K-means and Hierarchical Clustering

Schwartz

May 1, 2017

Best. Music. *EVAR*

1. Elliott Smith
2. Sufjan Stevens
3. Iron and Wine
4. Damien Rice

1. Die Antwoord
2. Kendrick Lamar
3. Dan le Sac Vs Scroobius Pip

1. Rufus Wainwright
2. Lyle Lovett
3. Julie Doiron

1. D'Gary
2. Kishi Bashi
3. Christine and the Queens
4. Beirut

1. Rage Against the Machine
2. System of a Down
3. Smashing Pumpkins

1. Beck
2. Cake
3. Beastie Boys

Objectives

- Review *Supervised* versus *Unsupervised*

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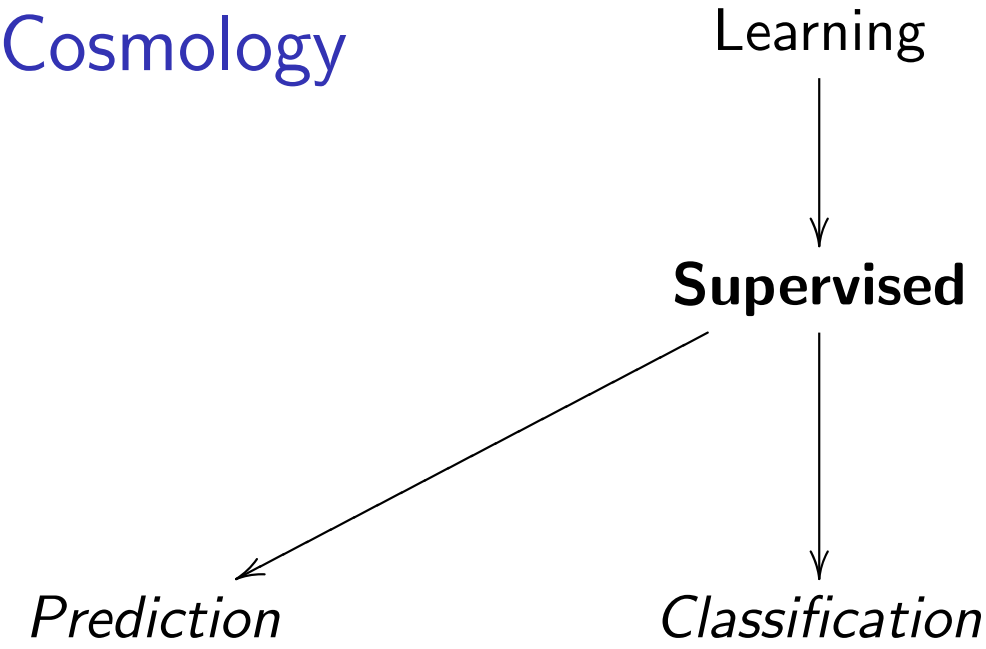
- Bayesian mixture models
- Expectation-Maximization (EM) algorithm

ML Cosmology



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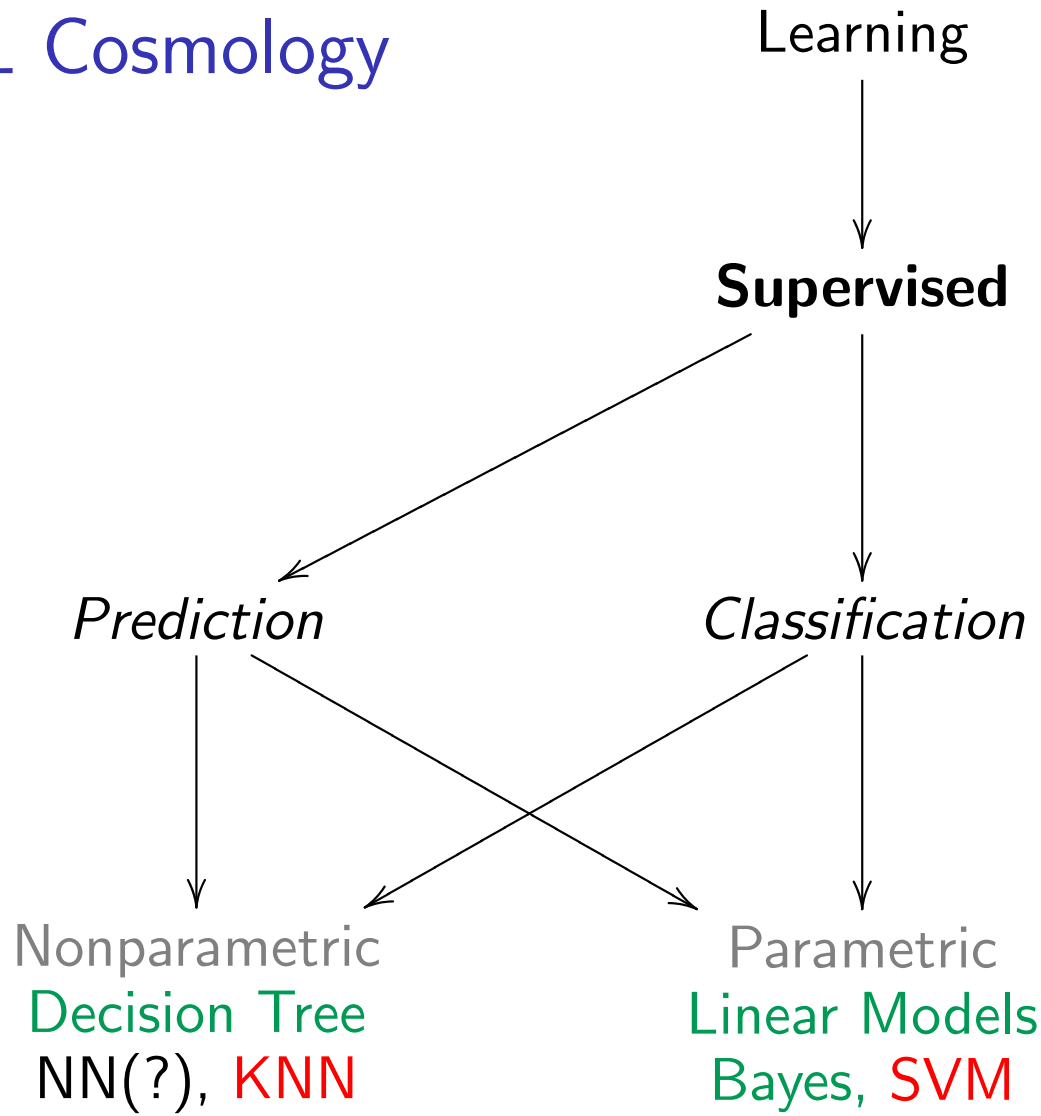

Continuous?
Categorical

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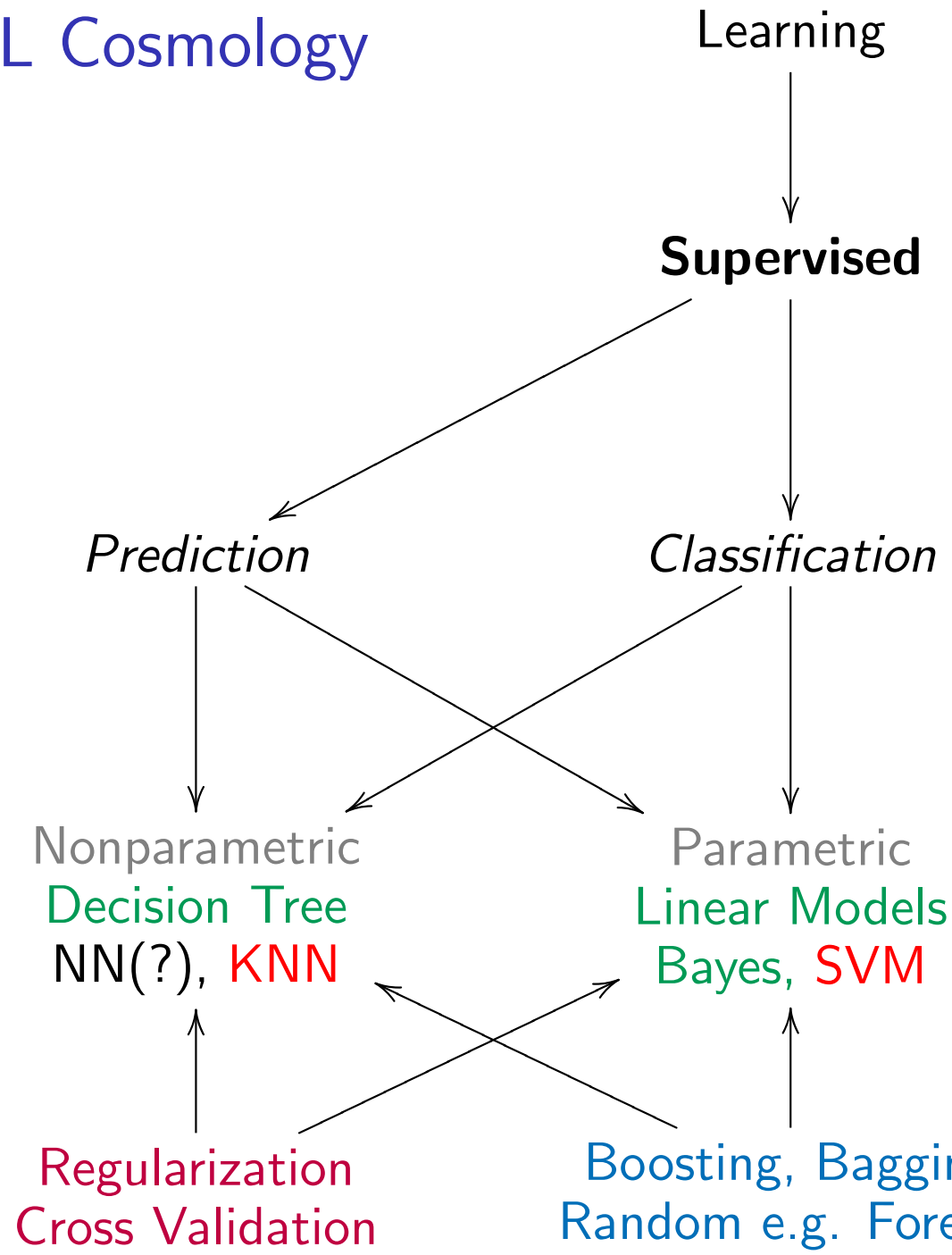

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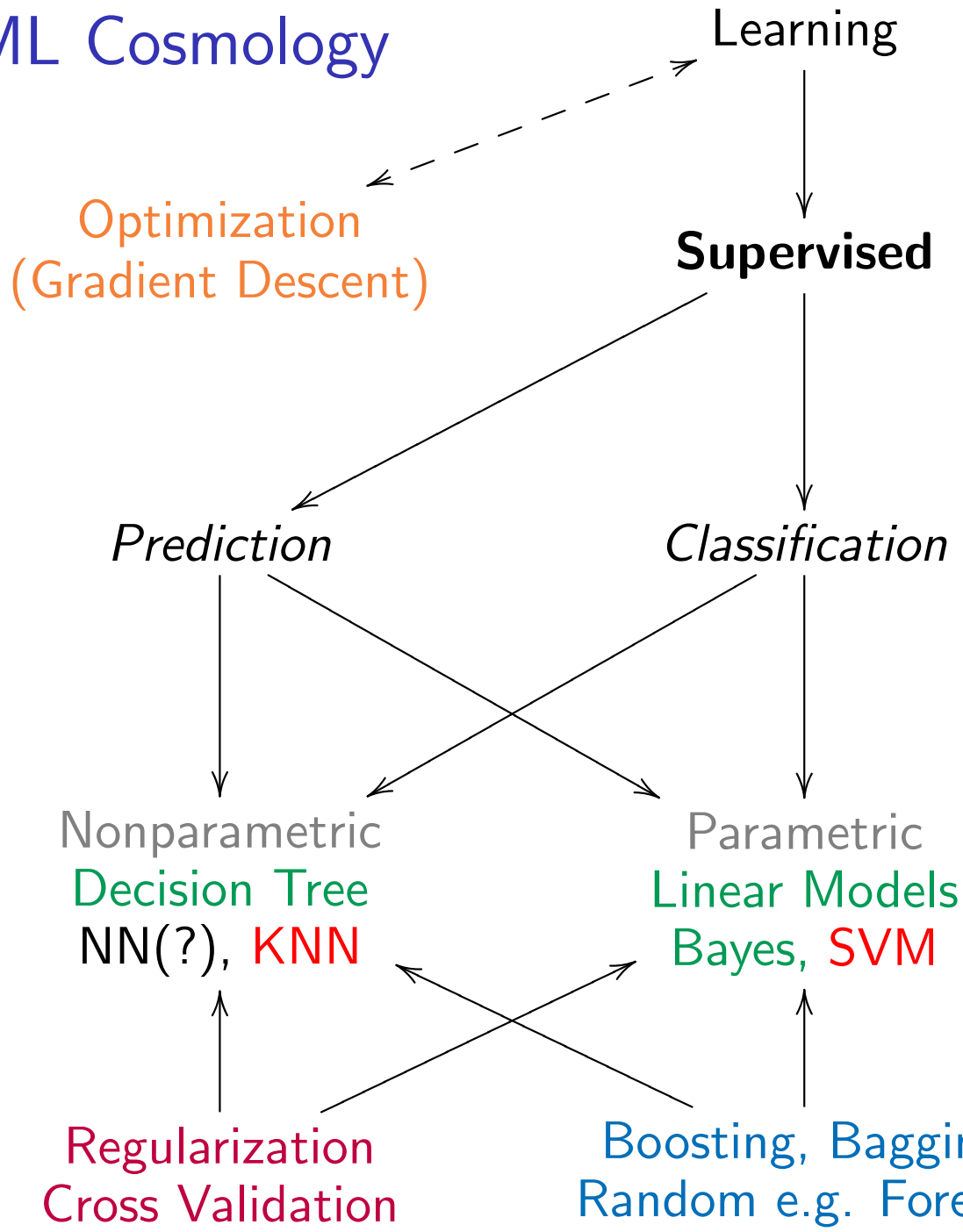

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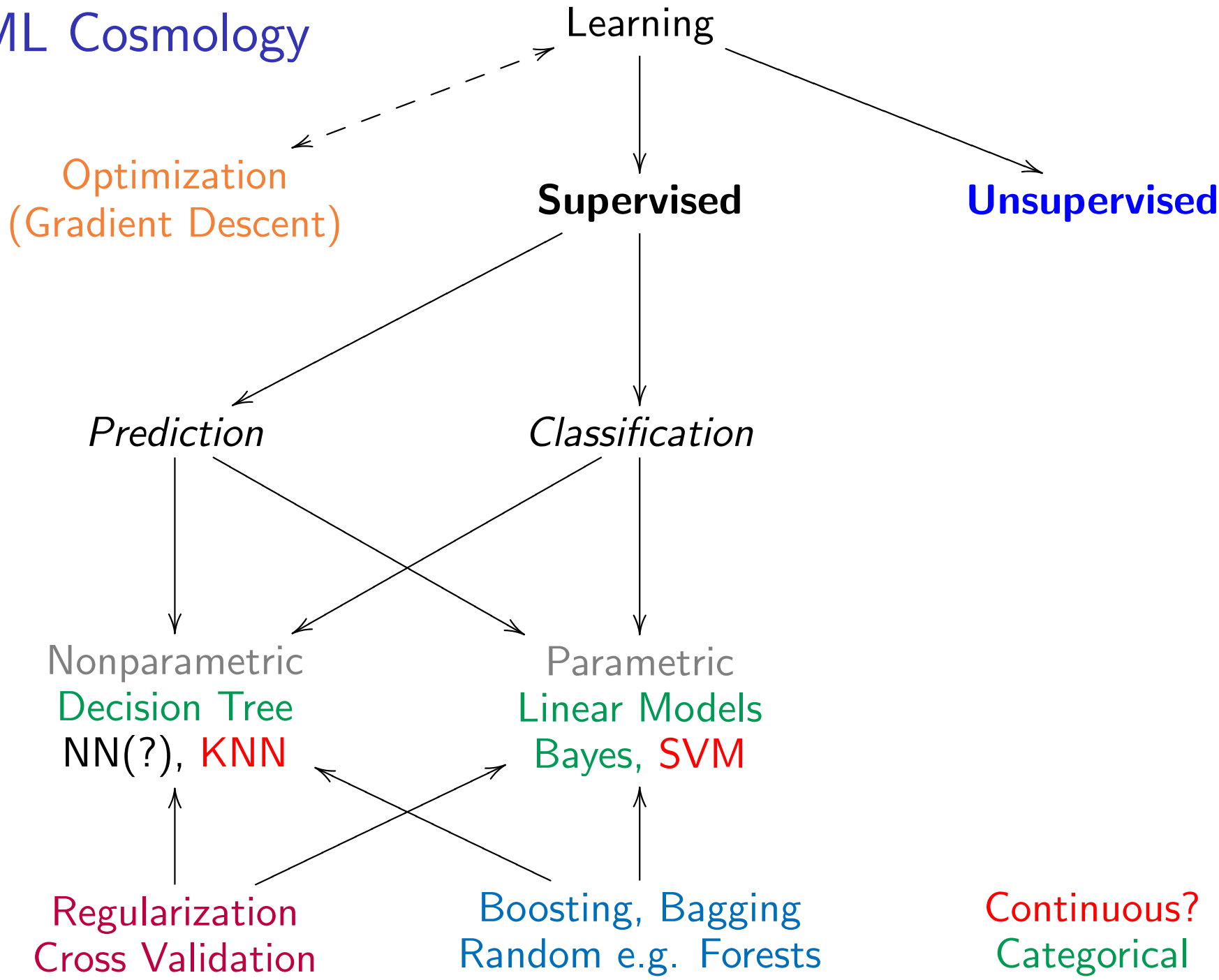

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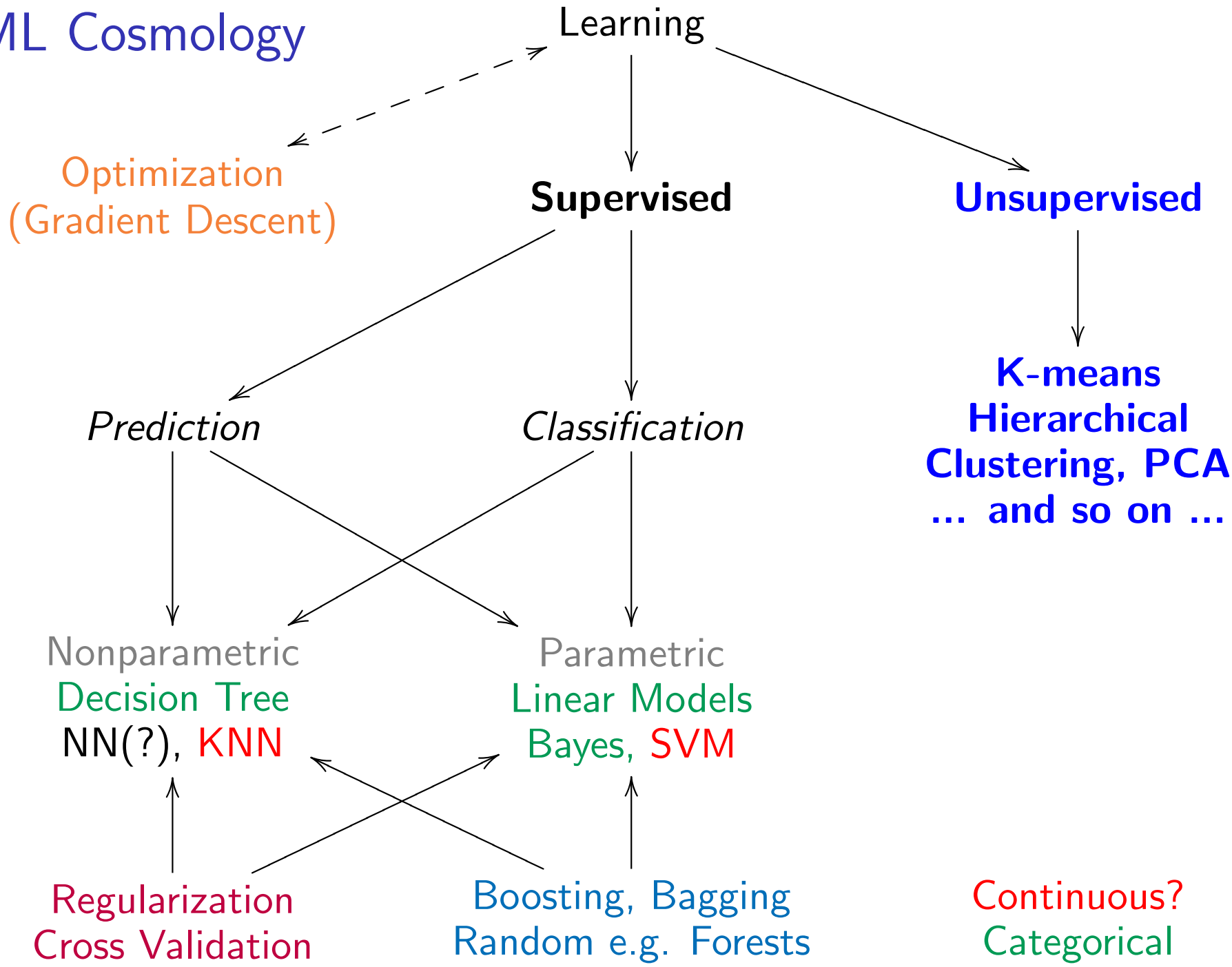
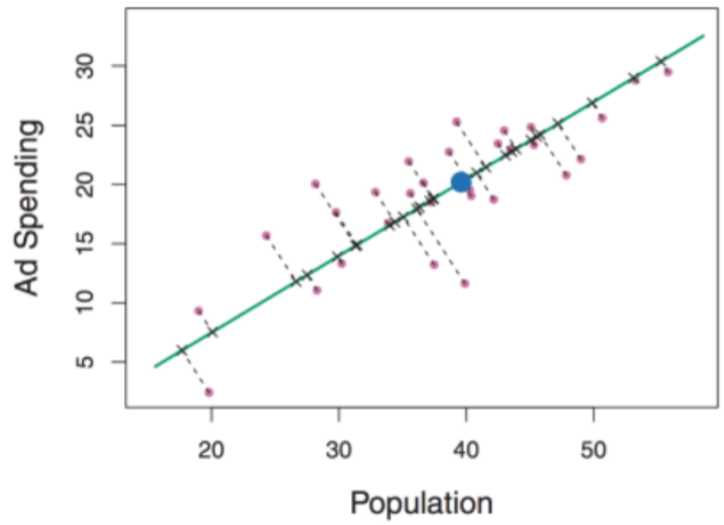
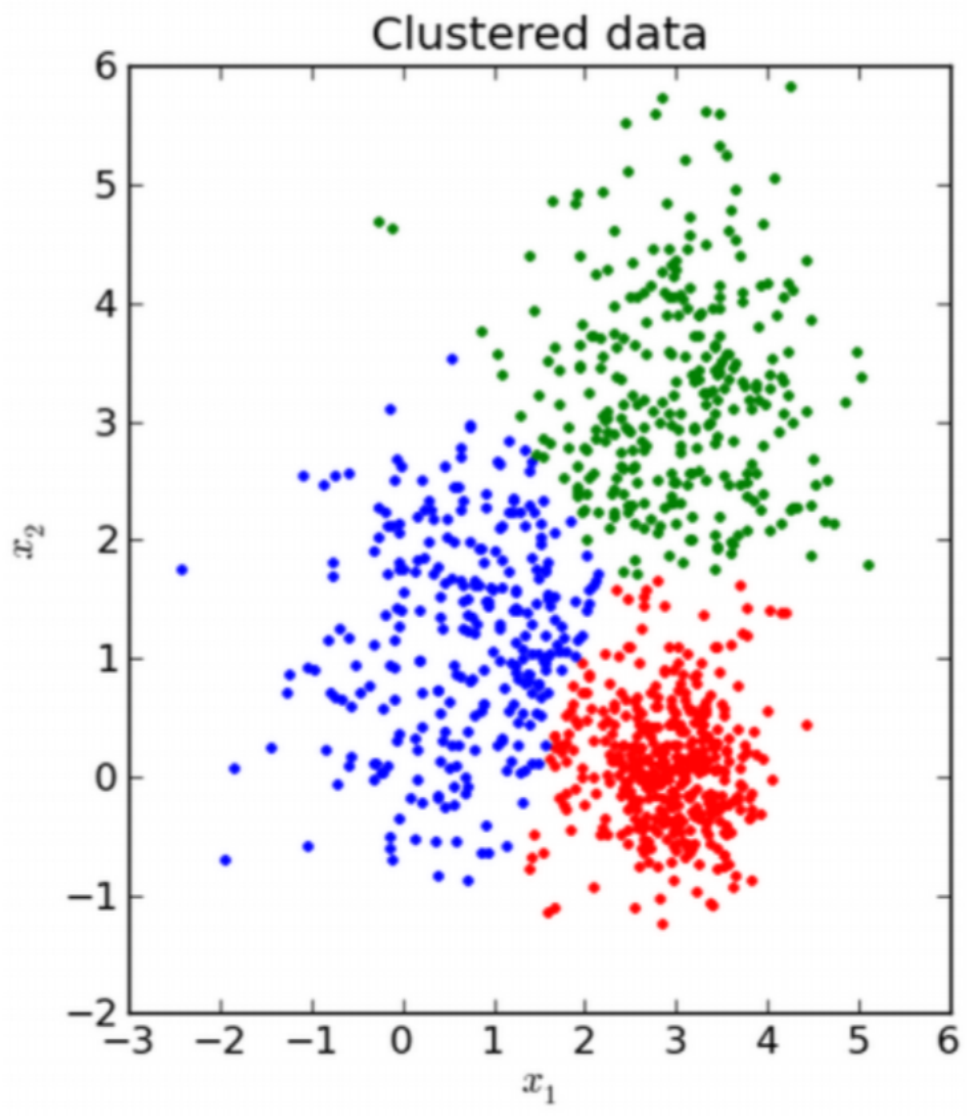


Unsupervised learning



Low dimensional representations of data capturing data variation

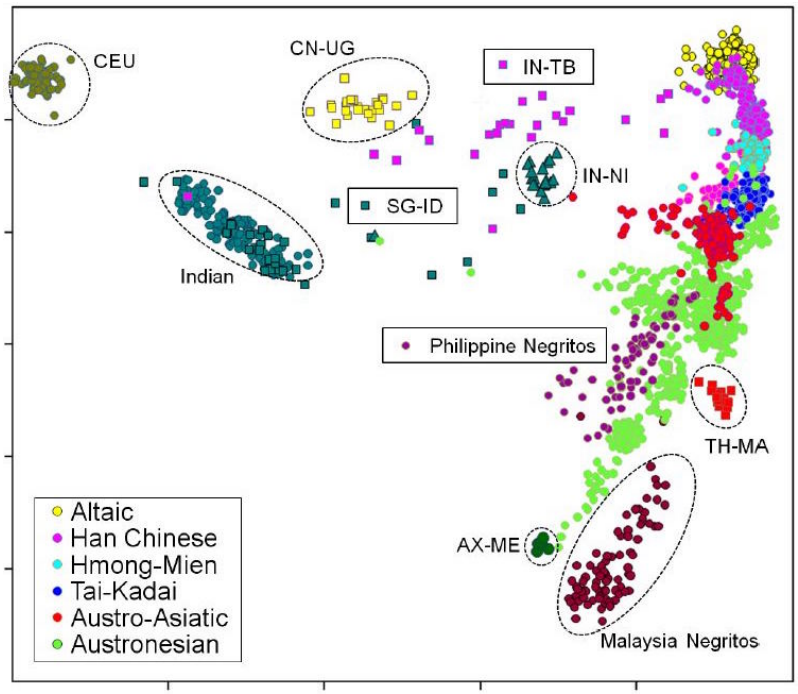


Homogeneous subgroups capturing data substructure

Supervised versus Unsupervised

Human Genetic Diversity in Asia

Two-Dimensional Representation



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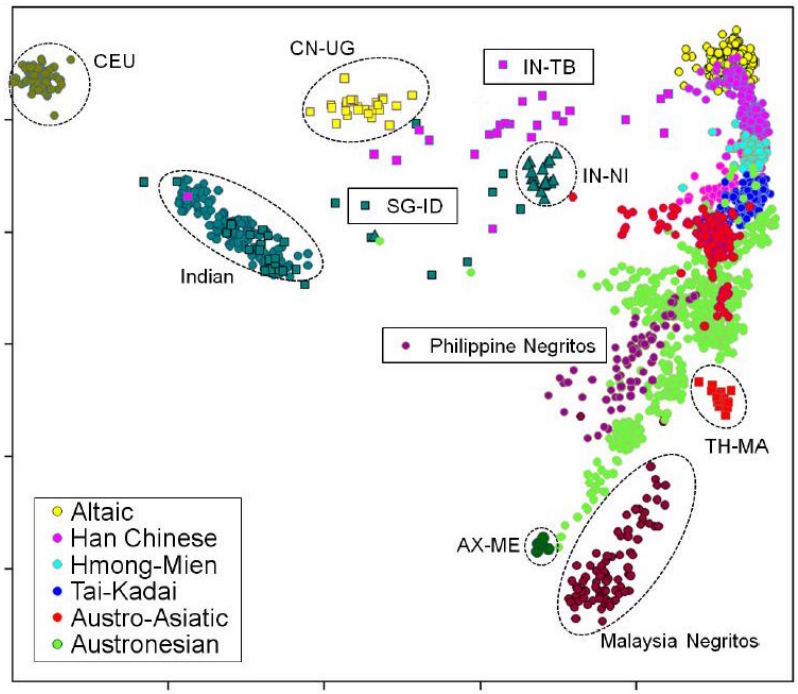

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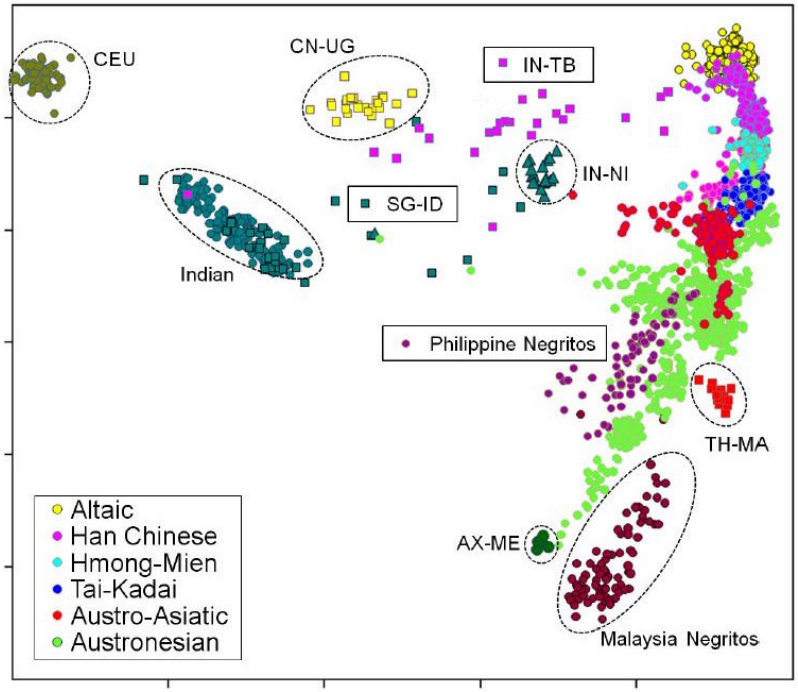

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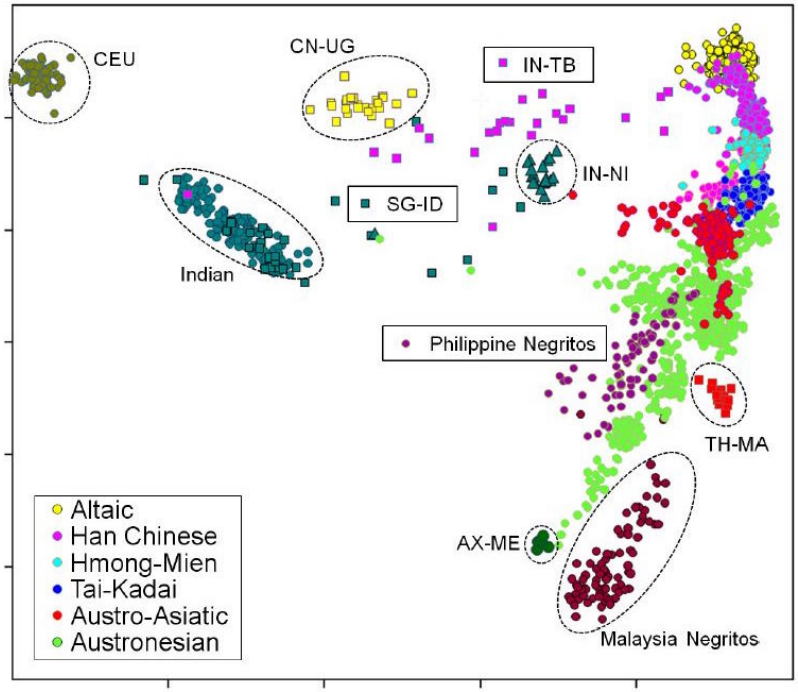

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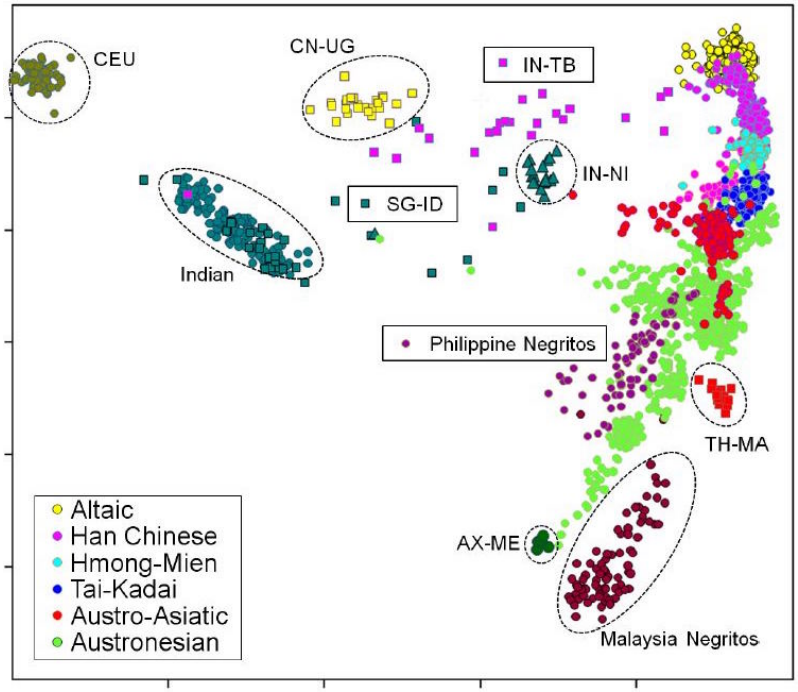

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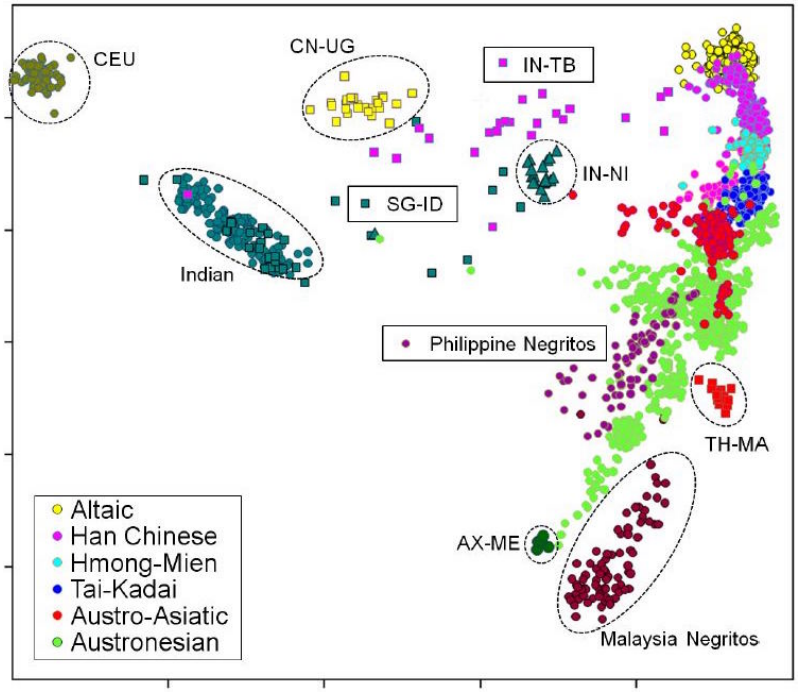

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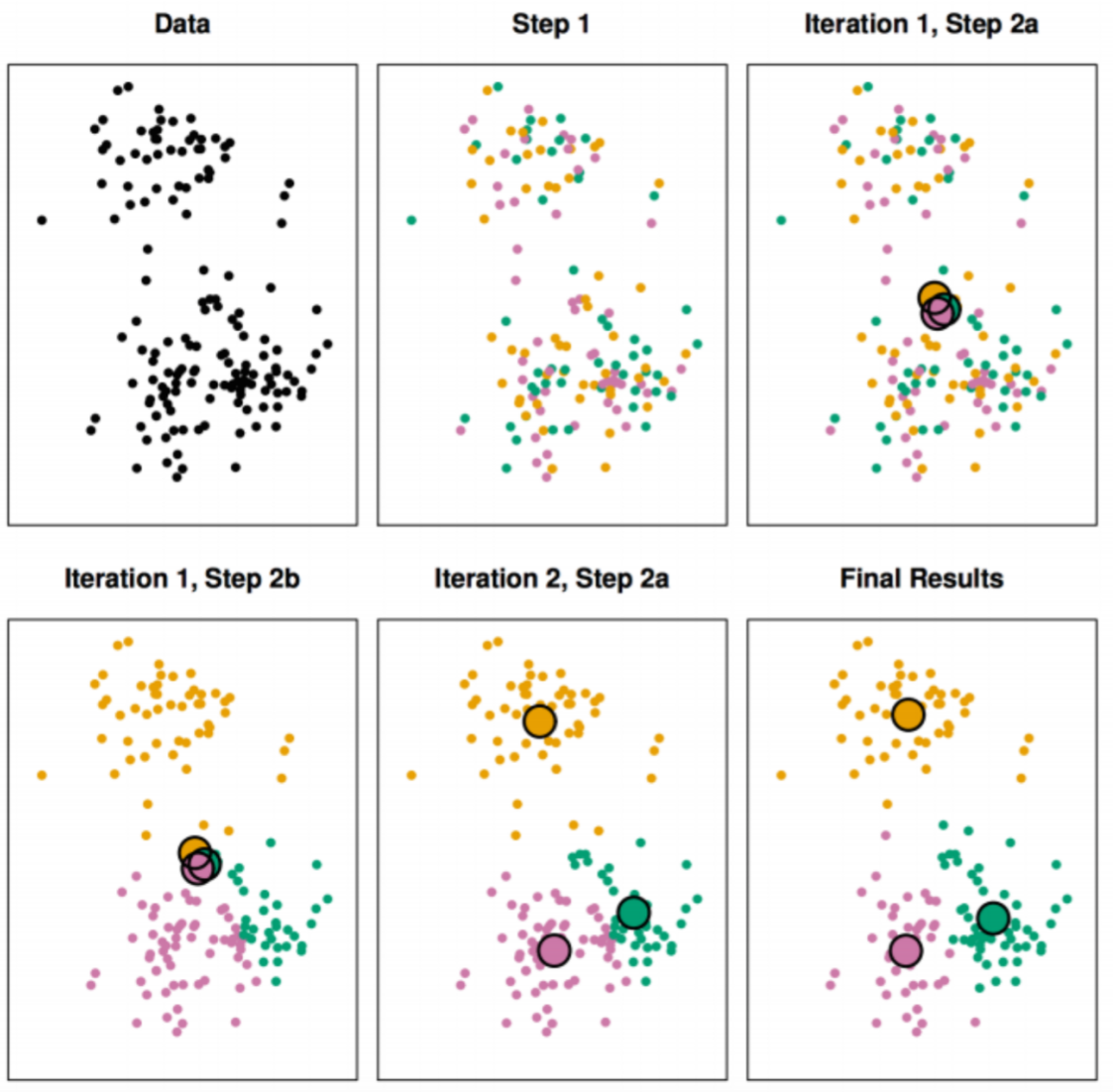
"Labels" are sometimes used to facilitate these objectives but unlike supervised learning, unsupervised learning is not necessarily immediately concerned with predicting outcomes (labels, targets, Y, dependent/endogenous variables)

K-Means

1. Choose *number of clusters*, K
2. Randomly assign data to each cluster k
3. Compute the centroid for each cluster k
4. Assign data to the cluster with the closest centroid
5. Return to step 3, unless the centroids have stabilized



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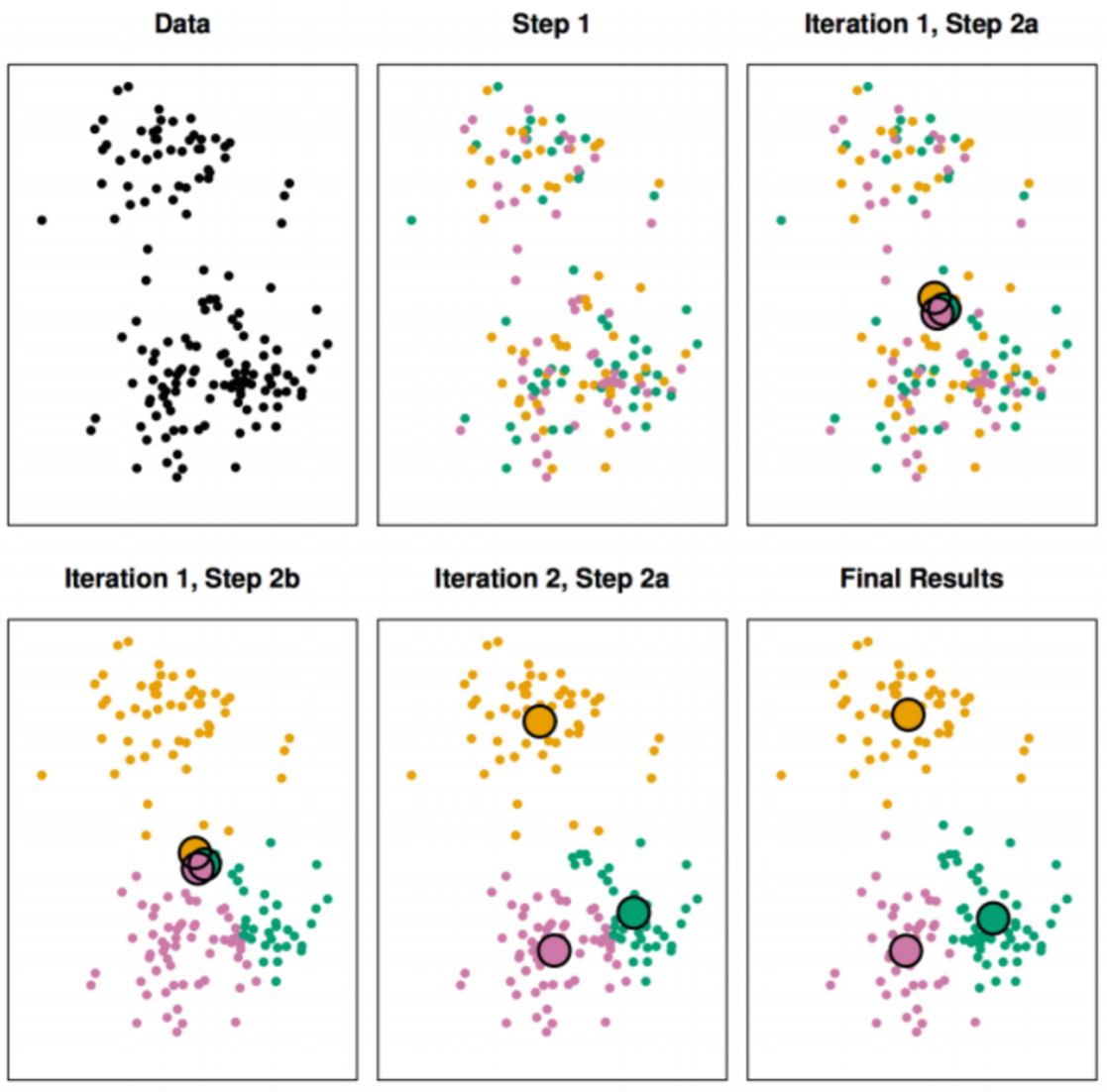
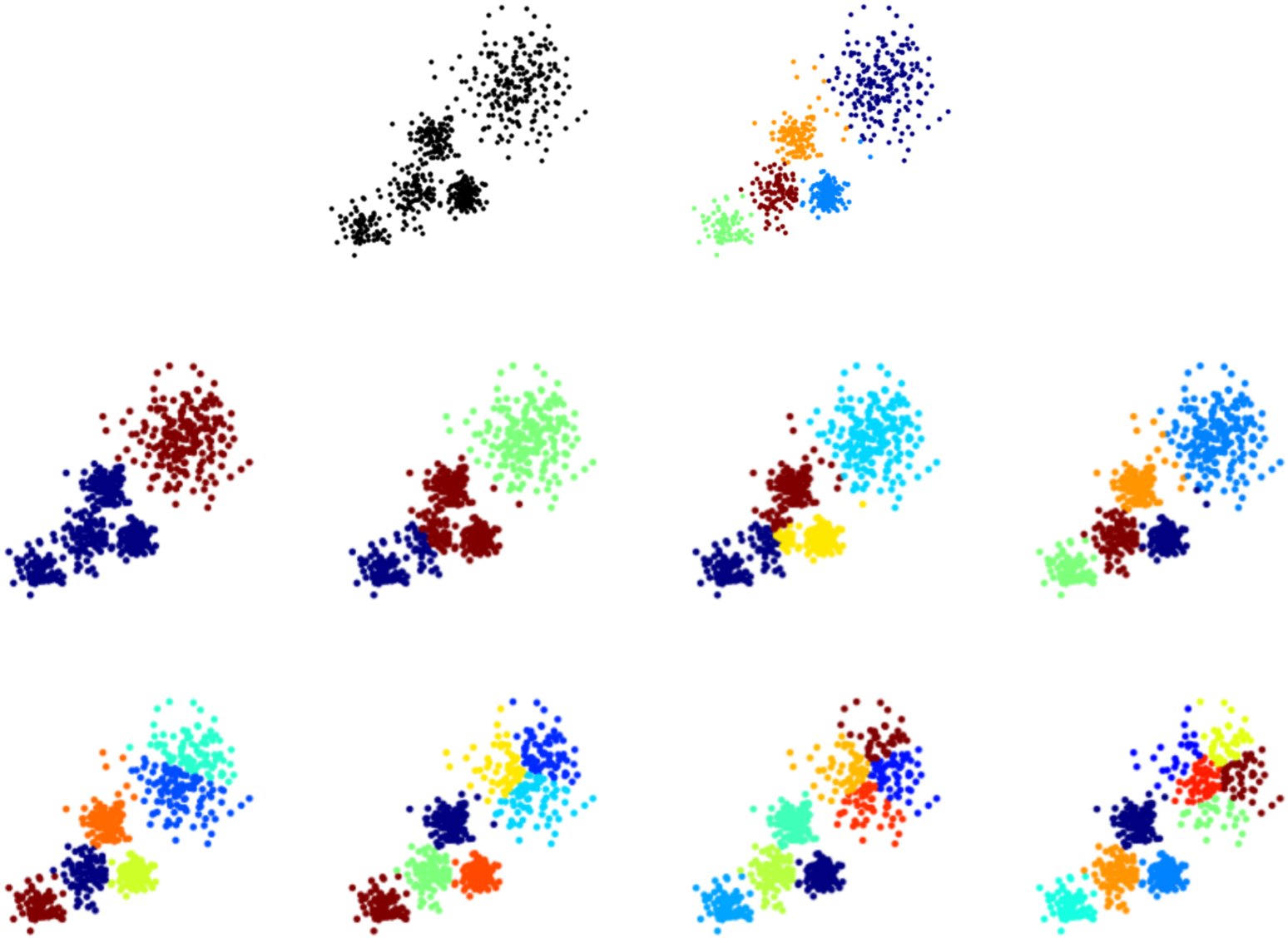


Other init. ideas?

K?



Elbow and Silhouette methods

For some clustering

$C_K(i) \mapsto \{1, 2, \cdots K\}$

clustering fit can be measured as

$$W(C_K) = \frac{1}{K} \sum_{C_K(i)=C_K(j)=k} ||x_i - x_j||^2$$

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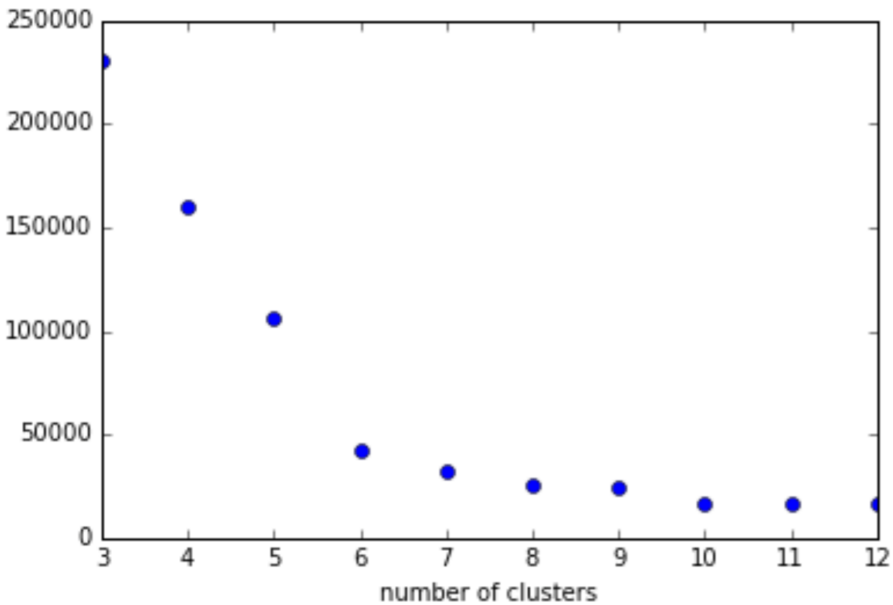

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$\bar{\Delta}_i^k$: x_i's dissimilarity in cluster k
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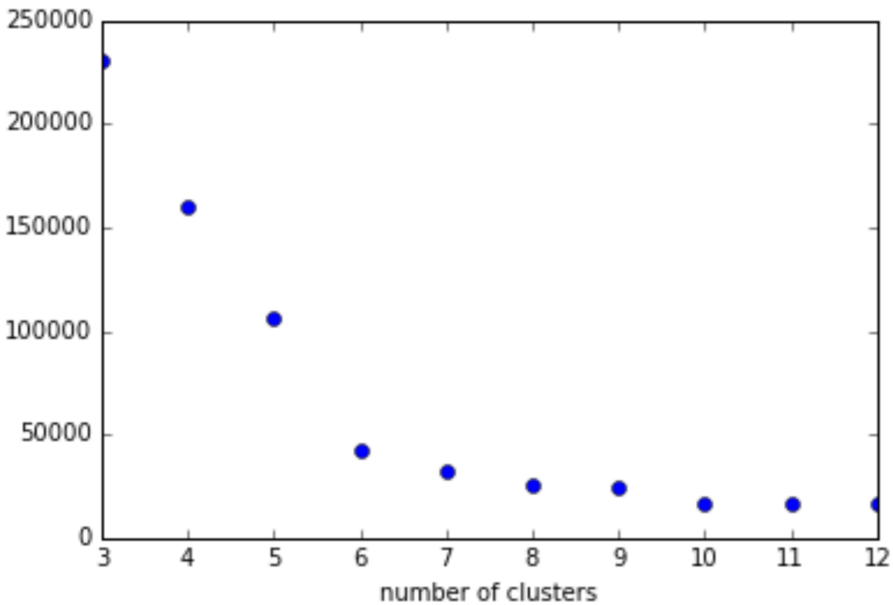

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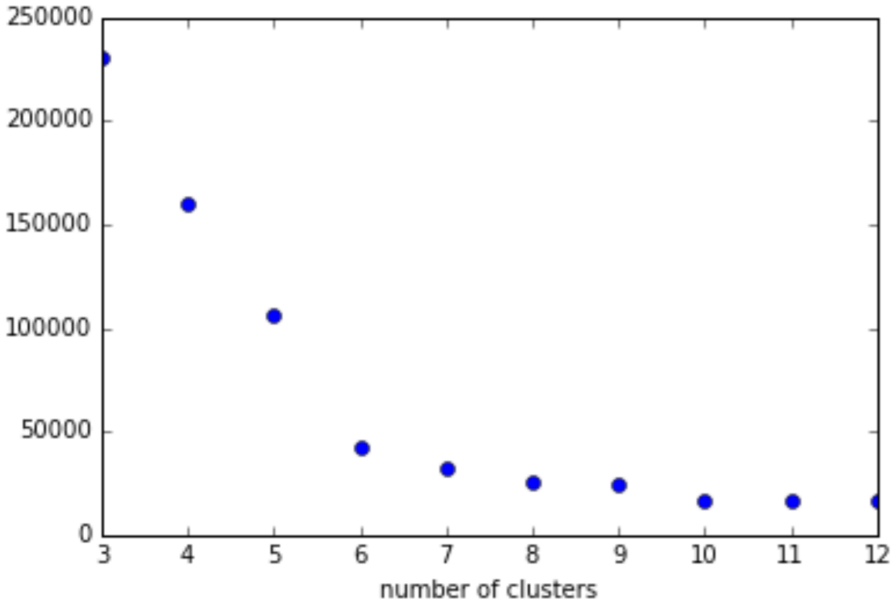
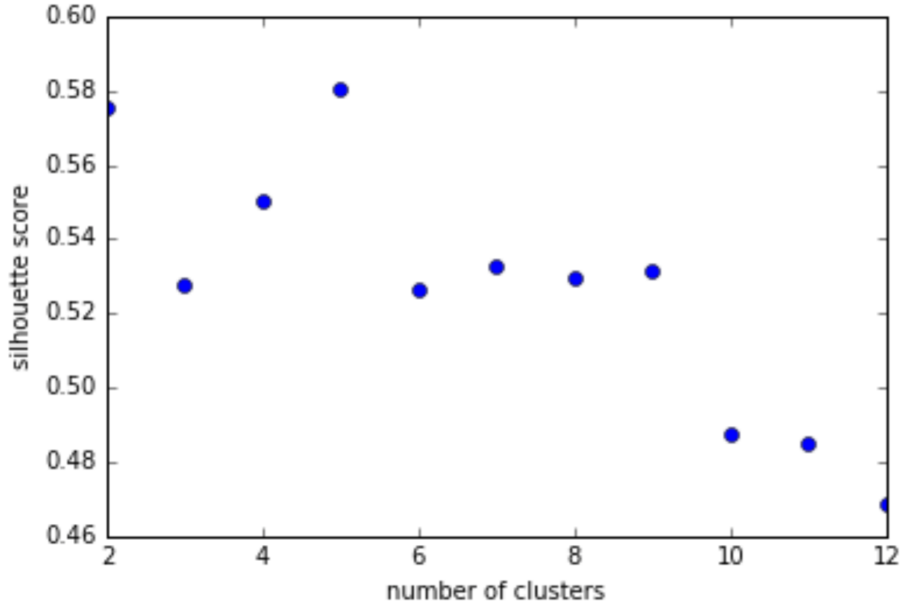


Compare average silhouette scores



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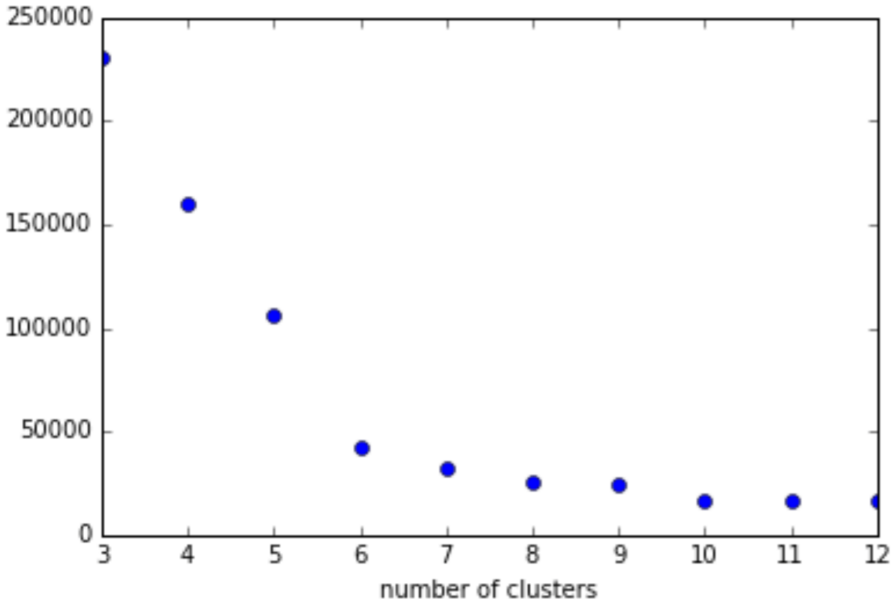
Does the scaling of the x matter?

Select based on diminishing returns



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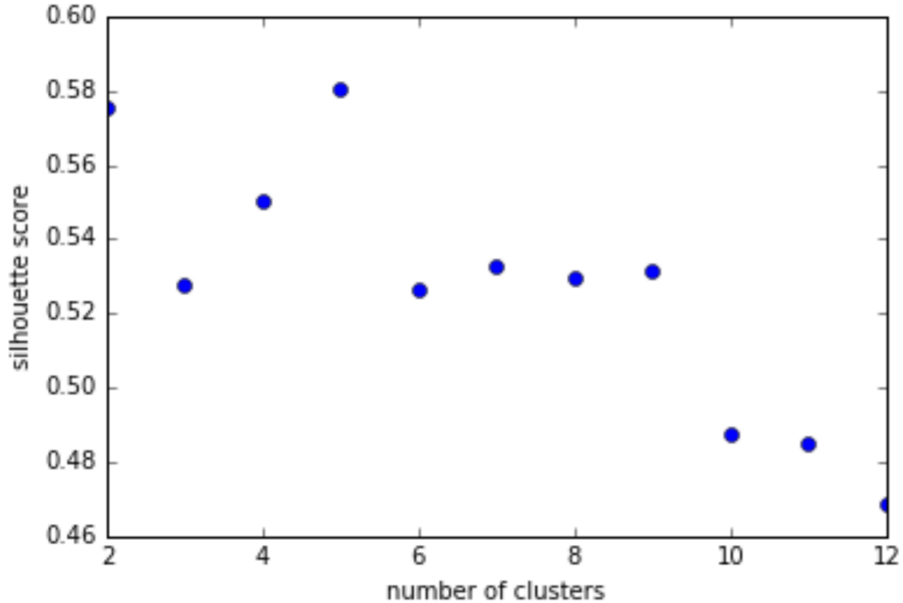

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Interpret silhouette scores?

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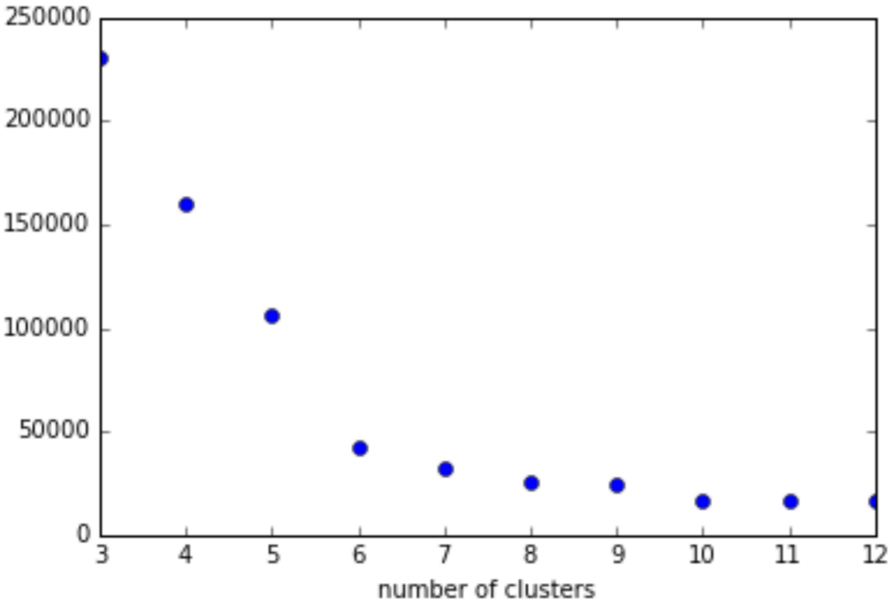

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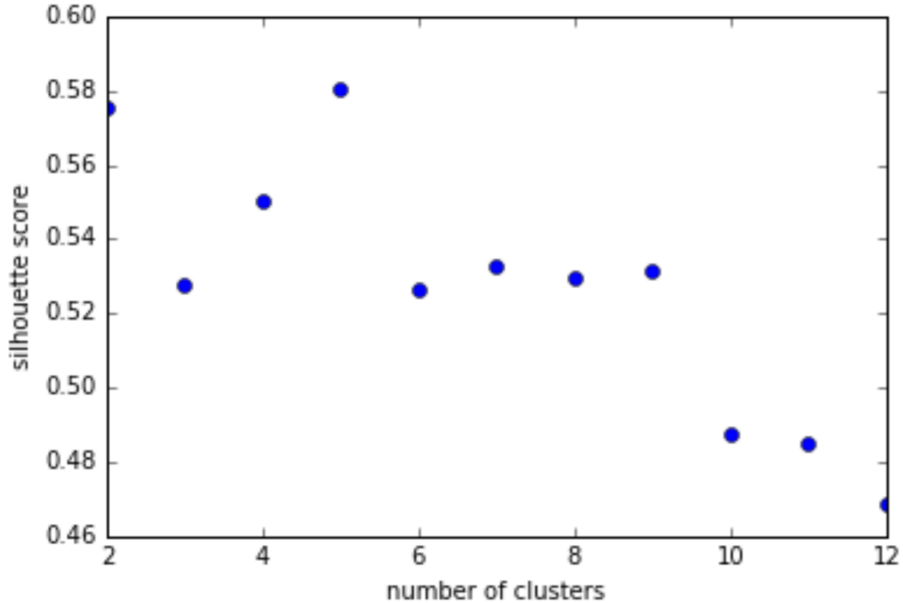

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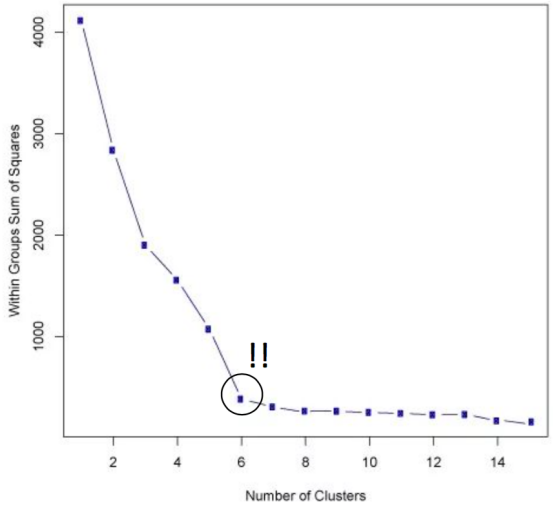

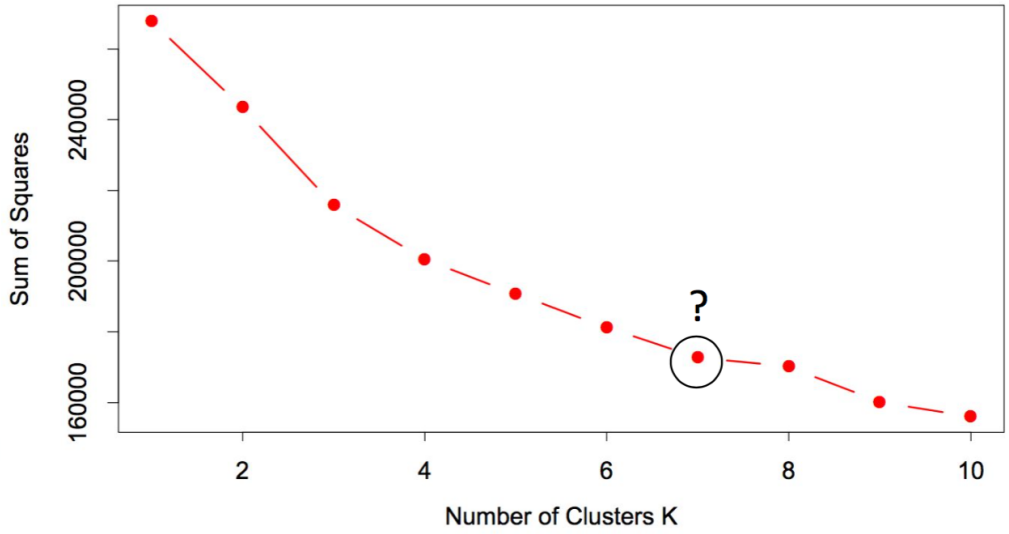

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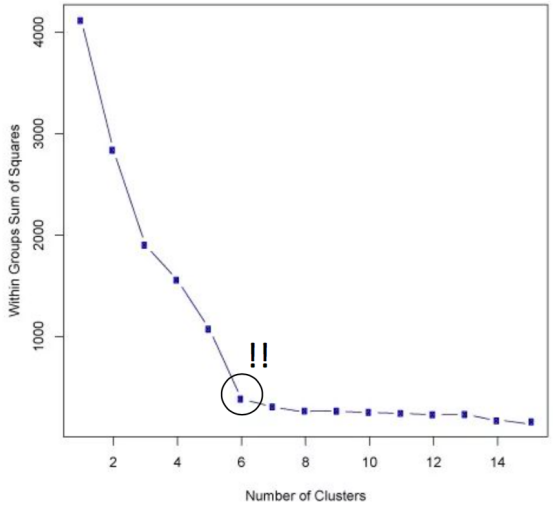

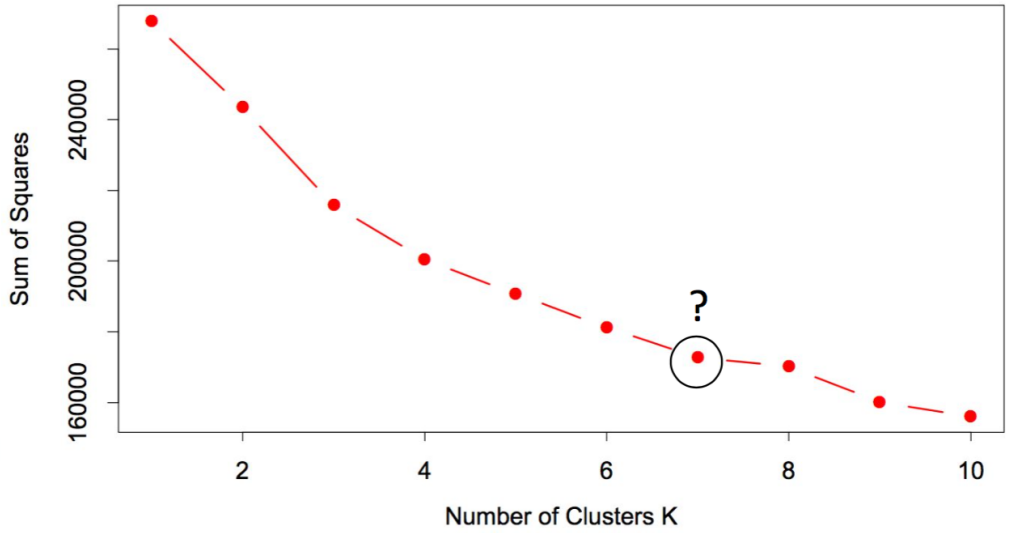

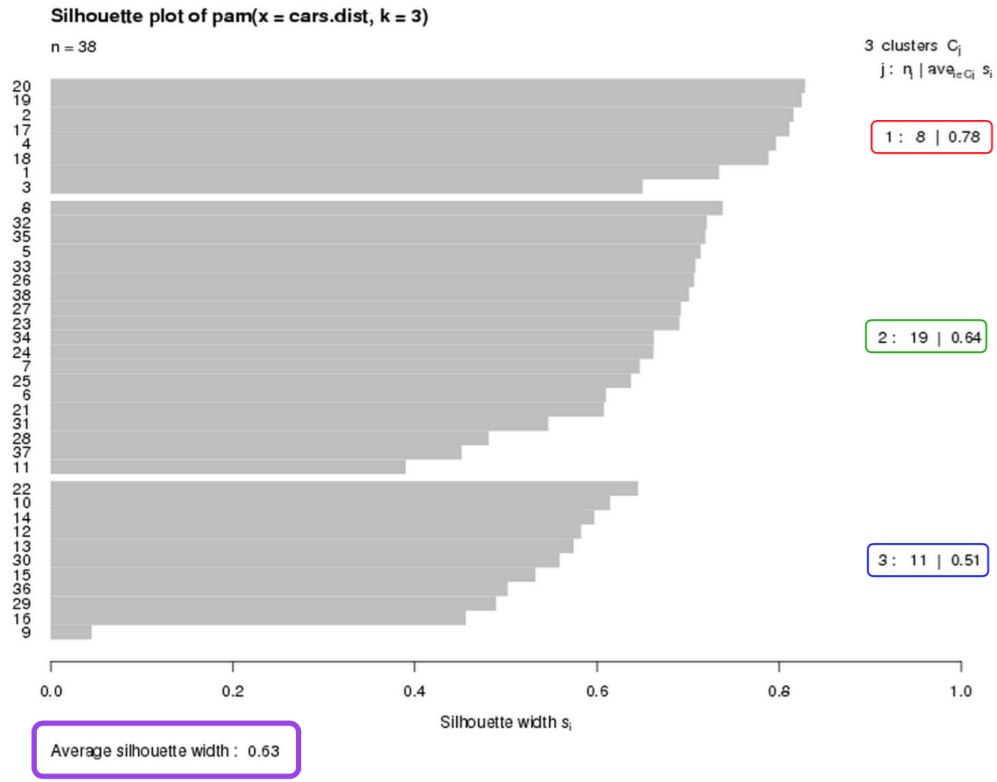

Guidelines for Overall Avg Silhouette

Range	Interpretation
0.71 – 1.0	Strong structure found
0.51 – 0.7	Reasonable structure
0.26 – 0.5	Structure weak/artificial
< 0.25	No substantial structure

The *permutation test*

- Students enrolling for a popular class get ids $i = 1, \cdots n$. As the class quickly fills, another section of the class is opened and students for that class are given ids $i = n + 1, \cdots 2n$.

```
array([ 1,  2,  3,  4,  5,  6,  7,  8,  9, 10, 11, 12, 13, 14, 15, 16, 17,
       18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34,
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       35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51,
       52, 53, 54, 55, 56, 57, 58, 59, 60])
```

- The classes have the same tests...

  How could we test which class is doing better?

```
array([59, 92, 93, 83, 92, 61, 84, 92, 70, 93, 76, 70, 84, 61, 75, 91, 73,
       67, 65, 64, 75, 80, 66, 56, 62, 53, 82, 69, 85, 94, 80, 86, 97, 99,
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- If the classes are doing equally well, the index doesn't matter

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       35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51,
       52, 53, 54, 55, 56, 57, 58, 59, 60])
```

- The classes have the same tests...
  How could we test which class is doing better?

```
array([59, 92, 93, 83, 92, 61, 84, 92, 70, 93, 76, 70, 84, 61, 75, 91, 73,
       67, 65, 64, 75, 80, 66, 56, 62, 53, 82, 69, 85, 94, 80, 86, 97, 99,
       77, 84, 94, 70, 80, 87, 77, 85, 95, 80, 88, 90, 85, 84, 92, 75, 83,
       89, 76, 95, 91, 97, 80, 70, 71, 94])
```

- If the classes are doing equally well, the index doesn't matter
  $\Longleftrightarrow$ We could *permute* the index and see if it matters

The *permutation test*

- Students enrolling for a popular class get ids $i = 1, \cdots n$. As the class quickly fills, another section of the class is opened and students for that class are given ids $i = n + 1, \cdots 2n$.

```
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       18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34,
       35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51,
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       67, 65, 64, 75, 80, 66, 56, 62, 53, 82, 69, 85, 94, 80, 86, 97, 99,
       77, 84, 94, 70, 80, 87, 77, 85, 95, 80, 88, 90, 85, 84, 92, 75, 83,
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       18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34,
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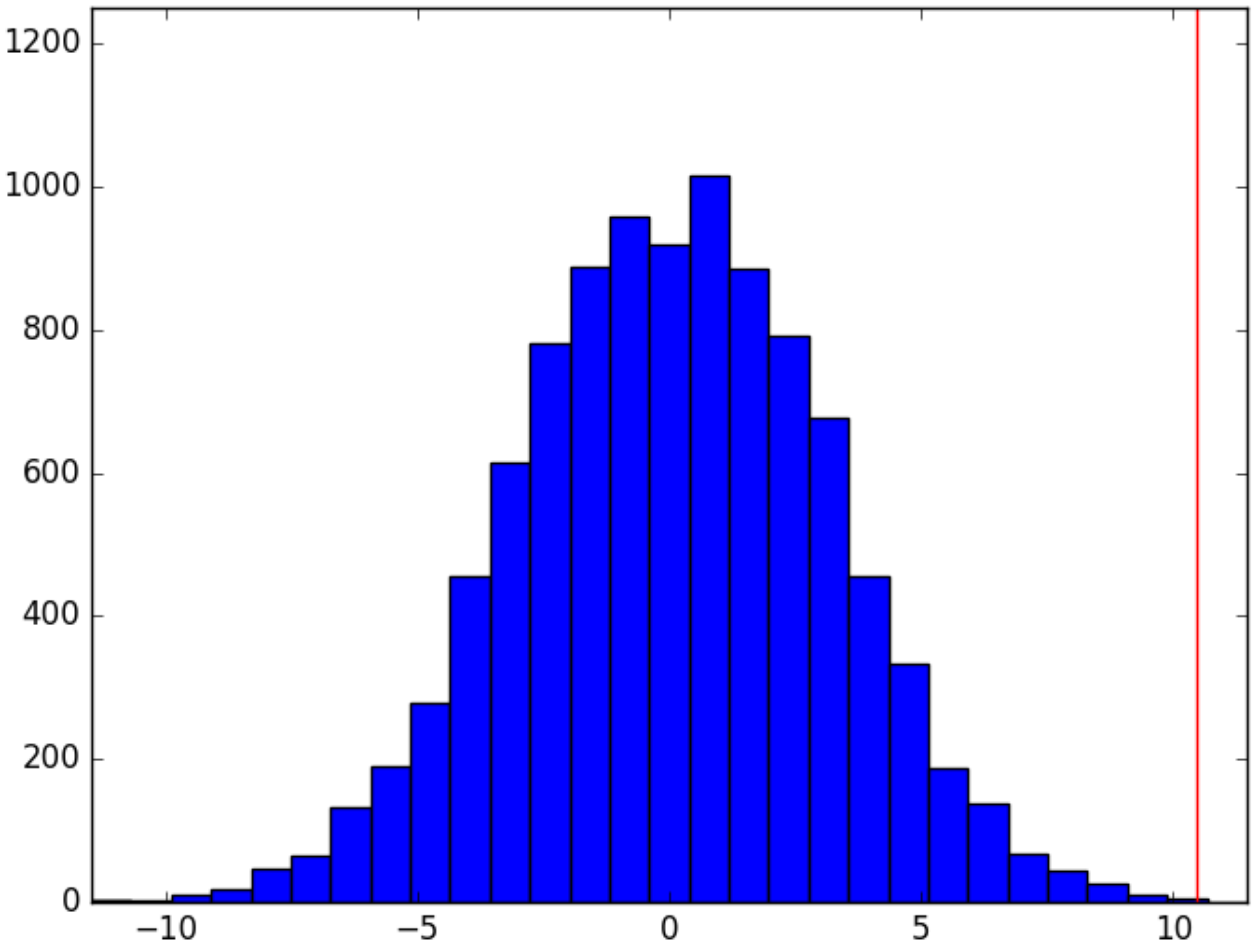
  $\Longleftrightarrow$ We could *permute* the index and see if it matters

1. Repeatedly permute the index and recalculating the test statistic each time
2. These samples approximate the test statistic distribution under the null
3. Compare the test statistic to this null distribution
   to suggest how "strange" the actual observed test statistic is if the null is true

The *permutation test* *(This is my* **favorite** *test, btw)*

1. Permute the ids (i.e., believe the null is true: ids don't matter)
2. Recalculate the test statistic each time (under null)
3. See how strange your observed statistic is compared to nulls



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- What was needed in the permutation test was really the distribution of the test statistic under the null hypothesis
- We used permutation to get it
  but that was just a means to an end
- We can test against *ANY* null distribution we wish to propose

The Gap statistic

Then choose the smallest K such that $\text{Gap}(K) \geq \text{Gap}(K+1) - s_{K+1}$

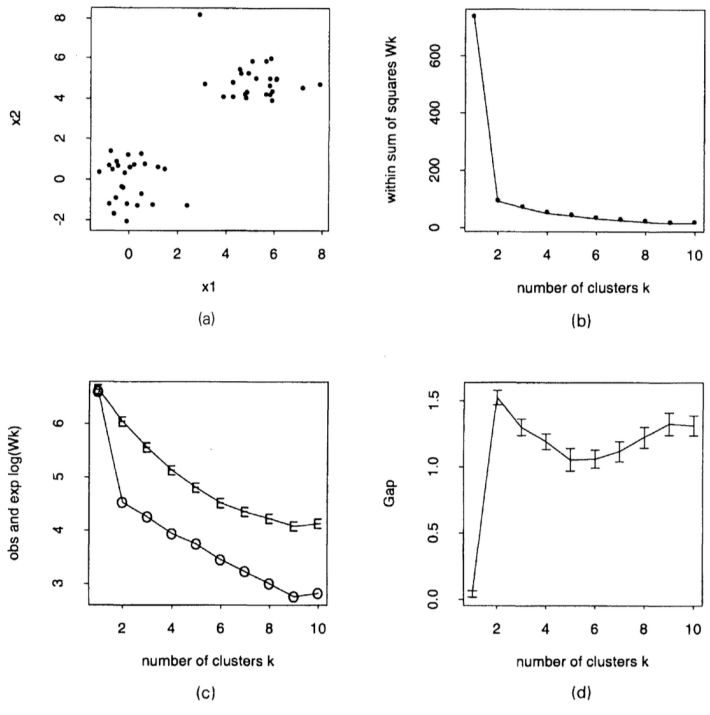

Fig. 1. Results for the two-cluster example: (a) data; (b) within sum of squares function W_k; (c) functions $\log(W_k)$ (O) and $\hat{E}_n^*\{\log(W_k)\}$ (E); (d) gap curve

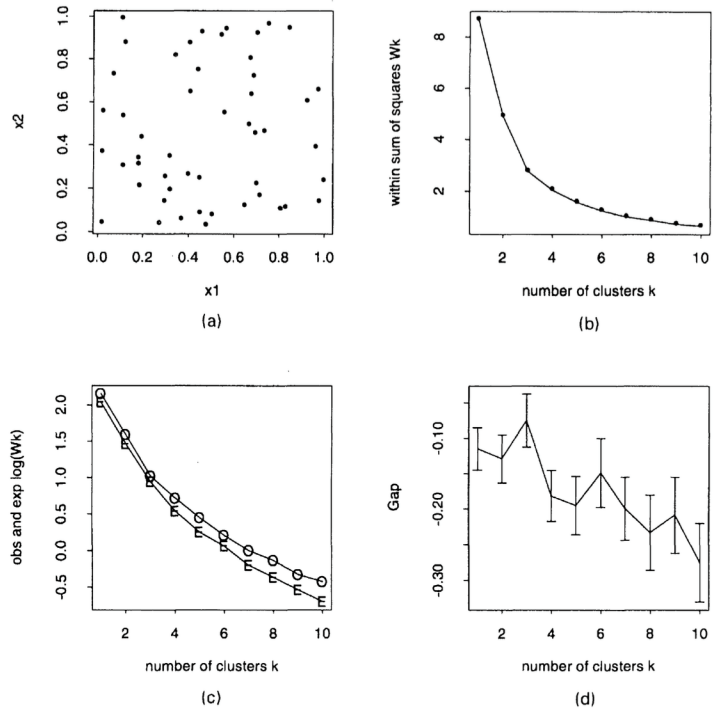

Fig. 2. Results for the uniform data example: (a) data; (b) within sum of squares function W_k; (c) functions $\log(W_k)$ (O) and $\hat{E}_n^*\{\log(W_k)\}$ (E); (d) gap curve

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For M null distribution samples, calculate

$$\text{Gap}(K) = \bar{l} - \log W^{(*)}(C_K) \quad \text{and} \quad s_K = \sqrt{\frac{1}{R}\sum_{j=1}^{M}\left(\log W^{(r)}(C_K) - \bar{l}\right)^2}$$

$$\text{where } W^{(r)}(C_K) = \sum_{C_K(i)=C_K(j)=k} ||x_i^{(r)} - x_j^{(r)}||^2 \quad \text{and} \quad \bar{l} = \frac{1}{R}\sum_{r=1}^{R}\log W^{(r)}(C_K)$$

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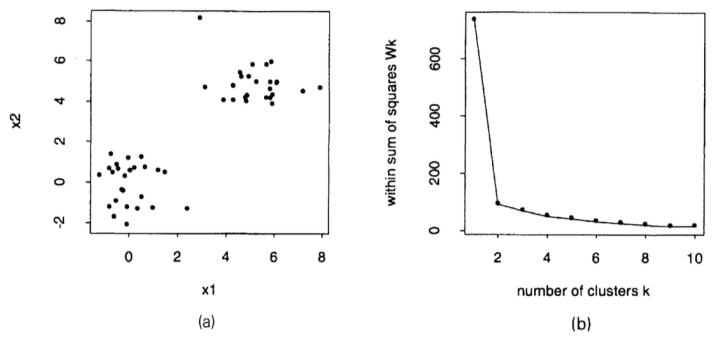

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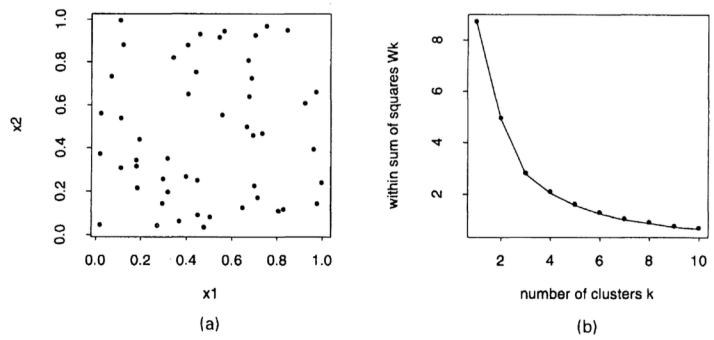

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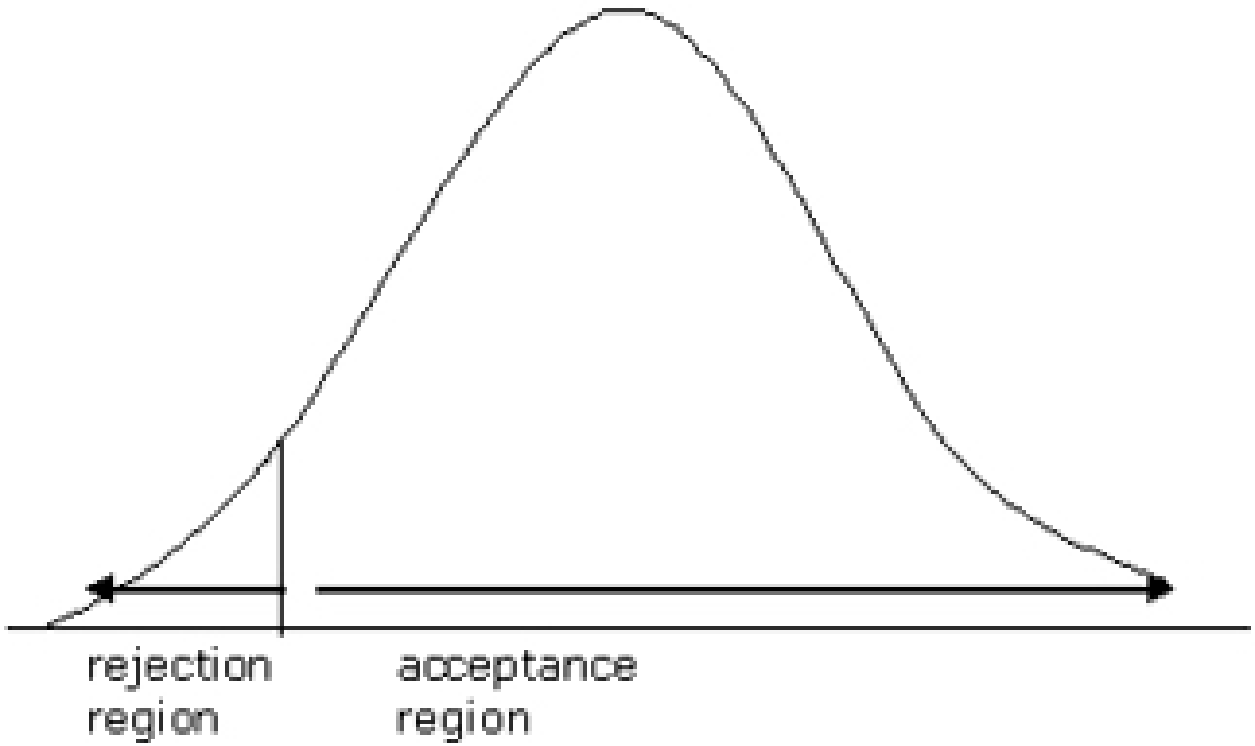

The curse of dimensionality

Just as nearest neighbors breaks down in high dimensional space...
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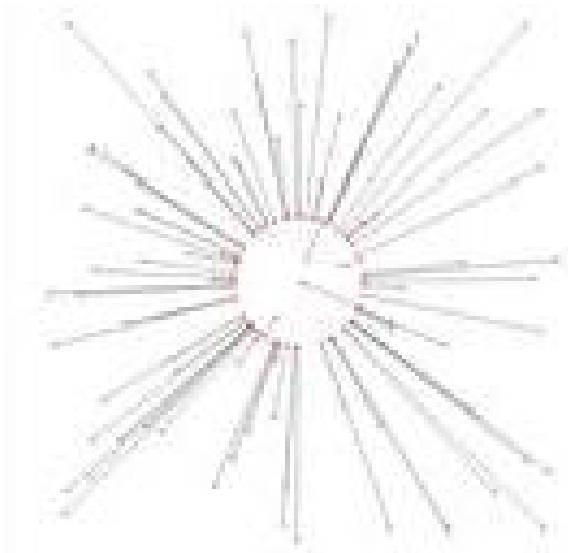
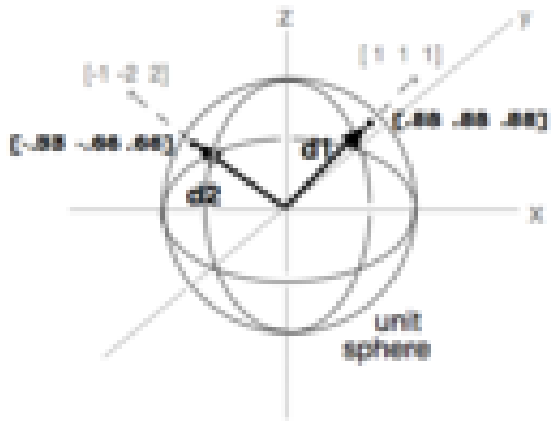
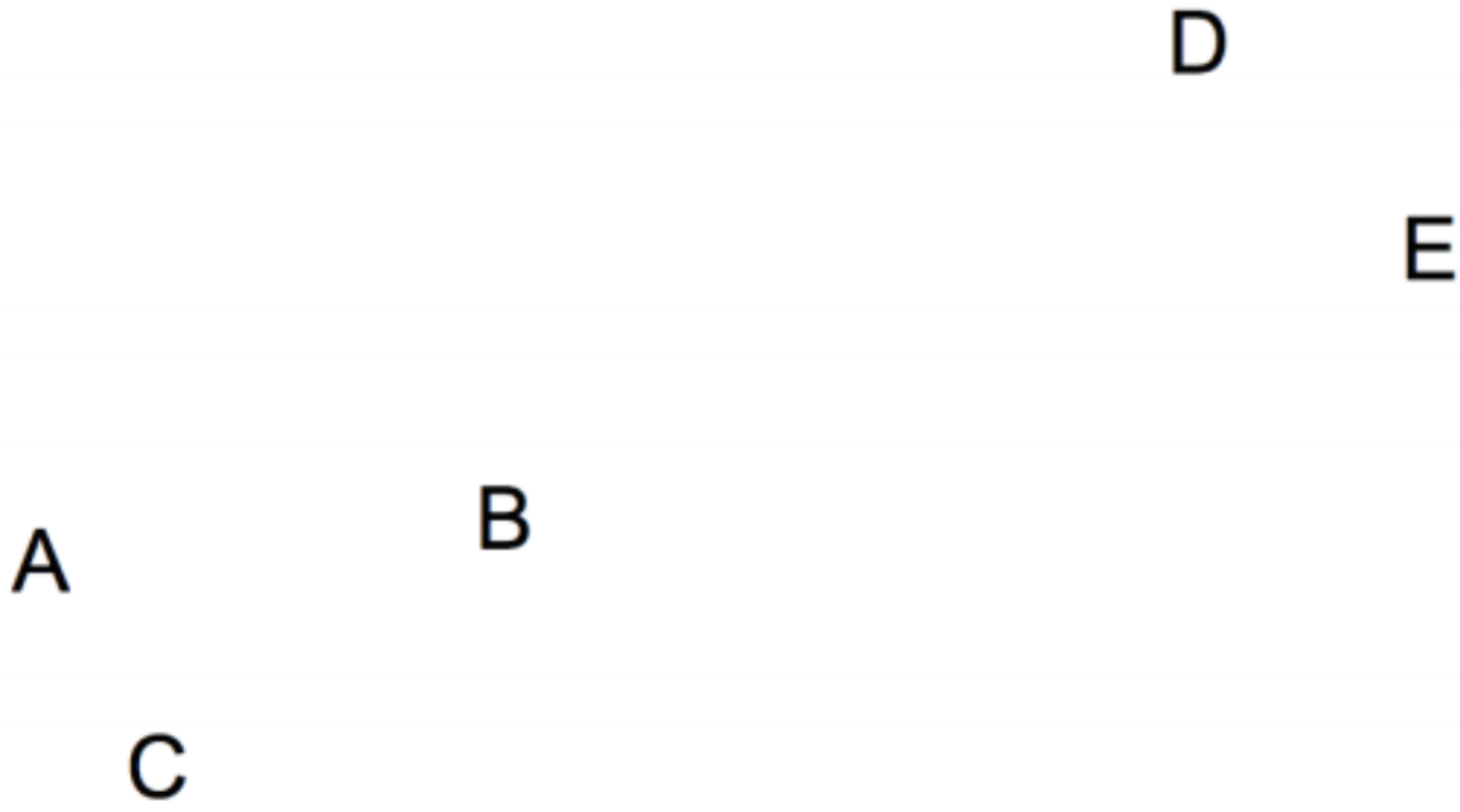
However, when we normalize a vector (e.g., turn bag-of-words into probabilities, or make it a unit vector) we project all vectors onto a *manifold* where "locality" and "neighborhoods" make sense again.

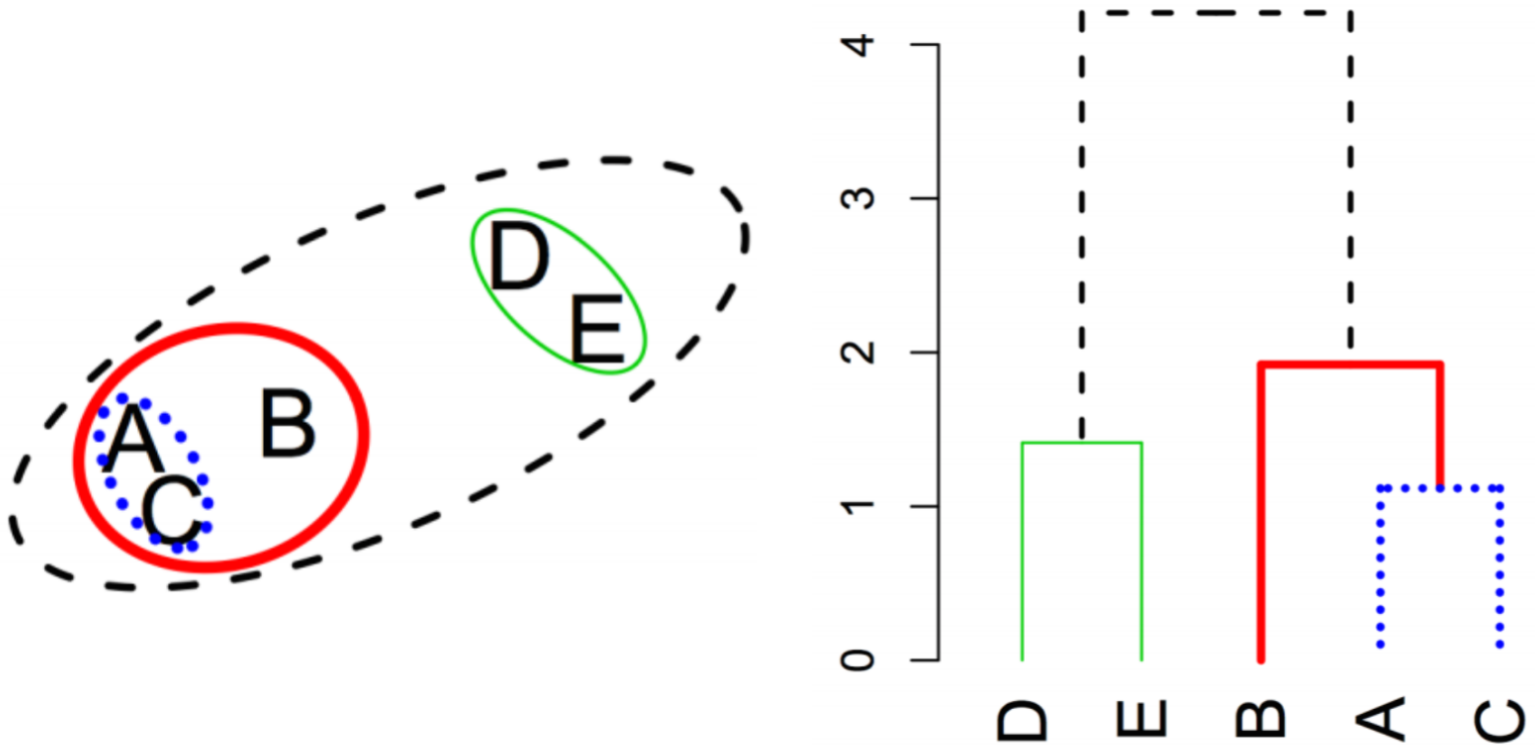




Challenge

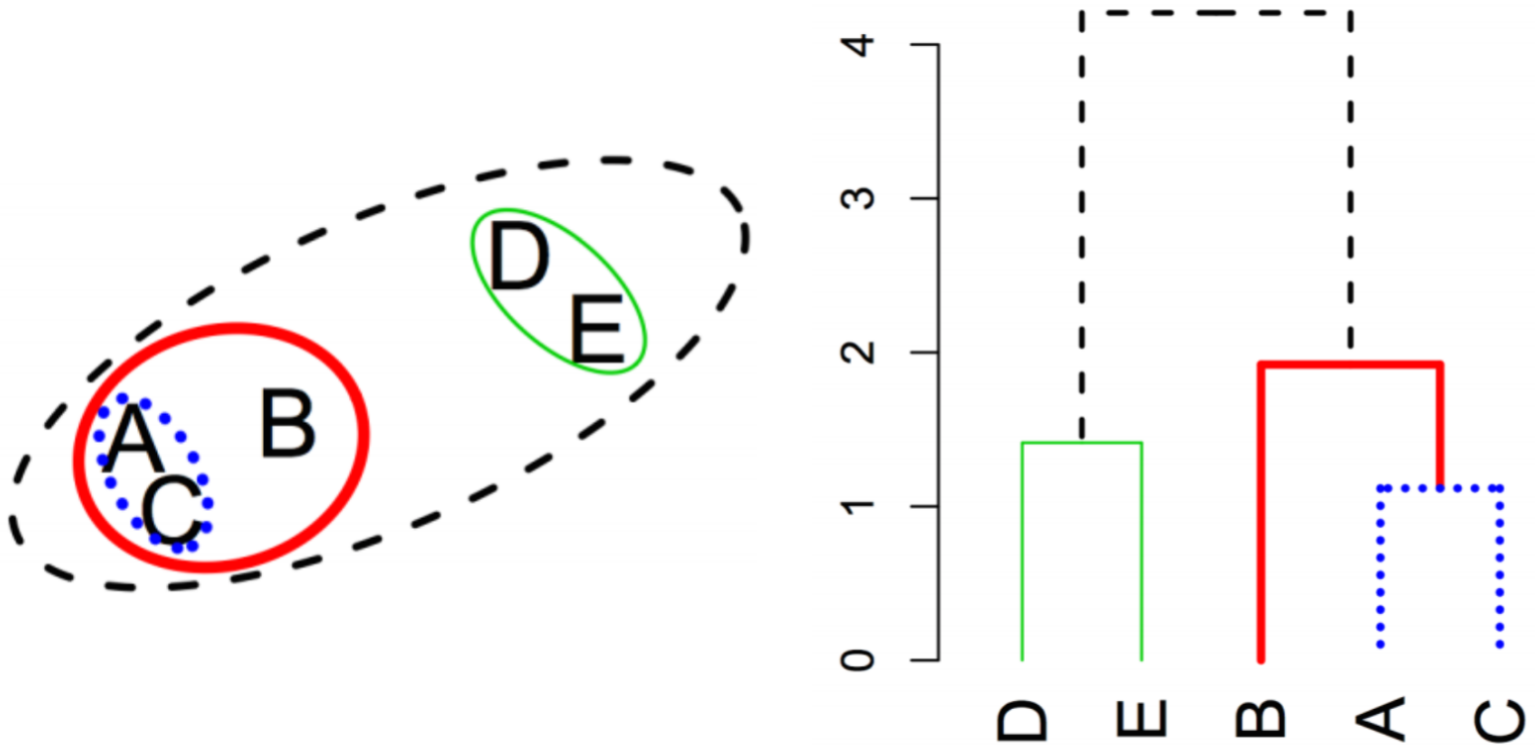


Hierarchical clustering



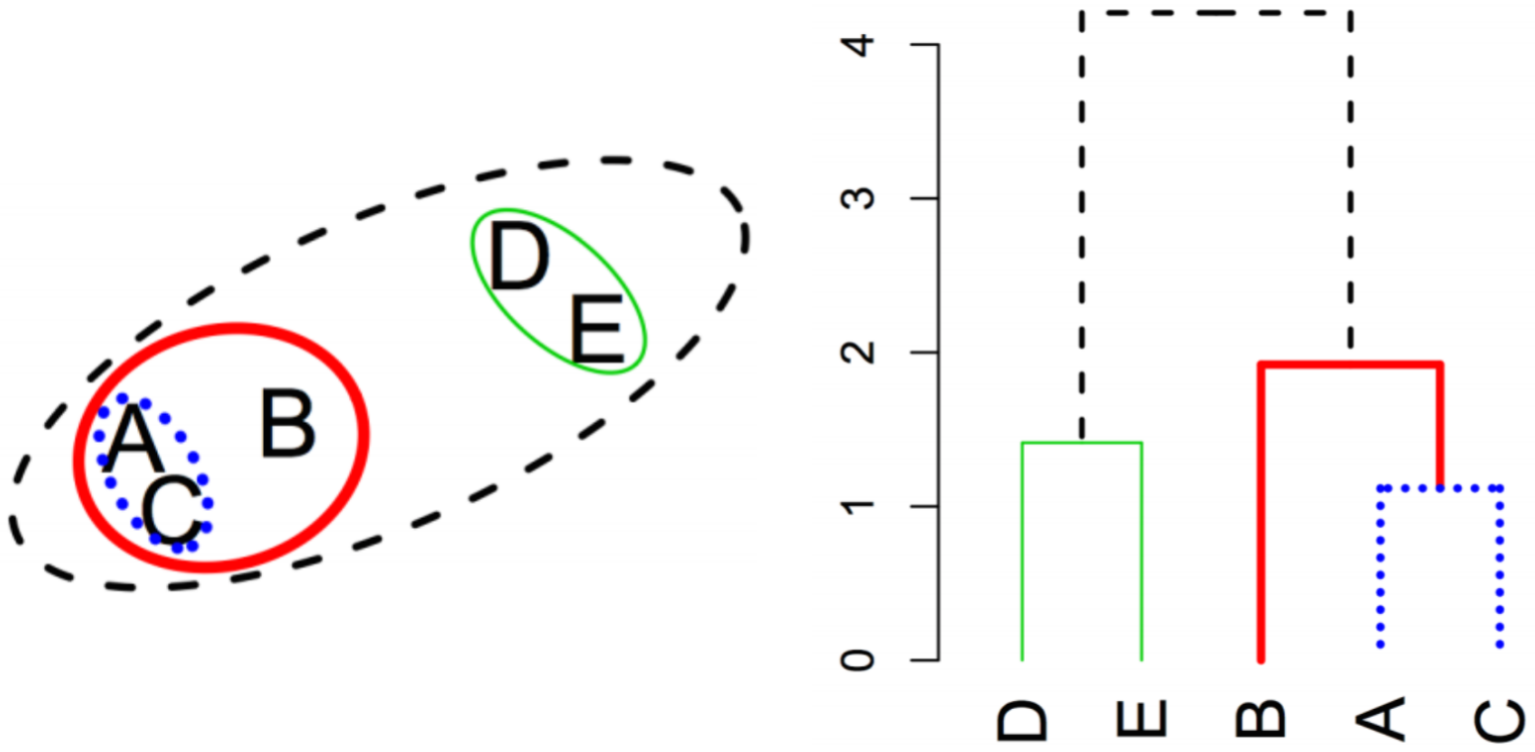
1. Assign each point to it's own cluster

Hierarchical clustering



1. Assign each point to it's own cluster
2. Computer pairwise cluster distances

Hierarchical clustering



1. Assign each point to it's own cluster
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3. Merge *closest two* clusters

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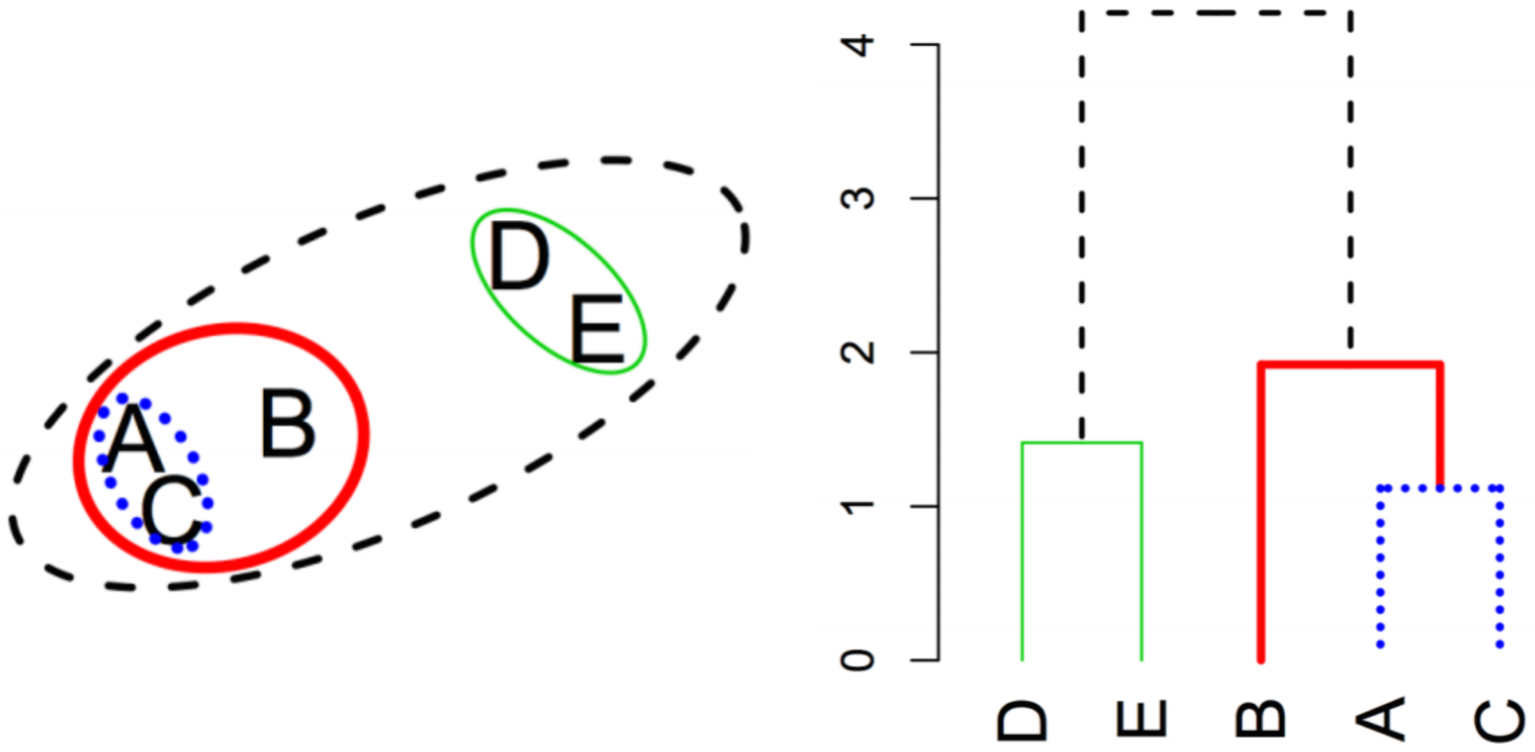

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4. Return to 2 until all clusters merged

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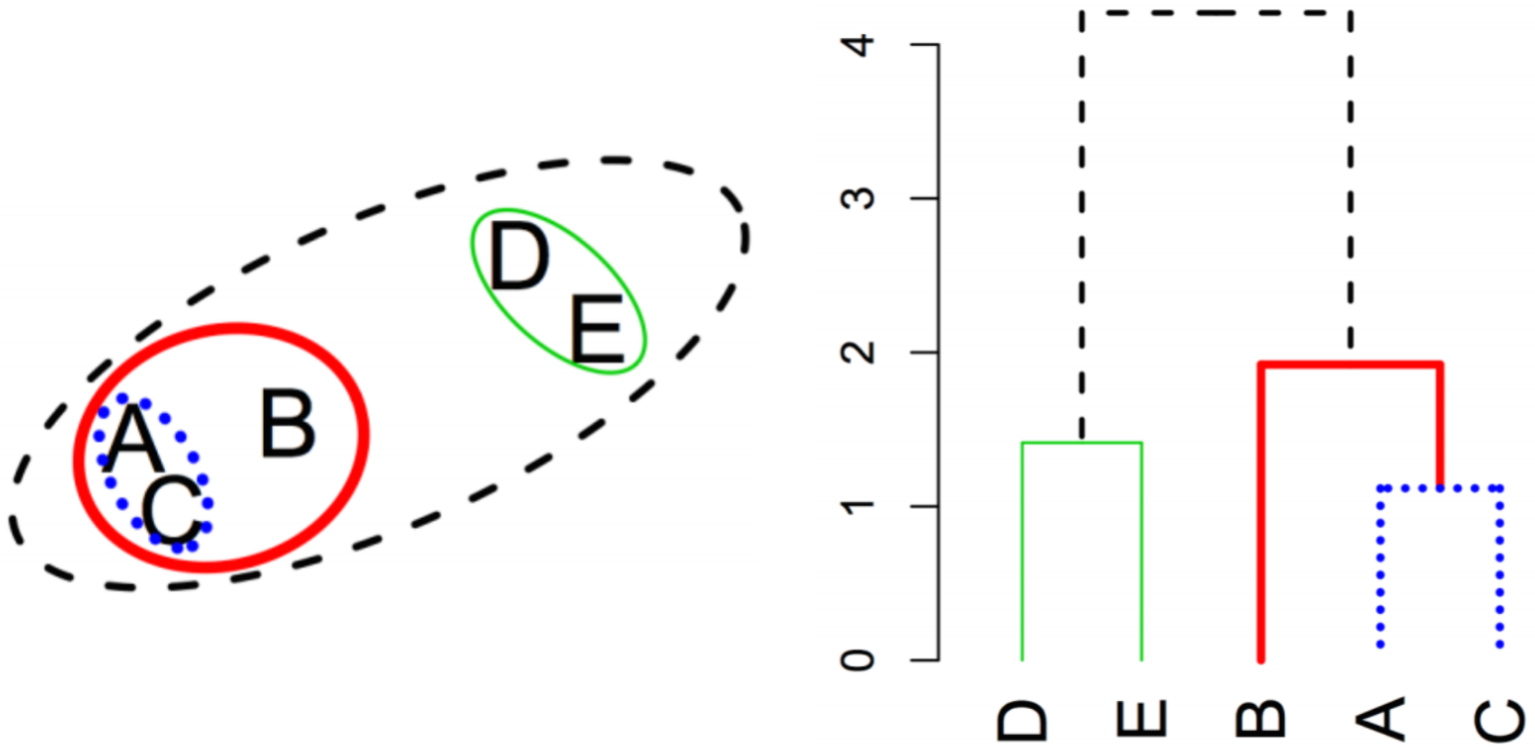

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- Single: minimum pairwise point dissimilarity
- Complete: maximum pairwise point dissimilarity
- Average: average pairwise point dissimilarity
- Centroid: centroid dissimilarity

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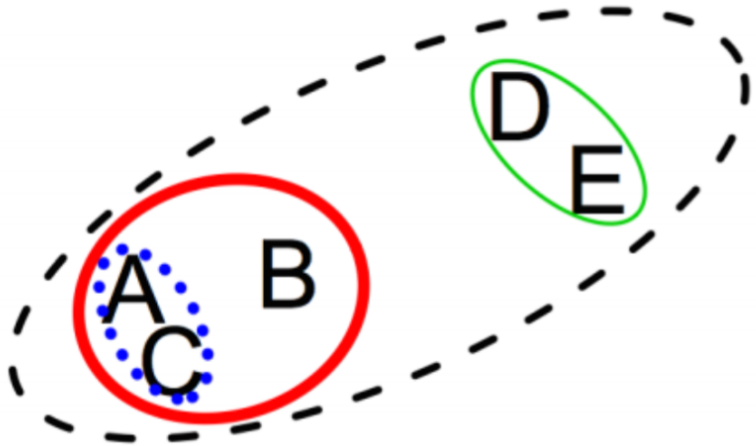

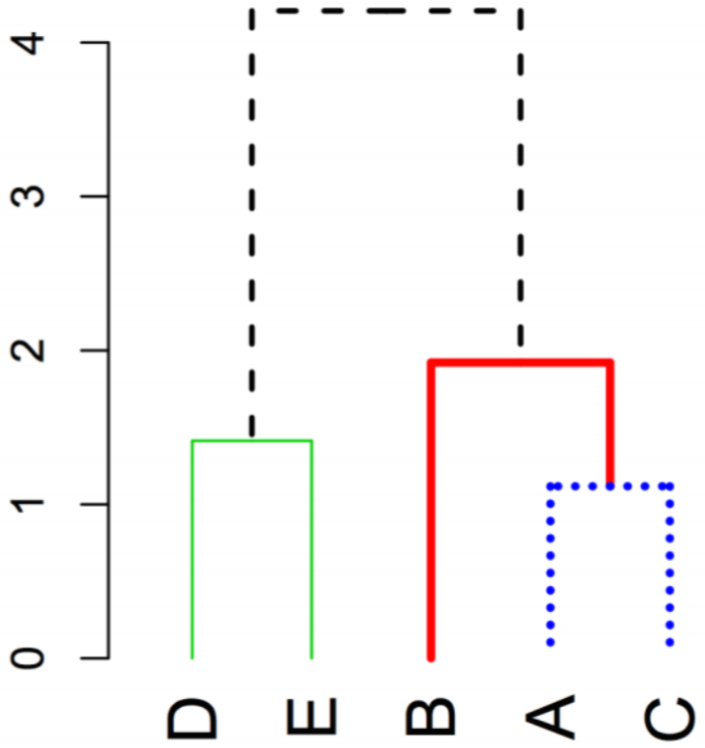

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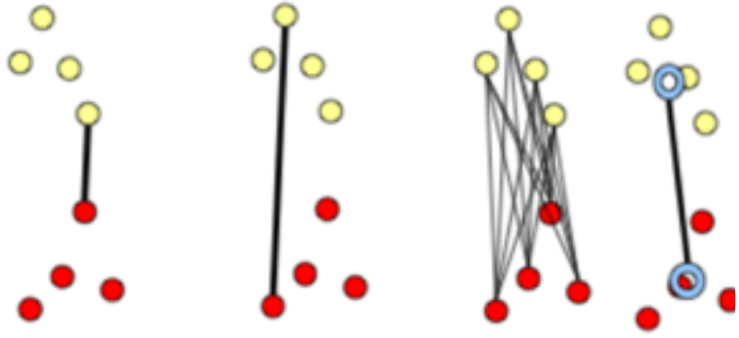

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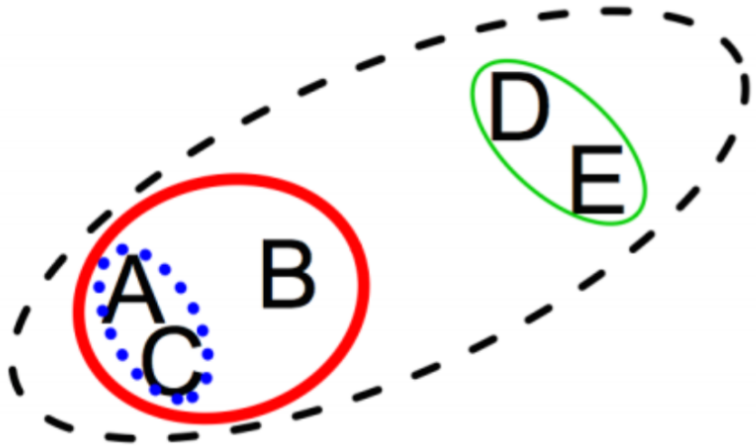

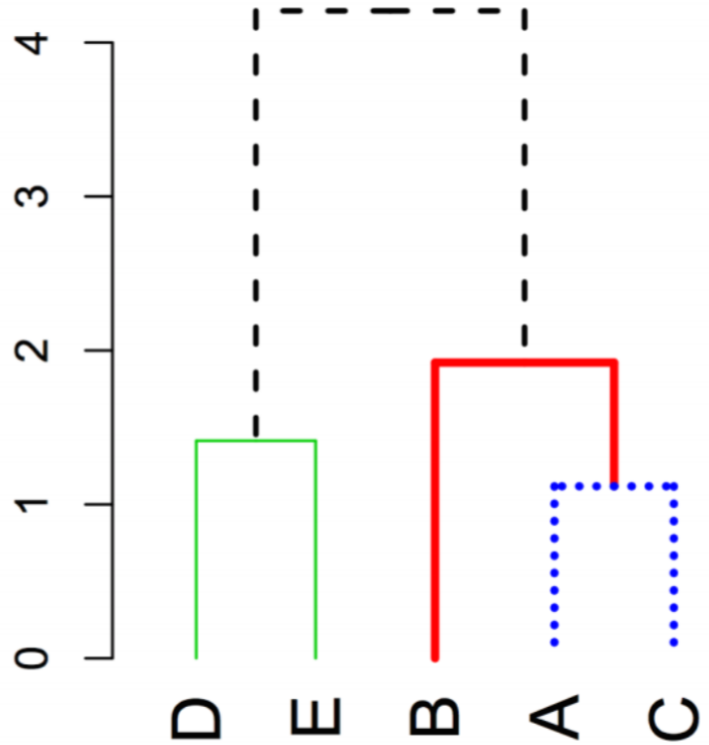

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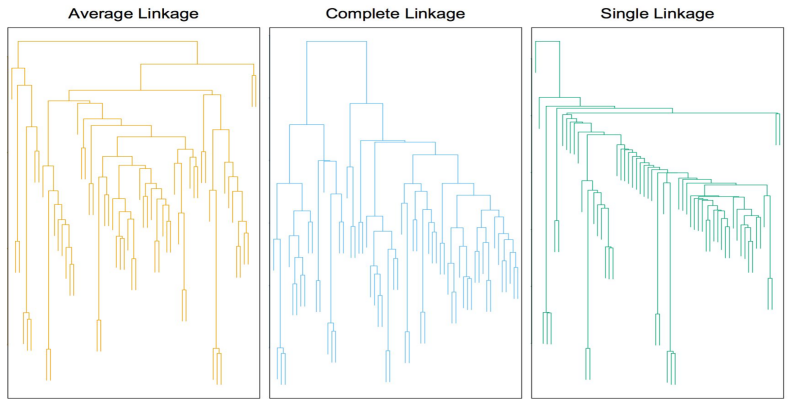
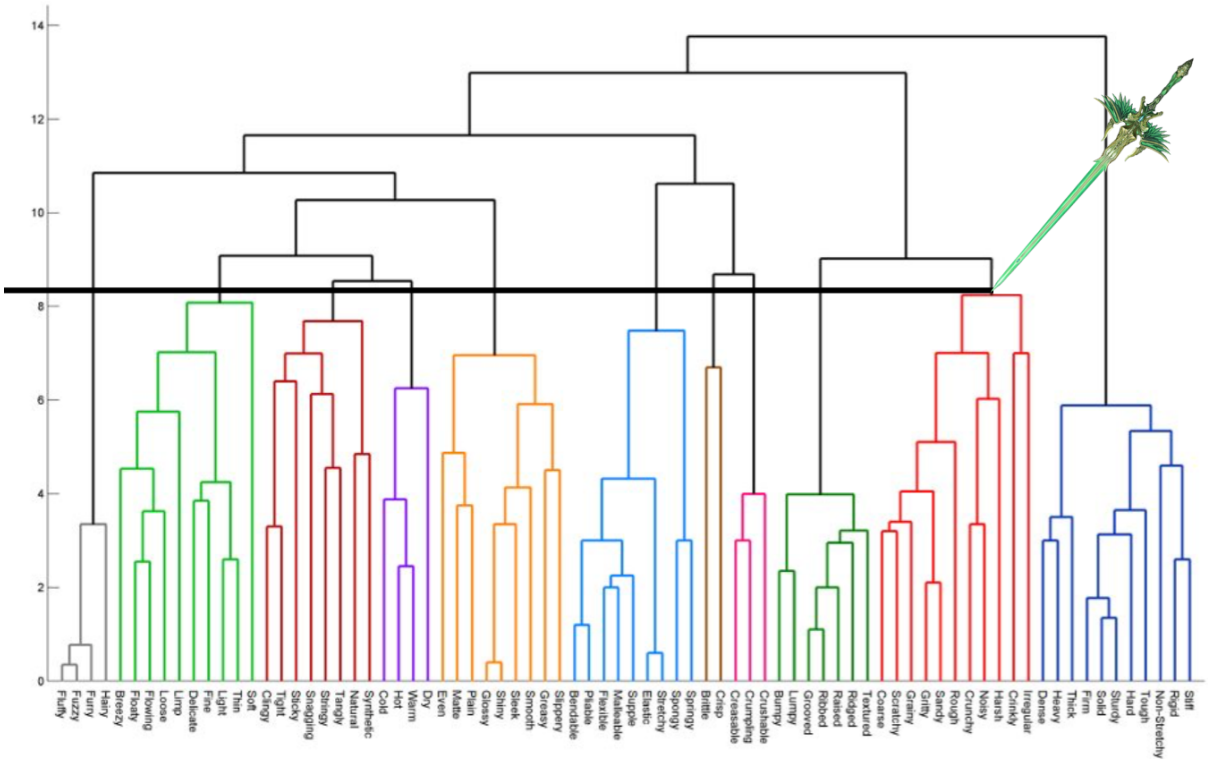
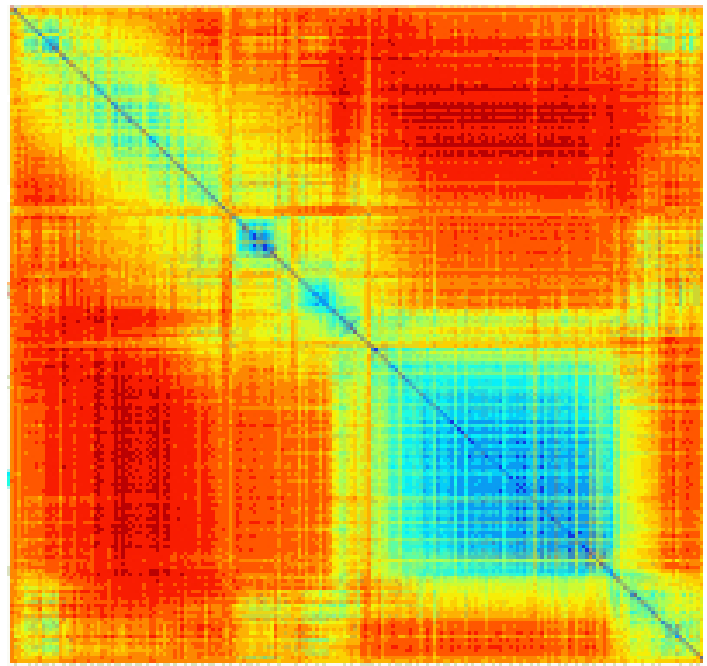


Hierarchical clustering

The number of clusters can be chosen like K-means

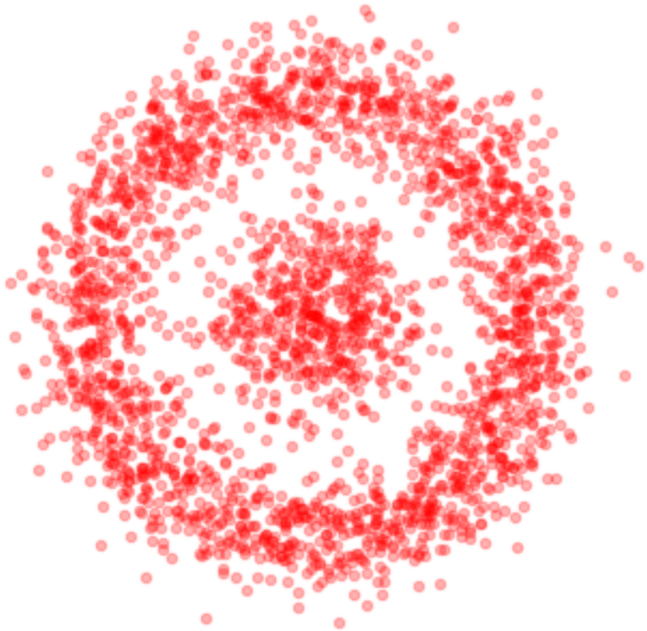
Or on the basis of distance dissimilarity between proposed merges





Unlike K-means, hierarchical clustering requires all pairwise comparisons so it doesn't scale gracefully with increasing data...

DBSCAN



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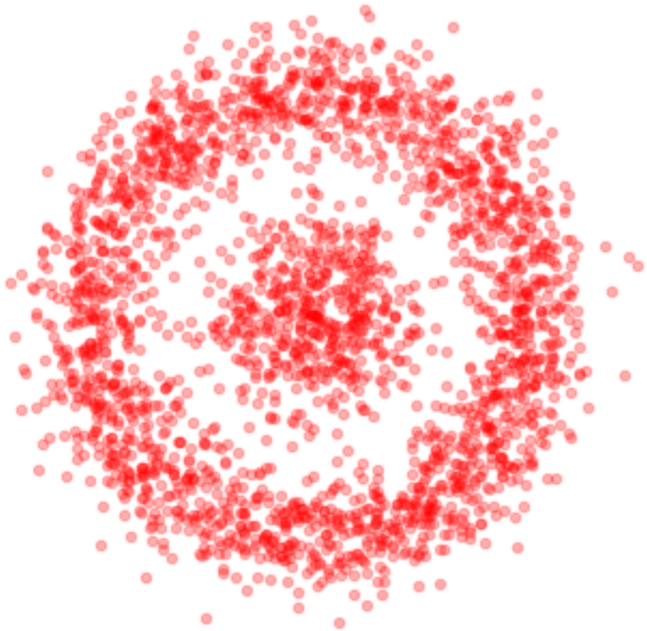
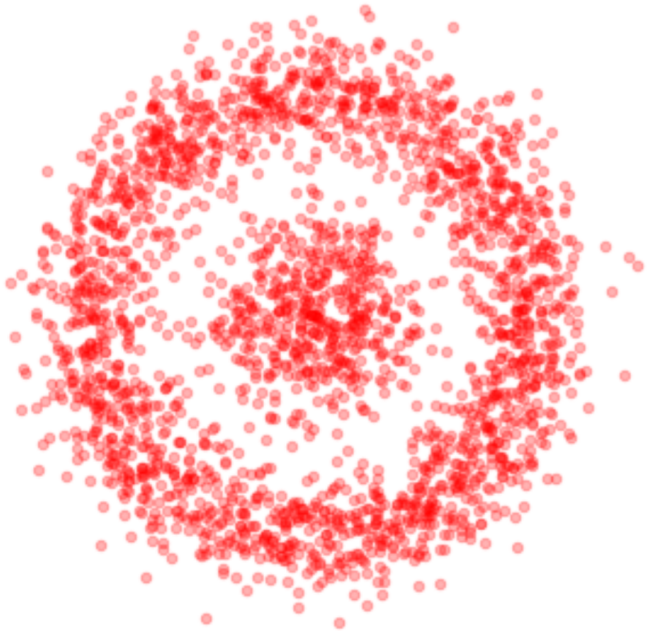


- Specify connection distance ϵ and minimum number of points for core m

DBSCAN



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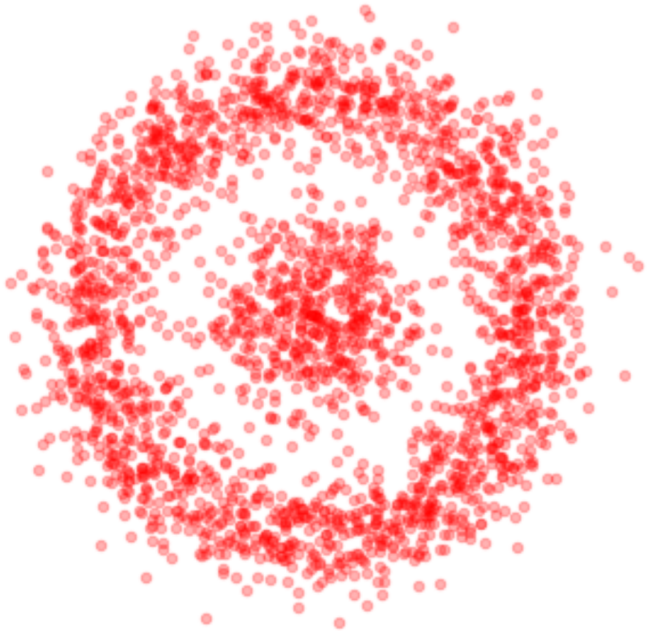

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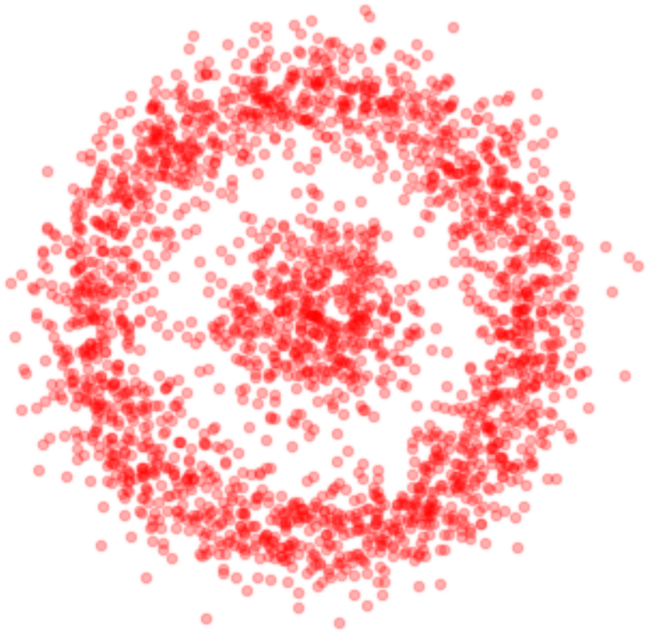

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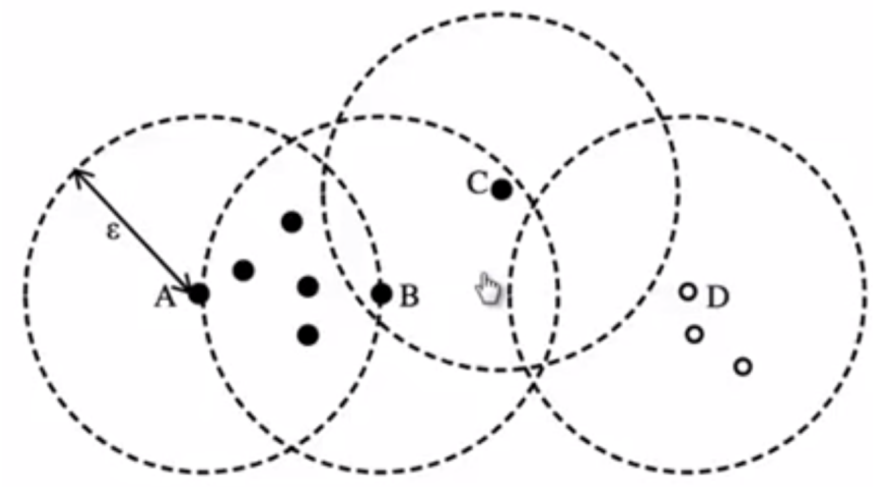

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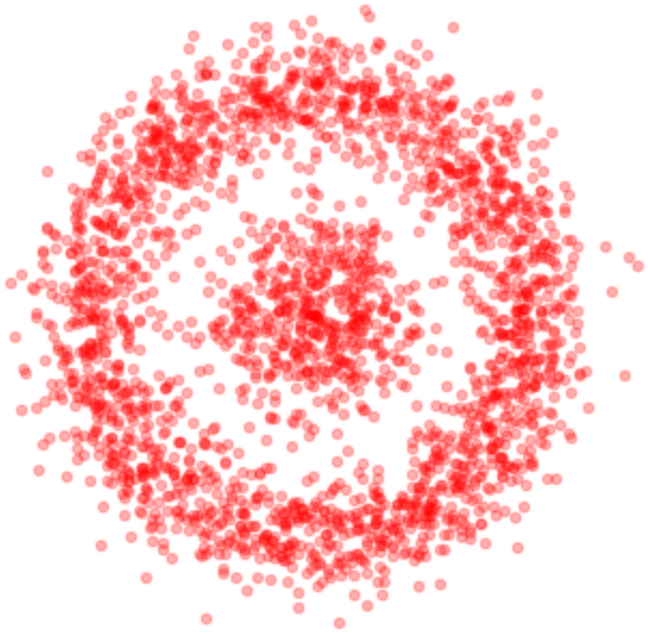

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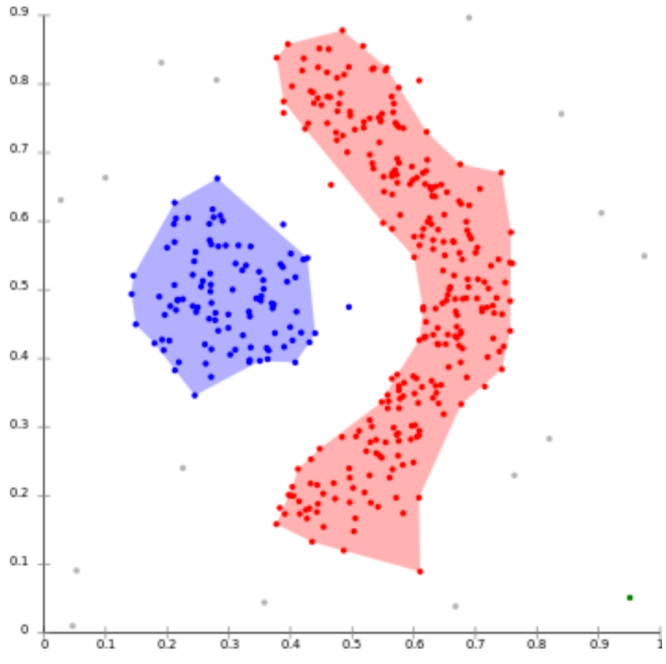
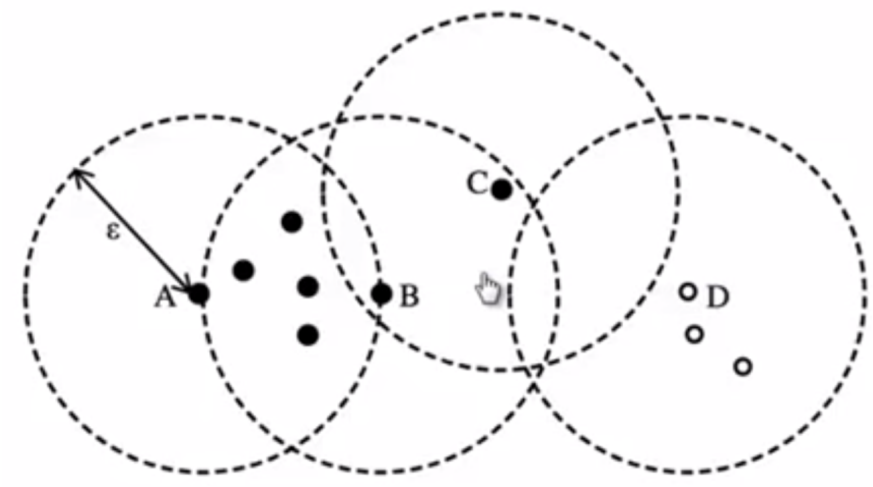
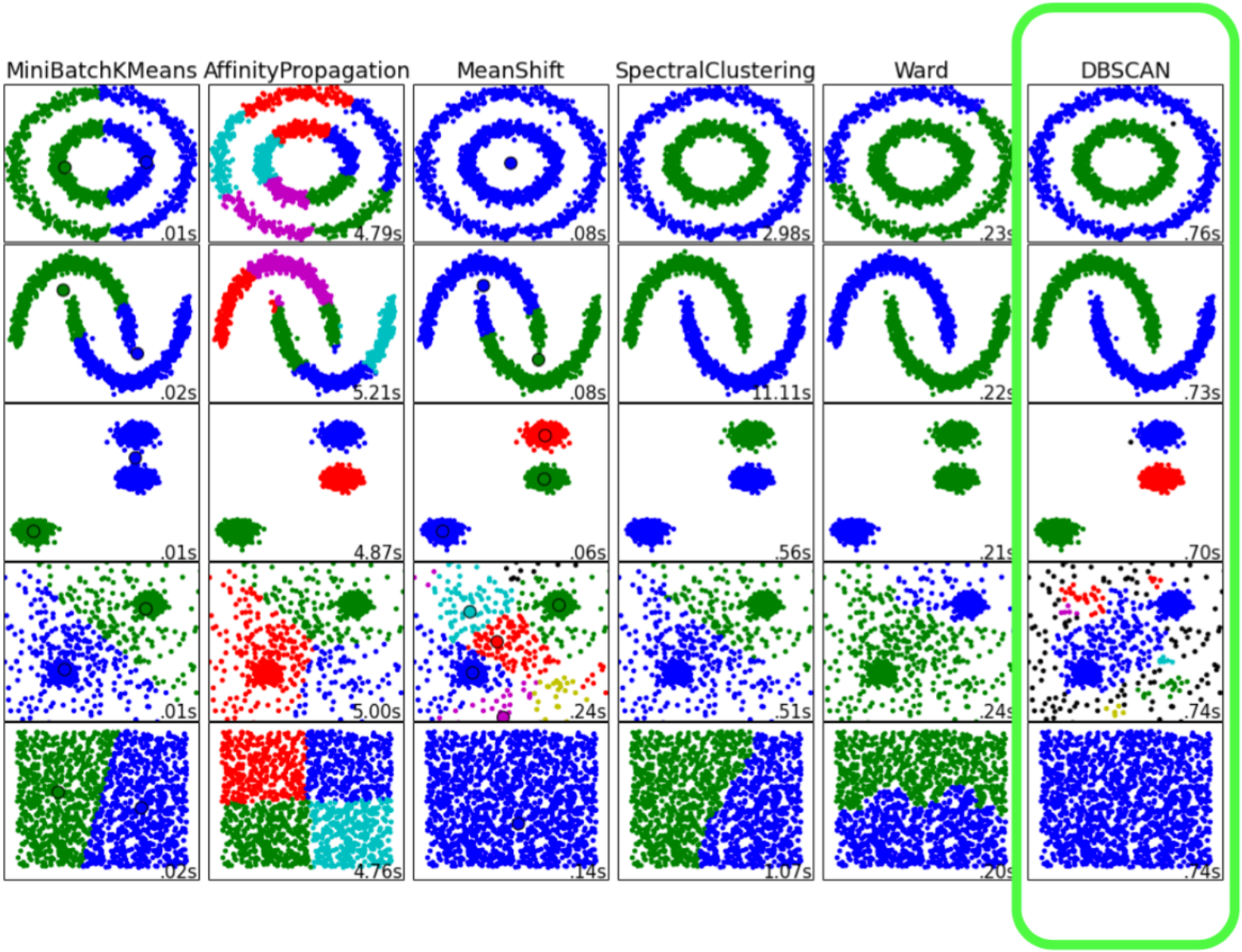


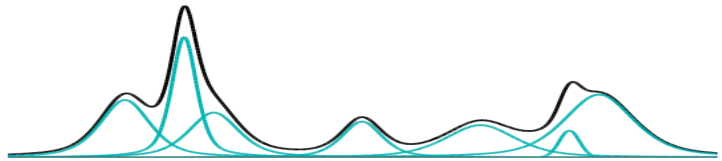


DBSCAN


MiniBatchKMeans
AffinityPropagation
MeanShift
SpectralClustering
Ward
DBSCAN
.01s
4.79s
.08s
2.98s
.23s
.76s
.02s
5.21s
.08s
11.11s
.22s
.73s
.01s
4.87s
.06s
.56s
.21s
.70s
.01s
5.00s
.24s
.51s
.24s
.74s
.02s
4.76s
.14s
1.07s
.20s
.74s

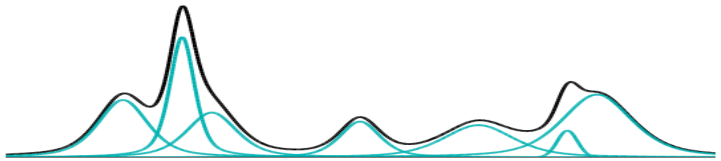
Bayesian Mixture Models *(xtra: my grad school jam)*

$$f(X_i|\boldsymbol{\mu}, \boldsymbol{\sigma^2}, \boldsymbol{\pi}, \boldsymbol{\pi}) = \sum_{k=1}^{K} \pi_k N(\mu_k, \sigma_k^2)$$

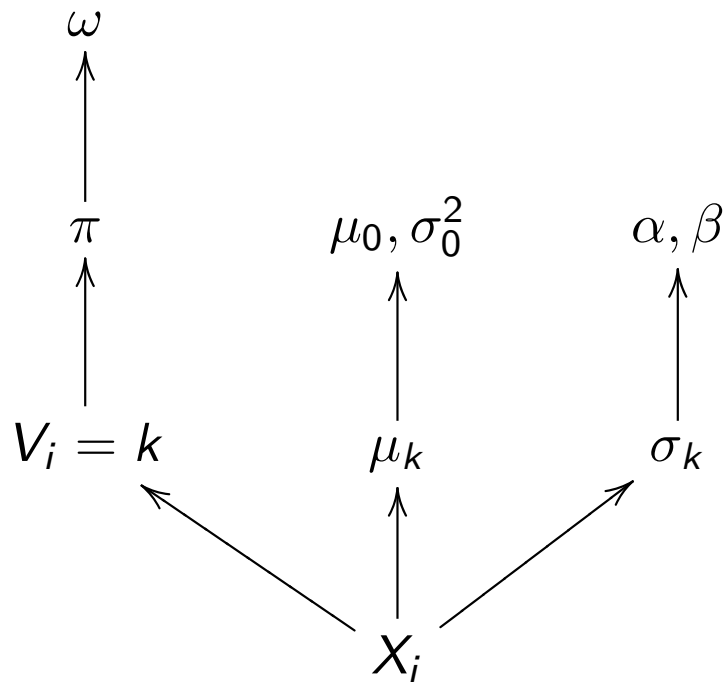


Bayesian Mixture Models *(xtra: my grad school jam)*

$$f(X_i|\boldsymbol{\mu}, \boldsymbol{\sigma^2}, \boldsymbol{\pi}, \boldsymbol{\pi}) = \sum_{k=1}^{K} \pi_k N(\mu_k, \sigma_k^2)$$

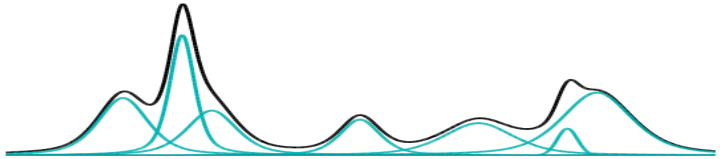


$$\begin{aligned}
f(X_i|V_i, \boldsymbol{\mu}, \boldsymbol{\sigma^2}, \boldsymbol{\pi}) &= N(\mu_k, \sigma_k^2) \\
\Pr(\boldsymbol{V}_i) &= \mathit{Multinomial}(\boldsymbol{\pi}, n = 1) \\
f(\boldsymbol{\pi}) &= \mathit{Dirichlet}(\boldsymbol{\omega}) \\
f(\mu_k) &= N(\mu_0, \sigma_0^2) \\
f(\sigma_k^{-2}) &= \mathit{Gamma}(\alpha, \beta)
\end{aligned}$$

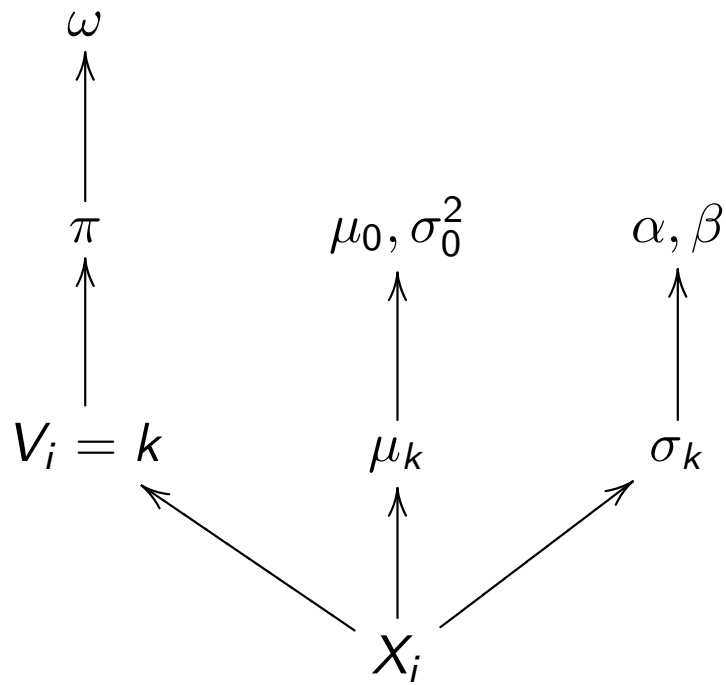


Bayesian Mixture Models *(xtra: my grad school jam)*

$$f(X_i|\boldsymbol{\mu}, \boldsymbol{\sigma^2}, \boldsymbol{\pi}, \boldsymbol{\pi}) = \sum_{k=1}^{K} \pi_k N(\mu_k, \sigma_k^2)$$



$$\begin{aligned}
f(X_i|V_i, \boldsymbol{\mu}, \boldsymbol{\sigma^2}, \boldsymbol{\pi}) &= N(\mu_k, \sigma_k^2) \\
\Pr(\boldsymbol{V}_i) &= \mathit{Multinomial}(\boldsymbol{\pi}, n = 1) \\
f(\boldsymbol{\pi}) &= \mathit{Dirichlet}(\boldsymbol{\omega}) \\
f(\mu_k) &= N(\mu_0, \sigma_0^2) \\
f(\sigma_k^{-2}) &= \mathit{Gamma}(\alpha, \beta)
\end{aligned}$$



$$\begin{aligned}
&f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma^2}, \boldsymbol{\pi}|\mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) = \\
&\prod_{i=1}^{n} \left[\left(\sum_{k=1}^{K} 1_{[V_{ik}=1]} \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{1}{2}\left(\frac{X_i - \mu_k}{\sigma_k}\right)^2} \right) \left(\prod_{k=1}^{K} \pi_k^{V_{ik}} \right) \right] \\
&\quad \times \left(\prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{1}{2}\left(\frac{\mu_k - \mu_0}{\sigma_0}\right)^2} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha-1} e^{-\beta\frac{1}{\sigma^2}} \right) \left(\frac{1}{\mathbf{B}(\omega)} \prod_{k=1}^{K} \pi_k^{\omega_k - 1} \right)
\end{aligned}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^{\mathbf{2}}, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

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Is made by cycling through the *full conditional distributions*

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

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$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

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A Gibbs sampler for the posterior

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$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

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$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

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$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *posterior sampling*

A Gibbs sampler for the posterior

$$f(\mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Is made by cycling through the *full conditional distributions*

$$f(\mu_k | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$f(\sigma_k^2 | \mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(V_{ik} = 1 | \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

$$\Pr(\boldsymbol{\pi} | \mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega})$$

$$\propto f(\mathbf{X}, \mathbf{V}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi} | \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \quad \text{(which is proportional to the joint distribution)}$$

Markov Chain Monte Carlo (MCMC) *full conditionals*

$$\Pr(V_{ik} = 1|\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\pi}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) \propto \pi_k \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{1}{2}\left(\frac{X_i - \mu_k}{\sigma_k}\right)^2}$$

$$f(\pi|\mathbf{V}, \boldsymbol{mu}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) = \mathit{Dirichlet}(\{\omega_k + n_k : k = 1, \cdots K\})$$

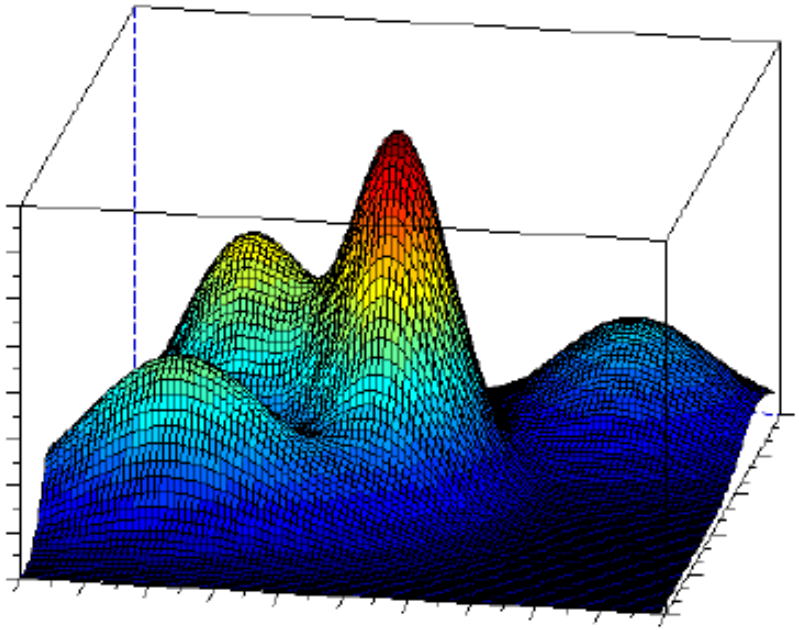
$$n_k = \sum_{V_{ik}=1} 1$$

$$f(\sigma_k^2|\mathbf{V}, \boldsymbol{\mu}, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) = \mathit{Gamma}\left(\frac{n_k}{2} + \alpha, \frac{1}{2}\sum_{V_{ik}=1}(X_i - \mu_k)^2 + \beta\right)$$

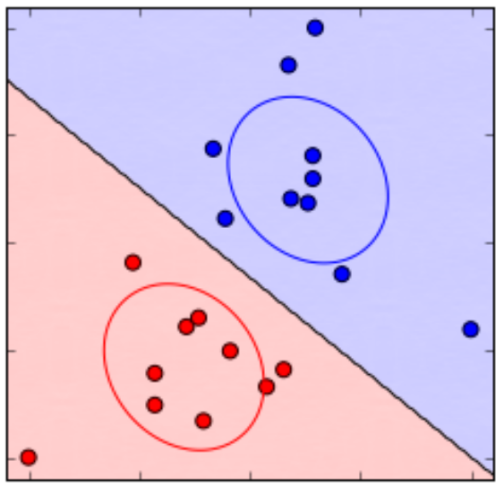
$$f(\mu_k|\mathbf{V}, \boldsymbol{\sigma}^2, \mathbf{X}, \mu_0, \sigma_0^2, \alpha, \beta, \boldsymbol{\omega}) =$$

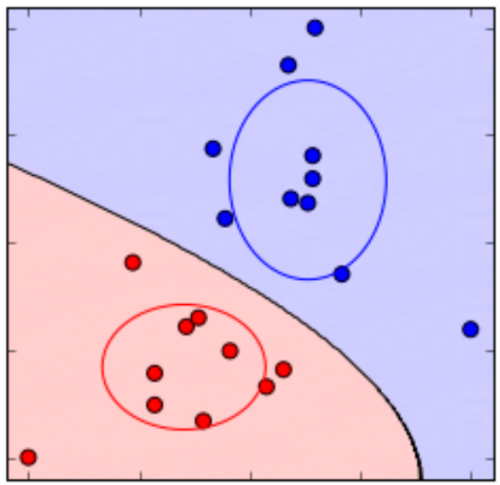
$$\mathrm{N}\left(\hat{\sigma}_k^2\left(\frac{\sum_{V_{ik}=1} X_i}{\sigma_k^2} + \frac{\mu_0}{\sigma_0^2}\right), \hat{\sigma}_k^2 = \left(\frac{n_k}{\sigma_k^2} + \frac{1}{\sigma_0^2}\right)^{-1}\right)$$

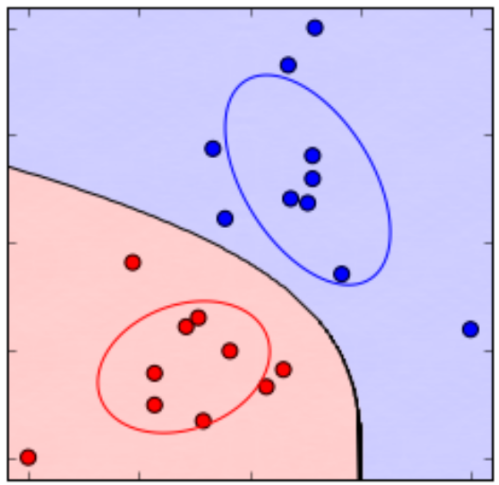
Mixture Models!



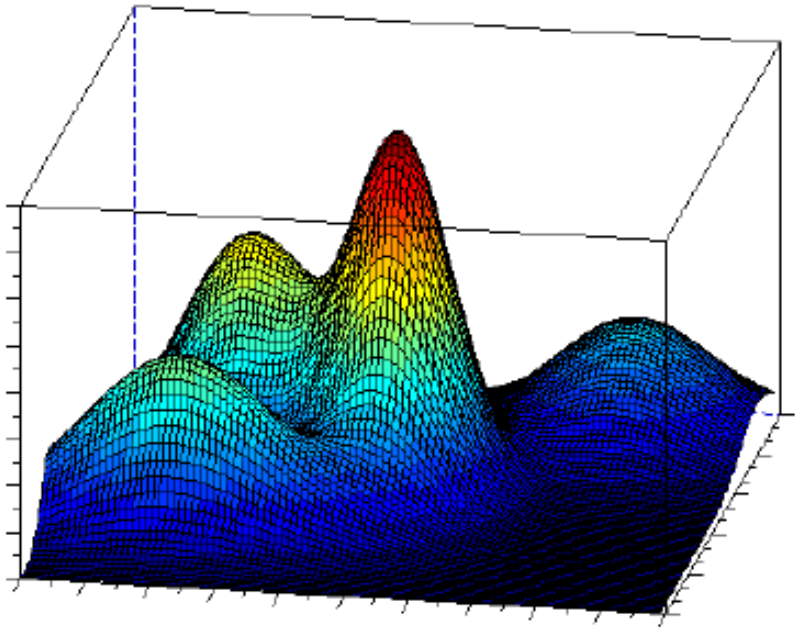
- Used to model "subpopulations"



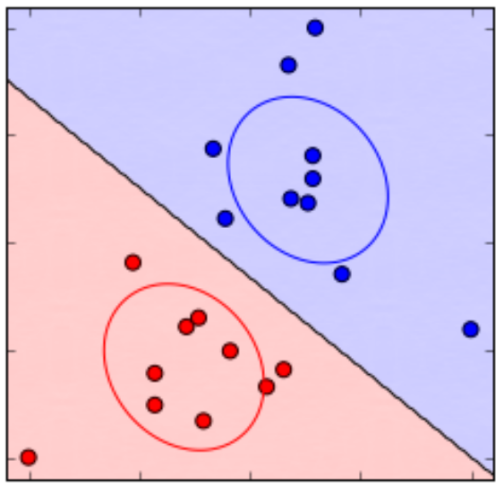


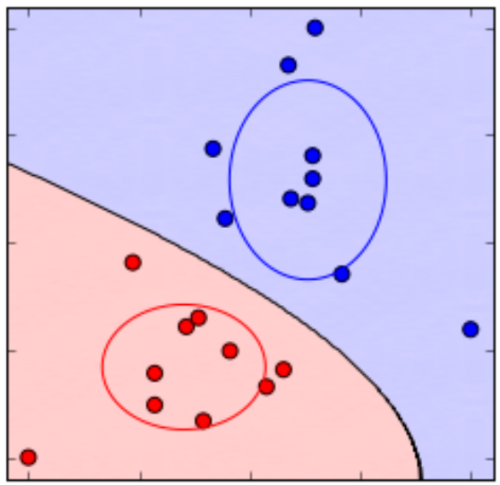


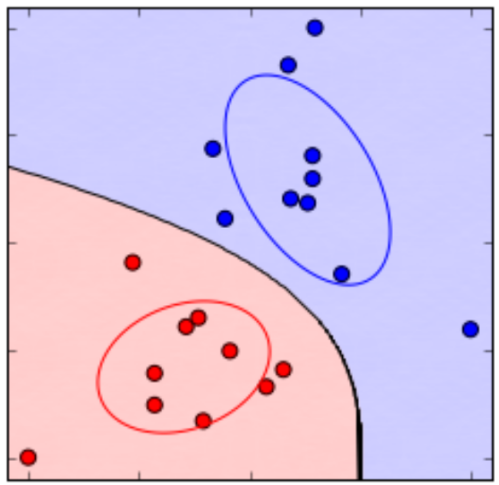
Mixture Models!



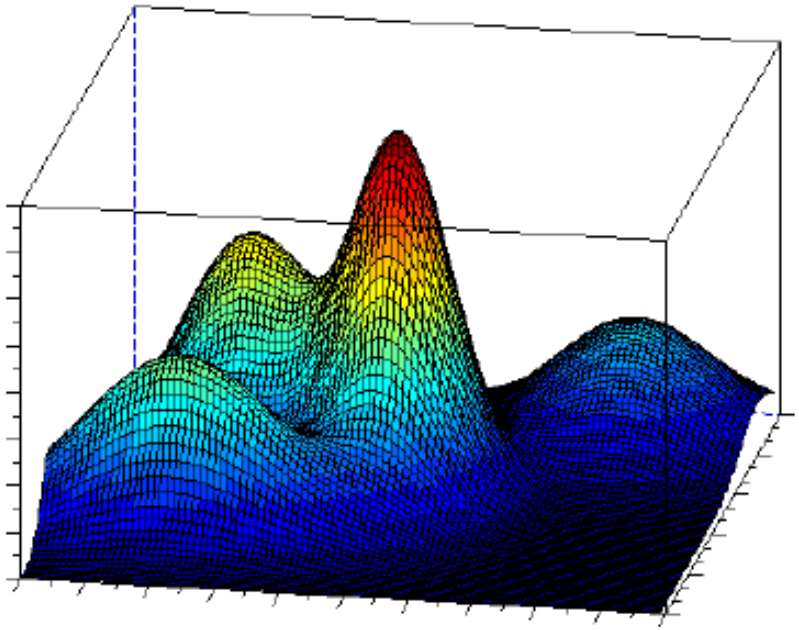
- Used to model "subpopulations"
- Or simply complex distributional shapes



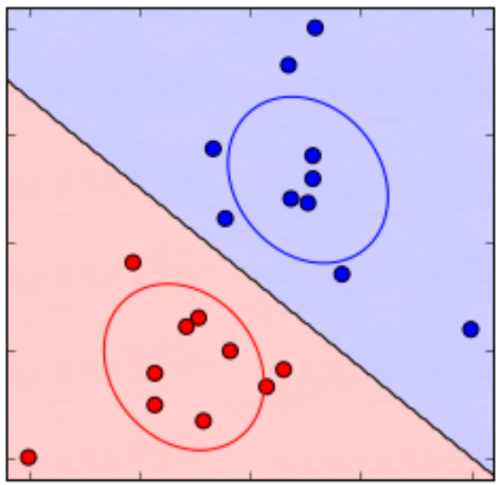


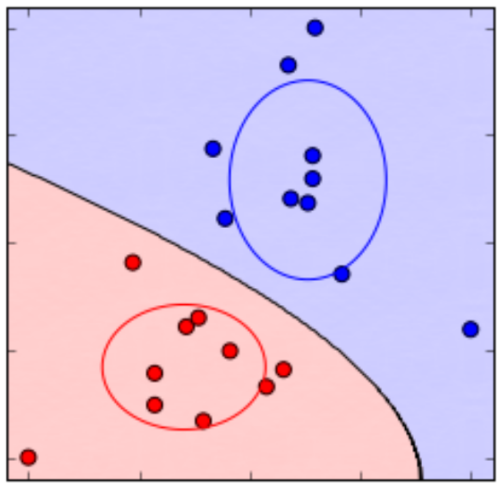


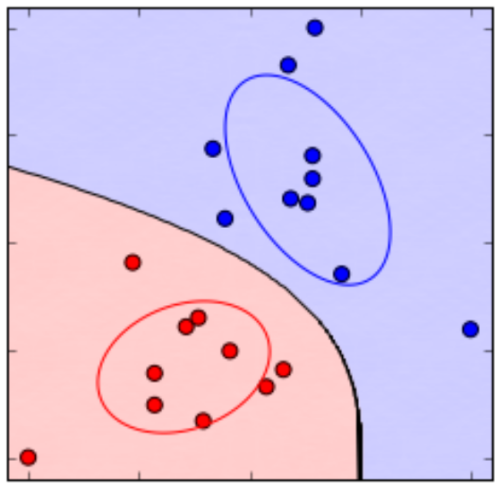
Mixture Models!



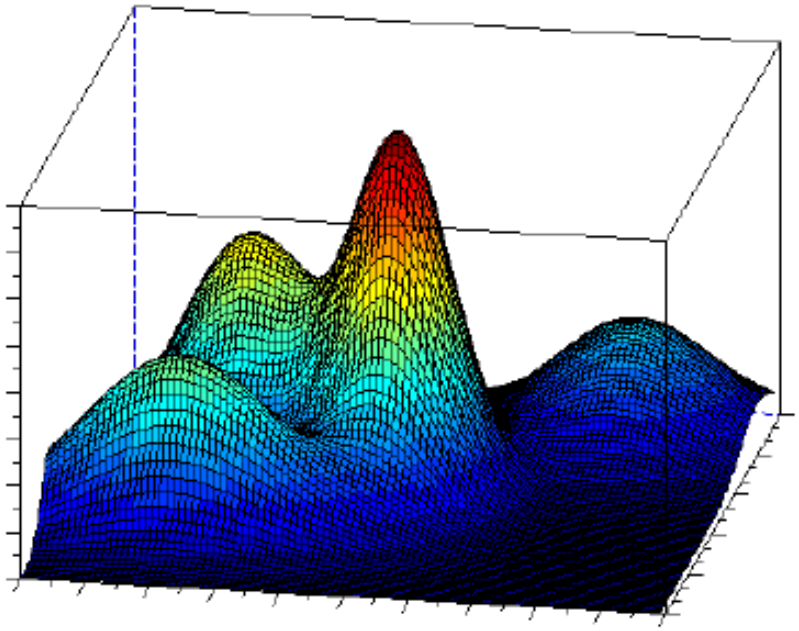
- Used to model "subpopulations"
- Or simply complex distributional shapes
- It's *almost nonparametric* like kernel density estimation







Mixture Models!



- Used to model "subpopulations"
- Or simply complex distributional shapes
- It's *almost nonparametric* like kernel density estimation
- Expectation-Maximization (EM) algorithm is another way to fit mixture models