Bayesian A/B Testing

Conjugate Priors

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2016

Overview

A/B Testing

- Frequentist - Review
- Bayesian - Overview

Bayesian A/B Testing

- Bayes' Theorem
- Likelihood
- Posterior
- Conjugate Prior

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Frequentist - Hypothesis Testing

Define a Metric	Declare null and alternative hypothesis.
Set Parameters	Significance, number of observations, etc.
Run Experiment	Make sure you follow it to a tee.
Compute Test Statistic	Make sure it's the approprite one.
Calculate P-Value	
Draw Conclusions	Reject H_0 in favor of H_A, or fail to reject.

Frequentist A/B Testing - Limitations?

- If one of the pages your testing appears to be obviously better, can you scrap the experiment and declare it the winner?
- At the end of an experiments, can you quantify how much better the winning pages is than the loosing page?

Example

"It's 95% likely that page A is better than page B."

- If, after you've begin your test, your boss comes to you with another version of the page and asks you to test it it too, can you update the test?

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Bayesian - Overview

Define a Metric	To quantify what "better" means.
Run Test	Collect data.
Continually Monitor Results	Can decide to stop at any time.
Suggest Next Step	Based on probabilities calculated.

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- **Bayes' Theorem**
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$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

$$P(B) = \sum_i P(B|E_i)P(E_i)$$

$$\sum E_i = A$$

$$P(B) = P(B \cap E)$$
$$= P(B \cap (E_1 \cup E_2 \cup \dots E_n))$$
$$= P((B \cap E_1) \cup (B \cup E_2) \dots (B \cup E_n))$$

$$\underbrace{P(B\cap E_1)}_{P(B|E_1)P(E_1)} + \underbrace{P(B\cap E_2)}_{P(B|E_2)P(E_2)} \dots + \underbrace{P(B\cap E_n)}_{P(B|E_n)P(E_n)}$$

Bayes' Theorem

updated belief

new data

parameter

$$P(\theta|y) = \frac{P(y|\theta)\times P(\theta)}{P(y)}$$

current belief of θ

$P(y|\theta)$ Likelihood

$P(\theta)$ Prior

$P(y)$ Normalizing Constant

Remember: $P(y) = \int P(y,\theta)d\theta$

This equation captures the essence of Bayesian modeling: that the uncertainty about some unknown parameter can be quantified.

Bayes' Theorem Distilled

All that mathiness can be boiled down to the straightforward idea that:

$$\textit{Posterior} \propto \textit{Likelihood} \times \textit{Prior}$$

$$P(B|A) \propto \frac{P(A|B)}{\cancel{P(A)}} P(B)$$

proportional

Back to A/B Testing

How, then, do we apply this supposedly straightforward idea of Bayes' Theorem to our statistics based question of A/B testing?

Let's first consider what we are really trying to do in the process of A/B testing.

Trying to determine which of options A & B is "better".

For example, trying to determine which page layout has a higher click-through rate.

Moving Towards Bayes

Considering that distilled version of Bayes' Theorem we saw earlier:

next prior

$$Posterior \propto Likelihood \times Prior$$

Let's attach some tangible ideas to these abstract notions of a "likelihood" and a "posterior" for the click-through rate example. For the next two slides let's consider the data associated with each of our pages separately.

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Likelihood

What are the possible outcomes for each observation in our data for a single page?

A click-through occurs or not, binary result.

How, then, are our data distributed; aka, what is the **likelihood**?

X $P(X=x) = p^x (1-p)^{n-x} \binom{n}{x}$

n # of trials

x # success

Likelihood

What are the possible outcomes for each observation in our data for a single page?

A click-through occurs or not, binary result.

How, then, are our data distributed; aka, what is the **likelihood**?

Binomial

$$P(X = k) = \binom{n}{k} p^k \times (1 - p)^{n-k}$$

$\rightarrow$ k successes in n trials.

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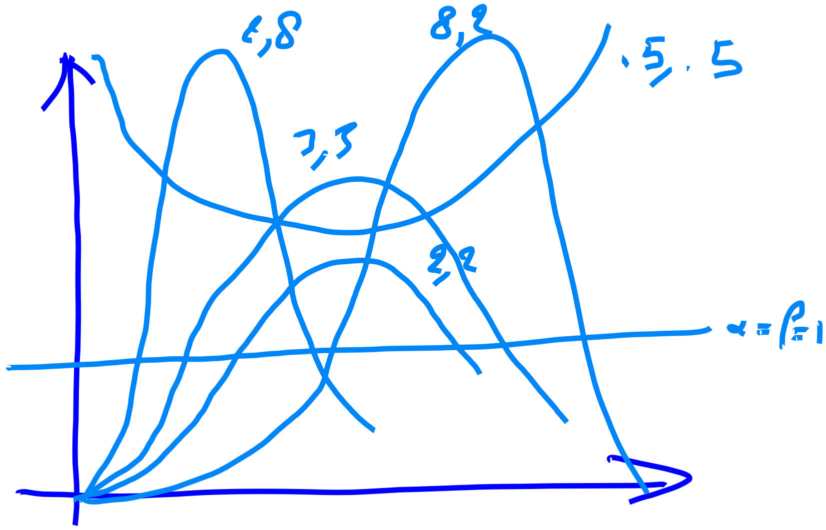
Posterior

Again, with respect to the data from a single page, what are the possible values that a click-through rate can take on?

Any real value in the open interval $[0, 1]$.

What distribution, then, should we use to model this click-through rate parameter; aka, what is the **posterior**?

$$X \sim \underbrace{Be(\alpha, \beta)}_{\text{shape parameters}} = \frac{1}{B(\alpha, \beta)} x^{\alpha - 1} (1 - x)^{\beta - 1}$$



$$X \sim B(\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$\alpha=\beta=1$ $X \sim \frac{1}{B(1,1)}$ uniform

distribution of prior	distribution of likelihood
Beta	Bernoulli/Binomial
Normal	Normal (σ know)
Gama	Poisson

Posterior

Again, with respect to the data from a single page, what are the possible values that a click-through rate can take on?

Any real value in the open interval $[0, 1]$.

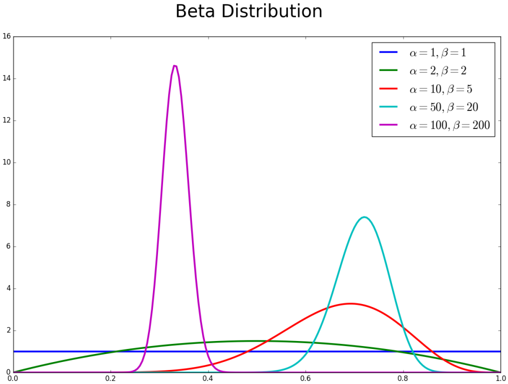
What distribution, then, should we use to model this click-through rate parameter; aka, what is the **posterior**?

Beta! Has support over the continuous interval $[0, 1]$.

Beta

$$X \sim Be(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Beta Distribution



$$E[X] = \frac{\alpha}{\alpha+\beta} \qquad Mode = \frac{\alpha-1}{\alpha+\beta-2}$$

Piecing it Together

Be(1,1) prior
run exp
Be(α_1, β_1) post prior
run exp
Be(α_n, β_n) post

Recalling the form of Bayes' Theorem:

$$\textit{Posterior} \propto \textit{Likelihood} \times \textit{Prior};$$

we now know what the likelihood and the posterior look like:

Likelihood $\propto p^k \times (1-p)^{n-k}$

Prior ~~Posterior~~ $\propto p^{\alpha-1}(1-p)^{\beta-1}$ Be(α, β)

What do you notice about these two distributions?

Posterior $p^{k+\alpha-1}(1-p)^{n-k+\beta-1}$

Be($k+\alpha$, $n-k+\beta$)

Piecing it Together

Recalling the form of Bayes' Theorem:

$$\textit{Posterior} \propto \textit{Likelihood} \times \textit{Prior};$$

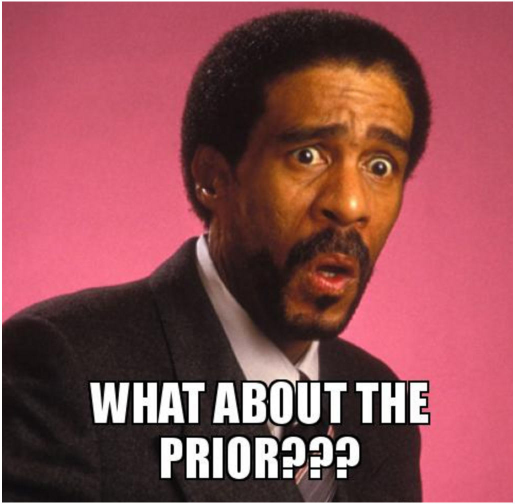
we now know what the likelihood and the posterior look like:

Likelihood	$\binom{n}{k} p^k \times (1-p)^{n-k}$
Posterior	$\frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$

What do you notice about these two distributions?

They have the same form!

Interlude



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Priors

Before we reason about what our prior, $P(\theta)$, should look like, let's discuss what a prior actually is.

Prior

The prior is a distribution that encodes our beliefs about the possible values that the parameter in question, θ, can take on before we collect any data.

Choosing a Prior

How do we choose our prior, then, since technically, the only requirement is that it is a distribution, aka $\int P(\theta) = 1$?

More specific to our current situation though, me might ask the question, what distribution would we like to use to encode our prior beliefs about θ?

To answer this question let's consider the functional form of the likelihood and posterior that we've visited already:

$$\begin{aligned} \mathit{Posterior} &\propto \mathit{Likelihood} \times \mathit{Prior} \\ \mathit{Beta} &\propto \mathit{Binomial} \times \mathit{Prior} \\ p^{\alpha-1}(1-p)^{\beta-1} &\propto p^k(1-p)^{n-k} \times \mathit{Prior} \end{aligned}$$

Conjugate Prior

In this case, the obvious distribution we should choose to encode our prior beliefs about $P(\theta)$ is the **beta**. Let's look at why this makes sense.

$$\textit{Posterior} \propto \textit{Likelihood} \times \textit{Prior}$$

$$\textit{Posterior} \propto \textit{Binomial} \times \textit{Beta}$$

$$\textit{Posterior} \propto p^k(1-p)^{n-k} \times p^{\alpha-1}(1-p)^{\beta-1}$$

$$\textit{Posterior} \propto p^{k+\alpha-1}(1-p)^{n-k+\beta-1}$$

Beta Conjugate Prior

This is the beta distribution! Just as we wanted our posterior to be.

$$p^{k+\alpha-1}(1-p)^{n-k+\beta-1}$$

What are the parameters of this distribution?

$$\alpha_1 = k + \alpha \qquad\qquad \beta_1 = n - k + \beta$$

A.k.a.

$$\textit{Posterior} \sim Be(k+\alpha, n-k+\beta)$$

Bayesian A/B Testing in Practice

$$\textcolor{red}{Posterior} \sim Be(k + \alpha, n - k + \beta)$$

How do we use this new-found tool?

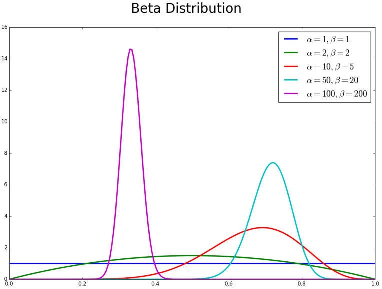
All we have to do is take the number of conversions, k, and the total number of views, n, for each of our pages and plug them into our posterior distribution.

What about α and β? What are the values of those parameters?

Back to the Prior

Those values α and β are the parameters for the beta distribution that we cleverly chose to describe our prior beliefs.

The values of α and β literally encode our prior beliefs about the values that θ could possibly take on. With this in mind, there are a couple of ways that we can choose these values:



- Choose an uninformative prior $\rightarrow \alpha = 1$ and $\beta = 1$.
- Act like you have observed some data before that represent data your experiment has to overcome.

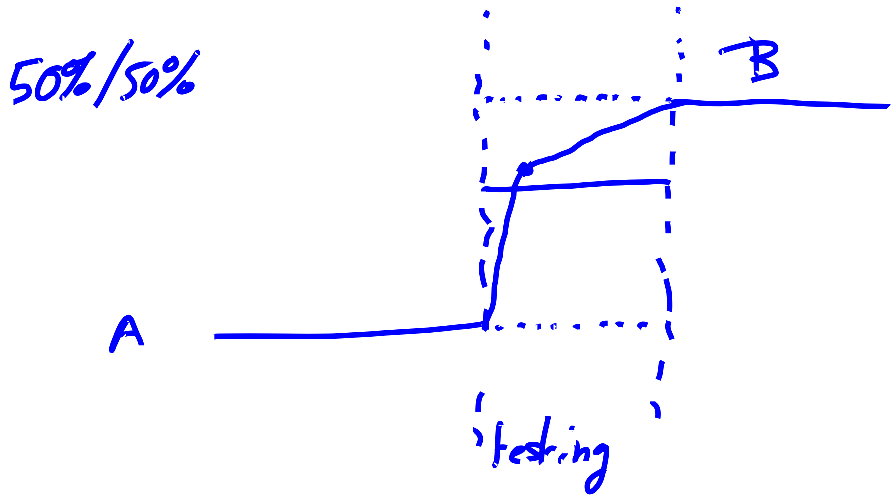
Bayesian A/B Testing in Code

site a prior posterior $Be(\alpha, \beta) \longrightarrow Be(k+\alpha, n-k+\beta)$

How do we actually perform a test with code, then? By simulation.

```
from numpy.random import beta

num_samples = 10000
alpha = beta = 1
site_a_simulation = beta(num_conv_a + alpha,
                         num_views_a - num_conv_a + beta,
                         size=num_samples)
site_b_simulation = beta(num_conv_b + alpha,
                         num_views_b - num_conv_b + beta,
                         size=num_samples)
```


50%/50%
A
B
testing

Bayesian A/B Testing in Code Cont.

What's the probability that site A has a higher conversion rate than site B?

```
np.mean(site_a_simulation > site_b_simulation)
```

What's the probability that site A has a 5% higher conversion rate than site B?

```
np.mean(site_a_simulation > (site_b_simulation + 0.05))
```

Visualizing the Simulation