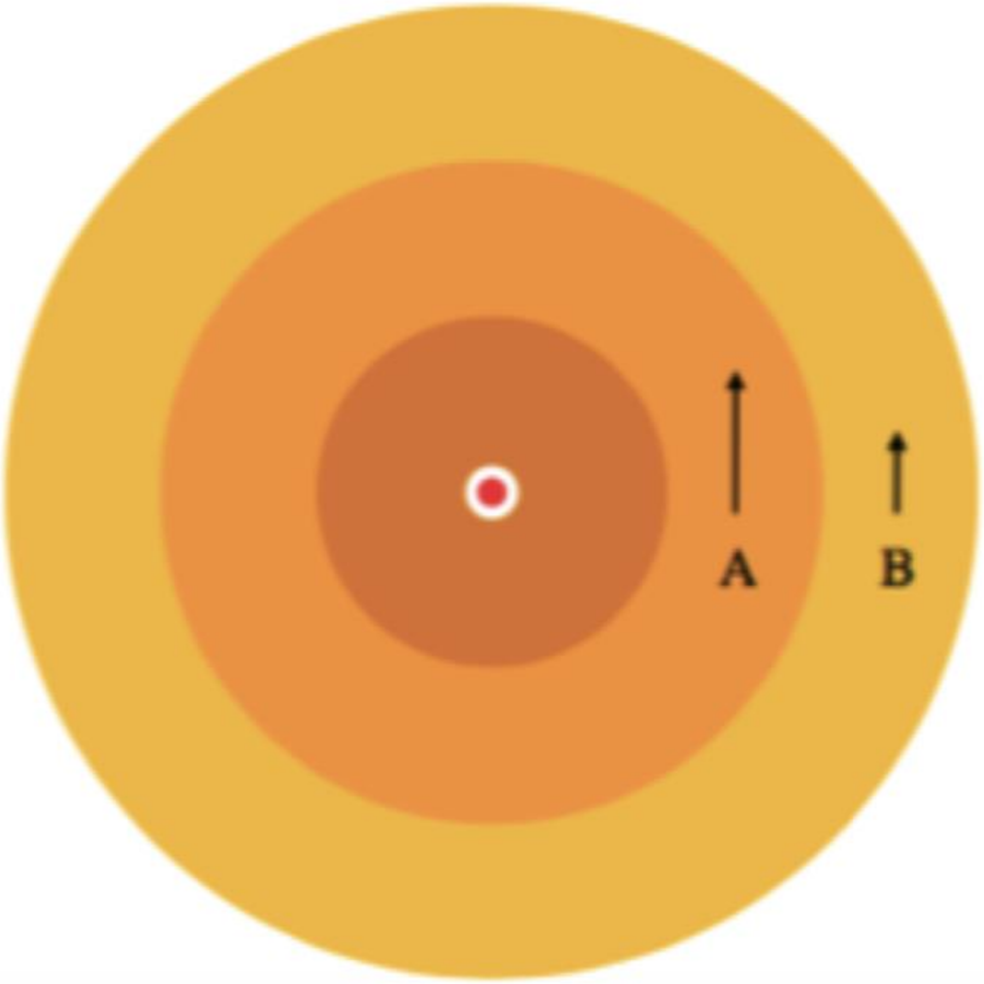
Shear + Viscosity = Heating



If there is shear between the fluid elements A and B, there will be exchange of bulk momentum between the fluids.

Along with it, the work of the viscous force leads to conversion of kinetic energy into heat.

In astrophysical contexts, the viscosity is not molecular but turbulent (more about it next module of the class), but the result is similar: massive viscous heating.

Viscous heating powers accretion disks

Reynolds number

Given the Navier-Stokes equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{f} + \nu \nabla^2 \boldsymbol{u}$$

Compare the ratio of inertial and the viscous term

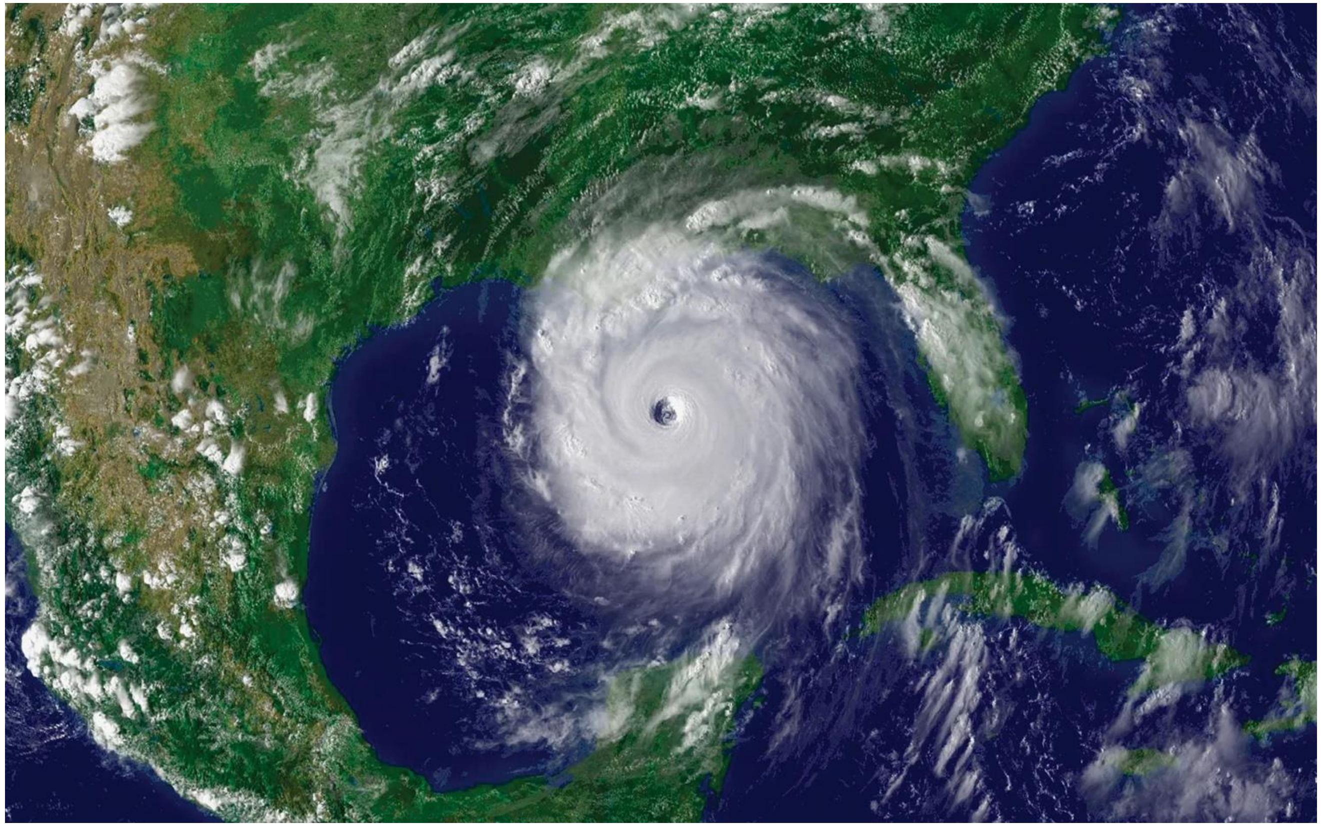
$$\frac{(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}}{\nu \nabla^2 \boldsymbol{u}}$$

The ratio of the two terms is a dimensionless quantity, the *Reynolds number*

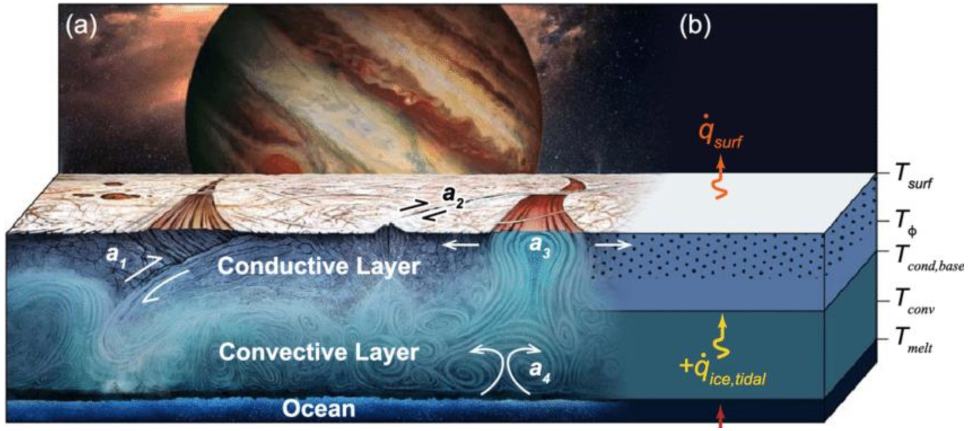
$$\boxed{\mathrm{Re} = \frac{UL}{\nu}}$$

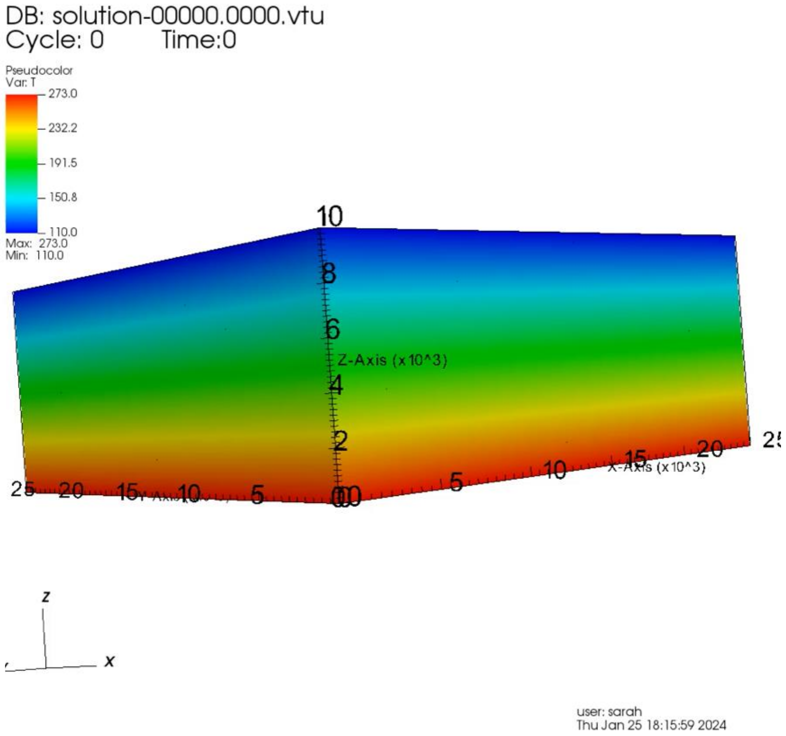
Reynolds number

The viscosity of air is 10^{-1} cm^2/s.
Usual reasonable values for the speed and length and estimate the Reynolds number of a hurricane.



Most fluids in the Universe are high Reynolds number
Notable exception: geophysics!





Equations of motion:

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{u},$$

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \boldsymbol{\nabla} p + \mathrm{Ra}\, T \hat{\boldsymbol{z}},$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) T = \nabla^2 T + q,$$

$$\sigma_{ij} = 2\eta \dot{\varepsilon}_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

Vorticity

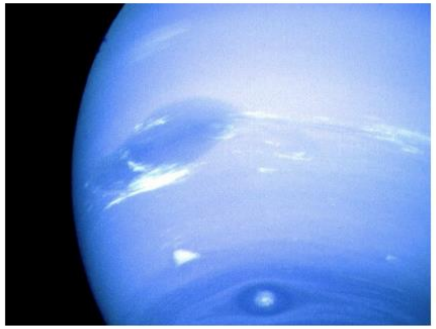
Vortices are
like entities plastered on the fluid, and moving with the flow.

We can define a quantity, the *vorticity*, the curl of the velocity

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$$

and show that under some circumstances that are often realized in nature, the vorticity is conserved.

Vortices – an ubiquitous fluid mechanics phenomenon



Vorticity Equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \underbrace{-(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}}_{\text{advection}} \underbrace{-\boldsymbol{\omega}(\nabla\cdot\boldsymbol{u})}_{\text{compression}} + \underbrace{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}}_{\text{stretching}} + \frac{1}{\rho^2}\underbrace{\nabla\rho\times\nabla p}_{\text{baroclinicity}} + \underbrace{\nu\nabla^2\boldsymbol{\omega}}_{\text{dissipation}}$$

The baroclinic term

$$\frac{1}{\rho^2}\nabla\rho\times\nabla p$$

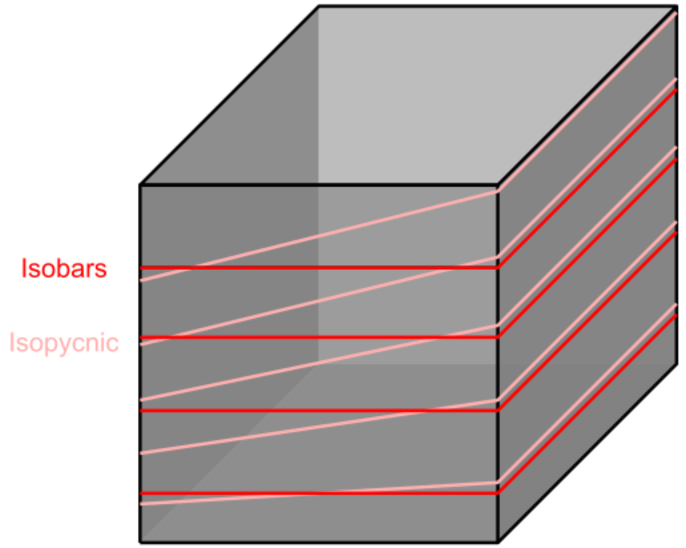
A ***barotropic*** fluid has $p = p(\rho)$. The surfaces of constant pressure and constant density are aligned, so the gradients point in the same direction.

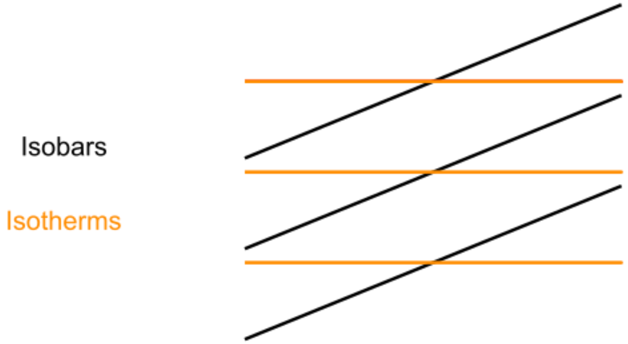
The word barotropic comes from *baro* (pressure) + *tropos* (direction). Meaning pressure gradient is in the same direction as the density gradient.

A ***baroclinic*** fluid has $p = p(\rho,T)$. The surfaces of constant pressure and constant density can be inclined, with the gradients pointing in different directions.

The word baroclinic comes from *baro* (pressure) + *inclination* (direction). Meaning pressure and density gradients are misaligned.

Baroclinic fluid





The baroclinic term

$$\frac{1}{\rho^2}\nabla\rho\times\nabla p$$

Notice that the polytropic equation of state $p = K\rho^\gamma$ is barotropic.
The interior of stars and giant planets are well-approximated by polytropes.

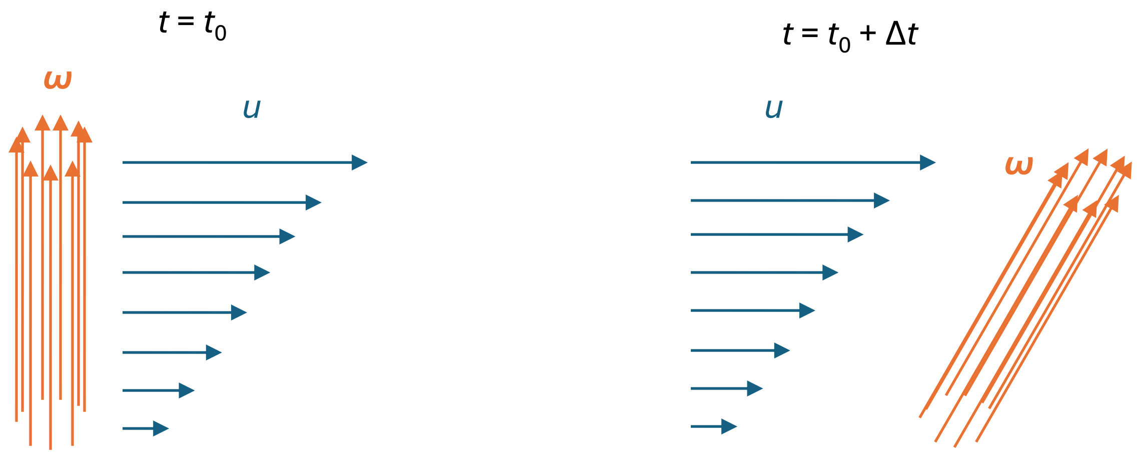
K is the entropy. For barotropic fluids, the entropy is constant.

The condition of baroclinicity then translates simply into: *varying entropy*.

An entropy gradient leads to vortex generation by *convection*.

The stretching term

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\, \boldsymbol{u}$$

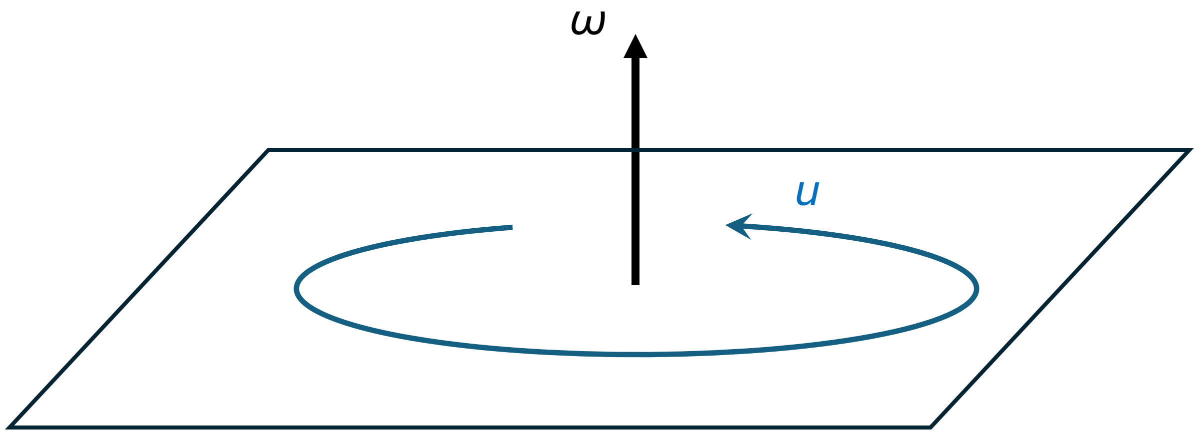


Stretching modifies vorticity along the direction of shear.
In essence, stretching is differential advection.

In components, one starts from pure y vorticity, and builds and x vorticity component.

The stretching term

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\,\boldsymbol{u}$$



For 2D, the vorticity always points out of the flow plane, so the stretching is zero.

Vorticity Equation

Cancels when:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \underbrace{-(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}}_{\textbf{advection}} - \overset{\textit{incompressible}}{\underbrace{\cancel{\boldsymbol{\omega}(\nabla\cdot\boldsymbol{u})}}_{\textbf{compression}}} + \overset{\textit{2D}}{\underbrace{\cancel{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}}}_{\textbf{stretching}}} + \overset{\textit{barotropic}}{\underbrace{\cancel{\frac{1}{\rho^2}\nabla\rho\times\nabla p}}_{\textbf{baroclinicity}}} + \overset{\textit{Re>>1}}{\underbrace{\cancel{\nu\nabla^2\boldsymbol{\omega}}}_{\textbf{dissipation}}}$$

Under these conditions:

$D\omega/Dt = 0$

Vorticity is conserved!