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LITTLE-$o()$ CALCULUS

LECTURE NOTES

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Contents

List of Figures

List of Tables

Preamble

The point of these notes is to introduce the main concepts of calculus using what is known as little-$o()$ notation, in the hope to provide a perspective that is simple, expressive, intuitive and correct.

Stitz-Zeager [www.stitz-zeager.com] and Boundless [www.boundless.com] are good backgrounds references.

Algebra is essential

I assume you are comfortable with algebra, including working with rational algebraic expressions, exponents and logarithms, function notation, absolute values, and inequalities.

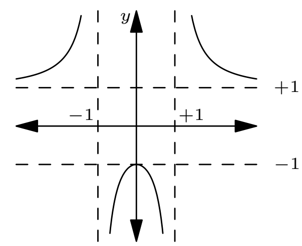

Figure 1: $y = f(x) = \frac{x^2+1}{x^2-1}$

For example, seeing

$$f(x) = \frac{x^2+1}{x^2-1}$$

And being asked to show

$$f(x+h) = \frac{x^2+1}{x^2-1} + \frac{2h \cdot (2x+h)}{(x-1)(x+1)(x+h+1)(x+h-1)}$$

and

$$|f(x)| < 2 \text{ if and only if}$$
$$x \in (-\infty, -\sqrt{3}) \cup (-\frac{1}{\sqrt{3}}, +\frac{1}{\sqrt{3}}) \cup (+\sqrt{3}, +\infty)$$

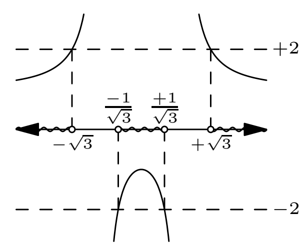

Figure 2: Where $|f(x)|$ is less than 2.

should seem (at worst) tedious, but not mysterious. Neither should the following[1]:

$$2^{\log_3(x)} = x^{\log_2(3)}.$$

If these are mysterious to you, then you need to learn college algebra.

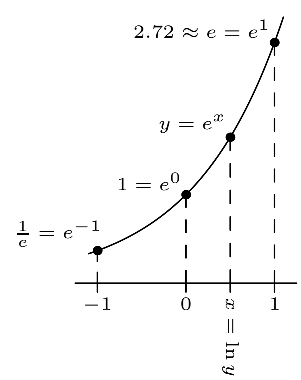

Figure 3: How $y = e^x$ and $x = \ln y$ are related, where $e \approx 2.72$ is Euler's constant.

[1] Recall $b^x = e^{x \ln b}$ and $\log_b x = \ln x / \ln b$

Trigonometry is helpful

It will be useful if you have some basic trigonometry, including measuring angles in radians, which is the only unit of angle we care about here. For example,

$$(\cos a)^2 + (\sin a)^2 = 1\,,$$
$$\cos a = \frac{1}{2} \text{ if and only if } a = \pm\frac{\pi}{3} + 2\pi n, \text{ for some integer } n\,,$$

and

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)\,,$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)\,,$$

should at least be familiar to the point of looking something up to remember the details.

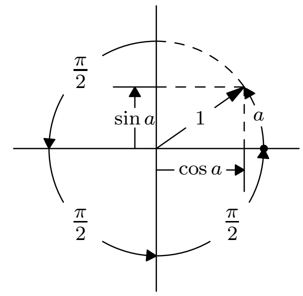

Figure 4: Defining $\cos a$ and $\sin a$ for radian angle a on the unit circle. Notice there are $2\pi \approx 6.28$ radians in a circle.

		6	1		5			
9				4		7		
8			7				2	
4								
		1				4		
				6				
	5		8		2	1		
	6				9			7
	4				6	9		5

Figure 5: A simple sudoku puzzle. Each row, column and 3×3 subgrid must contains all of the digits from 1 to 9.

Reading Tips

- Focus. The best environment for learning something mathematical is in a small group willing to help you work out a sudoku puzzle. If you can work out a sudoku puzzle in the place and with the people you study with, it is a good sign you can learn math as well. Learning, any learning, is for mono-taskers: you might multitask (usually poorly) on things you already understand. This, however, is something you are trying to understand.
- Drive, don't walk. Learning to use a computer algebra system (like Wolfram Alpha, wxMaxima or Maple), or at least a spreadsheet (like Google Docs, Libre Office or Microsoft Excel) can spare you from a lot of tedium compared to a crummy calculator.
- Understand first. Proofs are less important than understanding. If you are mostly interested in applications, don't worry as much about understanding every proof. Concentrate instead on why the ideas make sense and how you might use these ideas. If you are interested in mathematics itself, understanding the proofs is important, but useless without an intuition of why they make sense and how to use them.

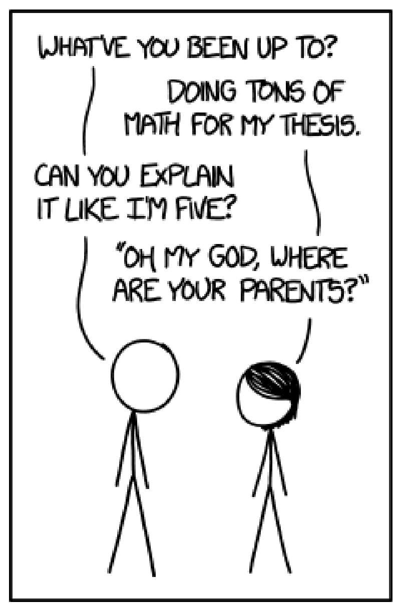

Figure 6: http://xkcd.com/1364

Basics

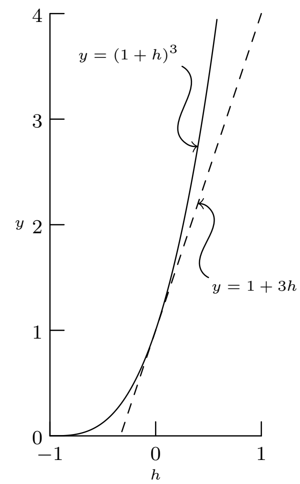

Figure 7: $(1+h)^3 = 1 + 3h + o(h)$.

Little-$o()$ notation is used to denote that some expression is small relative to some other value.

For example, if we say, as h approaches 0,

$$(1+h)^3 = 1 + 3h + o(h)\,.$$

We mean that, as the magnitude of h gets smaller, the magnitude of the difference between the left-hand side $L = (1+h)^3$ and the right-hand side $R = 1 + 3h$ is so small that, even when divided by h, it is still small.

We can check this ratio empirically:

h	$L = (1+h)^3$	$R = 1 + 3h$	$E = L - R$	$r = \left\lvert \frac{E}{h} \right\rvert$
0.1	1.331	1.3	0.031	0.31
−0.01	0.970 299	0.97	0.000 299	0.0299
0.001	1.003 003 001	1.003	0.000 000 300 01	0.003 001

Table 1: $(1+h)^3 = 1 + 3h + o(h)$.

Notice how the ratio $r = |E/h|$ shrinks as the magnitude of h shrinks. When we say an expression E is $o(h)$ as $h \to 0$, we mean the magnitude of $|E/h|$ gets small as the magnitude of h gets small.

The $o()$ notation allows for more general expressions for the relative comparison of smallness, such as $o(h^2)$ or $o(h \ln(h))$. In each case, we are saying that the expression in question is small in magnitude when divided by the expression in the $o()$ as the parameter (h in this case) gets small in magnitude.

Here are some other examples to help get the idea. You should check these with a calculator or spreadsheet. We give a formal definition of the notation at the end of this section.

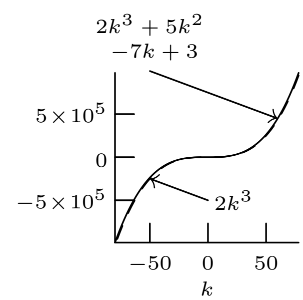

Figure 8: As $k \to \infty$, $2k^3 + 5k^2 - 7k + 3 = 2k^3 + o(k^3)$. Note how similar the curves are for large k.

As $k \to \infty$, $2k^3 + 5k^2 - 7k + 3 = 2k^3 + o(k^3)$.

This means, as the magnitude of k gets larger and larger, the cubic polynomial on the left is approximately the leading order term (term with the highest power of k) plus an error small relative to the size of that term.

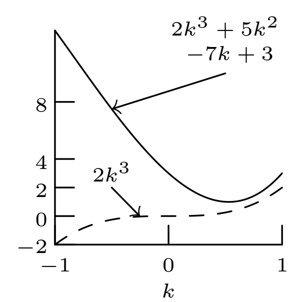

Figure 9: As $k \to \infty$, $2k^3 + 5k^2 - 7k + 3 = 2k^3 + o(k^3)$. Note how dissimilar the curves are for small k.

Notice how the error E grows, but it is still small when compared to $\varepsilon(k) = k^3$.

Table 2: $2k^3 + 5k^2 - 7k + 3 = 2k^3 + o(k^3)$.

k	$L = 2k^3 + 5k^2 - 7k + 3$	$R = 2k^3$	$E = L - R$	$\varepsilon(k) = k^3$	$r = \left\lvert \frac{E}{\varepsilon(k)} \right\rvert$
10	2433	2000	433	1000	0.433 00
−100	−1 949 297	−2 000 000	50 703	−1 000 000	0.050 70
1000	2 004 993 003	2 000 000 000	4 993 003	1 000 000 000	0.004 99

Euler's constant, e.

Euler's constant $e \approx 2.718\,281\,828\,459\,045$ is fundamental for exponents and logarithms, just as $\pi \approx 3.141\,592\,653\,589\,793$ is fundamental in trigonometry. It can be thought of as the value approximated by the expression $(1+h)^{1/h}$ as $h \to 0$.

In $o()$ notation[2]: as $h \to 0$, $(1+h)^{1/h} = e + o(h^0)$.

[2] Writing 1 as h^0 in the $o(h^0)$ may seem surprising, but it is a way of noting what parameter is getting small.

This means, as the magnitude of h gets smaller and smaller, the expression on the left is approximately Euler's constant $e = 2.718\,281\,828\,459\,045\ldots$, plus an error small relative to $h^0 = 1$.

Here is a tabular comparison for some small values of h:

Table 3: $(1+h)^{1/h} = e + o(h^0)$.

h	$(1+h)^{1/h}$	$r = \left\lvert \frac{(1+h)^{1/h} - e}{1} \right\rvert$
0.01	2.704 813 829	0.0135
−0.0001	2.718 417 755	0.000 136
0.000 001	2.718 280 469	0.000 001 36

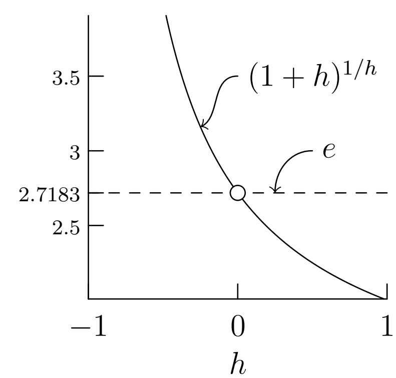

Figure 10: $(1+h)^{1/h} = e + o(h^0)$.

Sine.

As $h \to 0$, $\sin(x+h) = \sin(x) + \cos(x)h + o(h)$.

We will show this is true later, but, geometrically, it means that evaluating $y = \sin(x)$ near x is approximately a line[3] going through the point $(x, \sin(x))$ with slope $\cos(x)$.

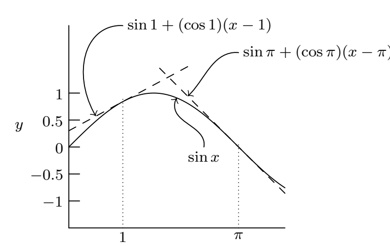

Figure 11: $\sin(x+h) = \sin(x) + \cos(x)h + o(h)$ for $x = 1$ and π.

[3] A line going through the point (x_0, y_0) with slope m is $y = y_0 + m \cdot (x - x_0)$.

This last example is really important. The idea that, near a given point x, many functions are well approximated by a line is a foundational idea of differential calculus.

Figure 12: $\sin(x+h) = \sin(x) + \cos(x)h + o(h)$ for $x = 1$ and π.

We call the slope of that approximating line $f'(x)$, so that $f(x+h) = f(x) + f'(x)h + o(h)$. For this example, we are saying that, if $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

The most important figure in calculus. For many functions, the value near a given point x is well approximated by a line called the tangent line of f (x) at x with slope f âĂš (x). As an equation: as h âĘŠ o , f (x + h) = f (x) + f âĂš (x)h + o (h). Such functions are called differentiable at x .

Always writing âĂIJas h âĘŠ o " is tedious. It will be clear from the o () notation which parameter we consider large or small.

Formalities E | is as small as we choose, An expression E is o (Îţ(h)) as h âĘŠ o , means the ratio r = | | Îţ(h) | provided we force the magnitude of h to be small enough (but not zero). Specifically, for E | < r . any bound r b > o , there exists an

h b > o , so that, if o < | h | < h b , then | | Îţ(h) | b

Redraw this in your notes so you remember the definition of little-o.

E | = o . If you are familiar with limit notation, this can be written as lim | | Îţ(h) | hâĘŠo In terms of large parameters, E = o (Îţ(k)) as k âĘŠ âĹđ is the same as E = o (Îţ(1/h)) as

h âĘŠ o by means of the substitution h = 1 /k.

E | = o . If you are familiar with limit notation, this can be written as lim | | Îţ(k) | kâĘŠâĹđ

Note that we divide E by Îţ (h) to get r in the above definition for some range of nonzero

values of h . So, for for any expression to be o (Îţ(h)), Îţ (h) must be defined and nonzero for some range of nonzero values of h . We therefore restrict Îţ (h) to such admissible functions:

For Îţ (h) to be admissible in o (Îţ(h)) notation, there must exist some h Îţ > o so that Îţ (h) is defined (finite) and nonzero for o < | h | < h Îţ .

All we ask of Îţ (h) to be admissible, is that there is a range near zero (but we donâĂŹt care about at zero), where it is defined (finite) and nonzero. This way we can divide E by Îţ (h) in this range to compute the ratio r = | E/Îţ(h)|.

We only use admissible Îţ (h) in these notes. In particular, Îţ (h) = | h | p is admissible for any value of p , and Îţ (h) = h p is admissible for any integer value of p . Summary

âŮŔ Little-o() notation is a way of describing an expression as small in comparison to some other value: saying E is o (Îţ(h)) as h âĘŠ o means the ratio r = | E/Îţ(h) | can

be made as small as desired provided the magnitude of h is small enough (but not zero).

âŮŔ Specifically, E is o (Îţ(h)) as h âĘŠ o means:

For any r b > o , there exists

h b > o , so that: E | < r . | If o < | h | < h b , then | Îţ(h) | b

âŮŔ E is o (Îţ(k)) as k âĘŠ âĹđ means E is o (Îţ(1/h)) as h = 1 /k âĘŠ o . âŮŔ For differentiable functions, the value f (x + h) near a given point x is well approximated by a line called the tangent line of f (x) at x with slope f âĂš (x). âŮŔ As h âĘŠ 0 , (1 + h) (1/h) = e + o (h 0), where e âĽĹ 2 .718 is EulerâĂŹs constant.

âŮŔ Because an expression E is divided by

Îţ (h) in the definition of o (Îţ(h)), Îţ (h) is admissible in o (Îţ(h)) only if it is defined and nonzero for small enough nonzero h : there must exist h Îţ > 0 so that Îţ (h) is defined (finite) and nonzero for 0 < | h | < h Îţ . âŮŔ We only use admissible Îţ (h) in these notes. In particular, Îţ (h) = | h | p is admissible for any value of p , and Îţ (h) = h p is admissible for any integer value of p.

Index