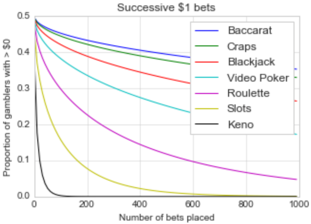
Probability

Schwartz

August 29, 2016

Beating the House

Game	House Advantage
Baccarat (no tie bets)	1.2%
Craps (pass/come)	1.4%
Blackjack (average player)	2.0%
Video Poker (average player)	0.5% - 3%
Roulette (double-zero)	5.3%
Slots	5.0%-10.0%
Keno (average)	27.0%



Blackjack can be legally beaten by keeping track of the probability of getting a high card (10,J,Q,K,A) compared to a low card (2,3,4,5,6). This is called *card counting*. In early 1979, four MIT students taught themselves card counting and along with a professional gambler and an investor who put up most of their capital ($5,000) went to Atlantic City for spring break. They went again in December and then recruited a few more MIT students as "students" for a "blackjack class". The "class" continued to visit Atlantic City intermittently until May 1980 (when the students graduated), during which time they increased their capital four-fold. At about the same time, Bill Kaplan returned to Cambridge after successfully running a blackjack team in Las Vegas. Kaplan earned his BA at Harvard in 1977 and was accepted into Harvard Business School but delayed admission while he ran the blackjack team. Kaplan ran his operation using funds he received upon graduation as Harvard's "outstanding scholar-athlete" and generated more than a 35 fold rate of return in less than nine months of play. Kaplan continued to run his Las Vegas blackjack team as a sideline while attending Harvard Business School but by the time of his graduation the players were so "burnt out" the team disbanded.

Objectives

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- Counting
- Random Variables
  - Marginal, Joint, and Conditional distributions
- Distributions
  - Representations: pmf/pdf/mgf/characteristic functions
  - Examples: Bernoulli, Binomial, Geometric, Multinomial, Poisson Uniform, Normal, χ^2, Gamma, Exponential, Beta
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- Exposure and comfort with a wide range of sophisticated statistical distribution theory concepts and notations

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- Pro $\implies$ Refresher
- Intermediate $\implies$ Solidify
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3. Can orient prediction/inference machine learning cosmology
   - gives a general theoretical framework to place methodologies

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a.k.a., *combinatorics* – the discipline of mathematics dedicated to *counting*

- Permutation: How many ways can you *permute* things

- Combination: How many ways can you *combine* things

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  - The number of k-sized subsets of m things ($k < m$) is

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{(n-k)!k!}$$

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- *How many ways can* $A \cdot B \cdot C$ *happen?*

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$$\binom{n-1}{k-1} \quad \binom{8}{2}$$

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$$\begin{aligned}
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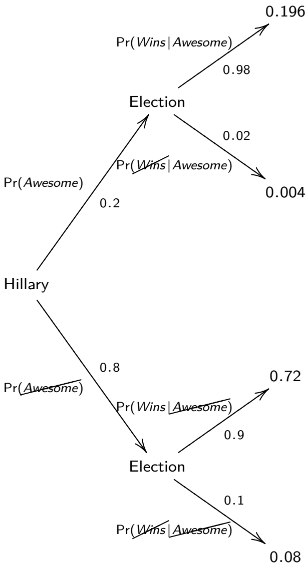

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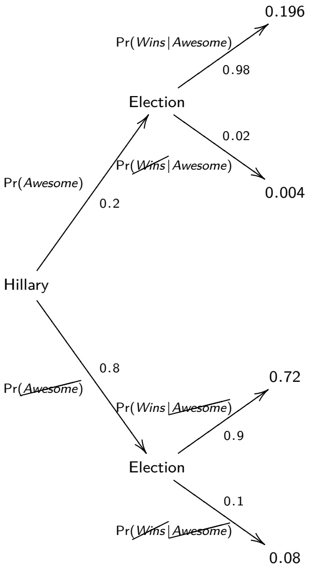

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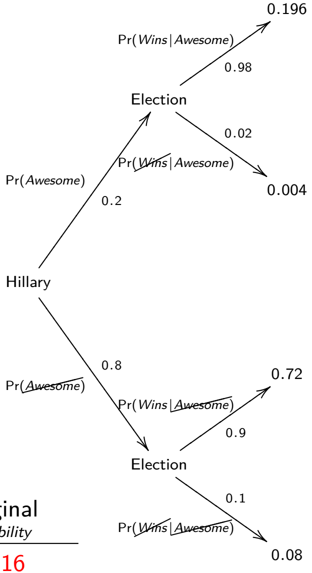

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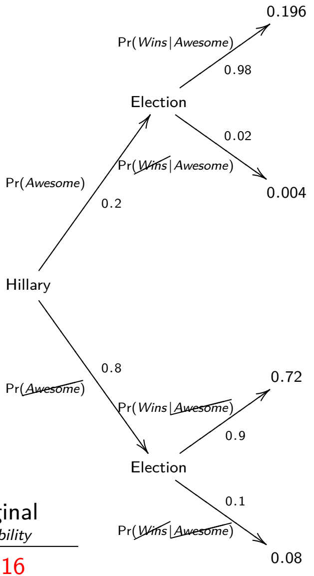

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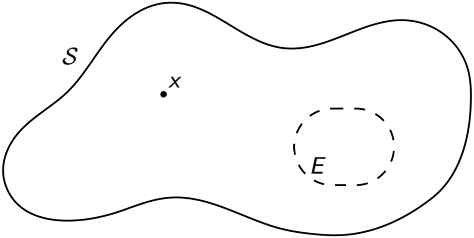

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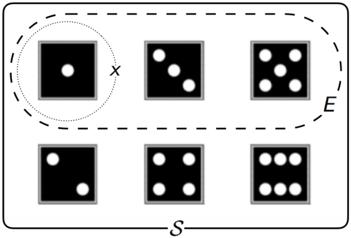

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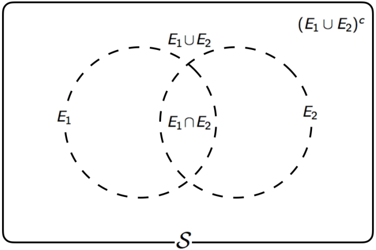

Venn Diagram

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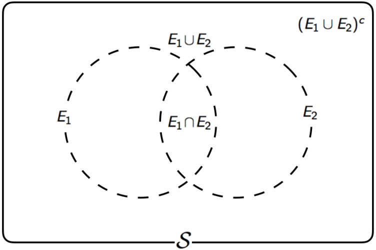

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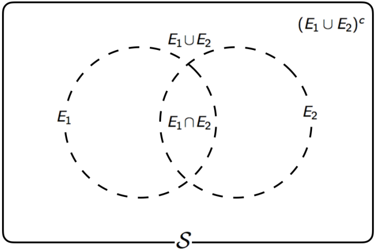

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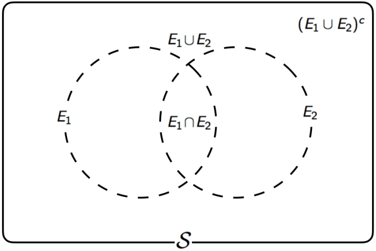

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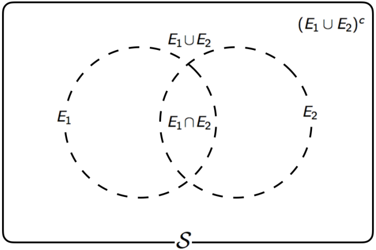

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- $\Pr(E \cap E^c) = 0$
- DeMorgan's Laws
  - $\Pr\left((E_1 \cup E_2)^c\right) = Pr\left(E_1^c \cap E_2^c\right)$
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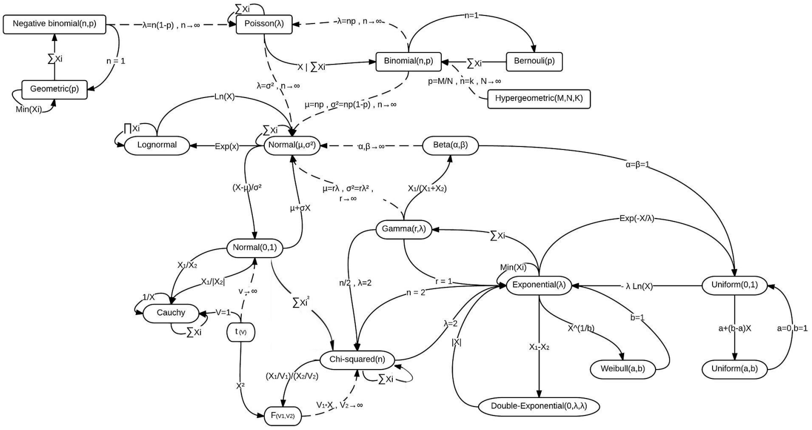
Distributions



Discrete Distributions: *Bernoulli*

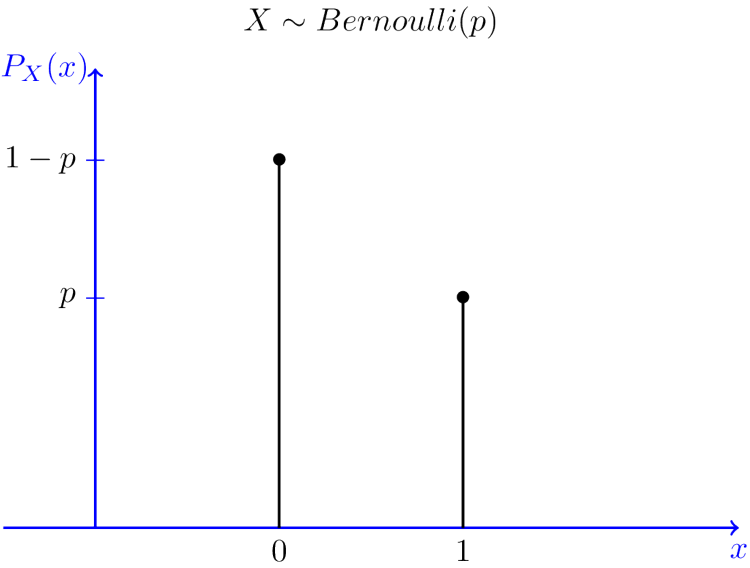
The “coin flip”

$$y \in \{0, 1\}$$

$$\Pr(Y = y|\theta) = \theta^y (1 - \theta)^{1-y}$$

$$\theta \in [0, 1]$$

Discrete Distributions: *Bernoulli*



Discrete Distributions: *Geometric*

The "how many times until"

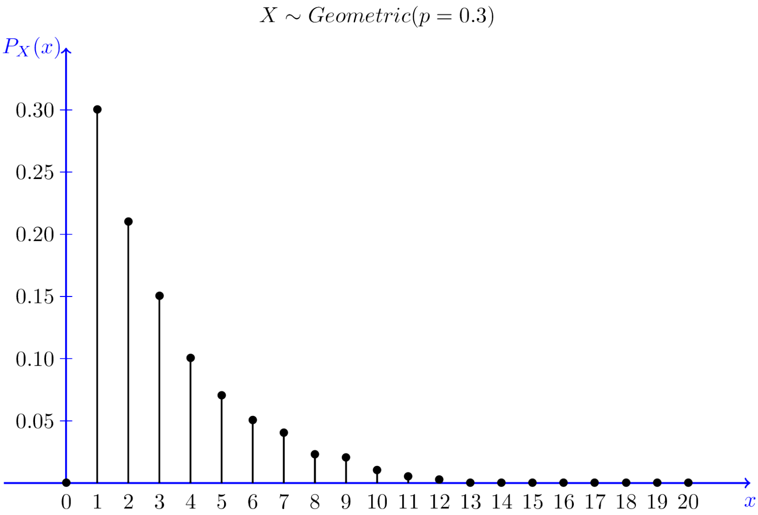
$$k \in \{0, 1, \cdots \infty\}$$

$$\Pr(X = k|\theta) = (1-\theta)^{k-1}\theta$$

$$\theta \in [0, 1]$$

"If at first you don't succeed, Try, try, try again" – William Edward Hickson

Discrete Distributions: *Geometric*



Discrete Distributions: *Binomial*

The "number of success in n trials"

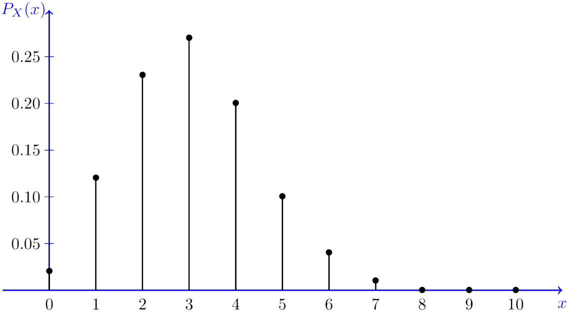
$$k \in \{1, 2, \cdots n\}$$

$$\Pr(X = k|\theta, n) = \begin{pmatrix} n \\ k \end{pmatrix} \theta^k (1-\theta)^{n-k}$$

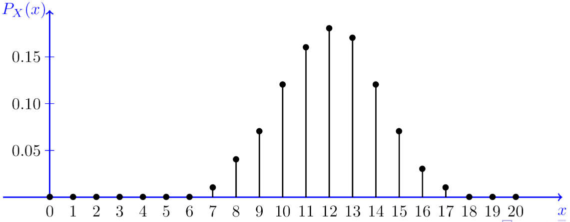
$$\theta \in [0, 1]$$

Discrete Distributions: *Binomial*

$X \sim Binomial(n = 10, p = 0.3)$



$X \sim Binomial(n = 20, p = 0.6)$



Discrete Distributions: *Multinomial*

The "fancy binomial"

$$\mathbf{x} = (k_1, x_2, \cdots, x_k)$$
$$x_j \in \{0, 1, \cdots m\} \text{ such that } \sum x_j = m$$

$$\Pr(\mathbf{X} = \mathbf{x} | \theta_1, \theta_2, \cdots \theta_k, m) = \frac{m!}{x_1! x_2! \cdots x_k!} \prod_{j=1}^{k} \theta_j^{x_j}$$

$$\theta_j \in [0, 1] \text{ such that } \sum \theta_j = 1$$

Discrete Distributions: *Poisson*

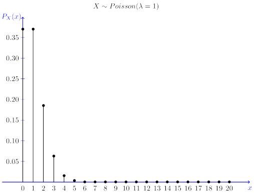
The "number of arrivals"

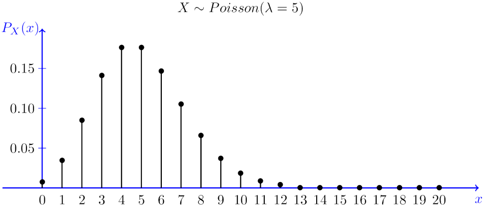
$$k \in \{0, 1, \cdots \infty\}$$

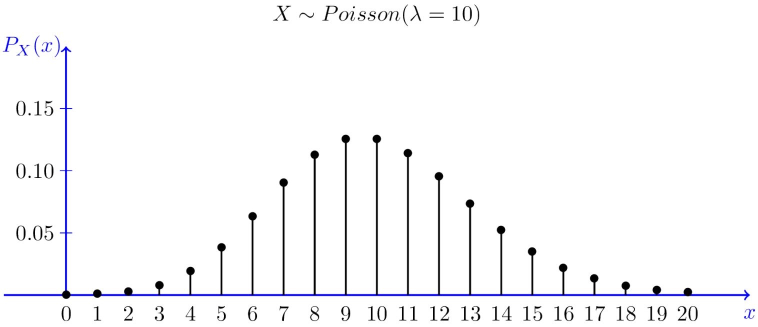
$$\Pr(X = k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\lambda \in \mathbb{R}^+$$

Discrete Distributions: *Poisson*

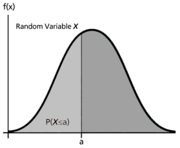




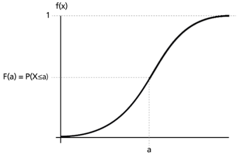


pmf's, *cdf*'s, and characteristic functions

1. We have thus far defined the distribution of a random variable by it's *probability mass function*



2. We can equivalently alternatively define the distribution of X by it's *cumulative distribution function*



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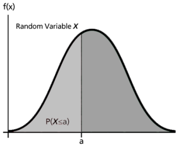

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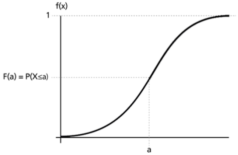

3. Yet another way to define the distribution of X is by its *moment generating function* or its *characteristic function*:

$$\mathrm{E}\left[tX\right], \text{ or } \mathrm{E}\left[itX\right], \text{ respectively}$$

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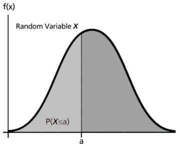

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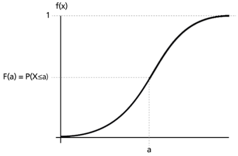

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Interestingly, the characteristic function of $X + Y$ for independent random variables X and Y is the product of the characteristic functions of X and Y

Discrete Distributions: *Poisson* + *Poisson*

- The characteristic function of a *Poisson* random variable is

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$$e^{\lambda_X\left(e^{it}-1\right)}e^{\lambda_Y\left(e^{it}-1\right)} = e^{(\lambda_X+\lambda_Y)\left(e^{it}-1\right)}$$

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- Quiz: name the distributions of $X+X$ and $Y+Y$ if

$$X \sim Bernoulli(\theta) \text{ and } Y \sim Binomial(\theta, n)$$

  with respective characteristic functions

$$1-\theta+\theta e^{it} \text{ and } \left(1-\theta+\theta e^{it}\right)^n$$

Discrete Distributions: *Poisson* ≈ *Bionomial*

- If $\lambda = n\theta$ then $\theta = \frac{\lambda}{n}$ so that

$$
\begin{aligned}
&= \binom{n}{k} \theta^k (1-\theta)^{n-k} \\
&= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{n \cdot (n-1) \cdots (n-k+1)}{n^k k!} \lambda^k \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \\
&\approx \frac{1}{k!} \lambda^k e^{-\lambda} 1 \text{ (as } n \to \infty) \\
&= \frac{\lambda^k e^{-\lambda}}{k!}
\end{aligned}
$$

Continuos Distributions: *Uniform*

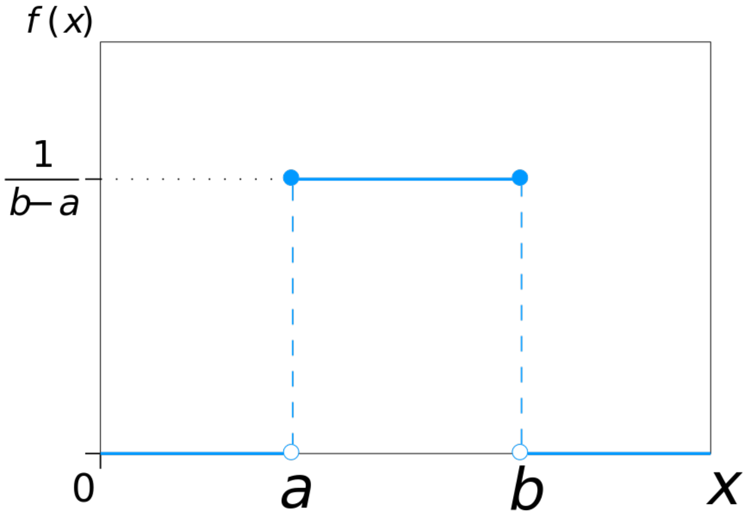
The "random continuous number"

$$u \in \mathbb{R}$$

$$f(X = u|a, b) = \frac{1}{b - a} 1_{[a,b]}(u)$$

$$a, b \in \mathbb{R}, a < b$$

Discrete Distributions: *Uniform*



Continuos Distributions: *Normal*

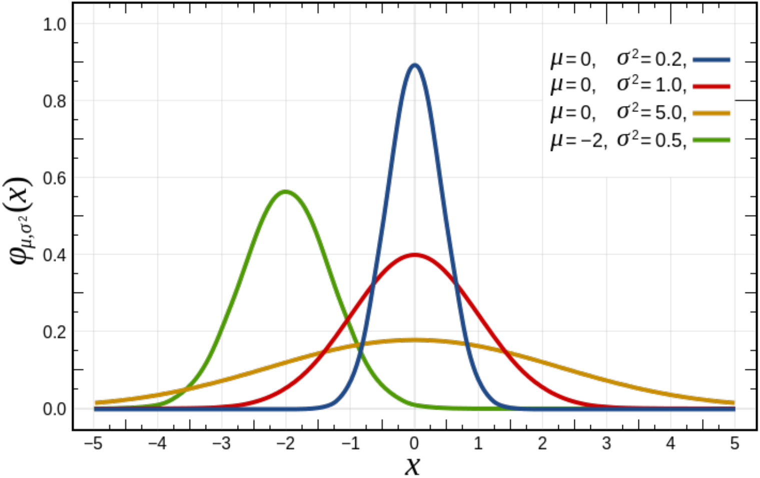
The "bell curve"

$$x \in \mathbb{R}$$

$$f(X = x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}^+$$

Continuos Distributions: *Normal*



Continuos Distributions: *Normal*²

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Continuos Distributions: *Normal*[2]

- If $X_j \sim Normal\left(\mu, \sigma^2\right)$ are normal random variables

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Continuos Distributions: *Normal*^{2}

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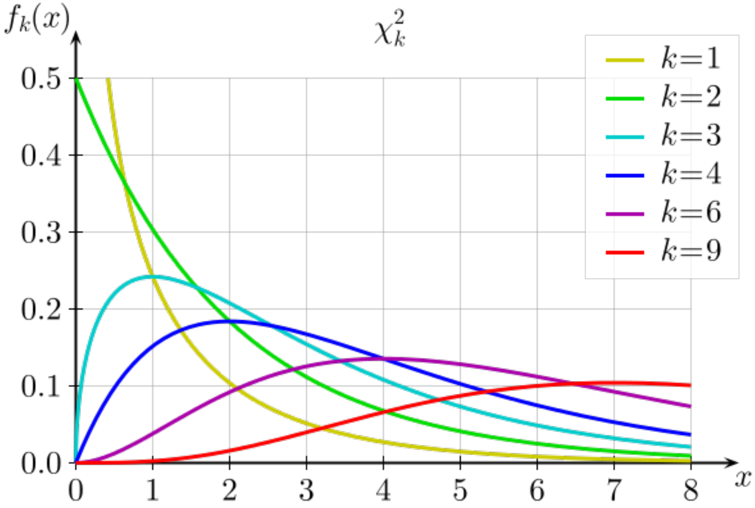
Continuos Distributions: *Normal*^{2} : χ^2_{df}



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What is the distribution of $X + Y$ if

$$X \sim Normal\left(\mu_X, \sigma_X^2\right) \text{ and } Y \sim Normal\left(\mu_Y, \sigma_Y^2\right)?$$

Continuos Distributions: *Normal* $\sim X_1 + X_2 + \cdots + X_n$

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The MGF of the sum of n *arbitrary* $i.i.d.$ random variables is

$$\left(1 + t\mathrm{E}[X] + \frac{t^2\mathrm{E}[X^2]}{2!} + \frac{t^3\mathrm{E}[X^3]}{3!} + \cdots\right)^n$$

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$$\begin{aligned}
& n\log\left(t\mathrm{E}[X] + \frac{t^2\mathrm{E}[X^2]}{2} + \cdots - \frac{t^2\mathrm{E}[X]^2}{2} - \cdots + \cdots\right) \\
&= n\log\left(t\mathrm{E}[X] + \frac{t^2(\mathrm{E}[X^2] - \mathrm{E}[X]^2)}{2!} + \cdots\right) \\
&= e^{tn\mathrm{E}[X] + \frac{1}{2}t^2 n(\mathrm{E}[X^2] - \mathrm{E}[X]^2) + n(\cdots)} \\
&\approx e^{tn\mathrm{E}[X] + \frac{1}{2}t^2 n(\mathrm{E}[X^2] - \mathrm{E}[X]^2)} \text{ as } n \to \infty
\end{aligned}$$

The binomial distribution with large n is approximately normal: why?

Continuos Distributions: *Gamma*

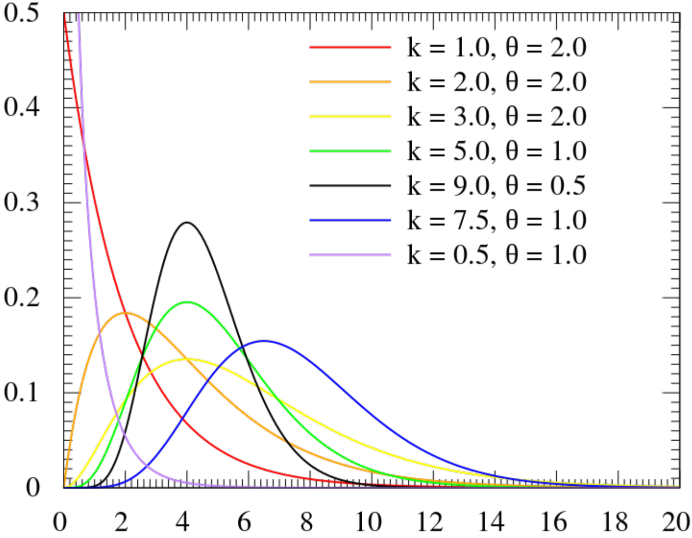
The “Bayesian model for variance”

$$x \in \mathbb{R}^+$$

$$f(X = x|\theta, k) = \frac{\theta^k}{\Gamma(k)} x^{k-1} e^{-x\theta}$$

$$\theta \in \mathbb{R}^+$$

Continuos Distributions: *Gamma*



Continuos Distributions: *Gamma ($\theta = 1/2$, Chi-squared)*

- We previously derived the χ^2_{df} distribution as the "sum of squared standard normal distributions"
- and noted its importance in hypothesis testing
- The χ^2_{df} is also a special case of the gamma distribution

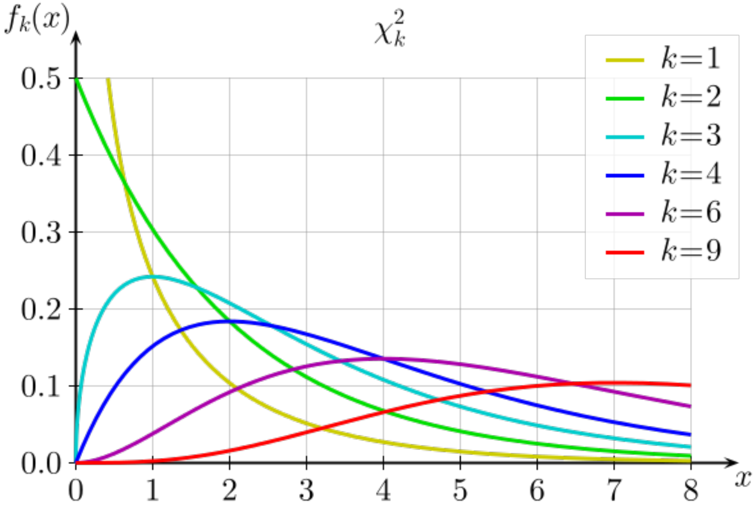
$$x \in \mathbb{R}^+$$

$$f(X = x|k) = \frac{\frac{1}{2}^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$$

$$k \in \mathbb{R}^+$$

- Bonus: if $X \sim \chi^2_v$ and $X \sim \chi^2_w$, then $\frac{\frac{1}{v}\chi^2_v}{\frac{1}{w}\chi^2_w} \sim F_{v,w}$

Continuos Distributions: *Gamma* $(\theta = 1/2, \chi^2_{df})$



Continuos Distributions: *Gamma (k=1, Exponential)*

- The *Exponential* is another special case of the gamma
- The Exponential is often used to model time to failure
- It has an interesting "ageless" property, however, in that

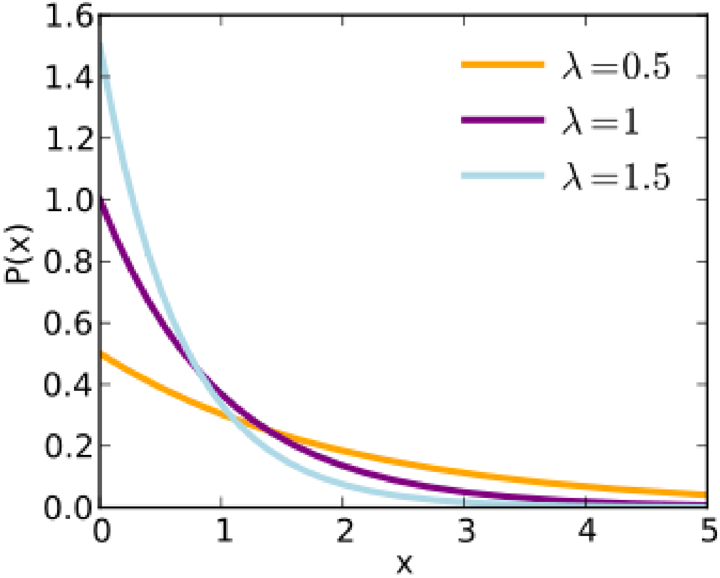
$$\Pr(X = x + c|x = 0) = \Pr(X = x + c|x) \text{ for any value of } x$$

$$x \in \mathbb{R}^+$$

$$f(X = x|\theta) = \theta e^{-x\theta}$$

$$\theta \in \mathbb{R}^+$$

Continuos Distributions: *Gamma (k=1, Exponential)*



Continuos Distributions: *Beta*

The "distribution for modeling random probabilities"

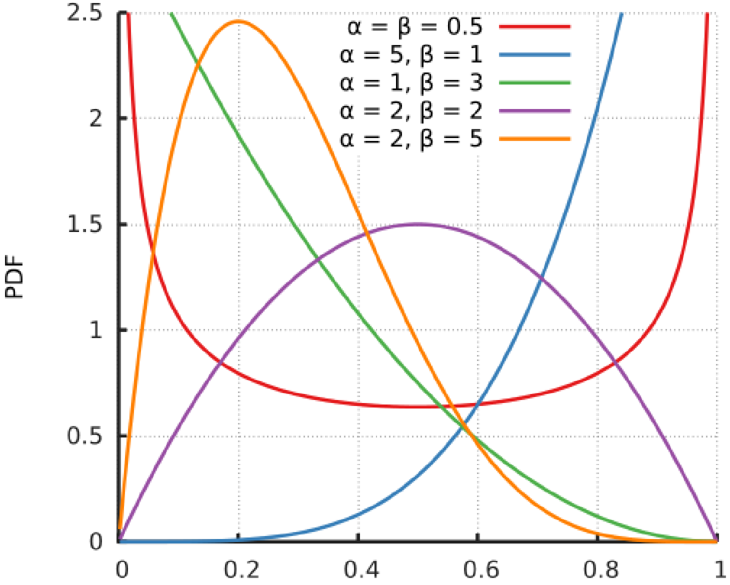
$$p \in [0, 1]$$

$$f(X = p|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$\alpha, \beta \in \mathbb{R}^+$$

$\alpha = \beta = 1$ results in a *uniform distribution* over the unit interval

Continuos Distributions: *Beta*



Joint Distributions

- A *joint distribution* is set of more than one random variable
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- We'll use a *multinomial distribution* for the *joint distribution*

$$\Pr(\mathbf{X} = \mathbf{x} | \theta_1, \theta_2, \cdots \theta_k, m) = \frac{m!}{x_1! x_2! \cdots x_k!} \prod_{j=1}^{k} \theta_j^{x_j}$$

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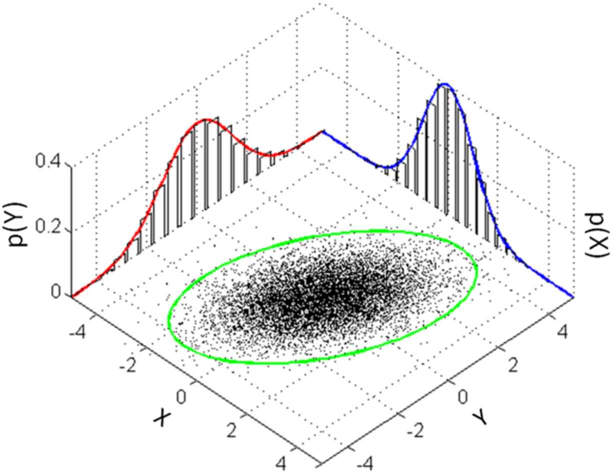
- *Joint distributions* are just a collection of random variables that may or may not have some dependencies on each other

Joint Distributions



It turns out this is a multivariate normal distribution – the marginals are themselves normal and strength of relationship between X and Y is determined by a correlation parameter ρ

Joint Distributions: *discrete*

- *Joint distributions* factor as *conditional* $\times$ *marginal* dist.'s

$$\Pr(\mathbf{X}_1, \mathbf{X}_2) = \Pr(\mathbf{X}_1 | \mathbf{X}_2) \Pr(\mathbf{X}_2)$$

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- *Marginal distributions* are derived from *joint distributions*

$$\Pr(\mathbf{X}_1) = \sum_{x_2 \in \mathcal{S}_{X_2}} \Pr(\mathbf{X}_1, X_2 = x_2)$$

Joint Distributions: *continuous*

- *Joint distributions* factor as *conditional* $\times$ *marginal* dist.'s

$$f(\mathbf{X}_1, \mathbf{X}_2) = f(\mathbf{X}_1|\mathbf{X}_2)f(\mathbf{X}_2)$$

- *Bayes theorem* is derived from joint distribution factoring

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- *Marginal distributions* are derived from *joint distributions*

$$f(\mathbf{X}_1) = \int_{\mathcal{S}_{X_2}} f(\mathbf{X}_1, X_2 = x_2)dx_2$$

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$$= \mathrm{E}[X^2] - \mathrm{E}[X]^2$$
- *The Covariance* of X and Y
$$\mathrm{Cov}[X, Y] = \int_{\substack{(x,y) \\ \in \mathcal{S}_{XY}}} (x - \mathrm{E}[X])(y - \mathrm{E}[Y]) \Pr(X = x, Y = y) d_{xy}$$
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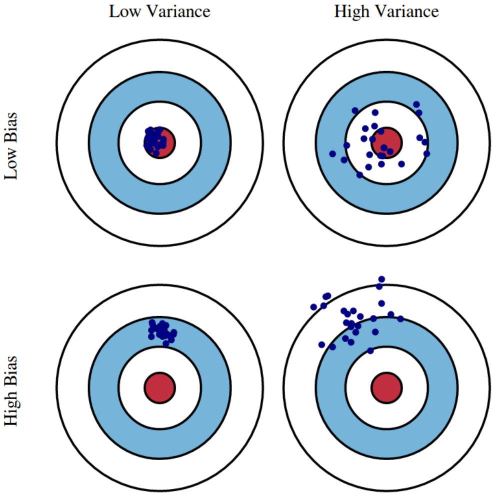
- The *Sample Correlation** $R_{XY} = \frac{S_{XY}}{\sqrt{S_X^2 S_Y^2}} \in [-1, 1]$

*not robust... correlation of ranks?

Why $n-1$?

$$
\begin{aligned}
E\left[\sum_{i=1}^{n}\left(x_i^2-\frac{1}{n}\sum_{j=1}^{n}x_j\right)^2\right] &= E\left[\sum_{i=1}^{n}\left(x_i^2-\frac{2x_i}{n}\sum_{j=1}^{n}x_j+\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)^2\right)\right] \\
&= E\left[\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j+\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j\right] \\
&= E\left[\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{j\neq i}x_ix_j+\frac{1}{n}\sum_{i=1}^{n}x_i^2+\frac{1}{n}\sum_{j\neq i}x_ix_j\right] \\
&= E\left[\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{i=1}^{n}x_i^2+\frac{1}{n}\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{j\neq i}x_ix_j+\frac{1}{n}\sum_{j\neq i}x_ix_j\right] \\
&= E\left[\frac{n-1}{n}\sum_{i=1}^{n}x_i^2-\frac{1}{n}\sum_{j\neq i}x_ix_j\right] = \frac{n-1}{n}\sum_{i=1}^{n}E\left[x_i^2\right]-\frac{1}{n}\sum_{j\neq i}E\left[x_ix_j\right] \\
&= \frac{n-1}{n}\sum_{i=1}^{n}(\sigma^2+\mu^2)-\frac{1}{n}\sum_{j\neq i}\mu^2 \quad (why?) \\
&= (n-1)(\sigma^2+\mu^2)-\frac{n^2-n}{n}\mu^2 = (n-1)\sigma^2
\end{aligned}
$$

Bias versus Variance of Estimators



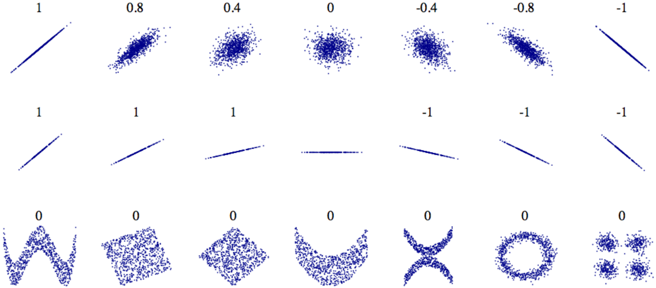
Covariance is not Correlation is not Causation

$$\text{Cov}[X, Y] \neq \text{Cor}[X, Y] = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}}$$

and "correlation *is not* causation"

or more generally, "association *is not* causation"

Conversely, *uncorrelated variables* **are not** necessarily *independent*



*Also, *mutually exclusive events* E_1 and E_2 quite *dependent* as opposed to independent