Summing Approximations

Warren MacEvoy

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Understanding asymptotic behavior of a series or function is useful.

Estimating Sums

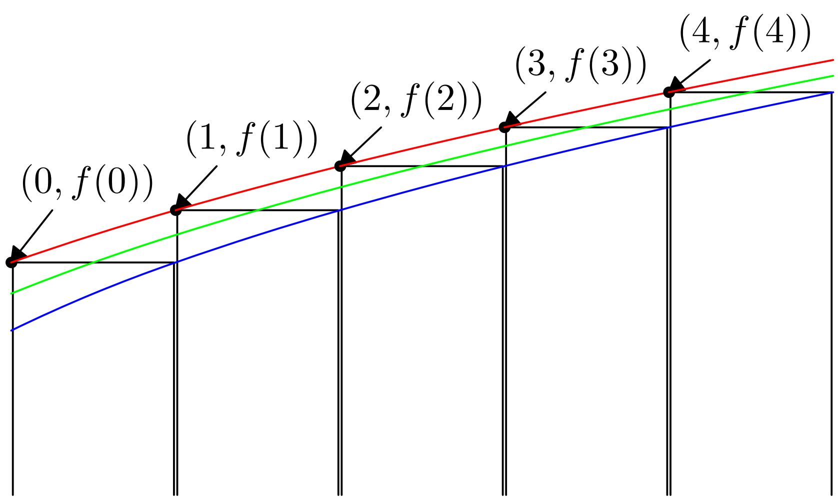

Figure 1: Comparing sum (black rectangles) with integrals of increasing f (red), f shifted -1/2 (green), and f shifted -1 (blue).

Summing is a linear operation, so

$$\sum_{k=0}^{n-1} [\alpha f(k) + \beta g(k)] = \alpha \sum_{k=0}^{n-1} f(k) + \beta \sum_{k=0}^{n-1} g(k)\,. \tag{1}$$

Summing powers of k:

$$\sum_{k=0}^{n-1} 1 = n\,. \tag{2}$$

$$\sum_{k=0}^{n-1} k = \frac{1}{2}n(n-1) = \frac{1}{2}n^2 + O(n)\,. \tag{3}$$

$$\sum_{k=0}^{n-1} k^2 = \frac{1}{3}n(n-1/2)(n-1) = \frac{1}{3}n^3 + O(n^2)\,. \tag{4}$$

$$\sum_{k=0}^{n-1} k^3 = \frac{1}{4}n^2(n-1)^2 = \frac{1}{4}n^4 + O(n^3)\,. \tag{5}$$

Generally, for $p > 0$,

$$\sum_{k=0}^{n-1} k^p \approx \frac{1}{p+1}(n-1/2)^{p+1} = \frac{1}{p+1}n^{p+1} + O(n^p)\,. \tag{6}$$

Estimating sums with integrals.
If $f(x)$ is increasing, then

$$\int_{a-1}^{b} f(x)\,dx \leq \sum_{k=a}^{b} f(k) \approx \int_{a-1/2}^{b+1/2} f(x)dx \leq \int_{a}^{b+1} f(x)\,dx \tag{7}$$