Hypothesis Testing

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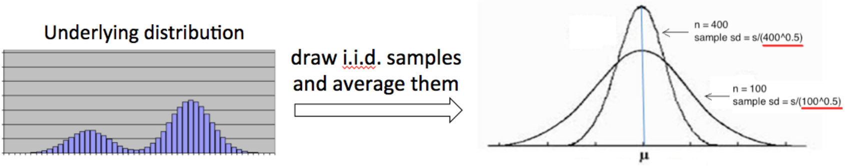
Galvanize

Morning Lecture Objectives

- The general steps of a statistical hypothesis test
- One-tail vs. Two-tailed tests
- Type-I and Type-II Error
- One-sample and Two-sample tests of mean
- One-sample and Two-sample tests of proportion

Central Limit Theorem

The CLT states that given certain conditions, the mean of a sufficiently large number of *i.i.d* random variables will be approximately normal, regardless of the underlying distribution



Central Limit Theorem

- Not only is the sample mean normally distributed, but the variance of the sample mean is smaller

$$\bar{X} \sim Normal\left(\mu, \frac{\sigma^2}{n}\right)$$

- As with any normal variable, we can derive a standard normal Z-score

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Hypothesis Testing

- Estimation
  - Parameter value is unknown
  - Goal is to find a point estimate and a confidence interval for likely values
- Hypothesis Testing
  - Parameter value is stated
  - Goal is to see if the parameter has changed

General Steps of Hypothesis Testing

1. State the null hypothesis (H_0) and alternative hypothesis (H_A)
2. Choose the significance level, α
3. Compute the appropriate test statistic
4. Compute the p-value under the assumption that H_0 is true
   - If p-value $\leq \alpha \longrightarrow$ Reject H_0 in favor of H_A
   - if p-value $> \alpha \longrightarrow$ Fail to Reject H_0

Null Hypothesis vs. Alternative Hypothesis

- Null Hypothesis (H_0)
  - Typically a measure of the status quo such as no effect
  - In terms of the parameter: $H_0 : \mu = 0$
- Alternative Hypothesis (H_A)
  - Usually states the effect the researcher hopes to detect
  - Example: Advertising causes 1% lift
  - In terms of the parameter: $H_A : \mu \neq 0$

Two-sided vs One-sided tests

- By default, we should compute a two-sided test which is more conservative:
  - For example: $H_0 : \mu = \mu_0$ vs $H_A : \mu \neq \mu_0$
  - Reject if test statistic in upper or lower tail
  - One half of p-value in each tail
- However, if we expect the effect to be in a specific direction, we can use a one-sided test:
  - Example: $H_0 : \mu \leq \mu_0$ vs $H_A : \mu > \mu_0$
  - Reject H_0 if test statistic in tail designated by H_A
  - P-value calculated based on direction specified in H_A

Two-sided vs One-sided tests

Direction	H_0	H_A	P-value
2-sided Test	$=$	$\neq$	One half of P-value in each tail
Left-Tail	$\geq$	$<$	All of P-value in left tail
Right-Tail	$\leq$	$>$	All of P-value in right tail

Type I and Type II Errors

- Type I error: Rejecting H_0 when it is true
- Type II error: Failing to reject H_0 when it is false

	H_0 is True	H_0 is False
Fail to Reject H_0	Correct Decision $(1-\alpha)$	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision $(1-\beta)$

Type I Error is also the Level of Significance

One Sample Test of Population Mean

Z-test used when σ^2 is known or $n \geq 30$

1. Default choice should be a two-sided test: $H_0 : \mu = \mu_0$ vs. $H_A : \mu \neq \mu_0$
2. Level of significance, α
3. Calculating standardized test statistic:

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu_0, \sigma^2) \xrightarrow{CLT} \bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

$$z_{ts} = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

4. P-value for a 2-sided test:

$$\text{P-value} = 2P(Z > |z_{ts}|)$$

One Sample Test of Population Mean

T-test used when σ^2 is unknown and $n < 30$

1. Example of a right-tail test: $H_0 : \mu \leq \mu_0$ vs. $H_A : \mu > \mu_0$
2. Level of significance, α
3. Calculating standardized test statistic:

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu_0, \sigma^2) \xrightarrow{CLT} \bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

$$t_{ts} = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} \sim t_{df} \text{ where } df = n - 1$$

4. P-value for a right-sided test:

$$\text{P-value} = P(T > t_{ts})$$

One Sample Test of Population Proportion

1. Example of a left-sided test: $H_0 : p \geq p_0$ vs. $H_A : p < p_0$
2. Level of significance, α
3. Calculating standardized test statistic:

$$X \sim Bin(n, p_0) \xrightarrow{CLT} \hat{p} = \frac{x}{n} \sim N\left(p_0, \frac{p_0(1-p_0)}{n}\right)$$

$$z_{ts} = \frac{\hat{p} - p_0}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

4. P-value for a left-sided test:

$$\text{P-value} = P(Z < z_{ts})$$

Two Sample Test of Difference in Population Means

1. $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_A : \mu_1 - \mu_2 \neq 0$
2. Level of significance, α
3. Calculating standardized test statistic:

$$\bar{X}_1 \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right) \text{ and } \bar{X}_2 \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$$

$$\longrightarrow \bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

$$t_{ts} = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim t_{df}$$

4. P-value for a 2-sided test:

$$\text{P-value} = 2P(T > |t_{ts}|)$$

Two Sample Test of Difference in Population Means

- With no assumptions about the population variances we utilize the formula on the previous slide with

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2}$$

  For simplicity, you can always choose the more conservative

$$df = min(n_1 - 1, n_2 - 1)$$

- Assumption of equal variance ($\sigma_1^2 = \sigma_2^2$), use the pooled variance estimator in the test statistic:

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$t_{ts} = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t_{df} \text{ where } df = n_1 + n_2 - 2$$

Two Sample Test of Difference in Population Proportions

1. $H_0 : p_1 - p_2 = 0$ vs. $H_A : p_1 - p_2 \neq 0$
2. Level of significance, α
3. Calculating standardized test statistic:

$$\hat{p}_1 = \frac{x_1}{n_1} \sim N\left(p_1, \frac{p_1(1-p_1)}{n_1}\right) \quad \hat{p}_2 = \frac{x_2}{n_2} \sim N\left(p_2, \frac{p_2(1-p_2)}{n_2}\right)$$

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}\right)$$

$$z_{ts} = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \text{ where } \hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

4. P-value for a two-sided test:

$$\text{P-value} = 2P(Z > |z_{ts}|)$$

Two Sample Test of Difference in Population Proportions

1. $H_0 : p_1 - p_2 = D$ vs. $H_A : p_1 - p_2 \neq D$
2. Level of significance, α
3. Calculating standardized test statistic:

$$\hat{p}_1 \sim N\left(p_1, \frac{p_1(1-p_1)}{n_1}\right) \text{ and } \hat{p}_2 \sim N\left(p_2, \frac{p_2(1-p_2)}{n_2}\right)$$

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}\right)$$

$$z_{ts} = \frac{\hat{p}_1 - \hat{p}_2 - D}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$

4. P-value for a two-sided test:

$$\text{P-value} = 2P(Z > |z_{ts}|)$$

Afternoon Lecture Objectives

- Chi-square Tests:
  - goodness-of-fit for a single categorical variable
  - independence between two categorical variables
- Multiple Comparisons
- Experimental vs. Observational Studies
- Confounding

χ^2 Goodness-of-Fit Test

- Used to compare the sample data of a categorical variable to the theoretical distribution
- O_i is observed counts and E_i are expected counts

$$\chi^2_{ts} = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_\nu \text{ where } \nu = k - 1$$

- An example might be to test if a dye is fair based on 120 rolls

Dye Face	1	2	3	4	5	6
O_i	18	24	15	21	23	19
E_i	20	20	20	20	20	20

χ^2 Test of Independence

- Used to compare two categorical variables under the assumption that they are independent
- $$\chi^2_{ts} = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \sim \chi^2_{\nu} \text{ where } \nu = (r-1)(c-1)$$
- Example comparing victim's race and application of death penalty

	Death Penalty		
	Yes	No	Totals
White	45	85	130
Black	14	218	232
Totals	59	303	362

Multiple Comparisons

- If a researcher wants to conduct multiple tests (i.e. make multiple comparisons), we need to adjust the individual α_I rate so that the overall experimental α_E rate remains at the desired level.
- Bonferroni correction to the individual rate is straightforward, easy to apply, but overly conservative

$$\alpha_I = \frac{\alpha_E}{m} \text{ where } m \text{ is number of comparisons}$$

Multiple Comparisons Example:

- Lets say you have 5 categories and you want to make all possible pairwise comparisons of the mean

$$\mu_i = \mu_j \quad \forall i \neq j$$

- This gives us 10 possible comparisons. If we use $\alpha_I = 0.05$ then the overall error rate would be

$$\alpha_E = 1 - (1 - \alpha_I)^{10} = (1 - 0.95^{10}) = 0.401$$

So we would have a 40% chance of falsely rejecting at least one of the null hypotheses

Confounding

- A confounding factor is an attribute that correlates with both the dependent variable and independent variable affecting the association between them
- An example of a confounding factor
    - Dependent variable: weight gain
    - independent variable: activity level
    - confounding factor: age

Experimental vs. Observational

- Experimental
  - Randomly assign subjects to treatments which minimizes confounding
  - Apply treatments to subjects
  - Can be used to establish causality
- Observational
  - Subjects self select into treatment groups
  - Might need to adjust for confounding factors
  - Cannot be used to establish causality