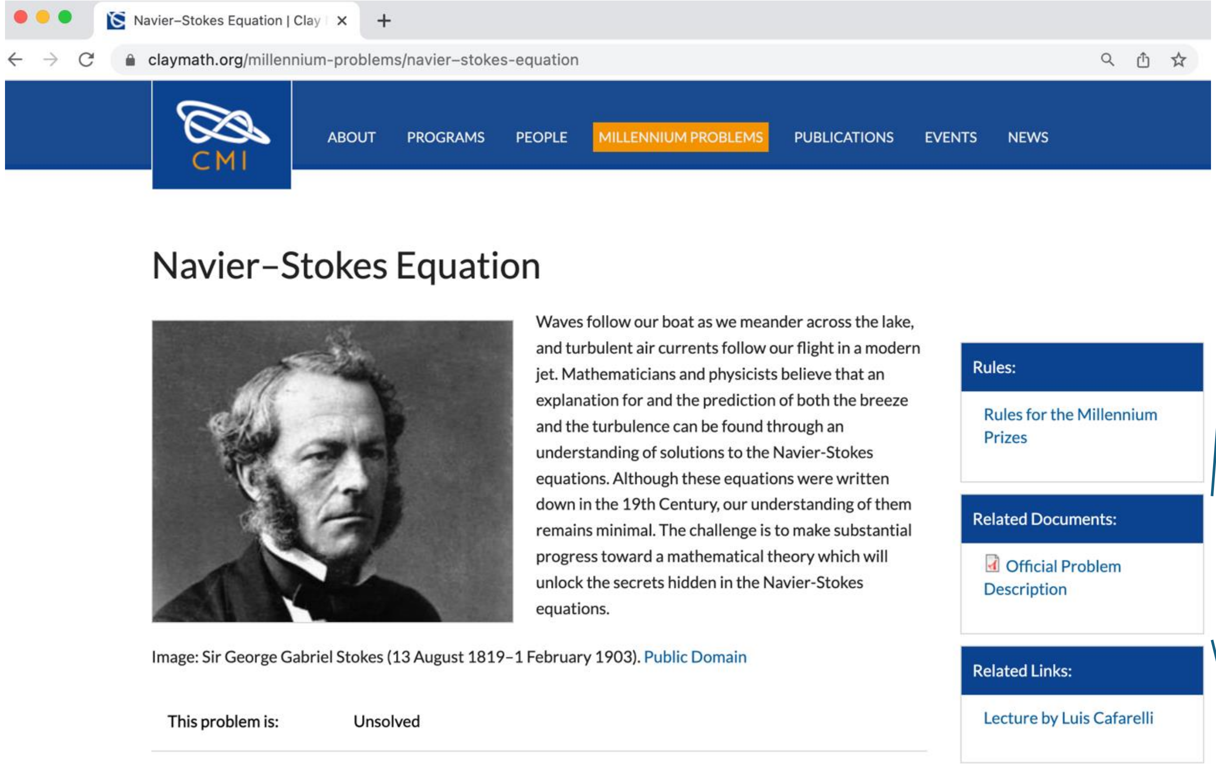
Corn starch solution is a shear-thickening non-Newtonian fluid.
Its viscosity increases with applied force. Too much force and it behaves like a solid.

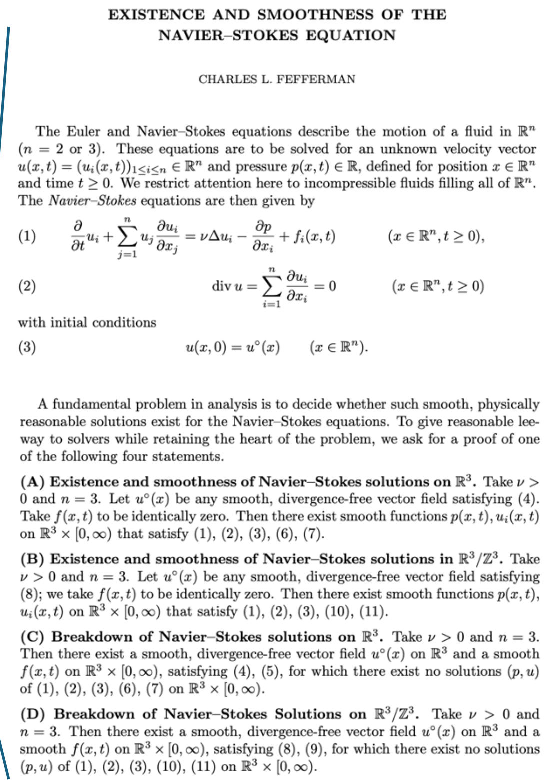
The origin of this behavior in non-Newtonian fluids is due to ***impurities*** in the fluid.

Grains, large macromolecules, etc, will impede the flow of the fluid molecules, and generate this force-dependent response as they cluster (shear-thickening) or break (shear-thinning) when force is applied.

Here we will concern ourselves with Newtonian fluids.

Millennium Prize ($1M cash prize) for solution of the Hydrodynamics Equations





EXISTENCE AND SMOOTHNESS OF THE NAVIER–STOKES EQUATION

CHARLES L. FEFFERMAN

The Euler and Navier–Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1\le i\le n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \ge 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The *Navier–Stokes* equations are then given by

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad (x \in \mathbb{R}^n, t \ge 0), \tag{1}$$

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x \in \mathbb{R}^n, t \ge 0) \tag{2}$$

with initial conditions

$$u(x,0) = u^\circ(x) \qquad (x \in \mathbb{R}^n). \tag{3}$$

A fundamental problem in analysis is to decide whether such smooth, physically reasonable solutions exist for the Navier–Stokes equations. To give reasonable leeway to solvers while retaining the heart of the problem, we ask for a proof of one of the following four statements.

(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6), (7).

(B) Existence and smoothness of Navier–Stokes solutions in $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (8); we take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (10), (11).

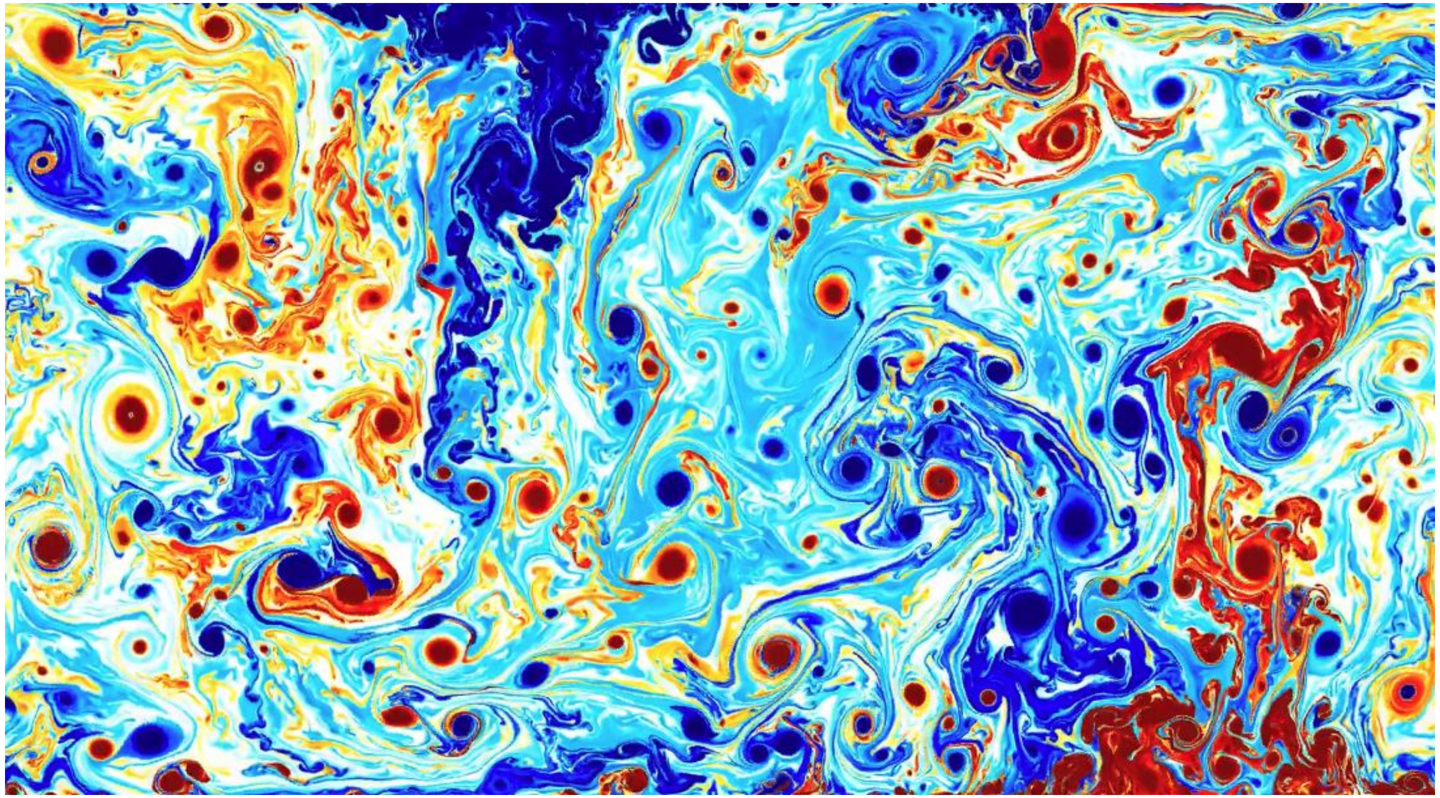
(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (4), (5), for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (8), (9), for which there exist no solutions (p,u) of (1), (2), (3), (10), (11) on $\mathbb{R}^3 \times [0,\infty)$.

Hydrodynamics

“The last unsolved problem of Classical Physics”

The nonlinear term



Fully developed turbulence

The Navier-Stokes equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \boldsymbol{f} + \nu \nabla^2 \boldsymbol{u}$$

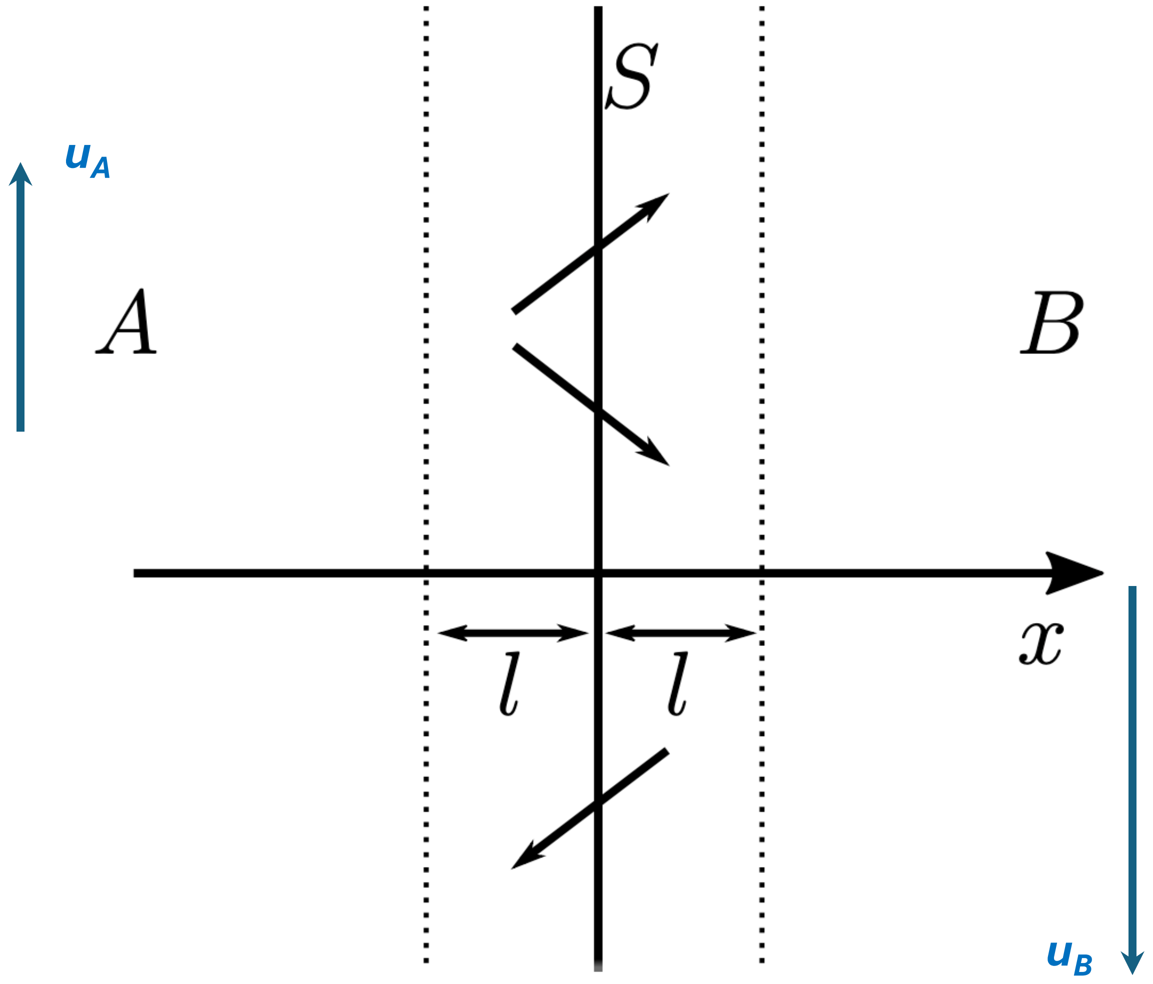
has a nonlinear term in velocity

$$(\boldsymbol{u} \cdot \nabla) \boldsymbol{u}$$

The momentum advects itself. All the complex behavior of hydrodynamical flows comes from this term. Ultimately, this comes from Newton's 1st law, the law of inertia: an object remains in motion unless a force is applied to it.

This term is also called the ***inertial*** term.

Microphysics of viscosity



Like with heat diffusion, particles exchanged between adjacent fluid elements A and B are the origin of viscosity.

Molecules are exchanged between layers whose thickness is of the order of the mean free path *l*.

If there is shear between the fluid elements A and B, there will be exchange of bulk momentum between the fluids.

Hence, the fluid parcels exchange both heat (heat diffusion) and momentum (viscosity).