Bayesian A/B Testing and the Multi-Armed Bandit

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Objectives

Morning objectives:

- Define and explain prior, likelihood, and posterior.
- Explain what a conjugate prior is and how it applies to A/B testing.
- Explain the difference between Frequentist and Bayesian A/B tests.
- Analyze an A/B test with the Bayesian approach.

Afternoon Objectives:

- Explain how multi-armed bandit addresses the tradeoff between exploitation and exploration, and the relationship to regret.
- Implement the multi-armed bandit algorithm.

Agenda

Morning:

- Review Bayesian statistics.
- Discuss an example of Bayesian A/B testing.
- Discuss conjugate priors.
- Compare to Bayesian and Frequentist approches.

Afternoon:

- What is a multi-armed bandit?
- How do we use one to do smarter A/B tests?

Bayes' Theorem

Recall **Bayes' Theorem**

$$P(x|\theta) = \frac{P(\theta|x)P(x)}{P(\theta)}$$

where

- $P(x|\theta)$ is the **posterior probability distribution** of hypothesis x being true, given observed data θ,
- $P(\theta|x)$ is the **likelihood** of observing θ given x,
- $P(x)$ is the **prior distribution** of x, and
- $P(\theta)$, the **normalizing constant**, is

$$P(\theta) = \sum_x P(\theta|x)$$

Example: Click-through rates

Consider two version of an ad on a website.

Which produces a higher click-through rate?

Each visit follows a Bernoulli distribution.

Use Bayesian analysis to find probability distributions of effectiveness.

Binomial Distribution (likelihood)

Likelihood of k successes out of n trials

$$\binom{n}{k} p^k (1-p)^{n-k}$$

- p: conversion rate (between 0 and 1)
- n: number of visitors
- k: number of conversions

Beta Distribution

Use the beta distribution for prior probabilities.

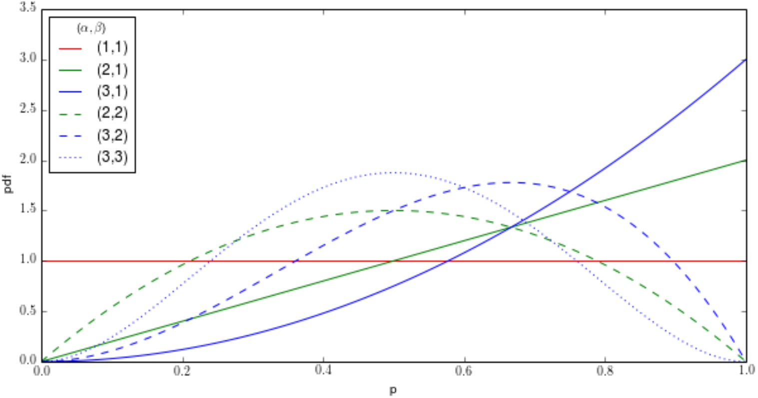
$$\frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}$$

where

- p: conversion rate (**between 0 and 1**)
- α, β: shape parameters
  - $\alpha = 1 +$ number of conversions
  - $\beta = 1 +$ number of non-conversions
- Beta Function (B) is a normalizing constant
- $\alpha = \beta = 1$ gives the *uniform distribution*
- mean is $\frac{\alpha}{\alpha+\beta}$

Beta Distribution

$$\frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}$$



Conjugate Priors

$$
\begin{aligned}
\textit{posterior} &\propto \textit{prior} * \textit{likelihood} \\
\textit{beta} &\propto \textit{beta} * \textit{binomial} \\
&= \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} * \binom{n}{k} p^k (1-p)^{n-k} \\
&\propto p^{\alpha-1}(1-p)^{\beta-1} * p^k (1-p)^{n-k} \\
&\propto p^{\alpha+k-1}(1-p)^{\beta+n-k-1}
\end{aligned}
$$

Result is a beta distribution with parameters $\alpha = \alpha + k$ and $\beta = \beta + n - k$

Conjugate Priors

A conjugate prior for a likelyhood is a class of functions such that if the prior is in the class, so is the posterior.

Likelihood	Prior
Bernoulli/Binomial	Beta distribution
Normal with known σ	Normal distribution
Poisson	Gamma

How important are these to do Bayesian statistics?

Frequentist vs. Bayesian

In both cases, we consider an ensemble of possible randomly generated universes.

Frequentist: The hypothesis is a fixed (though unknown) reality; we the observed data follows some random distribution

Bayesian: The observed data is a fixed reality; the hypotheses follow some random distribution.

Frequentist A/B testing

Frequentist procedure

- Choose n (number of experiments/samples) based on expected size of effect.
- Run **all** experiments and observe the data.
- The significance is probability of getting result (or more extreme) assuming no effect.
- Doesn't tell you how likely it is that a is better than b.

(aside: Wald sequential analysis)

Bayesian A/B testing

- No need to choose n beforehand.
- Update knowledge as the experiment runs.
- Gives probability of *anything you want*.

Why doesn't everyone like this better?

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What Is a Multi-Armed Bandit?

Each slot machine (a.k.a. one-armed bandit) has a difference (unknown!) chance of winning.

How do you maximize your total payout after a finite number of plays?

Assume all have the same payoff. ("binary bandits")

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Example: which website is easier to navigate?

Example: which drug is more effective?

Minimizing Regret

Regret is the difference of what we won and what we would expect with optimal strategy.

Exploration vs Exploitation

Traditional A/B testing: first find the best bandit (explore) then take advantage of it (exploit)

But we will loose money playing bandits we don't *think* are good.

Could we do better by exploring and exploiting at the same time?

How would *you* solve this?

Epilon-Greedy Algorithm

Usually choose the best so far, but sometimes (with probability ϵ) choose one randomly.

No "best" value, but $\epsilon = 0.1$ is typical.

What are the limitations?

Softmax

Choose a bandit randomly in proportion to the softmax function of the payouts, e.g.

If there are three bandits, A, B, and C, the probability of choosing A is

$$\frac{e^{p_A/\tau}}{e^{p_A/\tau} + e^{p_B/\tau} + e^{p_C/\tau}}$$

where

- p_A is the average payoff of bandit A so far (assume 1.0 to start).
- τ is the "temperature" (generally constant).

How does this behave in the extremes?

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How does this behave in the extremes?

- As $\tau \to \infty$, the algorithm will choose bandits equally.
- As $\tau \to 0$, it will choose the most successful so far.

What are the limitations?

UCB1 Algorithm

Choose a bandit to maximize

$$p_A + \sqrt{\frac{2 \ln N}{n_A}}$$

where

- p_A is the expected payout of bandit A.
- n_A is the number of times bandit A has played.
- N is the total number of trials so far.

This chooses the bandit for whom the Upper Confidence Bound is the highest.

Bayesian Bandit

Use Bayesian statistics:

- Find probability distribution of payout of each bandit thus far. (how?)
- For each bandit, sample from distribution.
- Choose bandit for whom the sample has highest expected payout.