Hypothesis Testing & Causal Inference

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Standards (1/2)

Today's standards:

- Given a dataset, state and test the null vs. alternative hypothesis, using the p-value for the difference of means or proportions
- Given a dataset, state and test the null vs. alternative hypothesis, using the p-value for Chi-square test of independence
- Describe a situation in which a one-tailed test would be appropriate vs. a two-tailed test
- State when to test hypothesis using:
  - z-test
  - t-test
  - two sample t-test (one-sided and two-sided)
  - two sample z-test (one-sided and two-sided)

Standards (2/2)

Today's standards:

- Define and state application of p-value, Type I Error, Type II Error, significance level, and power
- Account for the multiple hypotheses using Bonferroni correction
- Compute the difference of two independent random normal variables
- State when to use an A/B test to evaluate the efficacy of a treatment
- Design a simple A/B test

Objectives

Today's objectives:

- List key properties for experimental data
- Differentiate between experimental and observational data
- Perform hypothesis testing
- Design an A/B test to establish causality
- Measure treatment effects

Agenda

Today's plan:

1. Frequentist hypothesis testing
2. Key concepts: experimental vs. observational data
3. Experimental design

References

A couple references, ranked roughly by decreasing friendliness:

- Statistical Inference introduces basic probability and statistics
- All of Statistics: A Concise Course in Statistical Inference summarizes all things statistics
- A First Course in Design and Analysis of Experiments
- Experimental and Quasi-Experimental Designs for Generalized Causal Inference is a popular introduction to experimental design for social scientists
- Causal Inference for Statistics, Social, and Biomedical Sciences trenchantly explains the Rubin causal model
- Causality covers the structural causal model
- *The Design of Experiments* by R. A. Fisher is the classic reference, sadly out of print
- Sequential Analysis

Frequentist hypothesis testing

Frequentist hypothesis testing

To test a hypothesis:

1. State *null hypothesis*, H_0
2. State *alternative hypothesis*, H_1 (H_A)
3. Choose a *significance level*, α
4. Choose and compute appropriate test statistics
5. Compute *p-value* and 'reject' or 'fail to reject' H_0

Null hypothesis vs. alternative hypothesis

Null hypothesis (H_0):

- Typically, the status quo, such as no effect
- $H_0 : \mu = 0$

Alternative hypothesis (H_A)

- The alternative, such as advertising causes 1% lift
- $H_A : \mu \neq 0$ or $H_A : \mu \geq 0$
- Sometimes written as H_1

Statistics is conservative:

- Cannot '*accept a hypothesis*'
- Can only '*fail to reject*' it

Two-sided vs. one-sided tests

By default, we compute a *two-sided* test:

- Reject H_0 if test statistic is in upper or lower tail
- Compute p-value using probability of being in either tail

But, sometimes, we expect an effect to be in only one direction:

- Example: advertising should not decrease sales
- Use *one-sided* test
- $H_0 : \theta \leq \theta_0$ vs. $H_A : \theta > \theta_0$
- Reject H_0 if test statistic is in the wrong tail
- Compute p-value using the probability of being in only one tail

Type I and Type II errors

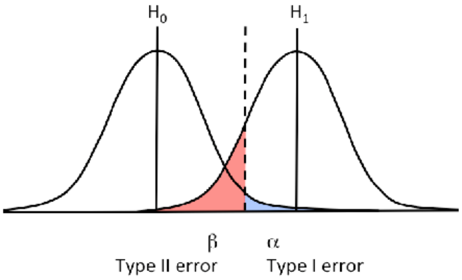
Type I error:

- Rejecting H_0 when it is true
- Example:
  - H_0 : defendant is innocent
  - Convicting someone who is innocent

Type II error:

- Failing to reject H_0 when it is false
- Example:
  - H_A : defendant is guilty
  - Acquitting someone who is guilty

H_0 vs. H_A



	H_0 is true	H_0 is false
Accept H_0	Correct Decision ($1-\alpha$)	Type II Error (β)
Reject H_0	Type I Error (α)	Correction Decision ($1-\beta$)

Figure 1: H_0 vs. H_A

Statistics

We compute statistics to perform inference and characterize parameters of interest:

- A *statistic*, $\Theta_n(X)$, is a function of data which characterizes some parameter of interest:
  - Depends on the n observations (rows)
  - Is a random variable
- A statistic, $\Theta_n(X)$, is *sufficient* for the parameter θ_0 if conditioning on it and the true parameter provides the same information as just conditioning on the statistic:

$$\Pr[x|\Theta_n(x), \theta_0] = \Pr[x|\Theta_n(x)]$$

Properties of statistics

A good statistic is usually unbiased and consistent:

- *Bias*:

$$bias = \mathbb{E}[\Theta_n(X)] - \theta_0,$$

where θ_0 is the 'truth'

- *Consistency*: a statistic is consistent if:

$$\underset{n \to \infty}{plim}\, \Theta_n(X) \to \theta_0$$

- *Robustness*: works well for a wide variety of distributions
- Will often accept some bias to decrease variance (Will discuss bias-variance trade-off in a couple weeks)

Significance level

Significance level is the cutoff for rejecting H_0:

- α is significance level
- $\alpha = \Pr[\texttt{reject } H_0 | H_0 \texttt{ is true}]$
- *Confidence level* is $(1 - \alpha) \times 100$, e.g., 95%

Example: significance level

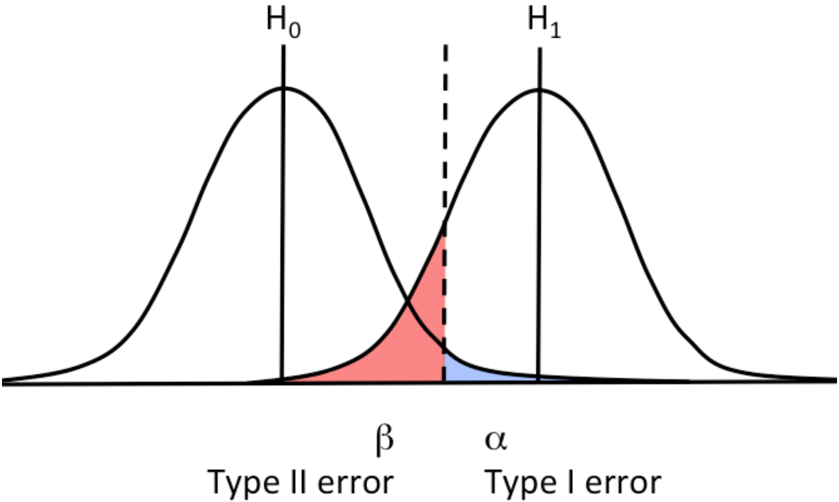


Figure 2:Hypothesis testing: H_0 vs. H_A

P-value

A *p-value* is the probability of observing data which is at least as extreme as what was observed:

- For a statistic $\Theta(X)$, $\texttt{p-value} = \Pr[\Theta(X) \geq \Theta(x)]$
- Large values of $\Theta(X)$ (small p-values) increase our belief that H_A is likely
- Reject H_0 if $\texttt{p-value} \leq \alpha$
- P-values can be controversial
- Beware of 'p-hacking' – manipulation to generate a significant result

Example: p-value for z-test

$$\texttt{p-value} = \Pr[Z < -|z| \texttt{ or } |z| < Z]$$

Confidence interval (CI)

To get a sense of the true value of the parameter of interest, compute a confidence interval:

Term	Symbol
Significance level	α
Parameter estimate	$\hat{\theta}$
Standard error	$\hat{\sigma}_\theta$
Critical z-value for $CI^{1-\alpha}$	$z_{1-\alpha/2}$

$$CI^{1-\alpha} = \left[\hat{\theta} - \hat{\sigma}_\theta \cdot z_{1-\alpha/2}, \hat{\theta} + \hat{\sigma}_\theta \cdot z_{1-\alpha/2}\right]$$

Note: 95% CI $\iff$ significance level $\alpha = 0.05$

More on confidence intervals

A couple things to note:

- Meaning of CI: if you compute CIs from multiple random samples from population, then 95% will contain the true, population value
- Popular values for $\alpha \in \{0.10, 0.05, 0.01\}$
- Use appropriate distribution to compute CI: e.g., for a t-statistic with ν degrees of freedom,

$$CI^{1-\alpha} = \left[\hat{\theta} - \hat{\sigma}_\theta \cdot t^{\nu}_{1-\alpha/2}, \hat{\theta} + \hat{\sigma}_\theta \cdot t^{\nu}_{1-\alpha/2}\right]$$

Getting the critical value

Can compute critical values using `scipy.stats` for any distribution:

```
import scipy as sp

# To determine shape parameters, see <dist>.shapes
>>> sp.stats.t.shapes
'df'
>>> alpha = 0.05
>>> df = 20
>>> sp.stats.t.ppf(1 - alpha / 2, df=df)
2.085963447265836
```

Power

Power is the probability of not making a Type II error, i.e., rejecting H_0 when H_A is true:

- $\beta = \Pr[\texttt{reject } H_A | H_A \texttt{ is true}]$
- β is similar to α, but if H_A is true
- $\textit{power} = 1 - \beta$
- An experiment with high power is more likely to reject H_0 when it is false
- Typically, set $\textit{power} = 1 - \beta = 0.80$

Example: power

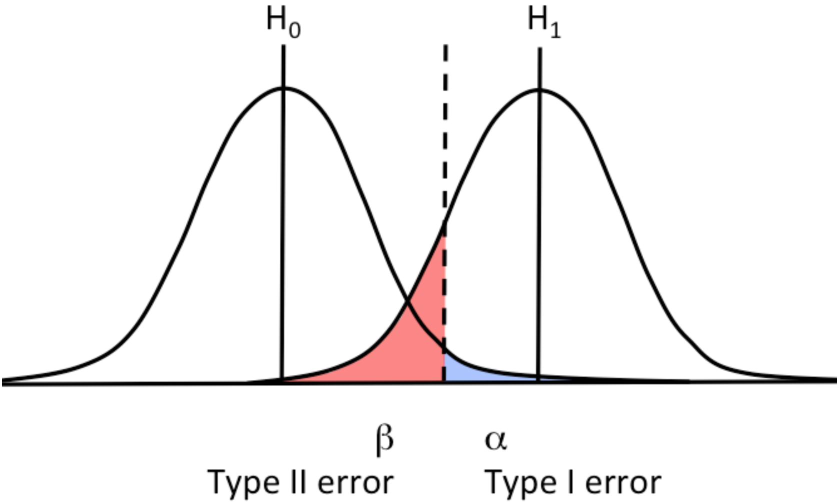


Figure 3: H_0 vs. H_A

Trade-off: significance level vs. power

You must trade-off significance level and power:

- Decreasing chance of Type I error will increase chance of Type II error
- Wise men recommend:

Term	Value
Significance level (α)	0.05
Confidence level	95%
Power ($1 - \beta$)	0.80

Factors affecting measurement of a signal

To increase probability of measuring a signal (rejecting H_0):

- Increase number of observations, n
- Increase effect size, i.e., $\theta_A - \theta_0$
- Decrease noise, σ^2

Common test statistics

Common test statistics:

- z-statistic
- t-statistic
- χ^2 for Wald test, score (LM) test, LR test
- F to test restrictions in linear regression

Example: regression parameter estimate

$$t = \frac{\hat{\beta}}{\hat{\sigma}(\hat{\beta})}$$

z-test

Use a z-test when the variance is known:

- $H_0 : \overline{x} = \mu$
- We test if the mean is μ, which could be known from past experiments
- *z-statistic*:

$$z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$$

- Sample variance is known: σ^2
- Compute p-value using Normal(0, 1)

t-test

Use a t-test when variance is unknown:

- $H_0 : \overline{x} = \mu$
- *t-statistic*:

$$t = \frac{\overline{x} - \mu}{s/\sqrt{n}}$$

- Use sample variance for denominator:

$$s^2 = \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \overline{x})^2$$

- Compute p-value using Student's t distribution
- Must specify *degrees of freedom*, ν:
  - Number of free parameters
  - $\nu = n - k$, where k is number of fitted parameters

Warning: `ddof`

Many Numpy functions compute population values by default:

- Example: `np.var(..., ddof=0, ...)` computes

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

- Must set `ddof=1` to get sample variance!

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

- `ddof` means 'delta degrees of freedom'
- In Pandas, `ddof` defaults to 1

Comparing two means: one sample

To compare a sample vs a known mean, μ_0, use the 1-sample t-statistic:

$$t = \frac{\overline{x} - \mu_0}{\sqrt{s^2/n}}$$

Then compute p-value:

```
import scipy as sp

(tstat, pval) = sp.stats.ttest_1samp(data, truth)
```

Unpaired: comparison of two random samples

To compare two independent samples, use the two-sample t-statistic:

$$t = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{s^2}}$$

$$s^2 = \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}$$

```
import scipy as sp

x = sp.stats.norm.rvs(loc=1, size=10)
y = sp.stats.norm.rvs(loc=1.1, size=12)
(tstat, pval) = sp.stats.ttest_ind(x, y)
# Returns: (1.2729753413788905, 0.21762566433145955)
```

Paired: comparison of paired samples

When you can pair data, use a paired t-test:

- Example: twin studies – each twin is assigned a different treatment
- Equivalent to a one-sample test on the paired differences
- Compute mean based on paired differences because samples are not independent

$$t = \frac{\overline{d} - \mu_0}{s_{diff} \big/ \sqrt{n}}$$

$$s_{diff}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (d_i - \overline{d})^2$$

$$d_i = x_{i1} - x_{i2}$$

- Use `sp.stats.ttest_rel()`

Review: Central Limit Theorem

How does the Central Limit Theorem motivate these tests?

Multiple hypothesis testing

Q: If you test 20 different button colors and button color has no effect, how many button colors would you expect to be significant at the 5% level on average?

Bonferroni confidence intervals

When testing multiple hypotheses together, we must be more conservative:

- Correct significance level to ensure overall significance remains the same
- Bonferroni: $\alpha \to \alpha / m$ if you have m tests:

$$\Pr[\cup(p_i \leq \frac{\alpha}{m})] \leq \sum_i \Pr[p_i \leq \frac{\alpha}{m}] \leq \alpha,$$

where p_i is the p-value for the i-th hypothesis

- Other corrections exist...

Key concepts: experimental vs. observational data

Models of causality: Rubin or Pearl

There are two main models of causality:

- Rubin causal model
- Structural causality model
  - Use to establish causality with observational data
  - See Judea Pearl's Causality

Potential outcomes notation (Neyman, 1923)

I adopt the notation of Imbens & Rubin:

- $Y_i(0)$ i's response if untreated
- $Y_i(1)$ i's response if treated
- $W_i \in \{0, 1\}$ indicates treatment status

Note: $Y_i(0)$ and $Y_i(1)$ may well have different distributions

Assumption: SUTVA

Stable unit treatment value assumption:

- Treatment is the same for all units
- Treatment of one unit does not affect the outcome of another
- Example: does aspirin cure headaches?
  - If you receive an aspirin, it has same effect on everyone
  - Giving you an aspirin, does not affect my headache

Key assumptions to establish causality

Assignment to treatment should be:

- *Individualistic*
  - A unit's probability of assignment is not affected by assignment status of other units
  - $p_i(X, Y(0), Y(1)) = q(X_i, Y_i(0), Y_i(1))$
- *Probabilistic*
  - Unit has non-zero probability of receiving either treatment
  - $0 < p_i(X, Y(0), Y(1)) < 1$
- *Unconfounded*
  - Assignment is independent of potential outcomes

Experimental vs. observational data

A classical random experiment:

- Is individualistic, probabilistic, and unconfounded
- Has a known *assignment mechanism*, $\Pr[W|X, Y(0), Y(1)]$
- If the assignment mechanism is unknown, the data is *observational*

Ceteris paribus

Ceteris paribus means 'other things equal':

- We cannot compare apples to oranges
- Attempt to establish causality by holding everything else fixed
- Or, randomizing so unobserved effects average to 0
- Condition on observables to establish causality, e.g.,

$$\mathbb{E}[\cdot|X = x]$$

Selection bias

We would like compute the treatment effect as

$$\tau = Avg_n[Y_i(1)] - Avg_n[Y_i(0)]$$

But, we do not observe response to counterfactual treatment. Thus, we would actually compute the **direct effect**,

$$Avg_n[Y_i(1)|W_i = 1] - Avg_n[Y_i(0)|W_i = 0]$$

Which is equivalent to **treatment effect** + **selection bias**:

$$\texttt{observed effect} = \underbrace{Avg_n[Y_i(1)|W_i = 1] - Avg_n[Y_i(1)|W_i = 0]}_{\texttt{direct effect}}$$
$$+ \underbrace{Avg_n[Y_i(1)|W_i = 0] - Avg_n[Y_i(0)|W_i = 0]}_{\texttt{selection}}$$

Selection bias

Selection bias occurs when:

- Treatment and control group have different distributions
- Unconfoundedness is violated:
  - Treatment status is correlated with responsiveness to treatment
  - Unobserved factors are correlated with outcomes and treatment status
  - E.g., smarter students are assigned to smaller classes
- Random assignment to treatment $\Rightarrow Y_i(0), Y_i(1) \perp W_i$
- Selection bias is everywhere – beware!

Example: selection bias

An MBA stack ranks zip codes by sales and advertises in the best performing zip codes:

- Is this a good idea?
- Can you establish causality?
- How would you measure the impact of the advertising campaign?

Why randomize?

To ensure that the treatment and control group have the same distribution:

- Block on observables to control for observable heterogeneity:
  - Stratified sampling
  - Clustered sampling
  - Systematic sampling
- Randomize over everything else
  - Should eliminate bias from unobserved heterogeneity (factors) on average
- Should ensure that our experiment has *internal validity*
- *External validity*: can we generalize our results to the world beyond our laboratory?

Experimental design

Review: significance vs. power

Q: What is the difference between significance and power?

Q: Which is more important when designing an experiment?

Q: How does changing the effect size, standard deviation, and sample affect power?

Overview

The goal of experimental design is to establish causality, estimate effect size, and avoid bias:

- Block on observables
- Randomize over everything else to avoid bias
- Distribution of treatment and control group should be the same

Power calculation

Always perform a power calculation to calculate number of observations needed to measure a signal:

- Make sensible guess about effect size and standard deviation … or run a pilot experiment
- Use $\alpha = 0.05$ and *power* $= 0.80$ unless you know better
- Usually, effect size is 'standardized,' i.e., divided by standard deviation
- Lift from advertising is often small, e.g., 1%
- For more complicated situations, compute power via Monte Carlo simulation

Power calculation

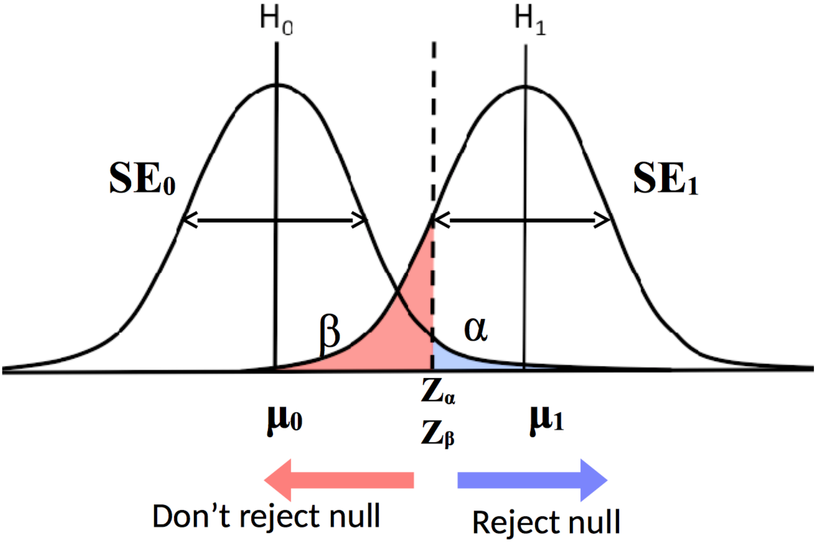


Figure 4:Power calculation (one-sided hypothesis)

Example: power calculation

How big do N_c and N_t need to be to measure an effect?

```
import statsmodels.stats.power as smp
import statsmodels.stats.api as sm

# Solve for number of observations needed
smp.zt_ind_solve_power(effect_size=0.01, alpha=0.05,
    power=0.80, alternative='two-sided')
# returns:  156977.21019023287

# Compute power for an design
smp.zt_ind_solve_power(effect_size=0.01, nobs1=10,
    alpha=0.05,ratio=1.0, alternative='two-sided')
# Returns:  0.050057277123711996
```

Check for balance

After designing your experiment, check for balance:

- Are distributions of exogenous covariates in different treatments the same?
- Are outcomes similar prior to treatment?
- Examine:
  - Moments of distribution (mean, standard deviation)
  - Compare distributions with Kolmogorov-Smirnov test
  - Train a logit model to predict if an observation is in the treatment or control group

Example: measure impact of advertising on click-through-rate (CTR)

Your engineering team ran an experiment where they changed the color of the checkout button from red to blue. How would you test if blue is better?

Data	Control	Treatment
Total visitors	N_C	N_T
Number of clicks	n_C	n_T

Questions:

- What is H_0?
- What is the CTR for each treatment?
- What is the effect size?
- What is the standard error?
- What test should you perform to test H_0?

Example: continued

Answer:

- CTR: $\widehat{ctr}_C = n_C / N_C$ and $\widehat{ctr}_T = n_T / N_T$
- $H_0 : ctr_C = ctr_T$, i.e., treatment has no effect
- Effect size: $\hat{\delta} = \hat{ctr}_T - \hat{ctr}_C$
- Use pooled sample proportion for standard error:

$$\widehat{ctr} = \frac{n_C + n_T}{N_C + N_T}$$

- Compute standard error for two independent samples:

$$\widehat{s}^2 = \widehat{ctr} \cdot (1 - \widehat{ctr}) \cdot \left(\frac{1}{N_C} + \frac{1}{N_T} \right)$$

Example: continued

- Test statistic:

$$z = \frac{\widehat{ctr}_C - \widehat{ctr}_T}{s}$$

- z-test – why is a t-test incorrect?

See Stat Trek for details

Pearson χ^2 test

For comparing counts in a table, O_{ij}, Pearson's χ^2 test for independence is easier:

$$U = \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

where the expected number of observations in a cell is:

$$E_{ij} = \frac{O_{i\cdot}O_{\cdot j}}{n}$$

Under H_0, $U \sim \chi^2_{\nu}$ where $\nu = (I-1)\cdot(J-1)$, i.e., reject H_0 if $U > \chi^2_{\nu,\alpha}$

Discussion based on All of Statistics

Intuition for Pearson's χ^2 test

Intuition for the χ^2 test:

- Use test to compare observed and expected cell counts under H_0
- $H_0 \Rightarrow p_{ij} = p_{i\cdot} \cdot p_{\cdot j}$
- So MLE estimator is:

$$\hat{p}_{ij} = \hat{p}_{i\cdot} \cdot \hat{p}_{\cdot j} = \frac{O_{i\cdot}}{n} \cdot \frac{O_{\cdot j}}{n}$$

Then the expected number of observations in each cell is:

$$E_{ij} = n\hat{p}_{ij} = \frac{O_{i\cdot} \cdot O_{\cdot j}}{n}$$

Example: Pearson's χ^2 test (1/3)

Is tonsillectomy related to Hodgkins disease?

	Hodgkins disease	No disease	$O_{i\cdot}$
tonsillectomy	90	165	255
no tonsillectomy	84	307	391
$O_{\cdot j}$	174	472	646

Example from All of Statistics

Example: Pearson's χ^2 test (2/3)

```
data = [ [ 90, 165 ], [ 84, 307 ] ]
chi2, p, ddof, expected = sp.stats.chi2_contingency(data,
    correction=True)
msg = """Test Statistic: {}
p-value: {}
Degrees of Freedom: {}"""

print( msg.format( chi2, p, ddof ) )
print( expected )
```

Example: Pearson's χ^2 test (3/3)

```
Test Statistic: 14.2651105944
p-value: 0.000158780892398
Degrees of Freedom: 1

[[  68.68421053  186.31578947]
 [ 105.31578947  285.68421053]]
```

Other measurement methods

Several methods to measure results, depending on type of data and experimental design:

- Regression/ANOVA, typically with dummy variables for treatment status
- Instrumental variables (IV)
- Difference-in-differences to control for heterogeneity
- Regression discontinuity design

Other types of experiments

Sometimes, we get lucky and Nature provides a randomization device which effectively creates experimental data:

- Field experiments: occur in field and not laboratory
- Natural experiments: ‘nature’ provides randomization
- More complex designs:
  - Multi-factor (A/B/C/...)
  - Latin squared

Example: natural experiment

A marketing manager runs an experiential marketing campaign on ten university campuses:

- How would you measure if advertising worked?
- What if the manager short-listed 50 campuses but could only obtain access to the chosen ten?
- What assumption(s) did you make?

Example: best practice

Consider this scenario:

- Collecting data is expensive.
- A manager collects data until the results appear significant and then terminates the experiment?

Is this a good idea? Hint: what are the random variables?

Wald sequential Analysis

Sequential analysis provides method to terminate an experiment once you have collected enough data:

- Treats experiment length as a random variable
- The correct way to terminate an experiment before a fixed time
- . . . this is not the same thing as 'terminating early'
- See reference for details
- Example: test quality of parts coming off an assembly line to compare two manufacturing processes

Summary

Q: What is the difference between Type I and Type II errors?

Q: How do you compute a confidence interval?

Q: To compare two click through rates, should you use a z-test or t-test?

Q: How can you establish causality?

Q: What assumptions must hold to run a classical random experiment?

Q: What is the difference between power and significance level? Which matters for inference? Which matters for designing an experiment?

Q: What is selection bias? How can I eliminate it?