

Logistic Regression and the ROC curve

Schwartz

July 22, 2017

Odd, even at best

Last year Leicester City was given 5000 to 1 odds to win the English Premier League. Actually, these are the longest odds *ever seen* for *any* top tier sporting league... *ever*. To put this in perspective, the current odds out of Vegas for “the most unlikely team to win the 2016/2017 NFL season” – woefully disastrous Cleveland* Browns – are 200 to 1.

Since the clubs inception in 1890, Leicester City has only managed to appear in the Premier league 10 seasons. They had only been promoted the previous season and just barely escaped relegation in their final match that season. Only five teams – Arsenal, Chelsea, Liverpool, Man. City, and Man. U. – have held the trophy for the past 21 seasons.

Only a few stout souls put money down on Leicester City last year. And when Leicester City (*literally against all odds*) won the premiership last season in absolutely stunning, unbelievable, and unprecedented fashion, those stout souls got paid. Everyone, that is, except for John Micklethwait. John M has made the same bet – 20 pounds (\$29) that Leicester will win their division – every August for the past 20 years. Every year, that is, except this one. Last year he moved from London to New York and missed placing his bet. That’s a pity for John M because if he had made his bet he would have won 100,000 pounds, or \$145,355.

Overall, \$3,000 was bet on Leicester City last season. The *unprecedented* \$15,000,000 payout nearly bankrupted the bookmakers. John M got \$0.

* Cleveland’s 52-year championship drought ended with the 2015/16 NBA season

Odds

$$\text{Odds} = \frac{p}{1-p} \implies p = \frac{\text{Odds}}{1 + \text{Odds}} = \frac{1}{1 + \text{Odds}^{-1}}$$

$$1 - p = \frac{1}{1 + \text{Odds}}$$

Objectives

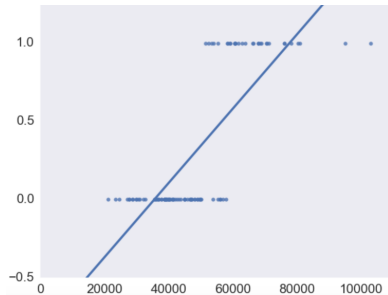
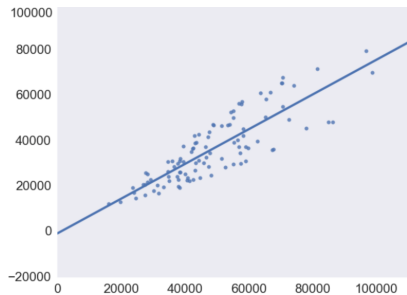
Morning

- ▶ Know why logistic regression is a thing:
 - ▶ Classification vs. Regression
 - ▶ Link functions
- ▶ Interpreting Logistic Regression
 - ▶ Fitted Values (probabilities)
 - ▶ Coefficients (log odds ratios)

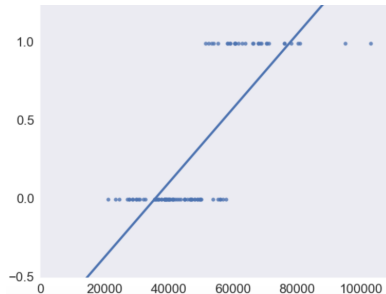
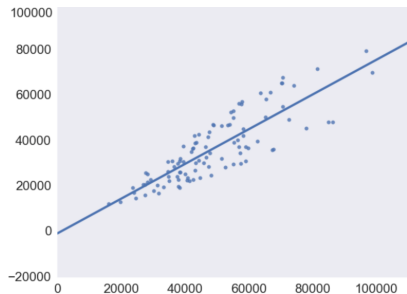
Afternoon

- ▶ T+, T-, F+, F- and other terminology
 - ▶ Confusion Matrices
- ▶ Thresholding Classification rules
 - ▶ ROC curves

Linear Regression

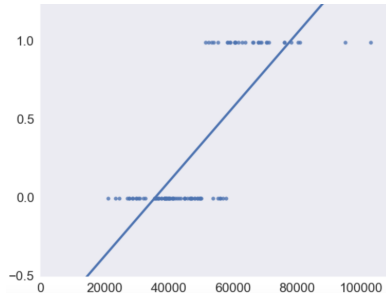
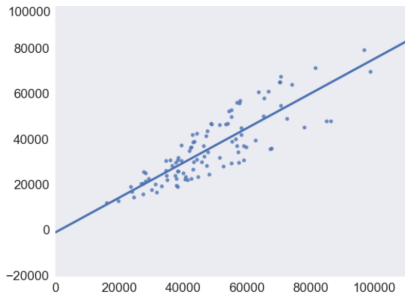


Linear Regression



Is this satisfactory
for predicting probability?

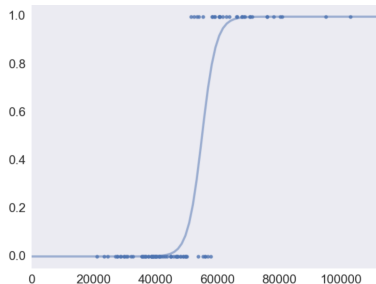
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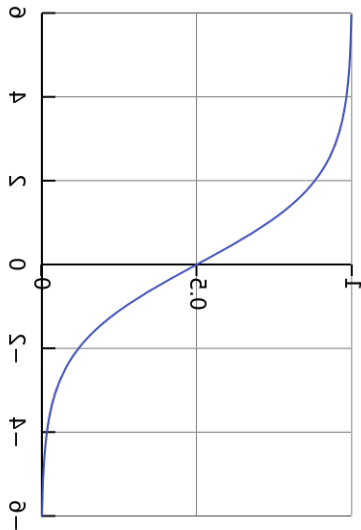
How about this instead \Rightarrow



Link functions

- The “logit”

$$g(p) = \log\left(\frac{p}{1-p}\right)$$



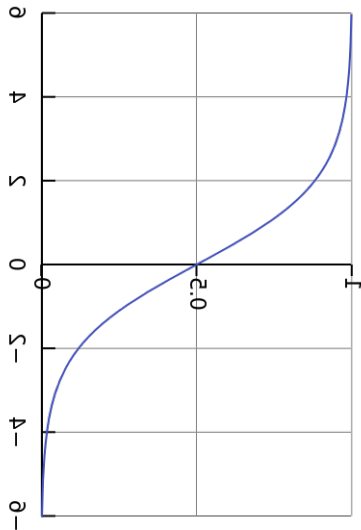
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$$p \in [0, 1] \mapsto Z \in \mathbb{R}$$



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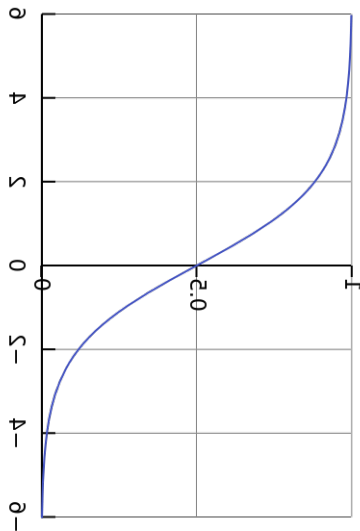
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But log odds go $-\infty$ to ∞ !



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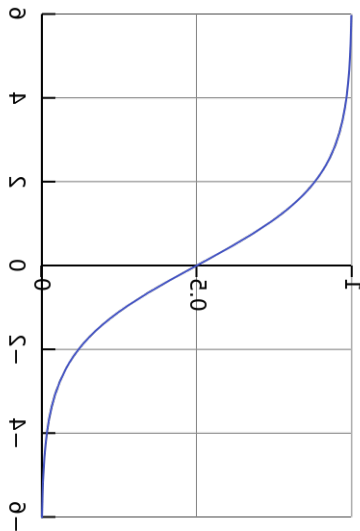
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Don't be at odds with odds!



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$$\hat{Y}_i = \Pr(Y = 1) = E[Y] = g^{-1}(Z)$$

because how else can Z stay between 0 and 1??

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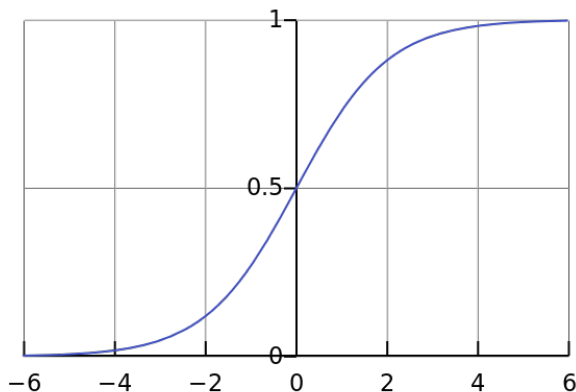
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So $g(p) = Z = \log\left(\frac{p}{1-p}\right) \in \mathbb{R}$ (which is called the logit function)

and $Z = \beta_0 + \beta_1 x_1 + \cdots + \beta_m x_m \in \mathbb{R}$ models the log odds

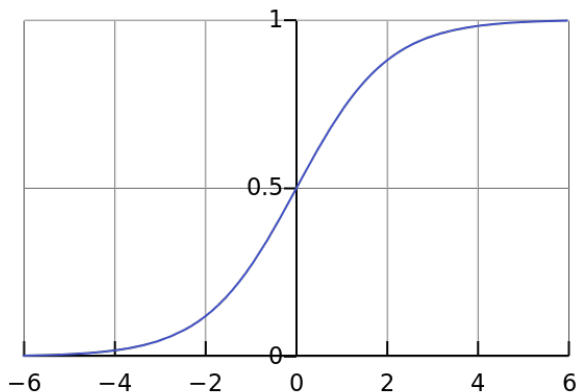
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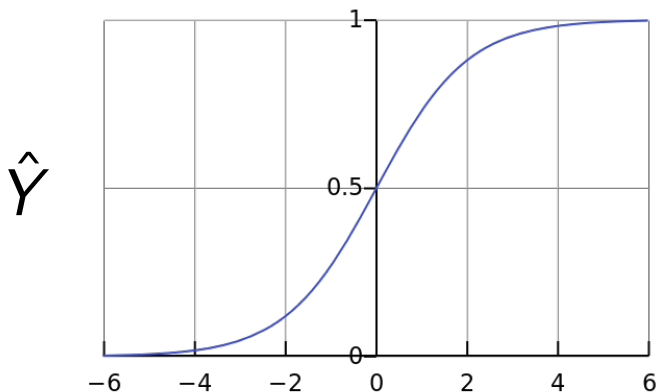


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there is a $\exp(\beta_j)$ multiplicative increase in *the odds*

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- ▶ I.e., *the odds* are linear in x on a multiplicative, i.e., odds increase with x on a *logarithmic* scale with base $\exp(\beta_j)$

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- ▶ I.e., *the odds* are linear in x on a multiplicative, i.e., odds increase with x on a *logarithmic* scale with base $\exp(\beta_j)$
- ▶ The *log odds* $\log \left(\frac{\Pr(Y=1|x)}{\Pr(Y=0|x)} \right)$ are on a linear scale
 $(\beta_0 + \beta_1 x_1 + \dots + \beta_n x_m)$

The Odds Ratio (OR)

- Equivalently, $\exp(\beta_j)$ is the *odds ratio (OR)* between 1-unit differences in x_j (e.g., 0 versus 1) when other x 's are constant

$$\exp(\beta_j) = \frac{\Pr(Y = 1|x_j + 1, x_{-j})/\Pr(Y = 0|x_j + 1, x_{-j})}{\Pr(Y = 1|x)/\Pr(Y = 0|x)}$$

since

$$\begin{aligned} & \frac{\Pr(Y = 1|x_j + 1, x_{-j})}{\Pr(Y = 0|x_j + 1, x_{-j})} \\ &= \exp(\beta_0) \exp(\beta_1 x_1) \cdots \exp(\beta_j(x_j + 1)) \cdots \exp(\beta_m x_m) \\ &= \exp(\beta_0) \exp(\beta_1 x_1) \cdots \exp(\beta_j x_j) \exp(\beta_j) \cdots \exp(\beta_m x_m) \end{aligned}$$

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- So β_j is the change in $\log(OR)$ for one unit changes in x_j ...

Logistic Regression *Likelihood* and *Deviance*

- Likelihood

$$f(\mathbf{Y}|\beta, \mathbf{x}) = \prod \left(\frac{1}{1 + e^{-\mathbf{x}_i^T \beta}} \right)^{Y_i} \left(\frac{1}{1 + e^{\mathbf{x}_i^T \beta}} \right)^{1-Y_i}$$

Logistic Regression *Likelihood* and *Deviance*

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► Deviance

$$D_M = -2 \left(\log f(\mathbf{Y}|\hat{\boldsymbol{\beta}}, \mathbf{x}) - \log f(\mathbf{Y}|\mathbf{Y}) \right)$$

$\underset{\text{approx.}}{\sim} \chi_{n-p-1}^2$

n = sample size

p = number of coefficients in model M

$f(\mathbf{Y}|\mathbf{Y})$ = saturated model (\mathbf{Y} perfectly predicted)

More Deviance

- In logistic regression

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[show this]

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[show this]

$$\stackrel{\text{approx.}}{\sim} \chi^2_{n-p-1}$$

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[show this]

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$$D_M = \frac{RSS}{\sigma^2} \quad \text{[show this]}$$

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[what are *residuals*?]

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[what are *residuals*?] [what are *residuals* in logistic regression?]

Fitting Logistic Regression

► MLE

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\Longleftrightarrow

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 - Coefficient standard errors can also be numerically estimated!

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Pseudo R^2

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Proportion of deviance explained

Pseudo R^2

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- ▶ http://www.ats.ucla.edu/stat/mult_pkg/faq/general/Pseudo_RSquareds.htm

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How else could you compare
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- ▶ Balancing observational comparison groups on *propensity scores* $\Pr(T|x)$ which controls bias from group covariate composition differences

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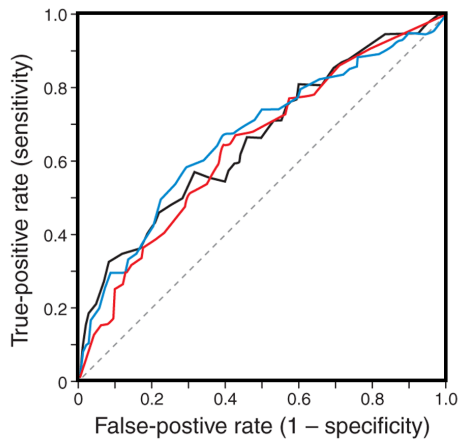
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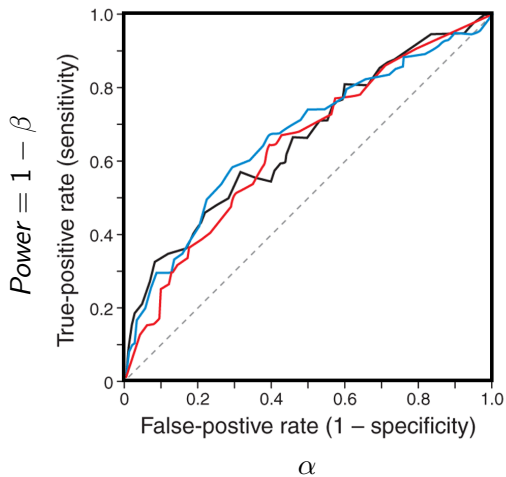
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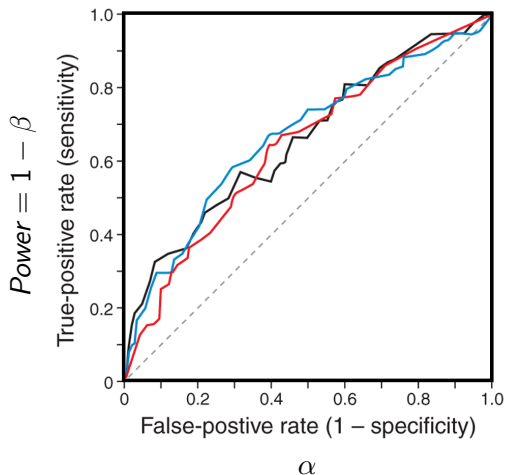
ROC/AUC



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<https://www.youtube.com/watch?v=JAQC59ArFJw>

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“Do we call hypotheses **accurately**?”

And just a one or two more...

		Predicted condition			
Total population		Predicted Condition positive	Predicted Condition negative	Prevalence $= \frac{\Sigma \text{Condition positive}}{\Sigma \text{Total population}}$	
True condition	condition positive	True positive	False Negative (Type II error)	True positive rate (TPR), Sensitivity, Recall $= \frac{\Sigma \text{True positive}}{\Sigma \text{Condition positive}}$	False negative rate (FNR), Miss rate $= \frac{\Sigma \text{False negative}}{\Sigma \text{Condition positive}}$
	condition negative	False Positive (Type I error)	True negative	False positive rate (FPR), Fall-out $= \frac{\Sigma \text{False positive}}{\Sigma \text{Condition negative}}$	True negative rate (TNR), Specificity (SPC) $= \frac{\Sigma \text{True negative}}{\Sigma \text{Condition negative}}$
Accuracy (ACC) = $\frac{\Sigma \text{True positive} + \Sigma \text{True negative}}{\Sigma \text{Total population}}$		Positive predictive value (PPV), Precision $= \frac{\Sigma \text{True positive}}{\Sigma \text{Test outcome positive}}$	False omission rate (FOR) $= \frac{\Sigma \text{False negative}}{\Sigma \text{Test outcome negative}}$	Positive likelihood ratio (LR+) = $\frac{\text{TPR}}{\text{FPR}}$	Diagnostic odds ratio (DOR) = $\frac{\text{LR+}}{\text{LR-}}$
		False discovery rate (FDR) $= \frac{\Sigma \text{False positive}}{\Sigma \text{Test outcome positive}}$	Negative predictive value (NPV) $= \frac{\Sigma \text{True negative}}{\Sigma \text{Test outcome negative}}$	Negative likelihood ratio (LR-) = $\frac{\text{FNR}}{\text{TNR}}$	

Thanks, Wiki!
(You're the besht!)