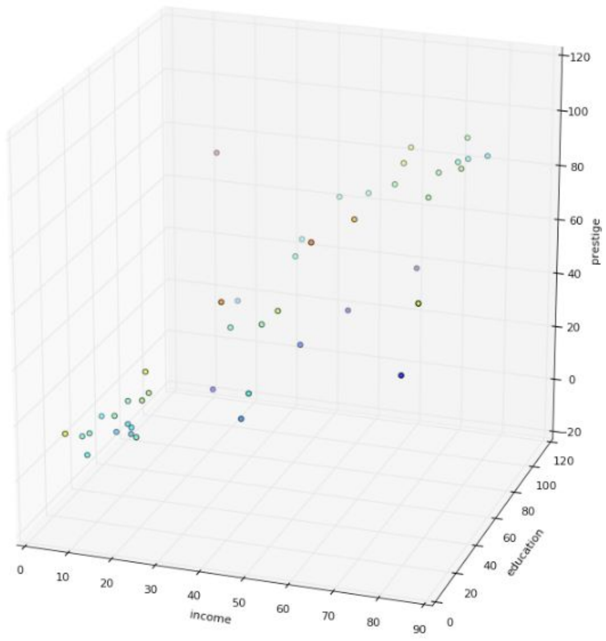
Linear Regression

DSI SEA5, jf.omhover, Sep 21, 2016



Linear Regression

DSI SEA5, jf.omhover, Sep 21, 2016

STANDARDS

- **Describe**, **interpret**, and **visualize** the model form of linear regression: Y = B0+B1X1+B2X2+....
- **Relate** Beta vector solution of Ordinary Least Squares to the cost function (residual sum of squares)
- **State** and troubleshoot the assumptions of linear regression model
- **Perform** OLS with statsmodels and interpret the output: Beta coefficients, p-values, R^2
- How can one **detect** outliers?

Linear Regression

DSI SEA5, jf.omhover, Sep 21, 2016

OBJECTIVES

- **Relate** linear regression with general machine learning
- **State** assumptions of linear regression
- **Estimate** a linear regression model
- **Evaluate** a linear regression model
- **Recognize and fix** common problems

What’s the Big Idea ?

Learning / Estimating FUNCTIONS

Reality VS Model : assumption, learning and error

REALITY

	type	income	education	prestige
accountant	prof	62	86	82
pilot	prof	72	76	83
architect	prof	75	92	90
author	prof	55	90	76
chemist	prof	64	86	90
minister	prof	21	84	87
professor	prof	64	93	93
dentist	prof	80	100	90
reporter	wc	67	87	52
engineer	prof	72	86	88
undertaker	prof	42	74	57
lawyer	prof	76	98	89

data

$$(x_1, y_1)$$

...

$$(x_n, y_n)$$

$$x \quad y$$

OBJECTIVE:
descriptive
predictive
normative
...

$$\sum (y_i - \hat{f}(x_i))^2$$

COST FUNCTION

MODEL

$$y = f(x) + \epsilon$$

take a function as an assumption

$$\hat{y} = \hat{f}(x)$$

Estimator of the function

Reality VS Model : LINEAR functions

REALITY

OBJECTIVE:
descriptive
predictive
normative
...

MODEL

	type	income	education	prestige
accountant	prof	62	86	82
pilot	prof	72	76	83
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dentist	prof	80	100	90
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data

$$(x_1, y_1)$$

...

$$(x_n, y_n)$$

$x \quad y$

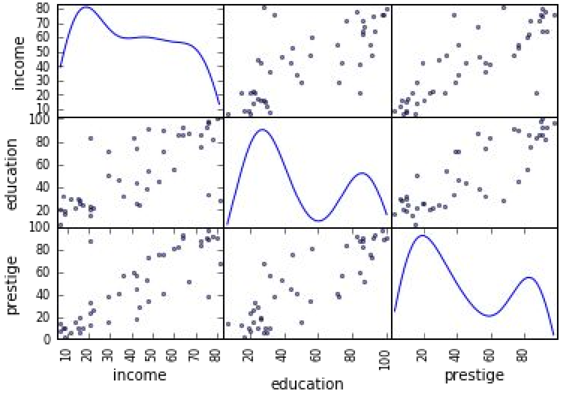
$$y \approx \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$$

We make the assumption that we have a linear relation

From reality to model : general process

REALITY

1) Having a data sample
Observing an underlying behavior

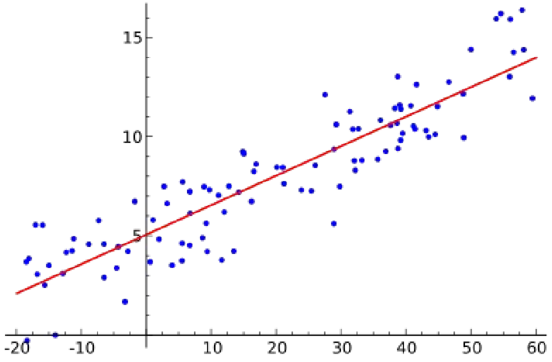


MODEL

2) Make an assumption
on the <u>model</u> underlying the data

$$y \approx \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$$

linear relation
(+ assumptions)



3) **<u>Find</u>** the instance of the model
that **<u>fits</u>** with data sample

Framing the problem : linear regression

REALITY

MODEL

	type	income	education	prestige
accountant	prof	62	86	82
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author	prof	55	90	76
chemist	prof	64	86	90
minister	prof	21	84	87
professor	prof	64	93	93
dentist	prof	80	100	90
reporter	wc	67	87	52
engineer	prof	72	86	88
undertaker	prof	42	74	57
lawyer	prof	76	98	89

data

X : independant variables
Y : dependant variable

DEFINE A CRITERIA
OPTIMIZE IT
OVER PARAMETERS

PROBLEM 1:
identify model class
verify assumptions

$$y \approx \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \ldots + \hat{\beta}_p x_p$$

PROBLEM 2:
find actual
coefficients/parameters
values

The Simple Linear Case

Framing the problem : SIMPLE linear regression

COST FUNCTION (Residual Sum of Squares)

$$RSS = \sum (y_i - \hat{f}(x))^2 = \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x)^2$$

O.L.S.

REALITY

MODEL

DATA

(x_1 , y_1)

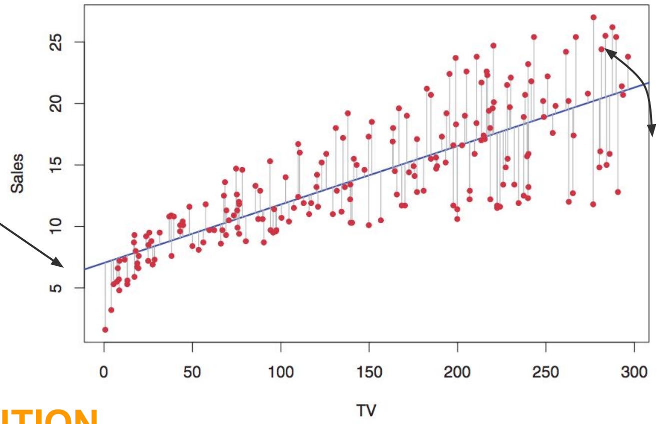
...

(x_n , y_n)

$x \quad y$

Y

$\hat{\beta}_0$



$\hat{\beta}_1$

X

model class

$$y \approx \beta_0 + \beta_1 x$$

PROBLEM

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

model instance
estimator
parameters

SOLUTION

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Figure from [ISL]

Evaluation as a model explaining the outcome

Residual Sum of Squares

$$RSS = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Total Sum of Squares

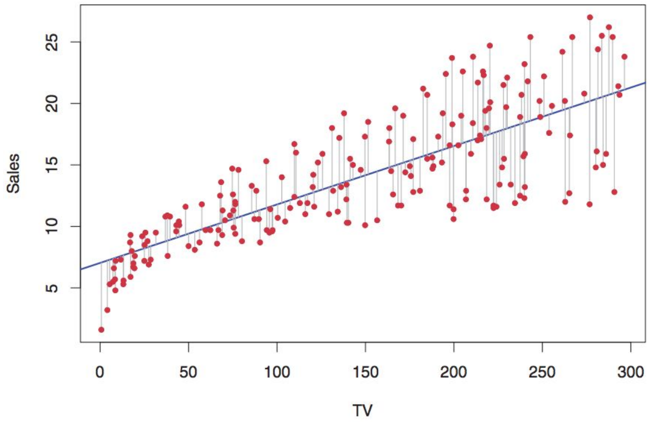
$$TSS = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

Explained Sum of Squares

$$ESS = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$$

R squared statistic

$$R^2 = \frac{TSS-RSS}{TSS}$$ in [0,1]



$$y = f(x) + \epsilon$$

class $$y \approx \beta_0 + \beta_1 x$$

instance $$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

$$y = \hat{\beta}_0 + \hat{\beta}_1 x + \epsilon$$

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \epsilon_i, \forall i < n$$

residual $$e_i = y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i$$

Figure from [ISL]

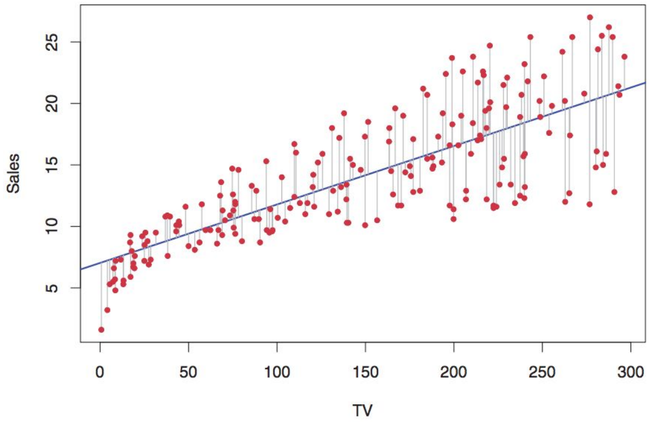
Evaluation as for hypothesis

Stating the null hypothesis

H0 : there is no relation between X and Y
H1 : there is some linear relation between X and Y

$$H_0 : \beta_1 = 0$$

$$H_1 : \beta_1 \neq 0$$



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residual $e_i = y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i$

Figure from [ISL]

The Multi-Linear Case

Framing the problem : multi-linear regression

COST FUNCTION (Residual Sum of Squares)

O.L.S.

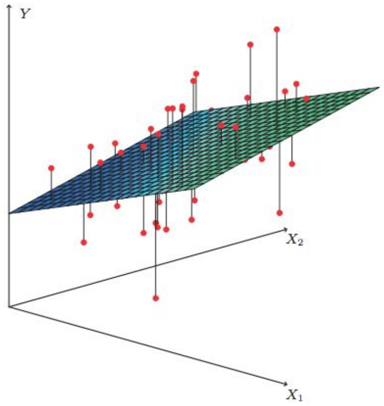
REALITY

MODEL

DATA

	type	income	education	prestige
accountant	prof	62	86	82
pilot	prof	72	76	83
lawyer	prof	76	98	89

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1p} \\ x_{21} & \cdots & x_{2p} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{np} \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = y$$



model class

$$y \approx \beta_0 + \beta_1 x_1 + \beta_1 x_2 + \ldots + \beta_p x_p$$

PROBLEM

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \ldots + \hat{\beta}_p x_p$$

model instance
estimator
parameters

SOLUTION

Figure from [ISL]

Framing the problem : multi-linear regression

COST FUNCTION (Residual Sum of Squares)

$$RSS(\beta) = (y - X\beta)^T (y - X\beta)$$

O.L.S.

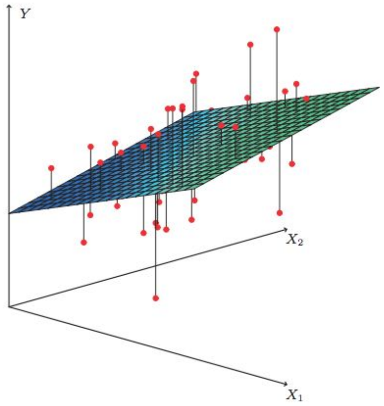
REALITY

MODEL

DATA

	type	income	education	prestige
accountant	prof	62	86	82
pilot	prof	72	76	83
lawyer	prof	76	98	89

$$X = \begin{bmatrix} 1 & x_{1,1} & \cdots & x_{1,p-1} \\ 1 & x_{2,1} & \cdots & x_{2,p-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n,1} & \cdots & x_{n,p-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = y$$



model class

$$y \approx X\beta$$

PROBLEM

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{p-1} \end{bmatrix}$$

$$\hat{y} = X\hat{\beta}$$

model instance
estimator
parameters

SOLUTION

$$\boxed{\hat{\beta} = (X^T X)^{-1} X^T y}$$

Figure from [ISL]

Evaluation as a model explaining the outcome

Residual Sum of Squares

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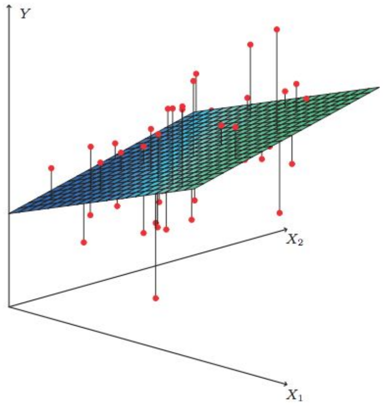


$$y = f(x) + \epsilon$$

class $y \approx X\beta$

instance $\hat{y} = X\hat{\beta}$

$$y = X\hat{\beta} + \epsilon$$

$$y_i = X\hat{\beta} + \epsilon_i$$

residual $e_i = y_i - X\hat{\beta}$

Figure from [ISL]

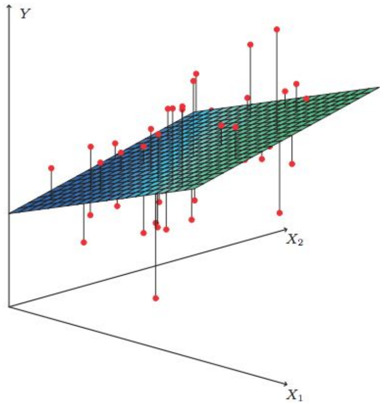
Evaluation as for hypothesis

Stating the null hypothesis

H0 : there is no relation between X and Y
H1 : there is some linear relation between X and Y

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$$

$$H_1 : \exists \beta_i \neq 0$$



$$y = f(x) + \epsilon$$

class $y \approx X\beta$

instance $\hat{y} = X\hat{\beta}$

$$y = X\hat{\beta} + \epsilon$$

$$y_i = X\hat{\beta} + \epsilon_i$$

residual $e_i = y_i - X\hat{\beta}$

Figure from [ISL]

Fitting a linear regression model in Python

```
y = prestige['prestige']
x = prestige[['income', 'education']].astype(float)
x['const'] = 1

prestige_model = statsmodels.api.OLS(endog=y, \
                                      exog=x).fit()

prestige_model.summary()
```

OLS Regression Results

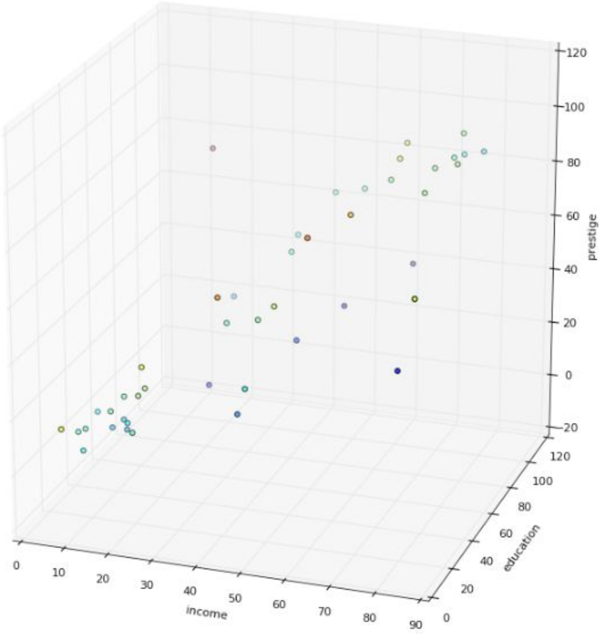
Dep. Variable:	prestige	**R-squared:**	0.828
Model:	OLS	**Adj. R-squared:**	0.820
Method:	Least Squares	**F-statistic:**	101.2
Date:	Tue, 20 Sep 2016	**Prob (F-statistic):**	8.65e-17
Time:	15:00:41	**Log-Likelihood:**	-178.98
No. Observations:	45	**AIC:**	364.0
Df Residuals:	42	**BIC:**	369.4
Df Model:	2		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[95.0% Conf. Int.]
income	0.5987	0.120	5.003	0.000	0.357 0.840
education	0.5458	0.098	5.555	0.000	0.348 0.744
const	-6.0647	4.272	-1.420	0.163	-14.686 2.556

Omnibus:	1.279	**Durbin-Watson:**	1.458
Prob(Omnibus):	0.528	**Jarque-Bera (JB):**	0.520
Skew:	0.155	**Prob(JB):**	0.771
Kurtosis:	3.426	**Cond. No.**	163.

Assumptions

Assumptions underlying Linear Regression



Linearity

Independence

Normal distribution of residuals

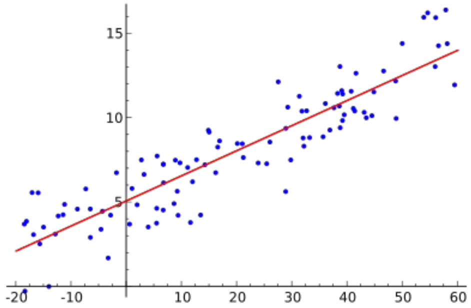
Homoscedasticity

Lack of multicollinearity

Assumptions // Linearity

Take the function component of your model

$$y = f(x) + \epsilon$$



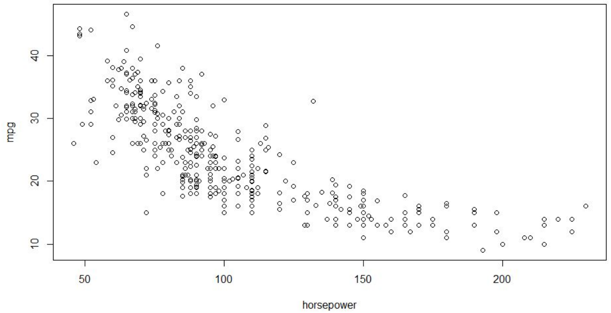
OK

This function is assumed to be linear.

$$y = \beta_0 + \beta_1 x_1 + \beta_1 x_2 + \ldots + \beta_{p-1} x_{p-1} + \epsilon$$

TROUBLESHOOTING

If it's not x, it might be 1/x or log(x) or x^2...



NOT OK

Assumptions // Linearity

Take the function component of your model

$$y = f(x) + \epsilon$$

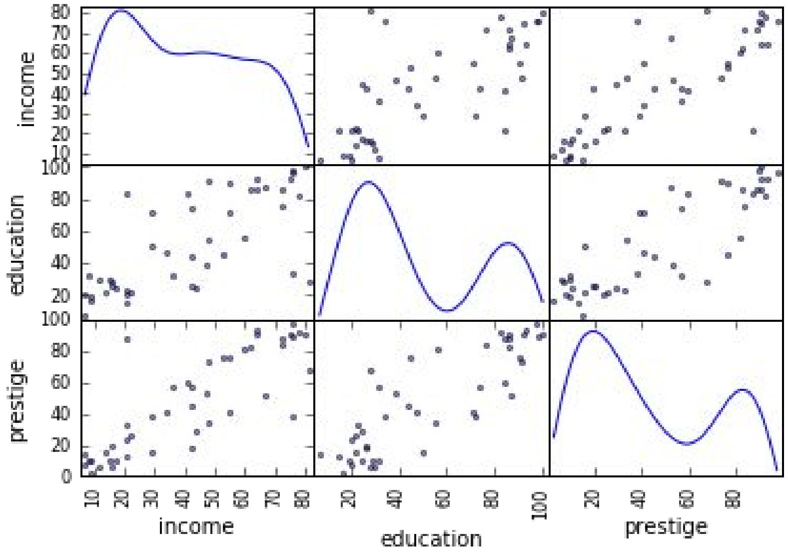
This function is assumed to be linear.

$$y = \beta_0 + \beta_1 x_1 + \beta_1 x_2 + \ldots + \beta_{p-1} x_{p-1} + \epsilon$$

TROUBLESHOOTING

If it's not x, it might be 1/x or log(x) or x^2…

Scatter plots




```
pd.scatter_matrix(df, diagonal = 'kde')
```

Assumptions // Linearity

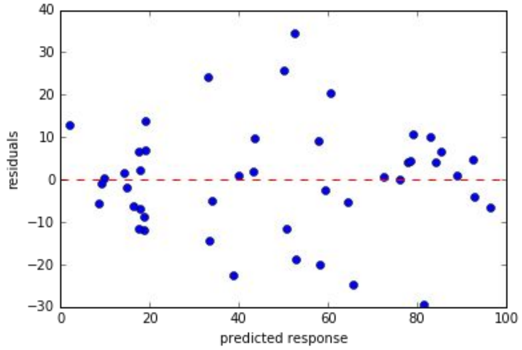
Take the function component of your model

$$y = f(x) + \epsilon$$

This function is assumed to be linear.

$$y = \beta_0 + \beta_1 x_1 + \beta_1 x_2 + \ldots + \beta_{p-1} x_{p-1} + \epsilon$$

$$e_i = y_i - X\hat{\beta}$$



TROUBLESHOOTING

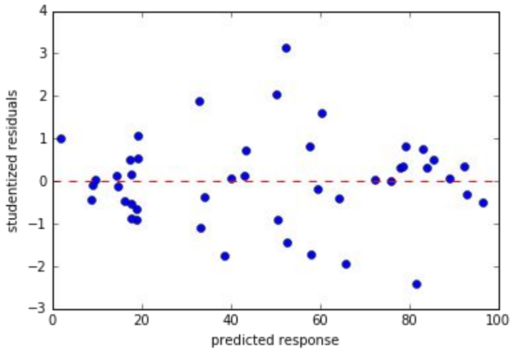
If it's not x, it might be 1/x or log(x) or x^2…

Scatter plots

Residual / Studentized residual plot

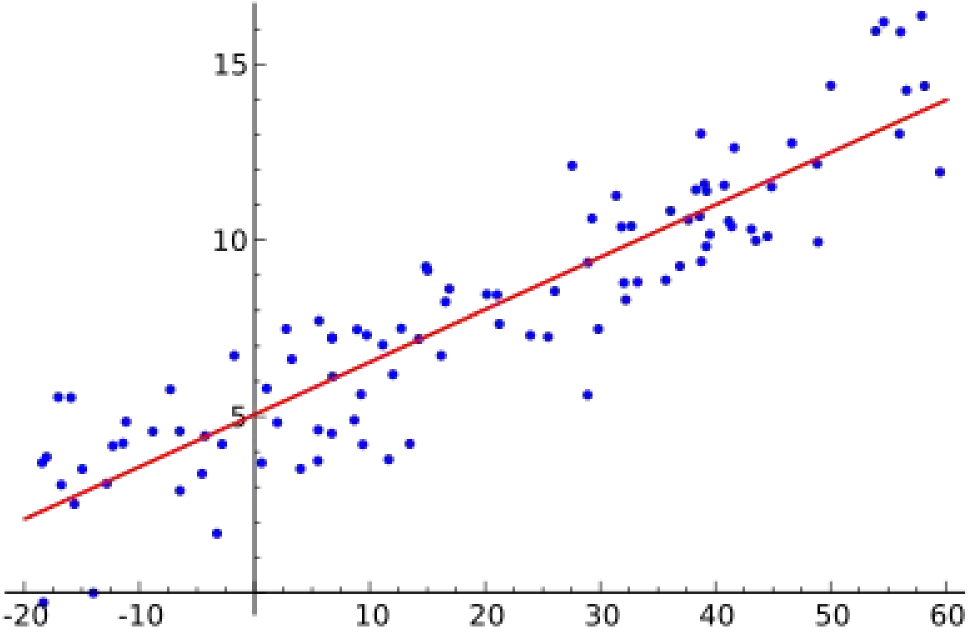
$$s_e^2 = \frac{1}{n-1} \sum e_i^2$$

$$r_i = \frac{e_i}{s_e^2}$$



Assumptions // Independence

Each observation is independent



Assumptions // Residuals are normally distributed

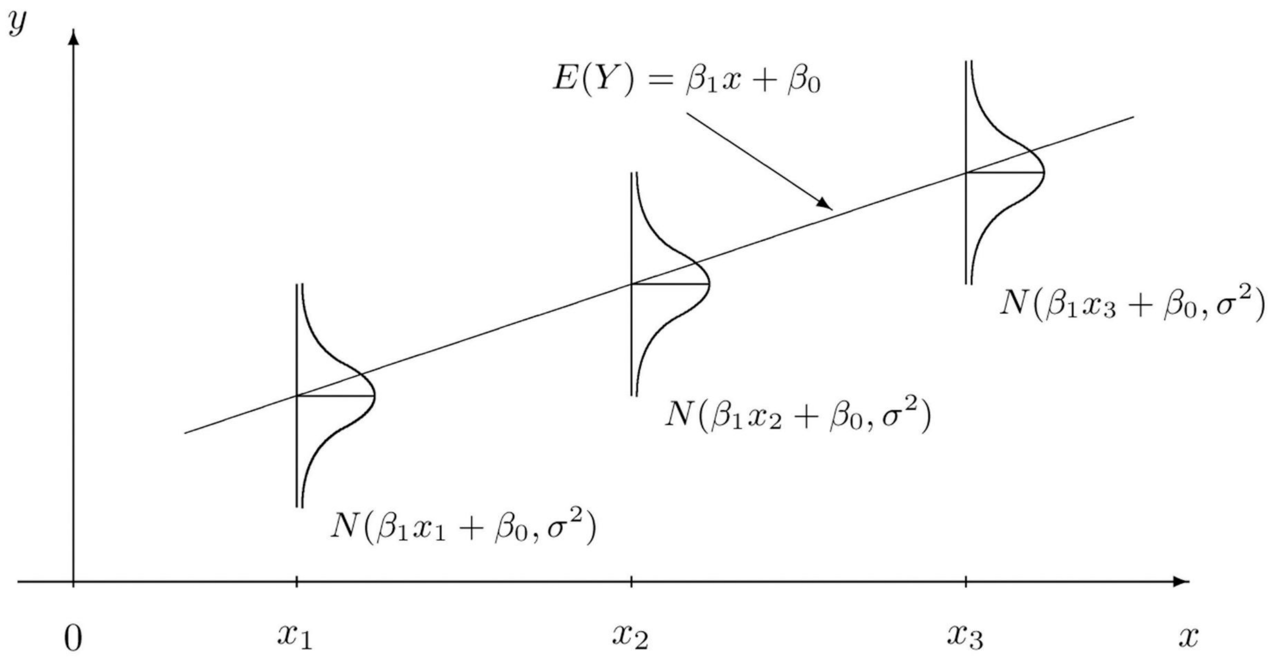


Figure from Tammy Lee's slides

Assumptions // Residuals are normally distributed

the quantile-quantile (q-q) plot

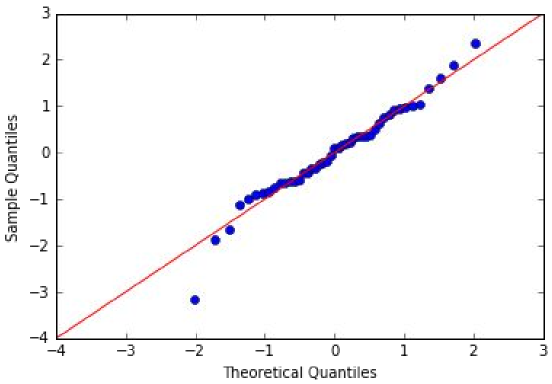
determining if two data sets come from populations with a common distribution.

plot of the quantiles of the first data set against the quantiles of the second data set.

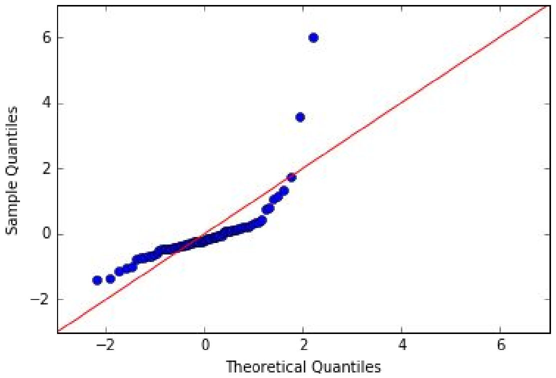
in our case, the second data set is based on the normal pdf

Qqplots should align on a 45-degree reference line

```
statsmodels.graphics.gofplots.qqplot(residuals
, dist=norm, line='45', fit=True)
```



OK



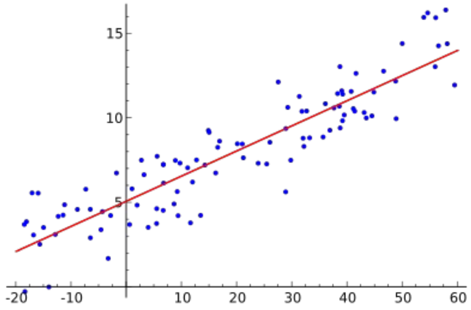
NOT OK

Assumptions // Homoscedasticity of residuals

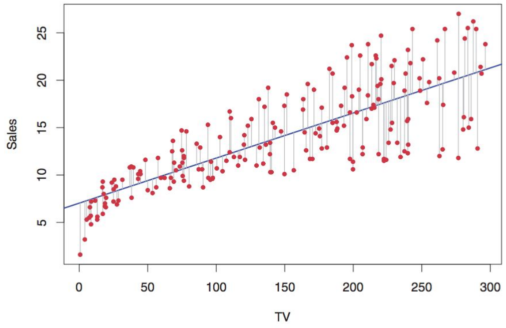
Same Variance, Constant Variance

$$y = X\hat{\beta} + \epsilon$$

the error term is the same
across all values of the independent variables



OK



NOT OK

TROUBLESHOOTING

Using the log of the response-y ?

Assumption // NO multicollinearity

We assume that there is

<u>no underlying linear relationship</u>

between independent variables

How to challenge that ?

Solution 1 : correlation matrix

	DJIA	S&P 500	Nasdaq	Canada	Mexico	Brazil	Stoxx 50	FTSE 100	CAC 40	DAX	IBEX	Italy	Netherlands	Sweden	Switzerland	Nikkei	Hang Seng	Australia
DJIA		0.97	0.85	0.57	0.56	0.52	0.52	0.48	0.51	0.56	0.49	0.50	0.50	0.42	0.42	0.09	0.11	0.07
S&P 500	0.97		0.91	0.62	0.58	0.55	0.50	0.47	0.50	0.55	0.48	0.50	0.49	0.41	0.41	0.09	0.11	0.05
Nasdaq	0.85	0.91		0.58	0.56	0.52	0.48	0.43	0.48	0.54	0.47	0.48	0.48	0.42	0.38	0.14	0.16	0.07
Canada	0.57	0.62	0.58		0.53	0.53	0.42	0.45	0.41	0.41	0.42	0.42	0.39	0.37	0.35	0.17	0.22	0.17
Mexico	0.56	0.58	0.56	0.53		0.56	0.42	0.42	0.44	0.43	0.43	0.44	0.39	0.38	0.38	0.17	0.25	0.17
Brazil	0.52	0.55	0.52	0.53	0.56		0.33	0.35	0.32	0.34	0.34	0.34	0.29	0.30	0.28	0.17	0.22	0.15
Stoxx 50	0.52	0.50	0.48	0.42	0.42	0.33		0.92	0.94	0.89	0.87	0.88	0.92	0.78	0.86	0.26	0.30	0.24
FTSE 100	0.48	0.47	0.43	0.45	0.42	0.35	0.92		0.86	0.80	0.80	0.82	0.84	0.73	0.78	0.26	0.30	0.26
CAC 40	0.51	0.50	0.48	0.41	0.44	0.32	0.94	0.86		0.89	0.88	0.89	0.92	0.78	0.84	0.28	0.32	0.25
DAX	0.56	0.55	0.54	0.41	0.43	0.34	0.89	0.80	0.89		0.83	0.84	0.86	0.75	0.77	0.26	0.29	0.21
IBEX	0.49	0.48	0.47	0.42	0.43	0.34	0.87	0.80	0.88	0.83		0.84	0.83	0.75	0.77	0.27	0.32	0.26
Italy	0.50	0.50	0.48	0.42	0.44	0.34	0.88	0.82	0.89	0.84	0.84		0.85	0.74	0.78	0.24	0.29	0.23
Netherlands	0.50	0.49	0.48	0.39	0.39	0.29	0.92	0.84	0.92	0.86	0.83	0.85		0.75	0.82	0.27	0.30	0.23
Sweden	0.42	0.41	0.42	0.37	0.38	0.30	0.78	0.73	0.78	0.75	0.75	0.74	0.75		0.75	0.29	0.33	0.27
Switzerland	0.42	0.41	0.38	0.35	0.38	0.28	0.86	0.78	0.84	0.77	0.77	0.78	0.82	0.75		0.29	0.32	0.29
Nikkei	0.09	0.09	0.14	0.17	0.17	0.17	0.26	0.26	0.28	0.26	0.27	0.24	0.27	0.29	0.29		0.52	0.49
Hang Seng	0.11	0.11	0.16	0.22	0.25	0.22	0.30	0.30	0.32	0.29	0.32	0.29	0.30	0.33	0.32	0.52		0.48
Australia	0.07	0.05	0.07	0.17	0.17	0.15	0.24	0.26	0.25	0.21	0.26	0.23	0.23	0.27	0.29	0.49	0.48	

Assumption // NO multicollinearity

We assume that there is

no underlying linear relationship

between independent variables

How to challenge that ?

Solution 2 : ???

We have an algorithm that and its name is…

Linear Regression !

$$X_1 = \alpha_2 X_2 + \alpha_3 X_3 + \cdots + \alpha_k X_k + c_0 + e$$

$$\mathrm{VIF} = \frac{1}{1 - R_i^2}$$

Rule of thumb, if VIF > 10, problem !

```
from statsmodels.stats.outliers_influence import variance_inflation_factor
variance_inflation_factor(x.values, index)
```

Outliers

What is an outlier ?

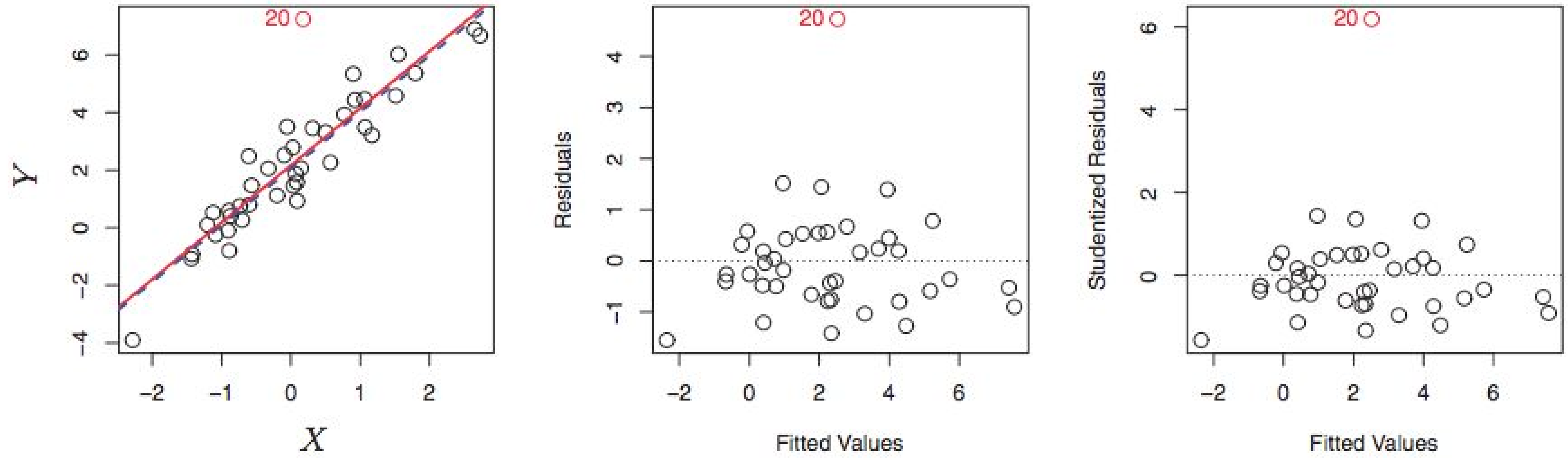
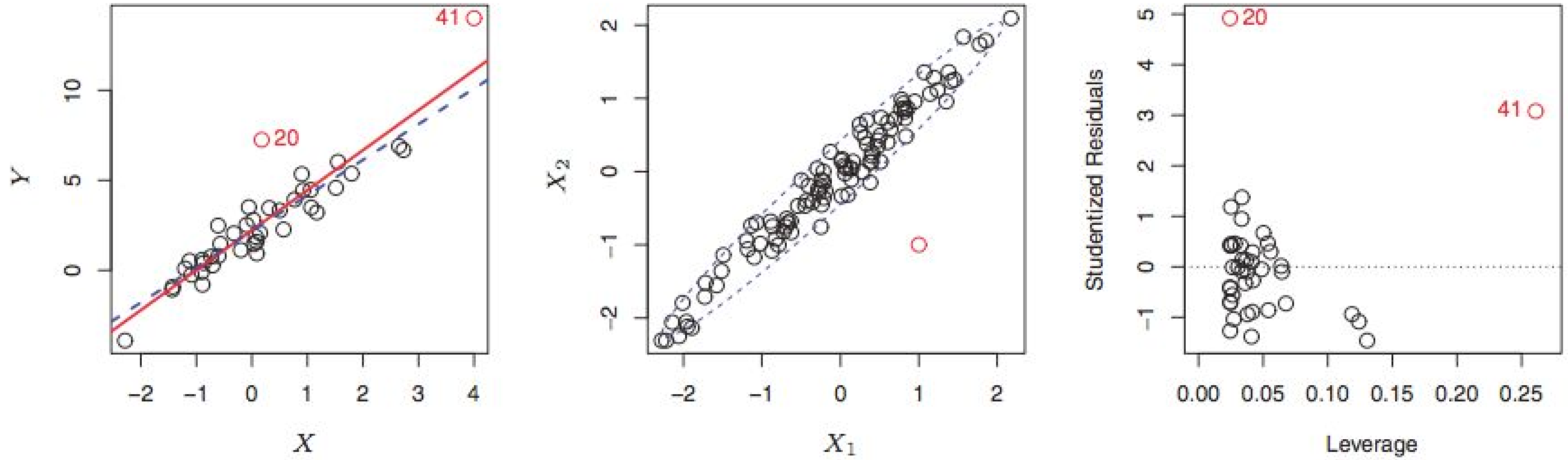


Figure from [ISL]

Leverage of data points



To measure influence of an outlier:

- Compute the 'hat matrix': $H = X^T(X^TX)^{-1}X$
- i-th element on the diagonal, $h_{ii} \equiv (H)_{ii}$, is i-th feature's 'leverage'
- An observation with a large residual may not have a lot of influence

Figure from [ISL], slide B. Skrainka

Linear Regression

DSI SEA5, jf.omhover, Sep 21, 2016

STANDARDS

- **Describe**, **interpret**, and **visualize** the model form of linear regression: Y = B0+B1X1+B2X2+....
- **Relate** Beta vector solution of Ordinary Least Squares to the cost function (residual sum of squares)
- **State** and troubleshoot the assumptions of linear regression model
- **Perform** OLS with statsmodels and interpret the output: Beta coefficients, p-values, R^2
- How can one **detect** outliers?

Useful snippets (soon on slack)

Ordinary Least Squares fit

```
import statsmodels

y = prestige['prestige']
x = prestige[['income', 'education']].astype(float)
x['const'] = 1

prestige_model = statsmodels.api.OLS(endog=y, exog=x).fit()
prestige_model.summary()
```

OLS Regression Results

Dep. Variable:	prestige	**R-squared:**	0.828
Model:	OLS	**Adj. R-squared:**	0.820
Method:	Least Squares	**F-statistic:**	101.2
Date:	Tue, 20 Sep 2016	**Prob (F-statistic):**	8.65e-17
Time:	15:00:41	**Log-Likelihood:**	-178.98
No. Observations:	45	**AIC:**	364.0
Df Residuals:	42	**BIC:**	369.4
Df Model:	2		
Covariance Type:	nonrobust		

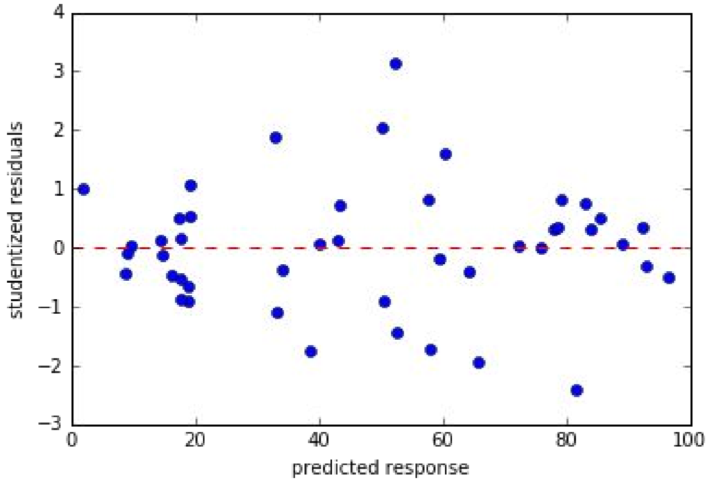
	coef	std err	t	P>|t|	[95.0% Conf. Int.]
income	0.5987	0.120	5.003	0.000	0.357 0.840
education	0.5458	0.098	5.555	0.000	0.348 0.744
const	-6.0647	4.272	-1.420	0.163	-14.686 2.556

Omnibus:	1.279	**Durbin-Watson:**	1.458
Prob(Omnibus):	0.528	**Jarque-Bera (JB):**	0.520
Skew:	0.155	**Prob(JB):**	0.771
Kurtosis:	3.426	**Cond. No.**	163.

Studentized residual plot

```
stdresids = prestige_model.outlier_test()['student_resid']

plt.plot(prestige_model.fittedvalues, resids, 'o')
plt.xlabel('predicted response')
plt.ylabel('studentized residuals')
plt.axhline(0, c='r', linestyle = '--')
```



QQPlot

```
statsmodels.graphics.gofplots.qqplot(residuals, dist=norm, line='45', fit=True)
```