MODEL ASSUMPTIONS & NORMAL EQUATIONS

(*) $$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i ,$$

β_0, β_1 ARE UNKNOWN, $\varepsilon_i \sim$ i.i.d $N(0, \sigma^2)$

$$\underset{\sim}{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \underline{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \underset{\sim}{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\underset{\sim}{Y} = \underline{X}\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}, \quad \underset{\sim}{\varepsilon} \sim N(0, \sigma^2 I)$$

(*) LEAST SQUARES SOLUTION:

MINIMIZE $MSE = \frac{1}{n}\sum_{i=1}^{n}\left[y_i - (\beta_0 + \beta_1 x_i)\right]^2$ TO

OBTAIN ESTIMATES OF β_0, β_1

NOTE: $MSE = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2$

IN EFFECT, MINIMIZE $SSE = \sum_{i=1}^{n} \varepsilon_i^2 = \underset{\sim}{\varepsilon}^T \underset{\sim}{\varepsilon}$

LET $Q = \sum_{i=1}^{n}\left[Y_i - (\beta_0 + \beta_1 X_i)\right]^2$

$$\frac{\partial Q}{\partial \beta_0} = -2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\frac{\partial Q}{\partial \beta_1} = -2\sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\left.\begin{aligned} \sum_{i=1}^{n} Y_i &= n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i &= \hat{\beta}_0 \sum_{i=1}^{n} x_i + \hat{\beta}_1 \sum x_i^2 \end{aligned}\right\} \text{NORMAL EQUATIONS}$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix} \Rightarrow \boxed{\underline{X}^T \underline{X} \underset{\sim}{\beta} = \underline{X}^T \underset{\sim}{y}}$$

<u>SOLUTION</u>

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{(\sum x_i)^2}{n}} \qquad \underset{\sim}{\hat{\beta}} = (\underline{X}^T \underline{X})^{-1} \underline{X}^T \underset{\sim}{y}$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \qquad [\Rightarrow \bar{Y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{X};\ \text{IMPLICATIONS?}]$$

<u>MULTIPLE</u> <u>REGRESSON</u>

$p-1$ FEATURES : $X_1, X_2, \ldots, X_{p-1}$

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_{p-1} x_{p-1,i} + \varepsilon_i$$

$$\underset{\sim}{Y}_{n\times 1} = \underline{X}_{n\times p} \underset{\sim}{\beta}_{p\times 1} + \underset{\sim}{\varepsilon}_{n\times 1}, \qquad \underset{\sim}{\varepsilon} \sim N(0, \sigma^2 I)$$

$$\underline{X} = \begin{bmatrix} 1 & x_{11} & x_{21} & \cdots & x_{p-1,1} \\ 1 & x_{12} & x_{22} & \cdots & x_{p-1,2} \\ \vdots & & & & \\ 1 & x_{1n} & x_{2n} & & x_{p-1,n} \end{bmatrix}_{n\times(p-1)}, \quad \underset{\sim}{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{p-1} \end{bmatrix}$$

LEAST SQUARES ESTIMATES

SOLVE $\underset{\sim}{X}^T \underset{\sim}{X} \underset{\sim}{\beta} = \underset{\sim}{X}^T \underset{\sim}{y}$

$$\hat{\underset{\sim}{\beta}} = \left(\underset{\sim}{X}^T \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}^T \underset{\sim}{y}$$

(*) $SSE = \underset{\sim}{\varepsilon}^T \underset{\sim}{\varepsilon} = (\underset{\sim}{Y} - \underset{\sim}{X}\underset{\sim}{\beta})^T (\underset{\sim}{Y} - \underset{\sim}{X}\underset{\sim}{\beta}) = \underset{\sim}{Y}^T \underset{\sim}{Y} - \underset{\sim}{\beta}^T \underset{\sim}{X}^T \underset{\sim}{Y}$

(*) $\hat{\underset{\sim}{Y}} = \underset{\sim}{X} \hat{\underset{\sim}{\beta}} = \underset{\sim}{X} \left(\underset{\sim}{X}^T \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}^T \underset{\sim}{Y} = H \underset{\sim}{Y}$

i) $H = \underset{\sim}{X}\left(\underset{\sim}{X}^T \underset{\sim}{X}\right)^{-1} \underset{\sim}{X}^T$: "HAT" MATRIX (PROJECTION)

ii) Let $H = [h_{ij}]$. THEN $0 \le h_{ii} \le 1$, $\sum_{i=1}^{n} h_{ii} = \text{rank}(\underset{\sim}{X})$

h_{ii} : LEVERAGE OF OBSERVATION i