Bayesian Hypothesis Testing

Natalie Hunt

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Objectives: answer the following:

- What is a prior, posterior, and likelihood?
- How do we apply Bayesian updating to A/B testing?
- What does the Beta distribution represent?
- What are some key differences between Bayesian and Frequentist Hypothesis Testing?

Review: frequentist p-values

- Remember the one-sentence definition of a p-value?

Review: frequentist p-values

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"The probability of observing data at least as extreme as the observation given the null hypothesis"

$$P(\text{data} \mid \text{null distribution})$$

$$P(y \mid \theta_0)$$

Wouldn't it be nice if, instead, we could give a probability of a ***parameter*** given the ***data***?

Wouldn’t it be nice...

Posterior Probability

Likelihood of Observations

Prior Probability

$$\Pr(\theta|y) = \frac{\Pr(y|\theta)\Pr(\theta)}{\Pr(y)}$$

Normalizing Constant

Review: Bayesian Inference

$$\overbrace{\Pr(\theta|y)}^{\text{Posterior Probability}} = \frac{\overbrace{\Pr(y|\theta)}^{\text{Likelihood of Observations}}\,\overbrace{\Pr(\theta)}^{\text{Prior Probability}}}{\underbrace{\Pr(y)}_{\text{Normalizing Constant}}}$$

- Coin example
  - ***y*** is a set of flips (heads or tails)
  - ***θ*** is the coin's probability of coming up heads for a single flip

Posteriors from yesterday's coin

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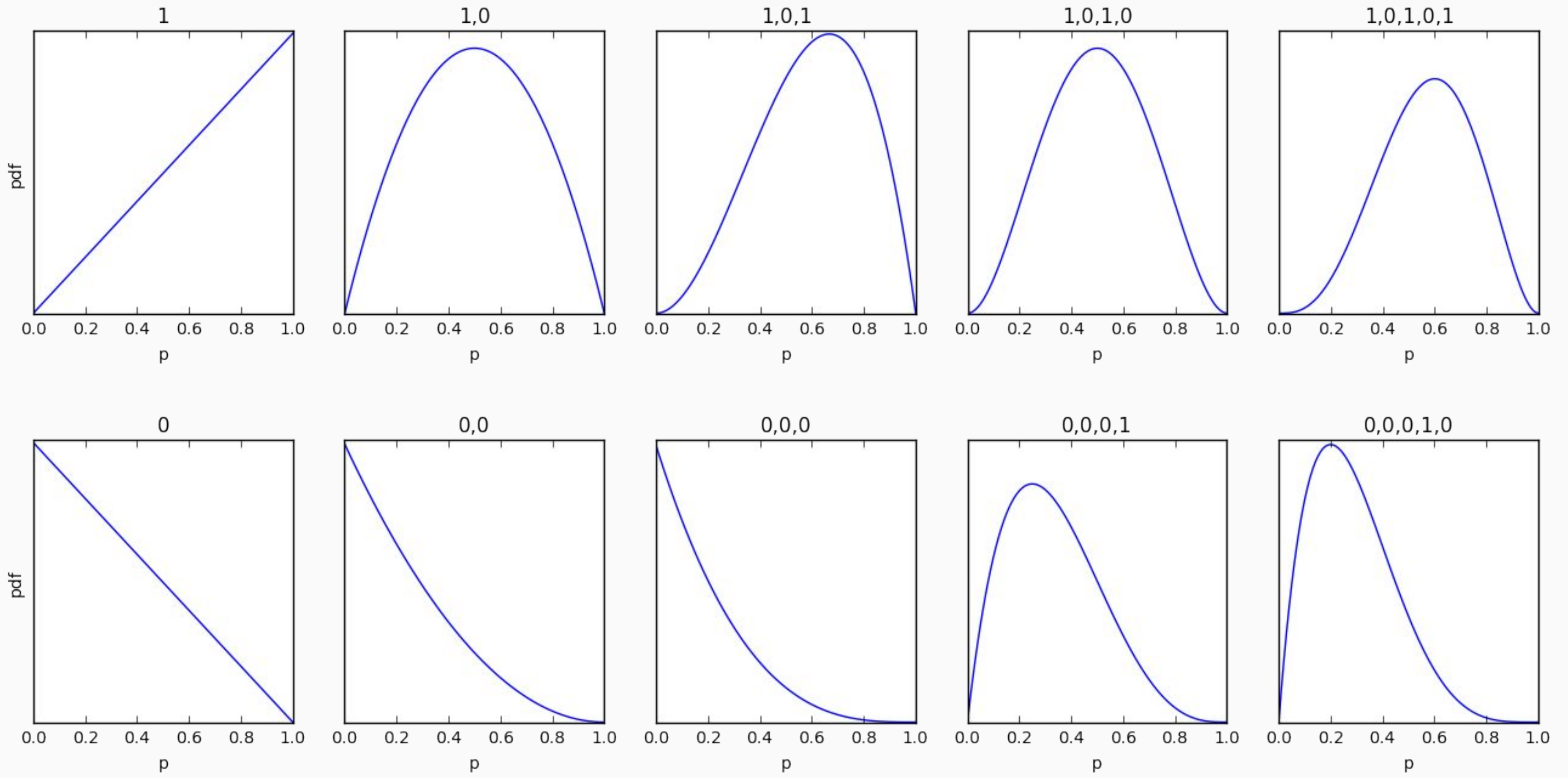

Review: Bayesian Inference

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- Click-through rate
  - ***y*** is a set of visits by unique users to a website, each of which either resulted in a click or not
  - ***θ*** is the probability of a click for a single visit
  - **Let's work with this example for the rest of the day**

Bayesian Inference: Distributions

$$Posterior \propto Likelihood \times Prior$$

- We're going to model each of these terms with an appropriate ***distribution***
- We'll see that it makes Bayesian updating easy and fun!
- Our goal is to find an analytical form for the ***posterior probability distribution*** over all the possible values of the ***true click-through rate*** p

Likelihood function

$$likelihood = P(y \mid p)$$

- ***y*** here represents a whole data set: “***n*** visits with ***k*** clicks”
- ***p*** is the probability of a click for a single visitor

What is the form of the likelihood function?

Likelihood function

$$likelihood = P(y \mid p)$$

- ***y*** here represents a whole data set: “***n*** visits with ***k*** clicks”

Binomial distribution

$$P(k \mid p; n) = \binom{n}{k} p^k (1-p)^{n-k}$$

Bayesian Inference

$$Posterior \propto Likelihood \times Prior$$

Binomial

Prior distribution

$$prior = P(p)$$

- We want to pick a distribution for ***p***, so it must be defined over [0,1]
- Hmm...

Prior distribution

$$prior = P(p)$$

- We want to pick a distribution for ***p***, so it must be defined over [0,1]
- Let's look at that binomial distribution again:

$$Binomial(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

- Can we make a distribution over ***p*** that has this same form?

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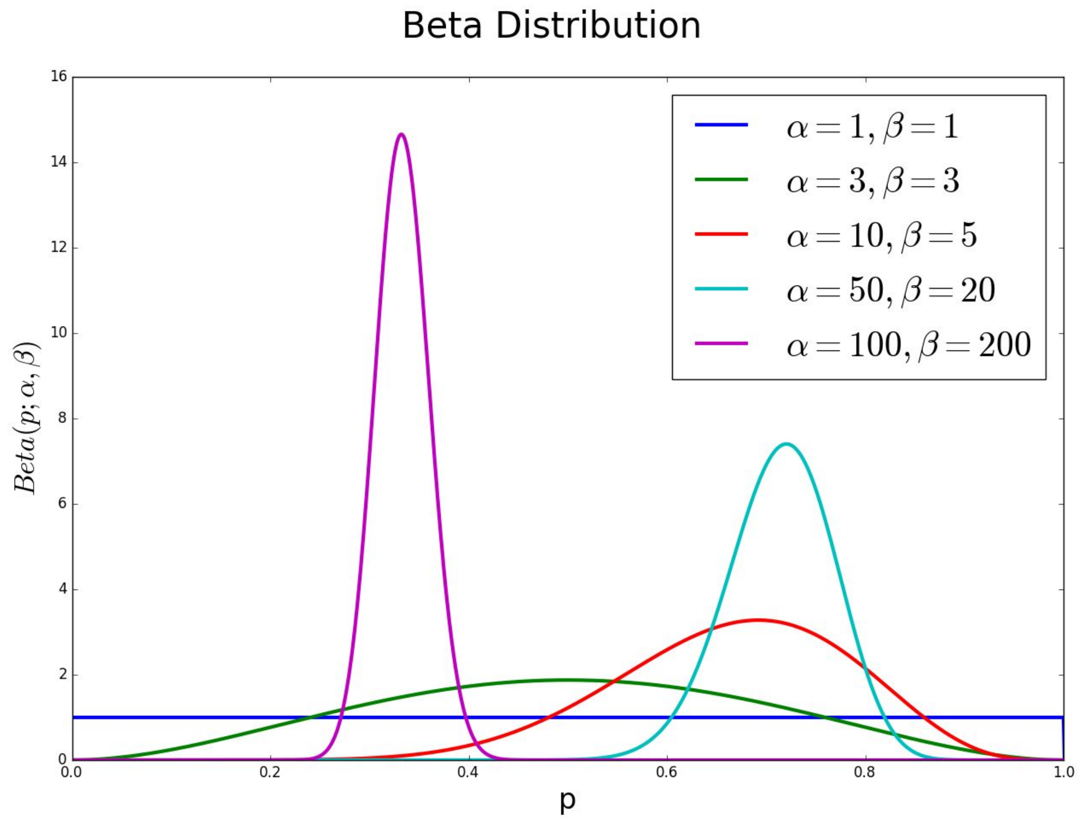
- Oh someone already made this one: the Beta distribution

$$Beta(p; \alpha, \beta) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}$$

Beta distribution



$$\mathrm{E}[p] = \frac{\alpha}{\alpha + \beta}$$

$$\mathrm{Mode} = \frac{\alpha - 1}{\alpha + \beta - 1}$$

- Our ***prior distribution*** is set by our choice of ***α*** and ***β***
- ***α=β=1*** is the uniform distribution

Bayesian Inference

$$Posterior \propto Likelihood \times Prior$$

	Binomial	Beta

Posterior

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$$posterior = P(p \mid y) = P(p \mid n, k)$$

$$posterior \sim \binom{n}{k} p^k (1-p)^{n-k} \times \frac{p^{\alpha-1}\,(1-p)^{\beta-1}}{B(\alpha,\beta)}$$

$$posterior \sim p^k (1-p)^{n-k} \times p^{\alpha-1}\,(1-p)^{\beta-1}$$

$$posterior \sim p^{\alpha+k-1}\,(1-p)^{\beta+n-k-1}$$

$$posterior = Beta(p; \alpha+k, \beta+n-k)$$

The posterior is a beta distribution with parameters ***α+k*** and ***β+n-k***

This means we can do all our Bayesian updates at once, instead of updating with one data point at a time!

Bayesian Inference

$$Posterior \propto Likelihood \times Prior$$

$\boxed{\text{Beta}}$ $\boxed{\text{Binomial}}$ $\boxed{\text{Beta}}$

Conjugate priors

$$Posterior \propto Likelihood \times Prior$$

Beta

Binomial

Beta

- ***Conjugate priors*** are pairs of distribution families for (likelihood, prior) such that the ***posterior*** belongs to the same parametric family as the ***prior***

Likelihood	Prior	Posterior
Normal	Normal	Normal
Poisson	Gamma	Gamma
Gamma	Gamma	Gamma
Binomial	Beta	Beta
Multinomial	Dirichlet	Dirichlet
Normal	Gamma	Gamma

In summary: ta-da! we have an easy Posterior

- If you start with the uniform distribution as a prior (which is the beta distribution with ***α=β=1***) then our posterior is a beta distribution with parameters

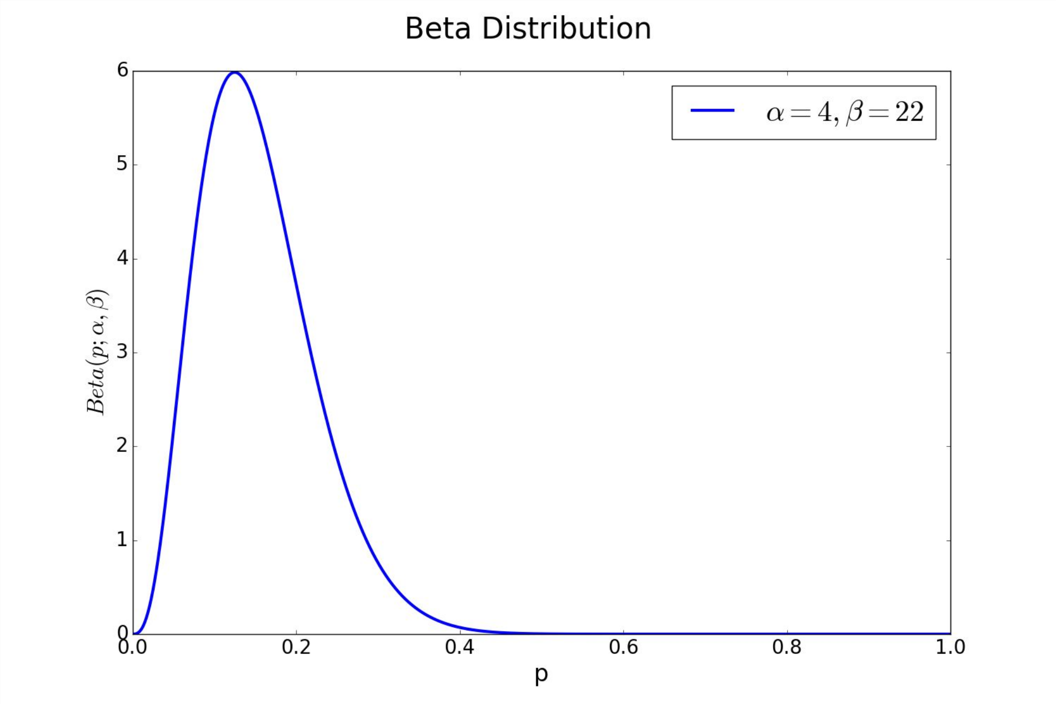
$$\alpha = 1 + k = 1 + (\#\text{ of successes})$$

$$\beta = 1 + n - k = 1 + (\#\text{ of failures})$$

$$Posterior = P(p \mid n, k) = Beta(p; \alpha, \beta) = \frac{p^{\alpha-1}\,(1-p)^{\beta-1}}{B(\alpha, \beta)}$$

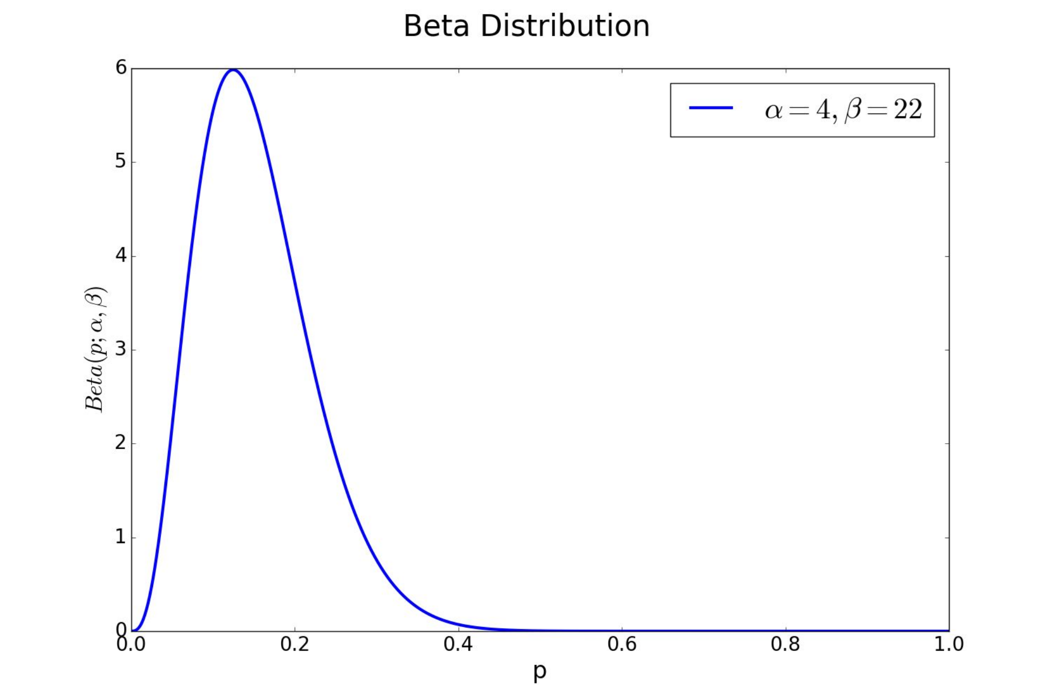
Example

- For example, if you had 24 trials with 3 successes, you'd have this distribution



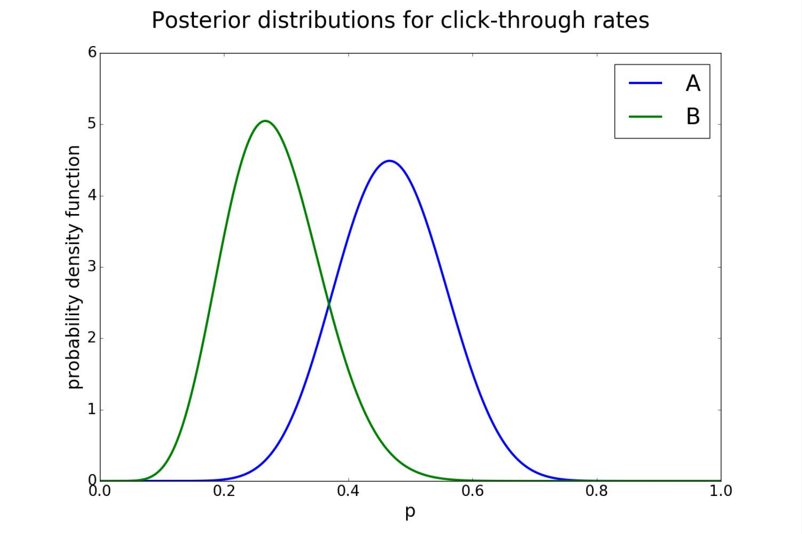
Statements you can make with this distribution

- "The probability that the true CTR is less than 0.15 is 53%"
- "There is a 95% probability that the true CTR lies between 0.045 and 0.312"
  - that's a **credible interval**



Bayesian A/B Testing

- Randomly send users to two versions of our site (A and B)
- Calculate/update the posterior distributions for each click through rate, **p_A** and **p_B**
- Say we end up with the two beta distributions on the right. How would you get the probability that **p_A** is greater than **p_B**?



Bayesian A/B Testing

- We **sample** from each distribution and see how often p_A is greater than p_B

```
# let's draw values from those distribution models
sample_size = 10000

# model for A, fed with the right values
A_sample = stats.beta.rvs(1 + clicks_A,
                          1 + views_A - clicks_A,
                          size=sample_size)

# model for B, fed with the right values
B_sample = stats.beta.rvs(1 + clicks_B,
                          1 + views_B - clicks_B,
                          size=sample_size)

# let's find out the probability that A is better than B
print np.mean(A_sample > B_sample)

# we can also find the probability that p_A is larger than p_B by 0.05
print np.mean(A_sample > (B_sample + 0.05))
```

Frequentist A/B Testing

- Define a metric (e.g., click through rate), null & alternative hypotheses
- Set the study parameters (significance level, power, number of observations)
- Run the test, **wait until it is done**, then analyze results
- Report p-value, confidence interval
- Reject or fail to reject the null hypothesis

Bayesian A/B Testing

- Define a metric (e.g., click through rate)
- Define a prior distribution of the metric
- Run the test, **continually monitoring results**
- **At any time** calculate the probability that CTR_A > CTR_B