Modeling Formal Languages in Grammatical Framework

On the Grammar of Proof

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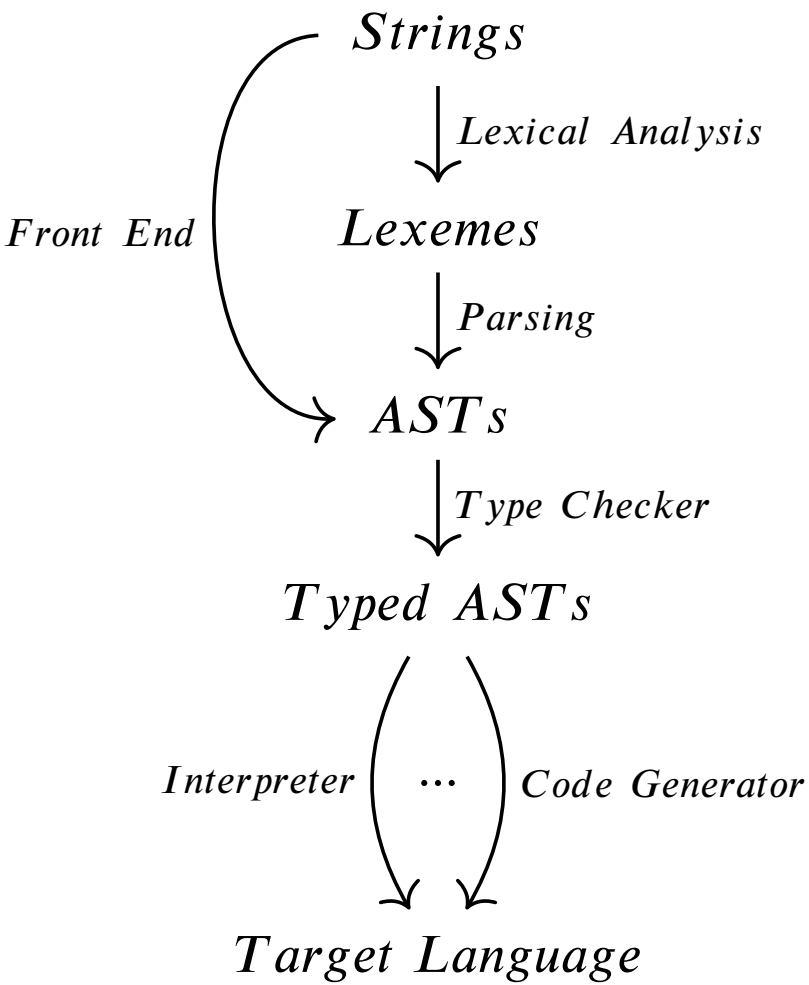
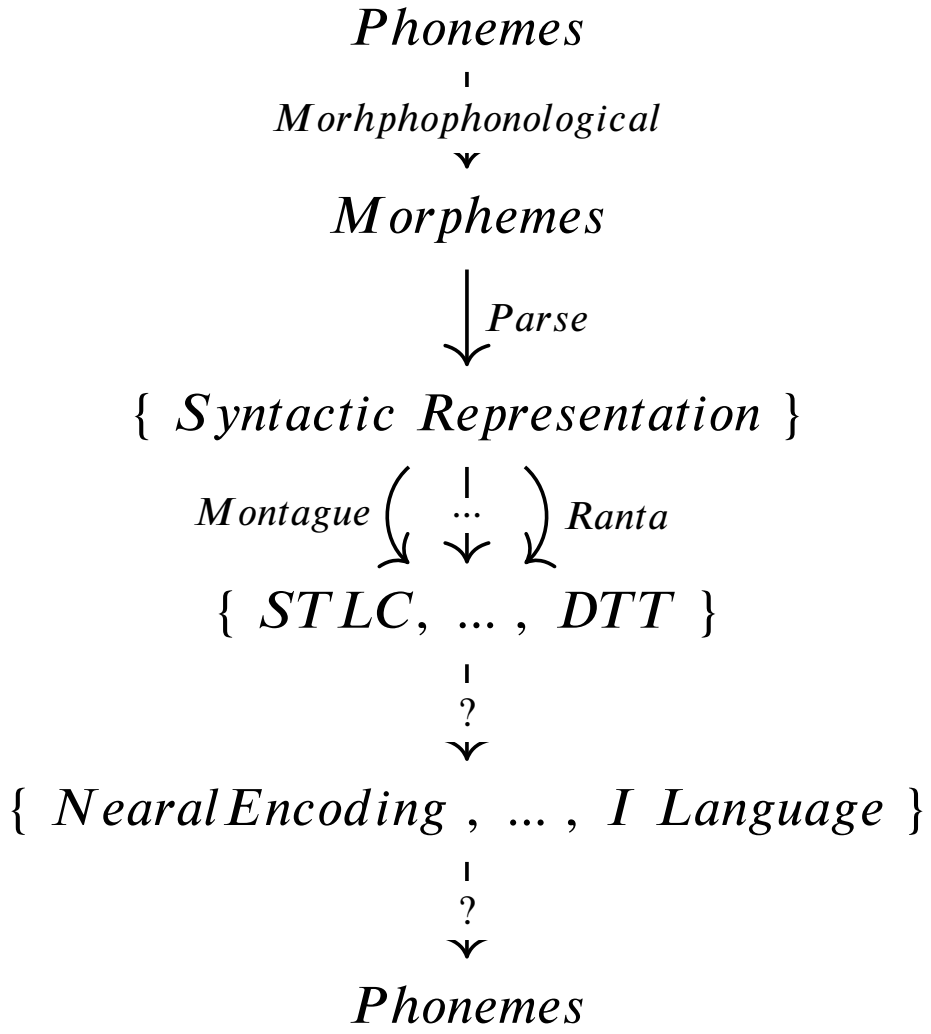
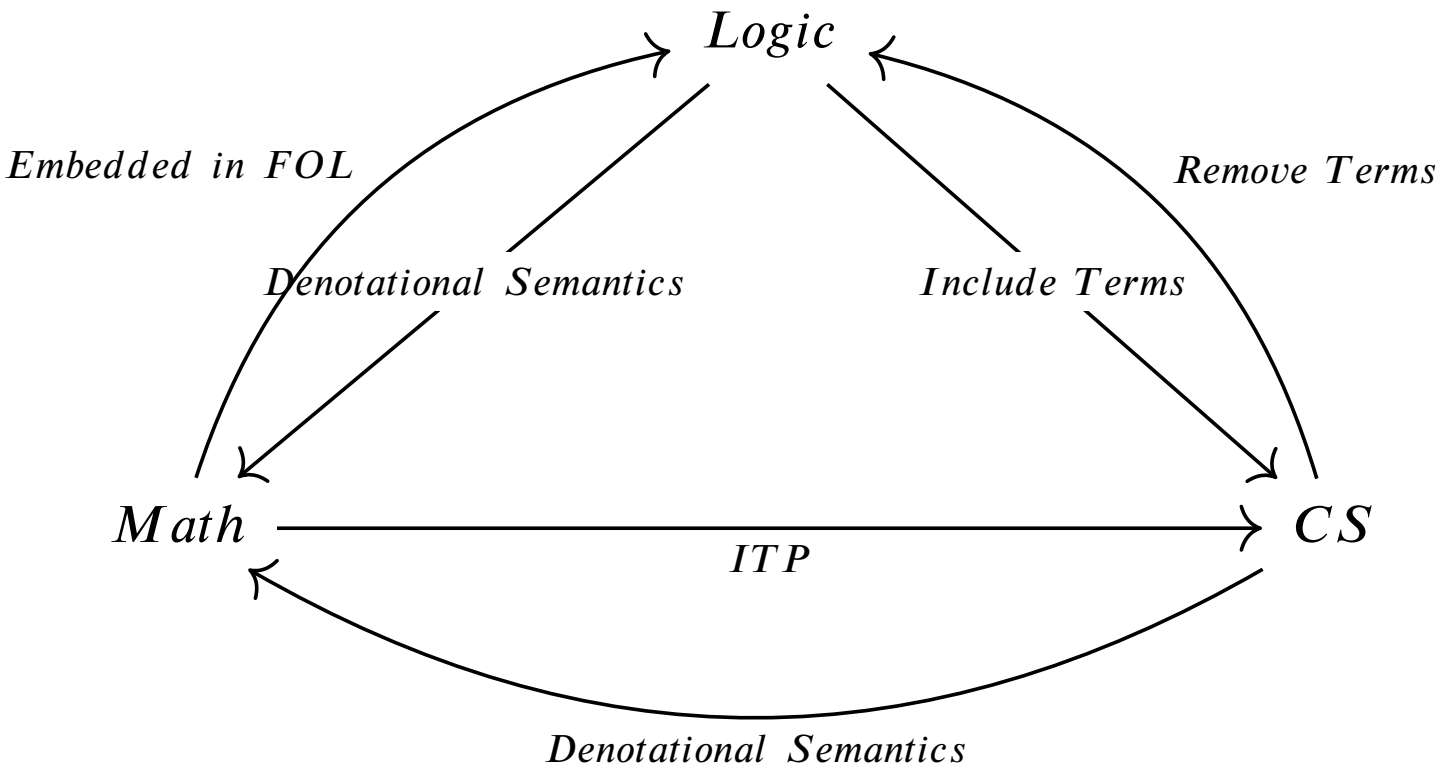
Abstraction Ladders





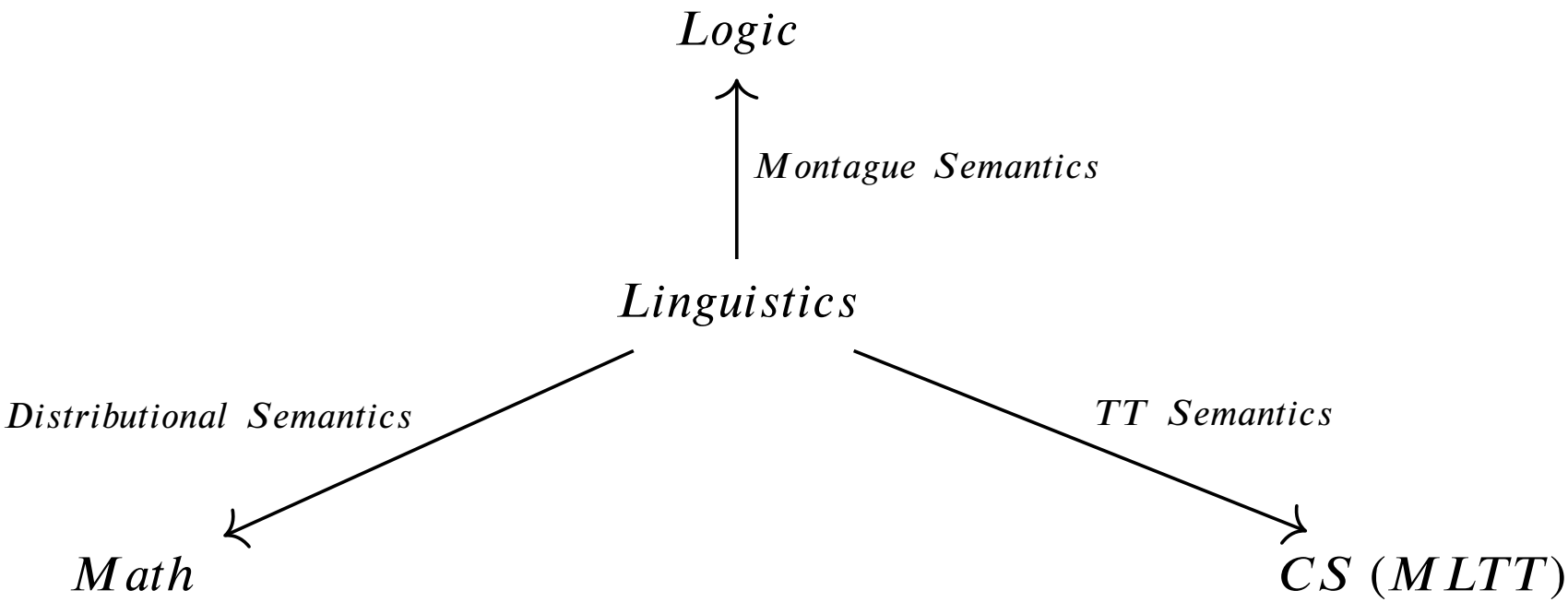
Computational Trinitarianism



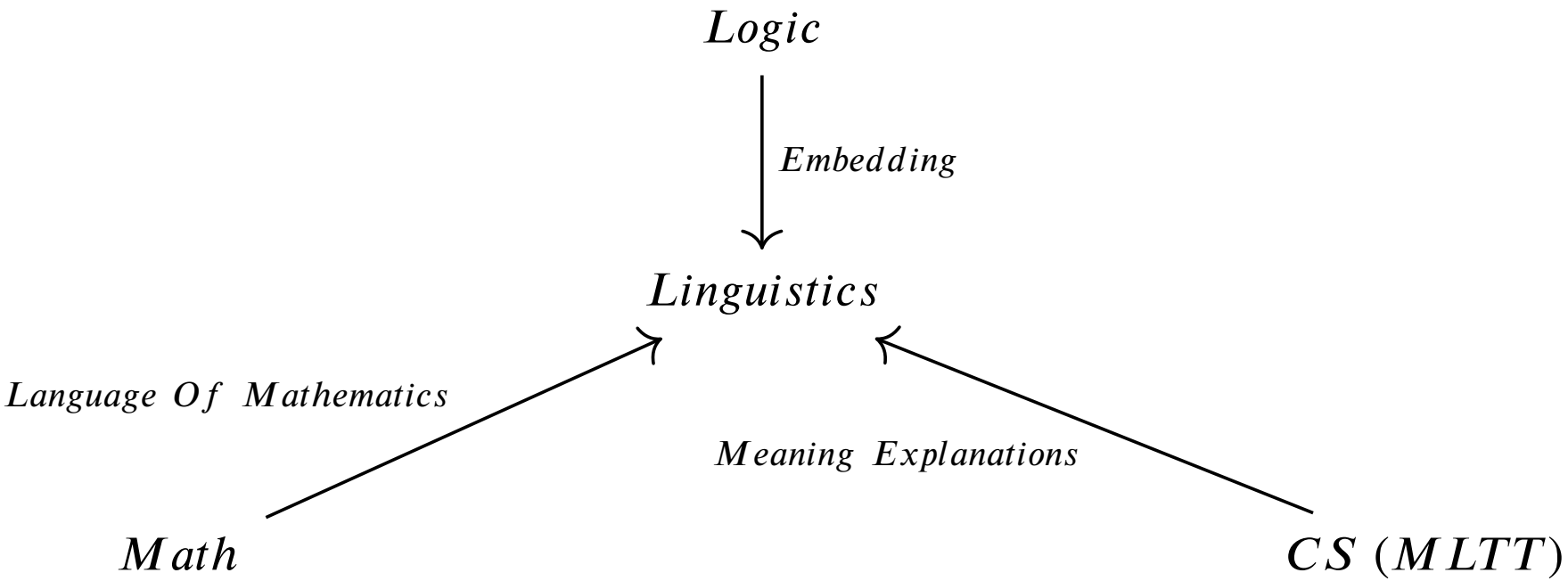
Interpretation Language

Observation 1.1

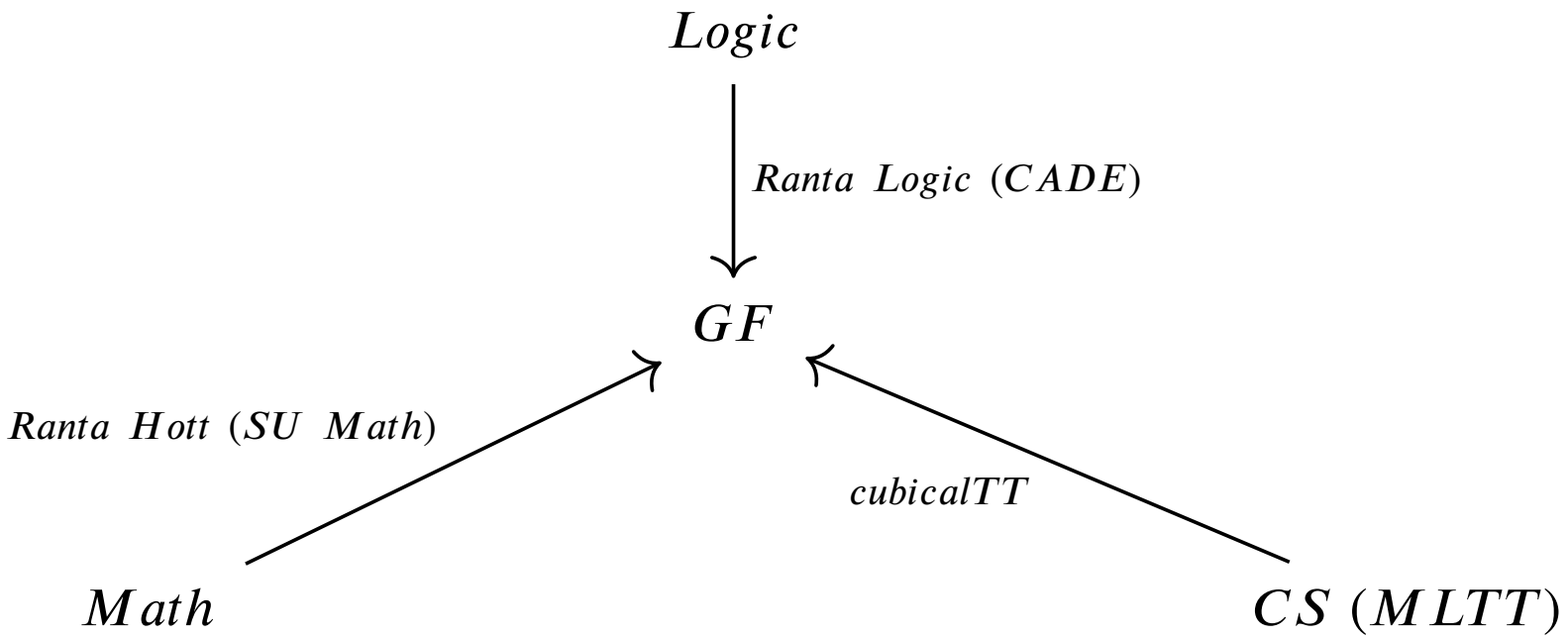
We acknowledge this is only semantic interpretations in these domains. One may decide on syntactic, pragmatic, or other ways in which to treat linguistics via these fields



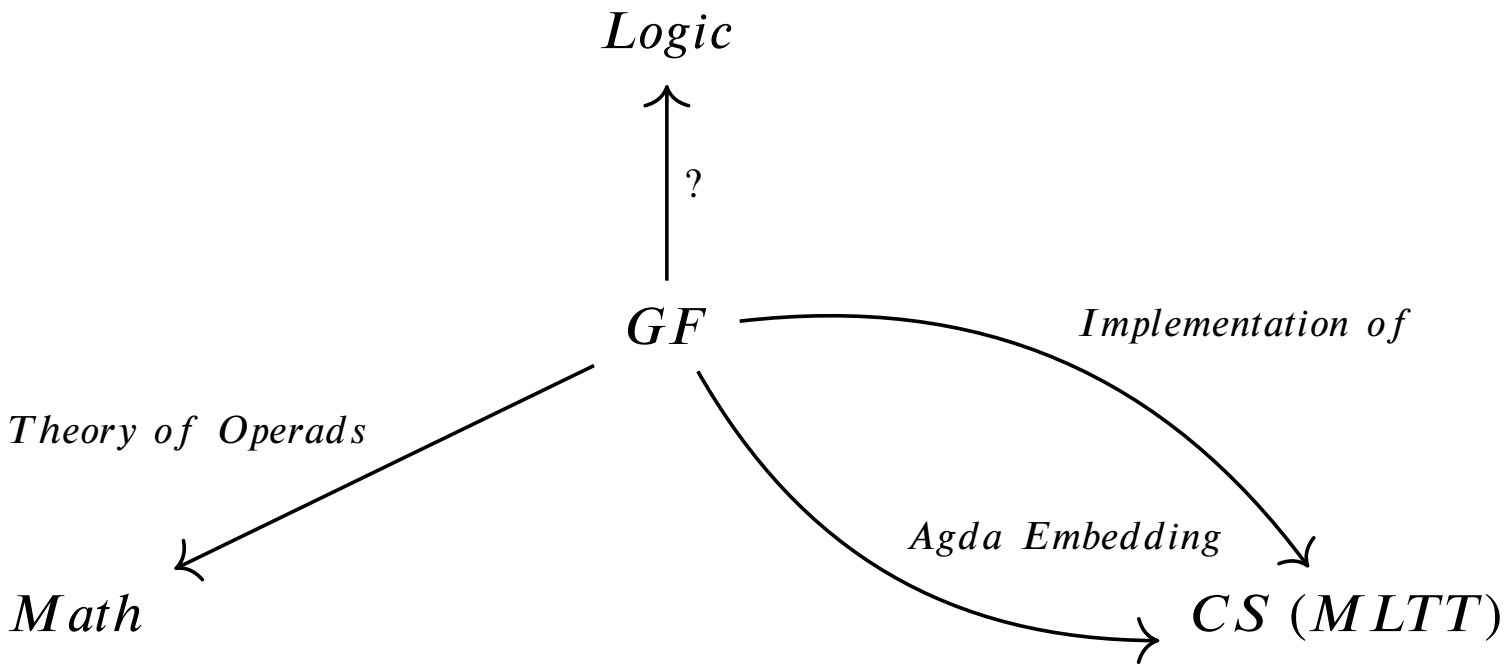
Trinitarian Linguistics



Trinitarian Grammars



Models of GF


Logic
?
GF
Implementation of
Theory of Operads
Agda Embedding
Math
CS (MLTT)

Remarks

- Trinitarian doctrine is in the “formal” space
- Trinitarian + Linguistics is partially formal, and very underexplored
- Introduces many philosophical concerns, perhaps a rereading of Wittgenstein should take place in this context

- Frege : Formal Proof, Predicate Logic
- Russel : Type Theory to resolve his paradox
- Brouwer : Constructivism

Mathematical logic and the relation between logic and mathematics have been interpreted in at least three different ways:

i. mathematical logic as symbolic logic, or logic using mathematical symbolism;
ii. mathematical logic as foundations (or philosophy) of mathematics;
iii. mathematical logic as logic studied by mathematical methods, as a branch of mathematics.

We shall here mainly be interested in mathematical logic in the second sense. What we shall do is also mathematical logic in the first sense, but certainly not in the third.

(Per Martin-Löf, Padua Italy, June 1980)

Syntactic Comparisons

First Order Logic

- $\forall$
- $\exists$
- $\supset$
- $\wedge$
- $\vee$
- $\neg$
- $\top$
- $\bot$
- $=$

Dependent Type Theory

- Π
- Σ
- $\rightarrow$
- $\times$
- $+$
- $\neg$
- $\top$
- $\bot$
- $\equiv$

Sets

- $\mathbb{N}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \to \mathbb{N}$
- $\{x | P(x)\}$
- $\varnothing$
- ?
- $\cup$
- ?

Types

- Nat
- $Nat \times Nat$
- $Nat \to Nat$
- $\Sigma x : \_.P(x)$
- $\bot$
- $\top$
- ?
- U_1

More Sets

- 1
- $(1, 0)$

Programs

- $suc\ \ zero$
- $(suc\ \ zero, zero)$

Judgments

Type Theoretic Judgments

- T is a type
- T and T' are equal types
- t is a term of type T
- t and t' are equal terms of type T

Mathematical Judgments

- P is a proposition
- P is true

- Notice that judgmental equality is uniquely type theoretic
- Judgments in type theory are decidable
- Truth (inhabitation) is not decidable
- More exotic judgments are available in TT, i.e. P is possible.

Important Differences

- The rules of the types make explicit that they are not equivalent to those of classical FOL
- An existential assertion in type theory requires data
- Excluded middle and double negation are not admitted in MLTT
- To be *not unhappy* is clearly of a different meaning than to be *happy*.
- This makes our approach to general translation of non-constructive mathematics *impossible* (at least such that it type-checks)

- One doesn't define logics, type systems in mathematics (e.g. metamathemeatics)
- Encoding things like rational and real numbers in type theory are
- already, category theorists and set theorists are at odds, (small and large categories), higher categories, which skeletons of categories are canonical, etc. incredibly difficult
- Additionally, intensional type theory comes with two distinct notions of equality, judgmental/definitional/computational and propositional equality

Example Donkey Anaphora

Interpret the sentence “every man who owns a donkey beats it” in MLTT via the following judgment :

$$\Pi z : (\Sigma x : man.\ \Sigma y : donkey.\ owns(x, y)).\ beats(\pi_1 z, \pi_1(\pi_2 z))$$

We judge $\vdash man$:type and $\vdash donkey$:type. type really denotes a universe

HoTT

- Equality is perhaps the most confusing detail for mathematicians
- Homotopy Type Theory is an all out coming to terms with what equality is in type theory
- Univalence Axiom : Equivalence is equivalent to equality
- Allows one to admit a topological interpretation of types
- Has led to Higher Inductive Types, where constructors can include equality types.

Interpretations of $t : \tau$

- Set theory : $n is a element of f N$
- Type theory : n is a term of type Nat
- Homotopy theory : n is a point in the space N
- Category theory : n is an arrow between the object N and itself
- Logic : n is a proof of the proposition Nat –broken

What is Agda?

- Implementation of MLTT
- Logical Framework
- Interactive proof development environment
- Inductive Types, Modules, Pattern Matching, more

Mathematical Declarations

- Theorem
- Proof
- Lemma
- Axiom
- Definition
- Example

Twin Prime Conjecture

Definition

A *twin prime* is a prime number that is either 2 less or 2 more than another prime number

Alternatively, we may state it as follows :

Definition

A *twin prime* is a prime that has a prime gap of two.

Definition

A *prime gap* is the difference between two successive prime numbers.

Theorem

There are infinitely many twin primes.

Twin Prime Conjecture in Agda

What is a Proof?

A proof is what makes a judgment evident

(Per Martin-Löf)

...there is a considerable gap between what mathematicians claim is true and what they believe, and this mismatch causes a number of serious linguistic problems.

(Mohan Ganesalingam)

Comparsion	Formal Proof	Informal Proof
Audience	Agda (and Human)	Human
Translation	Compiler	Human
Objectivity	Objective	Subjective
Historical	20th Century	<= Euclid
Orientation	Syntax	Semantics
Inferability	Complete	Domain Expertise Necessary
Verification	PL Designer	Human
Ambiguity	Unambiguous	Ambiguous

- One missed comparison from above : formal proof is an implementation, an informal proof is a specification
- Why? Historically, we think of semantics preceding (the abstract notion of a circle preceded its geometric understanding)
- although syntax oriented thinking may now be dominating the CS tradition)
- syntax oriented approach in Agda program,
- Both are necessary in the end, especially for big proofs.
- Propositions or theorem statements are *intentionally unambiguous*

...when it comes to understanding the power of mathematical language to guide our thought and help us reason well, formal mathematical languages like the ones used by interactive proof assistants provide informative models of informal mathematical language. The formal languages underlying foundational frameworks such as set theory and type theory were designed to provide an account of the correct rules of mathematical reasoning, and, as Gödel observed, they do a remarkably good job. But correctness isn't everything: we want our mathematical languages to enable us to reason efficiently and effectively as well. To that end, we need not just accounts as to what makes a mathematical argument correct, but also accounts of the structural features of our theorizing that help us manage mathematical complexity.

(Avigad)

Syntactic Completeness

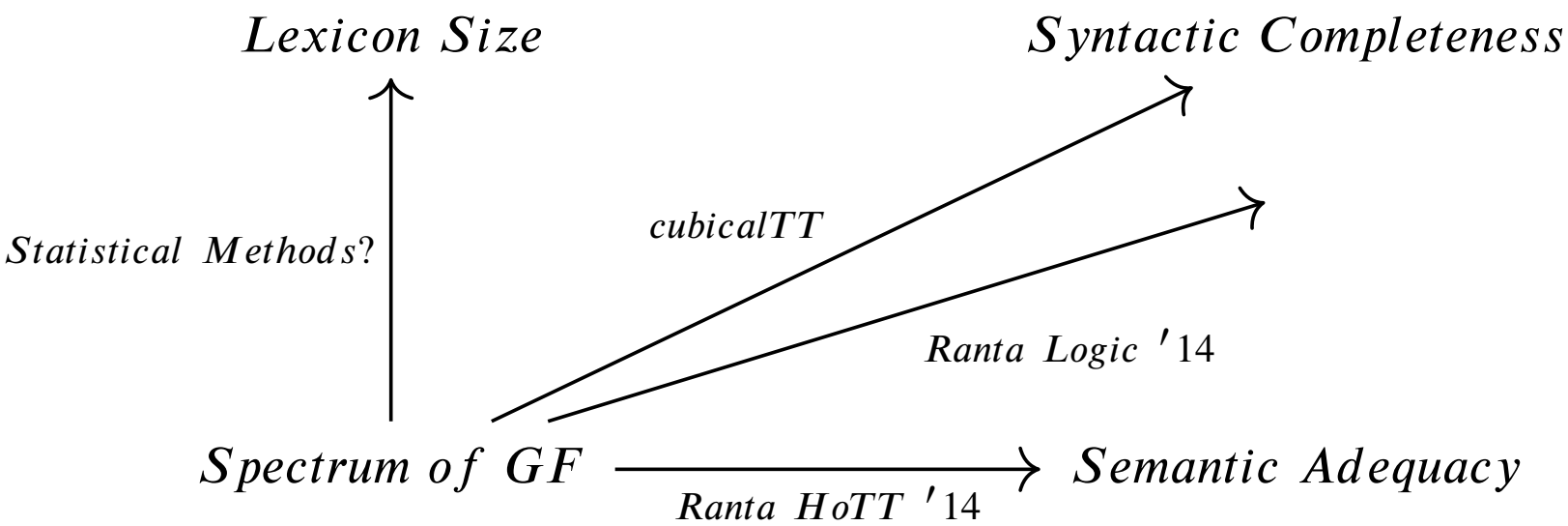
Given an utterance or natural language expression that a mathematician might understand, does the GF grammar emit a well-formed, syntactically complete expression in the target logic or programming language?

- The intended meaning manifests differently for different people - there may be no absolute intention
- Many possible syntax's - beyond just alpha conversion (think Coq) Perhaps the details will have to change depending on the tools available
- Starting Point : math as developed in interactive theorem prover

Semantic Adequacy

Given a well formed syntactic expression in, for instance, Agda, one can ask if the resulting NL expression generated by GF is *semantically adequate*, and coherent to a "fluent speaker"

- Think expressive adequacy in logic
- In dispute among mathematicians
- Kind of like a "Turing test for our approach"
- depends historical developments as our language develops, both internally (cite gangaselem) and more interestingly, as a society.
- Starting Point : [**?**] mathematics


Lexicon Size
Syntactic Completeness
Statistical Methods?
cubicalTT
Ranta Logic '14
Spectrum of GF
Semantic Adequacy
Ranta HoTT '14

Concrete vs Abstract

Comparison of concrete vs. abstract perspective when designing a GF Grammar
Abstract : Capture more semantic content (from a NL perspective) more “freedom of expression” simpler makes easier to work with in Haskell (where tactics would come into play in Coq)
Concrete : Simpler semantic space, easier for designing PL for it eliminates ambiguity. focus on this may mean more feasibility for real implementation
As will be seen, a smaller abstract syntax leans towards syntactic completeness with a larger linearization syntax (in terms of the complexity of the lincats effects the complexity of the entire linearization space)

Logic ’11

- Ranta Cade 2011
- Application grammar for logical propositions defined over some mathematical domain (integral arithmetic, euclidean geometry)
- Doesn’t cover proofs
- Uses a core grammar and extended, translate between them via PGF extension

Ideas

- One can use theorems from FOL, to simplify the NL expression (semantics preserving normalizer)
- Serves as a basis for other logics

Core

- minimal necessary expressivity
- syntactically complete
- $\forall x(Nat(x) \supset Even(x) \vee Odd(x))$
- "for all x, if x is a natural number then x is even or x is odd"
- ambiguous parses : Catalan explosion with n conjunctions
- need normal form

Semantically Inadequate

"is it the case that the sum of 3 and the sum of 4 and 10 is prime and 9999 is odd"

Core Syntax

construction	**symbolic**	**verbal**
negation	$\sim P$	*it is not the case that P*
conjunction	$P \,\&\, Q$	*P and Q*
disjunction	$P \vee Q$	*P or Q*
implication	$P \supset Q$	*if P then Q*
universal quantification	$(\forall x)P$	*for all x, P*
existential quantification	$(\exists x)P$	*there exists an x such that P*

Extended

- much more expressive
- semantically adequate
- "every natural number is even or odd"
- increases both number of categories and functions
- also need for more complicated linearization categories
- complex PGF backend to keep this syntactically complete
- questions about scalability

Extended Syntax

construction	**symbolic**	**verbal (example)**
atom negation	$\overline{A}$	*x is not even*
conjunction of proposition list	$\&[P_1, \ldots, P_n]$	*P, Q and R*
conjunction of predicate list	$\&[F_1, \ldots, F_n]$	*even and odd*
conjunction of term list	$\&[a_1, \ldots, a_n]$	*x and y*
bounded quantification	$(\forall\, x_1, \ldots, x_n : K)P$	*for all numbers x and y, P*
in-situ quantification	$F(\forall\, K)$	*every number is even*
one-place predication	$F^1(x)$	*x is even*
two-place predication	$F^2(x, y)$	*x is equal to y*
reflexive predication	$\mathit{Refl}(F^2)(x)$	*x is equal to itself*
modified predicate	$\mathit{Mod}(K, F)(x)$	*x is an even number*

Translation

$[\![-]\!] : Extended \to Core$

- relatively simple (in the sense that it should be deterministic)
- More or less uses the same logical structure from the "standard view"
- Core syntax as a model for extended

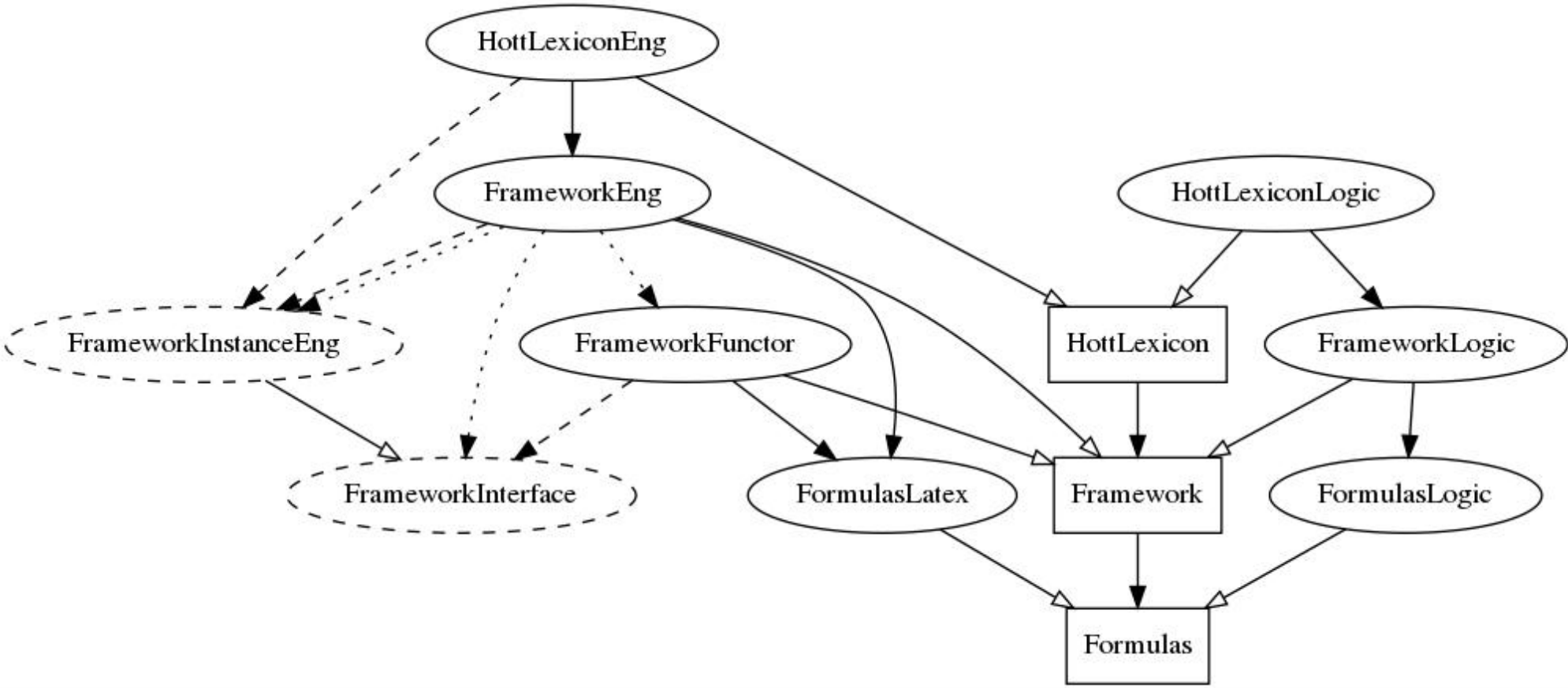
$[\![-]\!] : Core \to Extended$

- Flattening a list
  $x\ and\ y\ and\ z \mapsto x,\ y\ and\ z$
- Aggregation
  $x\ is\ even\ or\ x\ is\ odd \mapsto x\ is\ even\ or\ odd$
- In-situ quantification
  $\forall\ n \in Nat,\ x\ is\ even\ or\ x\ is\ odd \mapsto every\ Nat\ is\ even\ or$
- Negation
  $it\ is\ not\ that\ case\ that\ x\ is\ even \mapsto is\ not\ even$
- Reflexivitazion
  $x\ is\ equal\ to\ x \mapsto x\ is\ equal\ to\ itself$
- Modification
  $x\ is\ a\ number\ and\ x\ is\ even \mapsto x\ is\ an\ even\ number$

Ranta's (unpublished) talk at Stockholm Math Seminar

- Case study for text of *real* mathematics writing
- Purely GF translation
- Complex abstract structure including latex, metadocument structure, etc.
- Homotopy Type Theory (HoTT) specific lexicon
- Includes proofs, but AST not similar to PL
- Includes both expressions and propositions (breaks curry howard)
- Semantically Adequate but syntactically incomplete

HoTT Grammar Modules



Framework.gf

```
cat
  Paragraph ; -- definition, theorem, etc
  Definition ; -- definition of a new concept
  Assumption ; -- assumption in a proof -- let ...
  [Assumption]{1} ;  -- list of assumptions in one sentence
  Conclusion ; -- conclusion in a proof -- thus P
 Prop ; -- proposition,sentence or formula, A is contractible
  Sort ; -- set, type, etc corresponding to a common noun
 Ind ; -- individual element corresponding to a singular term
  Fun ; -- function with individual value
  Pred ; -- predicate: function with proposition value
  [Ind] ; -- list of individual expressions -- 1, 2 and 3
 UnivPhrase ; -- universal noun phrase -- for all x,y : A
  ConclusionPhrase ; -- conclusion word -- hence
  Label ; -- name/number of definition, theorem, etc
  Title ; -- title for theorem, definition, etc
```

Formulas.gf

```
cat
  Exp ;            -- formal expression
                   -- x + y = z
  Var ;            -- variable
                   -- x
  [Var]{1} ;       -- list of variables
                   -- x, y, z
  [Exp]{1} ;       -- list of expressions
                   -- 1_{A}, A
  Format ;         -- line other than content
                   -- \begin{document}
  MathEnv ;        -- math environment
                   -- $ ... $
```

Comparative Syntax

- We work with the definition of contractability, the notion that a type (or space) is actually a point, i.e. up to equality, there is only object.
- We show the rendered latex, a pidgin agda syntax (after some concrete modifications), and the Agda code

Definition: A type A is contractible, if there is $a : A$, called the center of contraction, such that for all $x : A$, $a = x$.

```
isContr ( A : Set ) : Set =
  ( a : A ) ( * ) ( ( x : A ) -> Id ( a ) ( x ) )
```

- We also show that a notion of a map being an equivalence, that of the every element in the codomain having a contractible image (think bijection)
- notice the error in the Pidgin case

Definition: A map $f : A \to B$ is an equivalence, if for all $y : B$, its fiber, $\{x : A \mid f\,x = y\}$, is contractible. We write $A \simeq B$, if there is an equivalence $A \to B$.

```
Equivalence ( f : A -> B ) : Set =
  ( y : B ) -> ( isContr ( fiber it ) ) ; ; ;
  fiber it : Set = ( x : A ) ( * ) ( Id ( f ( x ) ) ( y ) )
```

```
Equivalence : (A B : Set) → (f : A → B) → Set
Equivalence A B f = ∀ (y : B) → isContr (fiber' y)
  where
    fiber' : (y : B) → Set
    fiber' y = Σ A (λ x → y ≡ f x)
```

GF as a programming language parser

[fragile]

- How can a PL parser be bootstrapped to linearize to natural language ?
- Syntactically Adequate out the box
- There exists many existing programs to test on, with hopefully, less overhead work needed
- In dependently typed language, almost everything is an expression
- No distinguishing syntactically between types and terms
- this makes the parser easy to write for the PL but difficult for the NL
- All precedence information at the concrete level

```
cat
  Exp ;
  Var ;
  Tele ;
  LTele ;
  [Var]{1} ;
  [Tele]{2} ;
  [LTele]{2} ;
```

```
fun
  Dtype , Dterm : Var -> Exp -> Decl ;

  Earr : Exp -> Exp -> Exp ;
  Epi : [Tele] -> Exp -> Exp ;
  Eid : Exp -> Exp -> Exp -> Exp ; -- for type
  Eid2 : Exp -> Exp -> Exp ;
  Enat , Euni : Exp ;

  Evar : Var -> Exp ;
  Elam : [LTele] -> Exp -> Exp ;
  Eapp : Exp -> Exp -> Exp ;

  Erefl : Exp ;
  Eidind : Exp -> Exp -> Exp -> Exp -> Exp -> Exp ;

  Ezer : Exp ;
  Esuc : Exp -> Exp ;
  EsucEta : Exp ;
  Enatind : Exp -> Exp -> Exp -> Exp -> Exp ;
```

```
data ℕ' : Set where
  zero' : ℕ'
  suc' : ℕ' → ℕ'

data _≡'_ {A : Set} (a : A) : A → Set where
  refl' : a ≡' a

two : ℕ'
two = suc' (suc' zero')

double : ℕ' → ℕ'
double zero' = zero'
double (suc' n) = suc' (suc' (double n))

four : ℕ'
four = double two

_+'_ : ℕ' → ℕ' → ℕ'
zero' +' y = y
suc' x +' y = suc' (x +' y)

2+2=4 : (two +' two) ≡ four
2+2=4 = refl
```

Associativity of natural numbers

```
Theorem: For any n, m and p,
  n + (m + p) = (n + m) + p.
Proof: By induction on n.
  First, suppose n = 0. We must show that
    0 + (m + p) = (0 + m) + p.
  This follows directly from the definition of +.
  Next, suppose n = S n', where
    n' + (m + p) = (n' + m) + p.
  We must now show that
    (S n') + (m + p) = ((S n') + m) + p.
  By the definition of +, this follows from
    S (n' + (m + p)) = S ((n' + m) + p),
  which is immediate from the induction hypothesis. Qed.
```

```
ap : (f : A → B) → a ≡ a' → f a ≡ f a'
ap f refl = refl

associativity-plus : (m n p : ℕ) → (( m + n )+ p) ≡ (m + (n + p))
associativity-plus zero n p = refl
associativity-plus (suc m) n p = ap suc (associativity-plus m n p)
```

```
natind : {C : ℕ -> Set} -> -- predicate
         C zero ->         -- base case
         ((n : ℕ) -> C n -> C (suc n)) -> --IH
         (n : ℕ) -> C n
natind base step zero = base
natind base step (suc n) = step n (natind base step n)
```

```
associativity-plus-ind :
  (m n p : ℕ) →
  ((m + n) + p) ≡ (m + (n + p))
associativity-plus-ind m n p =
  natind
    baseCase
    (λ n₁ ih → simpl n₁ (indCase n₁ ih))
    m
  where
    baseCase : (zero + n + p) ≡ (zero + (n + p))
    baseCase = refl
    indCase : (n' : ℕ) → (n' + n + p) ≡ (n' + (n + p)) →
                         suc (n' + n + p) ≡ suc (n' + (n + p))
    indCase = (λ n' x → ap suc x )
    simpl : (n' : ℕ)
                  → suc (n' + n + p) ≡ suc (n' + (n + p))
                  → (suc n' + n + p) ≡ (suc n' + (n + p))
    simpl n' x = x
```

Ideal natural language proof
My natural proof Comparison

Issues, TODO

- Get list categories correct
- Length of list issues, how to get it to properly linearize to two sytnaxes with unique list idioms
- Properly deal with all the *applications* in the NL case
- Multiple concrete syntaxes for syntactic nuance

A minimal Dependently Typed PL in GF

BNFC

- Backus-Naur form Converter (BNFC) : a cousin of GF for that combines the concrete and abstract into a single grammar file.
- supports precedence out of the box. this is done via a seperate GF module
- BNFC -> GF is a pretty seamless translation
- Everything from Göteborg

cubicalTT

- Existing grammars for experimental PLs, like cubicalTT
- Treats equality natively, Univalence becomes a theorem
- Experimental, hot
- cubicalTT became cubical Agda

Future Work

Convert mathematicians. More interaction between PL communities - Lean, Agda, Coq , Prl - These languages each represent a huge field of interrelated and independent research.
Linguistic analysis. Gangaselem's work is both practical and philosophical, both directions need a lot more research
A mathematician objected that it can't be difficult to prove a 2 is prime in Agda. This is because they in some sense are mixing definitional and propositional equality.
Prop equality <-> isomorphism via univalence, but even moreso after cubical
HITs <-> new proofs, new discoveries more akin to semantic feeling
A FORMAL SYSTEM FOR EUCLID'S ELEMENTS
https://www.cambridge.org/core/journals/review-of-symbolic-logic/article/abs/formal-system-for-euclids-elements/07CA7E5F8E1C5C2EB632E1005CBE7BEF
Different intersecting ideas of models, syntax vs semantic
Krasimir : bigger grammar, begins resembling RGL (but we also need to investigate where compile time optimizations are necessary for this) We need an RGL to take care of things that satisfy both the