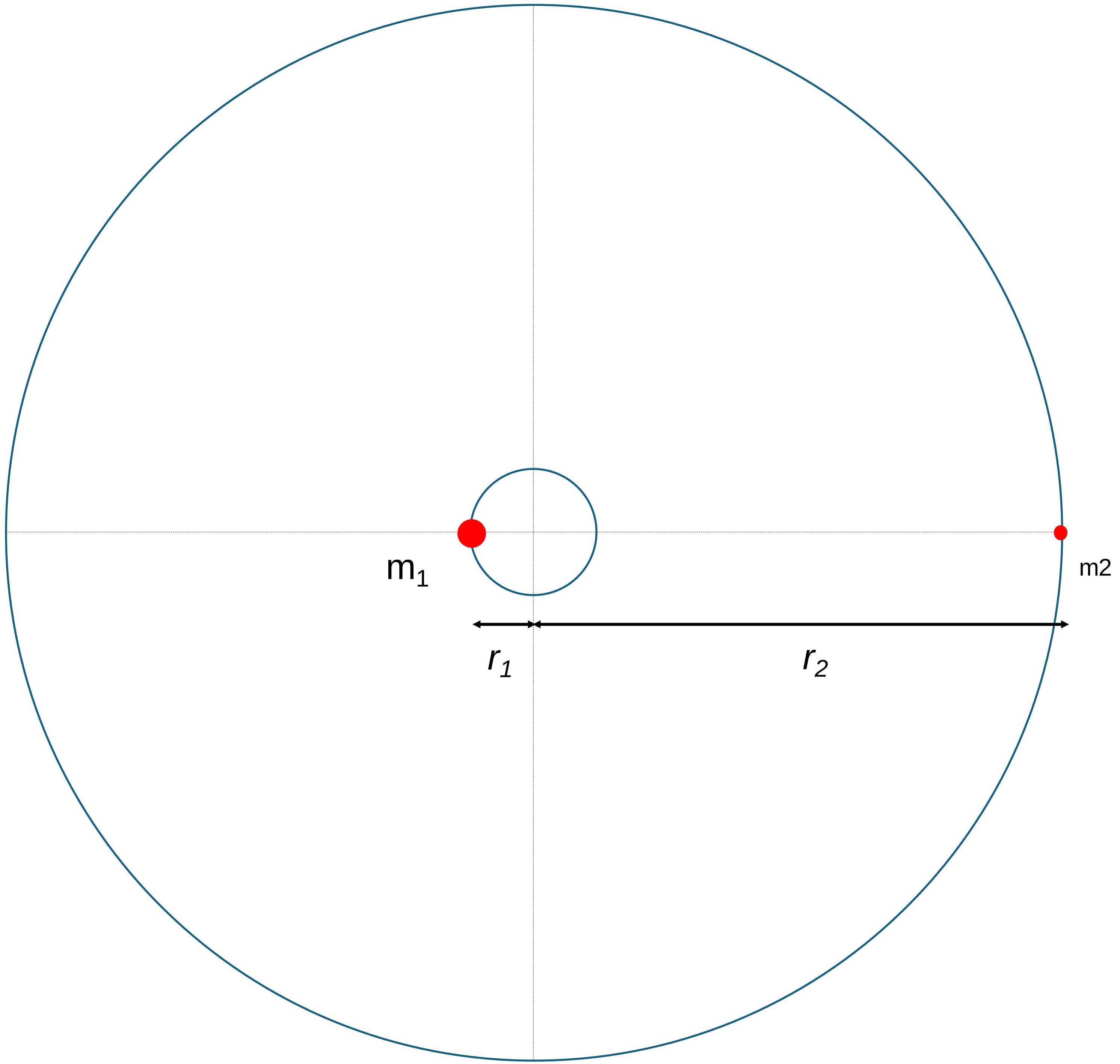
Time integration

Start simple – the circular two-body problem.

$$\begin{aligned}\frac{dx_i}{dt} &= v \\ \frac{dv_i}{dt} &= -G\sum_{j\neq i}\frac{M_i}{|r_i - r_j|^3}\left(r_i - r_j\right)\end{aligned}$$

The problem does not have spatial derivatives, only time derivatives. We want to advance in time from interval t_0 to t_1:

$$\begin{aligned}x_i &= x_i + \int_{t_0}^{t_1}\frac{dx_i}{dt}dt \\ v_i &= v_i + \int_{t_0}^{t_1}\frac{dv_i}{dt}dt \\ t &= t + \int_{t_0}^{t_1} dt\end{aligned}$$

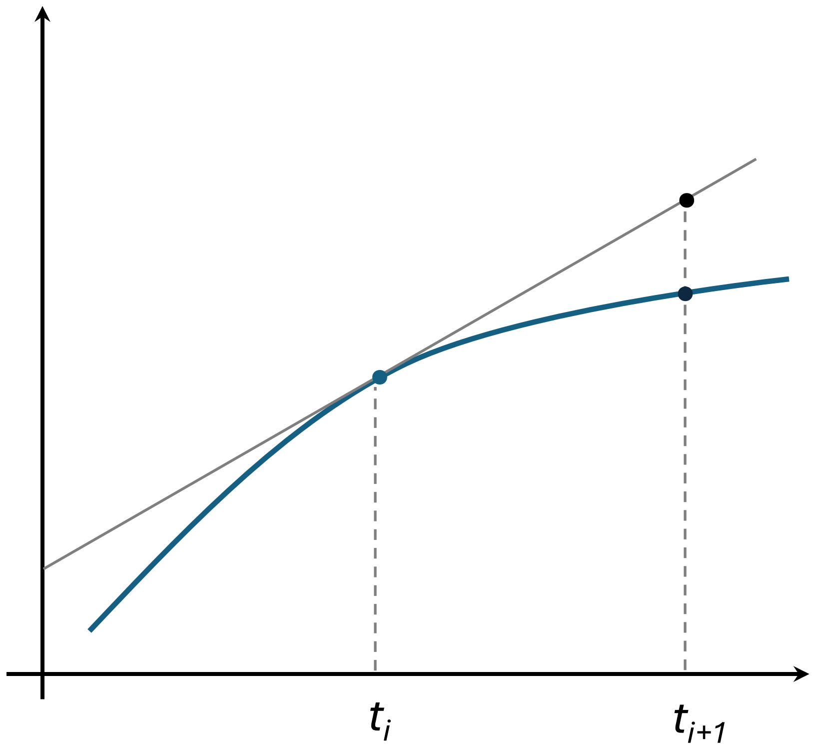


Euler Timestepping

"Naïve" scheme:

Start from the present point, calculate the time derivative

$$\boxed{\begin{aligned} x_i &= x_i + v_0 \Delta t \\ v_i &= v_i + a_0 \Delta t \\ t &= t + \Delta t \end{aligned}}$$



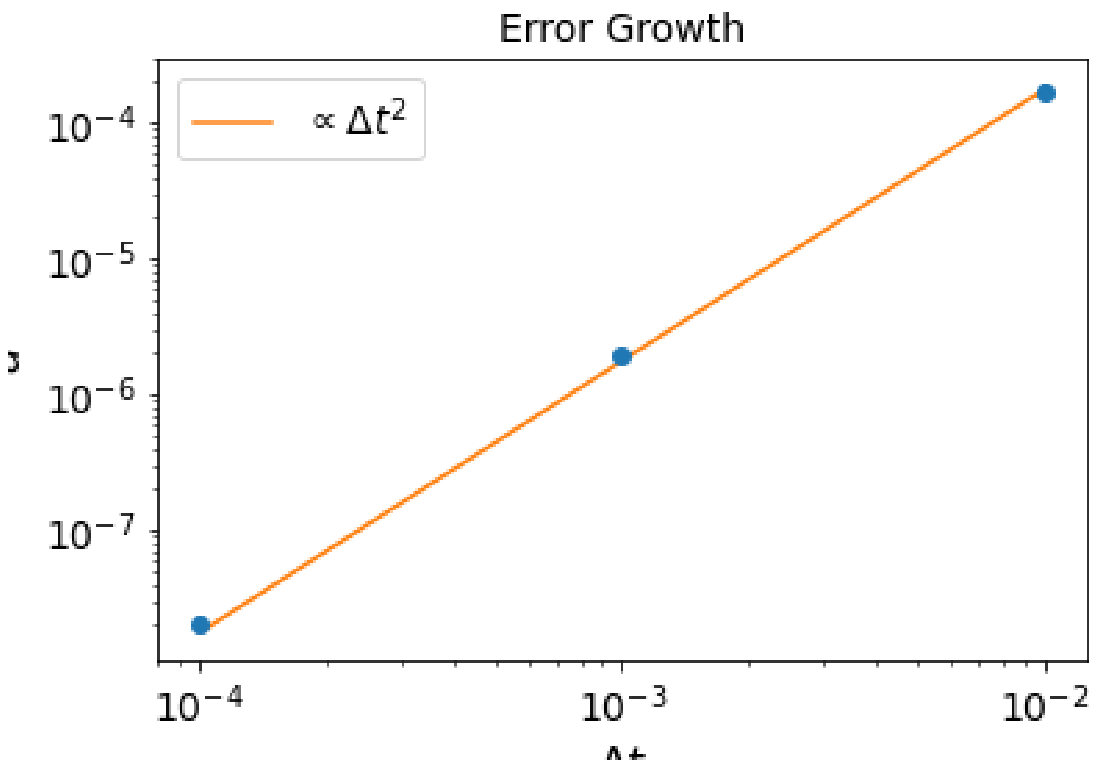
Compared to the Taylor series, this scheme is clearly a truncation at the first order term

$$x(t + \Delta t) = x(t) + \frac{dx}{dt}\Delta t + \frac{1}{2}\frac{d^2x}{dt^2}\Delta t^2 + \frac{1}{3!}\frac{d^3x}{dt^3}\Delta t^3$$

$$\begin{aligned} x_i &= x_i + v_0 \Delta t + O(\Delta t^2) \\ v_i &= v_i + a_0 \Delta t + O(\Delta t^2) \\ t &= t + \Delta t \end{aligned}$$

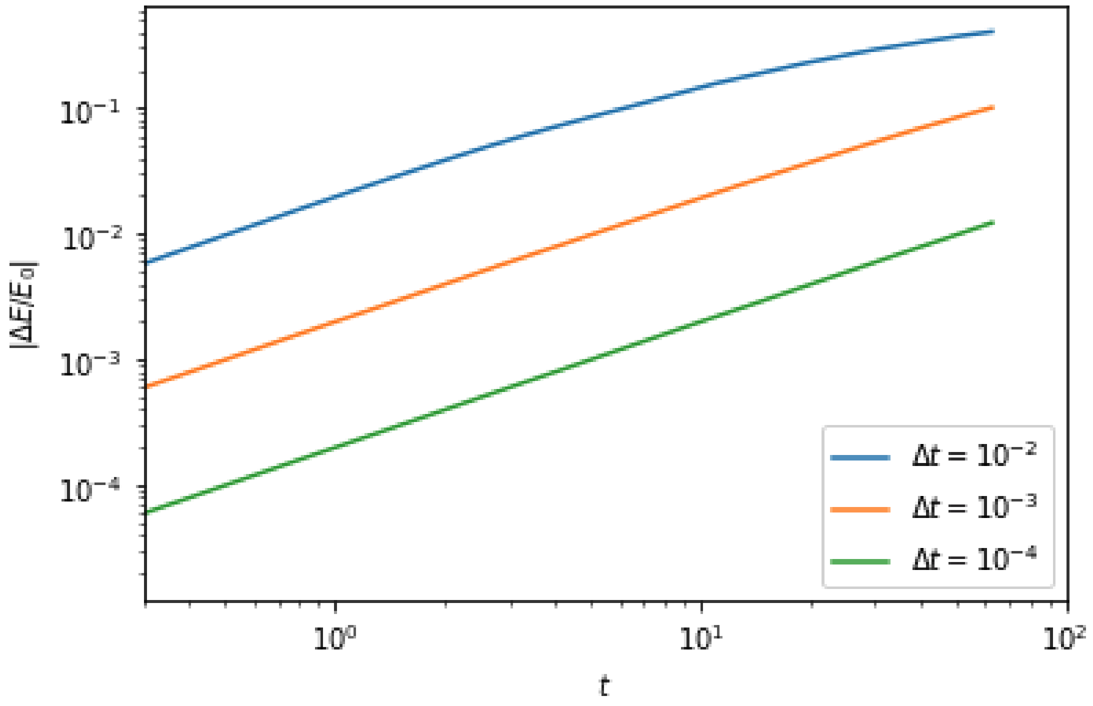
Energy Error: Euler

This approximation has local truncation error of second order. Meaning the energy error in a single time step is quadratic



To reach a solution at time T it takes N= T/Δt timesteps, so the order of the accumulated error is

$$N\ \mathcal{O}(\Delta t^2) = \mathcal{O}(\Delta t)$$



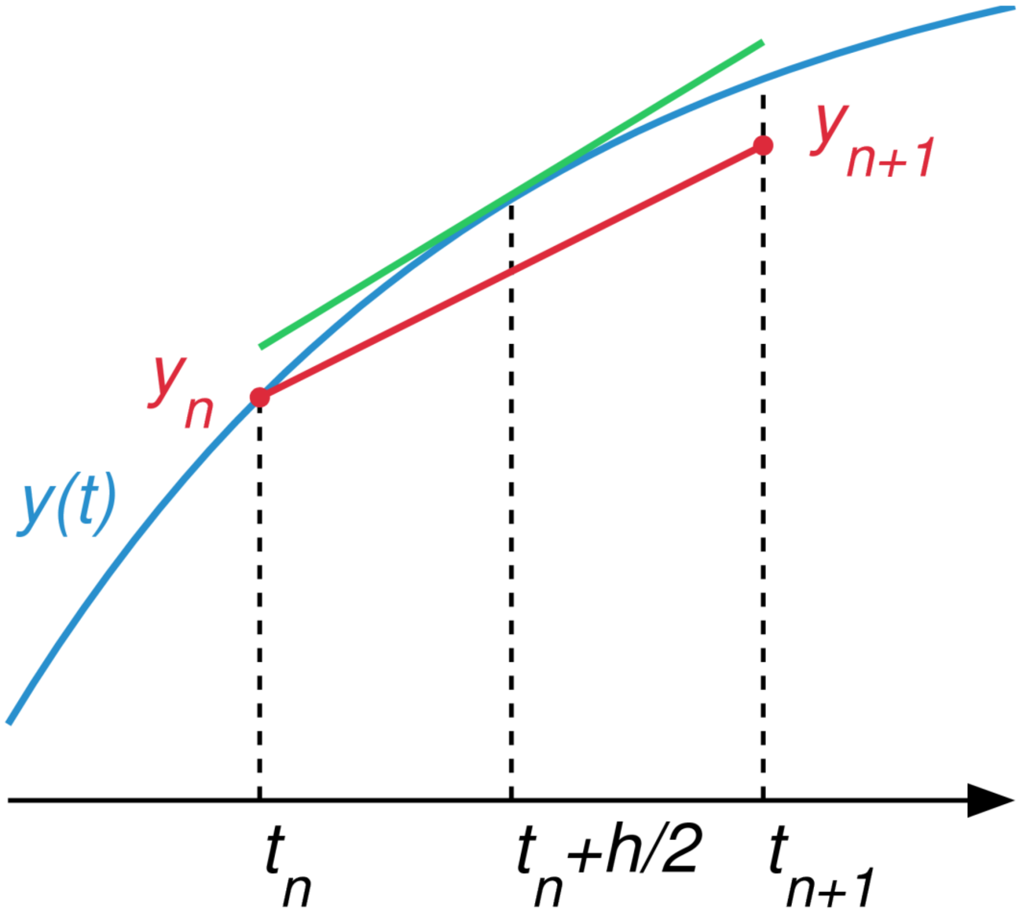
Midpoint Method

The midpoint method uses the slope of the derivative at the midpoint between present and future to achieve better accuracy.

The method is equivalent to the trapezoidal rule for integration

$$\int_t^{t+\Delta t} f(t)dt \approx \left(\frac{f(t_{n+1}) + f(t_n)}{2}\right)\Delta t = f(t_{n+1/2})\Delta t$$

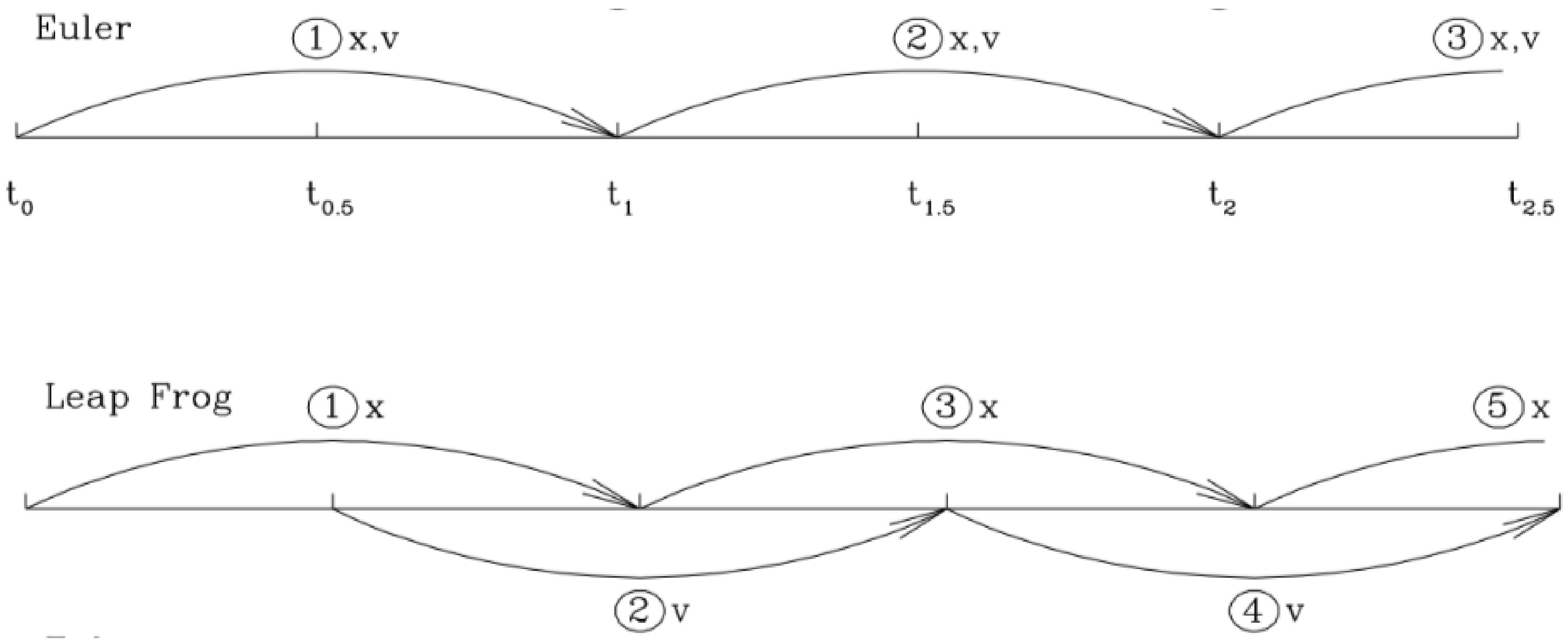
This method is second-order accurate $O(\Delta t^3)$ (see lecture notes for proof), and it used as a step in 4^{th} order accurate schemes like leapfrog and kick-drift-kick.



Leapfrog

The leapfrog integration uses the midpoint rule to offset the position and velocities by half a timestep (hence leapfrog). The velocities are always calculated at half timesteps, and the positions (and hence accelerations) always at integer timesteps.

$$v_{i+1/2} = v_{i-1/2} + a_i \Delta t$$
$$x_{i+1} = x_i + v_{i+1/2} \Delta t$$



Order of Leapfrog

Consider $i = 0$

$$v_{1/2} = v_{-1/2} + g_0 \Delta t$$
$$x_1 = x_0 + v_{1/2} \Delta t$$

and the second derivative $x'' = g$

$$\boxed{x_1 - 2x_0 + x_{-1} = g_0 \Delta t^2 + \epsilon}$$

Where ε is the truncation error. Expanding in Taylor the points x_1 and x_{-1}

$$x_1 = x_0 + \frac{dx}{dt}\Delta t + \frac{d^2x}{dt^2}\frac{\Delta t^2}{2} + \frac{d^3x}{dt^3}\frac{\Delta t^3}{3!} + O\left(\Delta t^4\right)$$
$$x_{-1} = x_0 - \frac{dx}{dt}\Delta t + \frac{d^2x}{dt^2}\frac{\Delta t^2}{2} - \frac{d^3x}{dt^3}\frac{\Delta t^3}{3!} + O\left(\Delta t^4\right)$$

Summing them

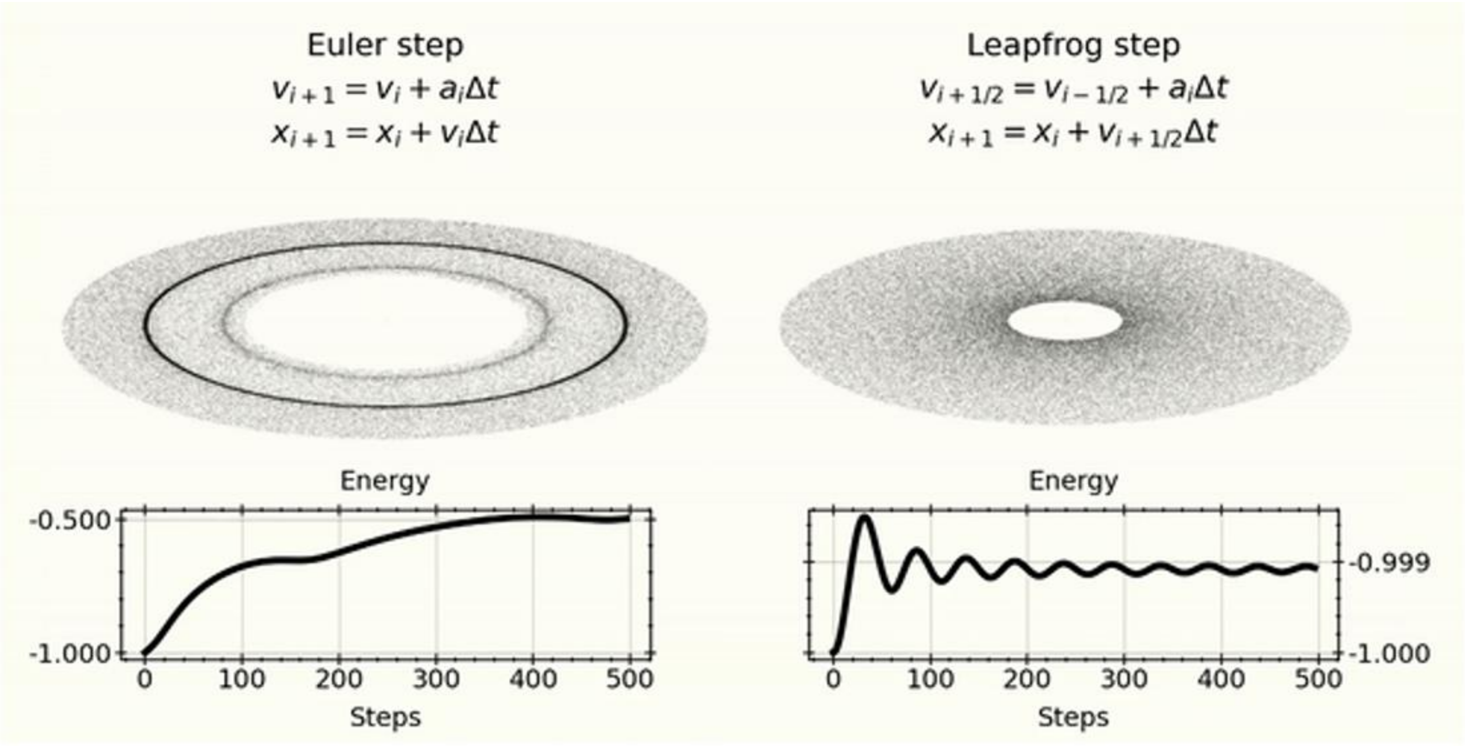
$$\boxed{x_1 + x_{-1} = 2x_0 + \frac{d^2x}{dt^2}\Delta t^2 + O\left(\Delta t^4\right)}$$

Comparing both, we conclude that because the 3rd order term cancels in the sum, the truncation error is of order $O(\Delta t^4)$

Time Reversibility and Energy Error

The leapfrog algorithm is time-reversible, which leads to the property that the energy error is *bounded*.

The time-reversibility is because it if you start from the future and set $\Delta t \rightarrow -\Delta t$, you trace back the same steps to reach the past. The Euler algorithm does not have this property.



Kick-Drift-Kick

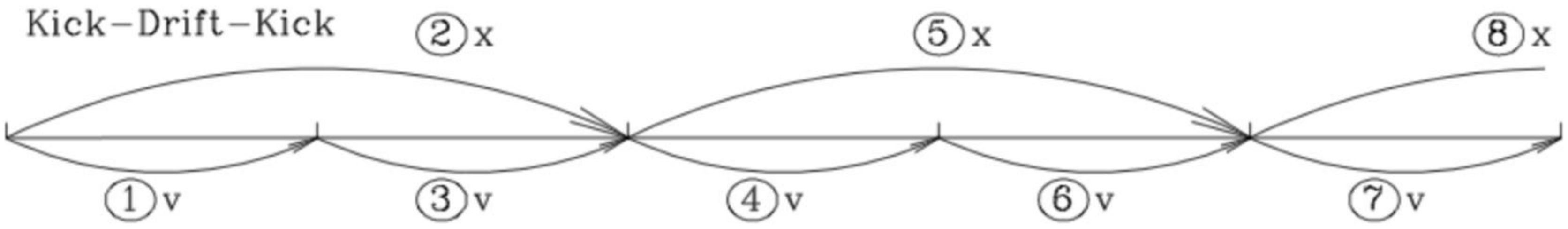
The *kick-drift-kick* (KDK) algorithm uses a version of leapfrog, but brings the velocity to the same instant as the space integration. The algorithm is

- A half time step in velocity using Euler
- A full step in position using the midpoint velocity
- A second half time step in velocity using "modified" Euler (using the future acceleration now that the future position is known)

$$v_{i+1/2} = v_i + a_i \frac{\Delta t}{2} \qquad \text{(kick)}$$

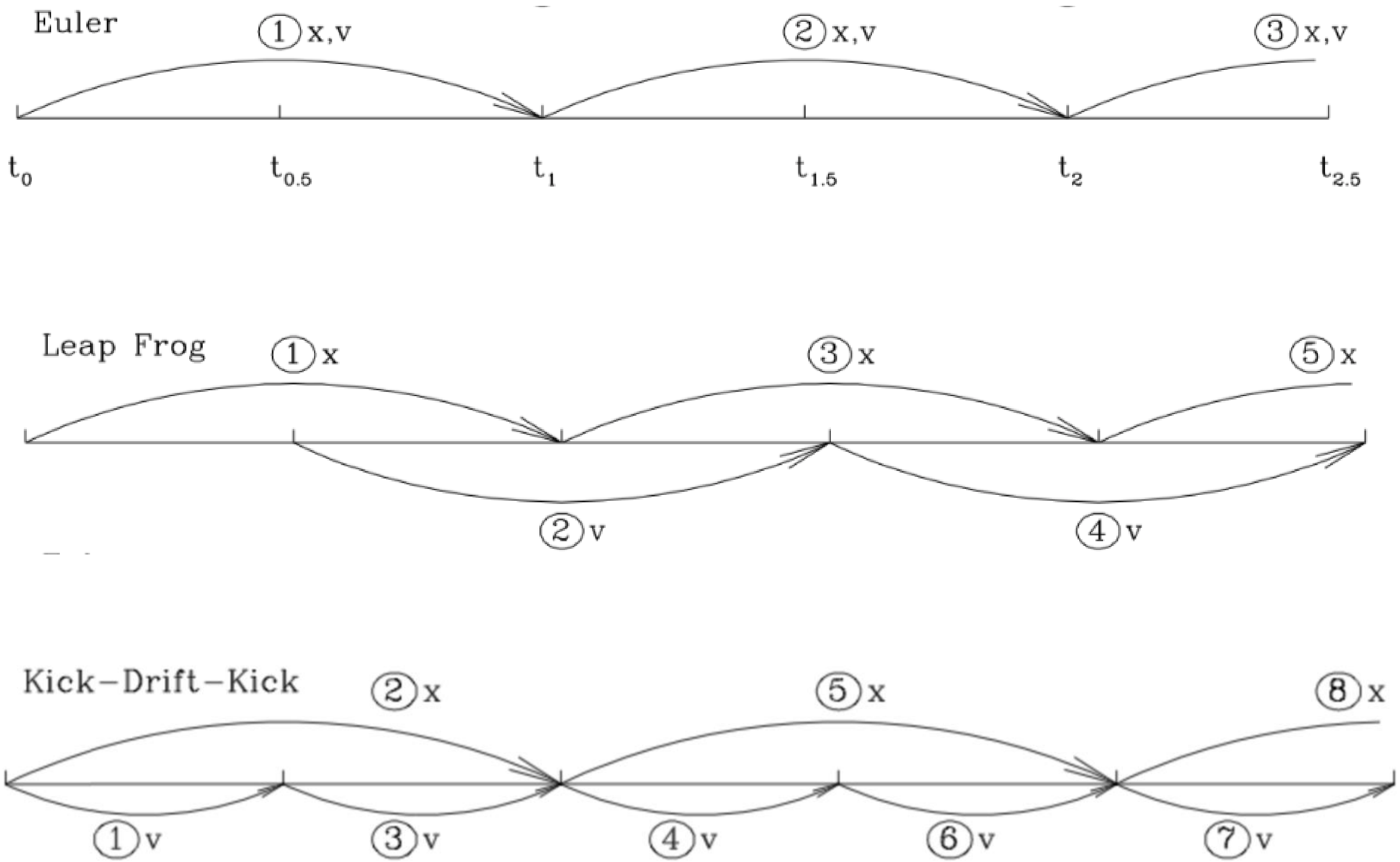
$$x_{i+1} = x_i + v_{i+1/2} \Delta t \qquad \text{(drift)}$$

$$v_{i+1} = v_{i+1/2} + a_{i+1} \frac{\Delta t}{2} \qquad \text{(kick)}$$



KDK is also 4th order accurate, and also time-reversible

Graphical Summary



Energy Error: KDK

This approximation has local truncation error of fourth order. Meaning the energy error in a single time step is quartic.

Unlike Euler, where the error grows in time, the error in KDK is bounded. Because of the high-order, machine precision is quickly reached as the timestep decreases.