Bayesian Inference

(afternoon)

Objectives

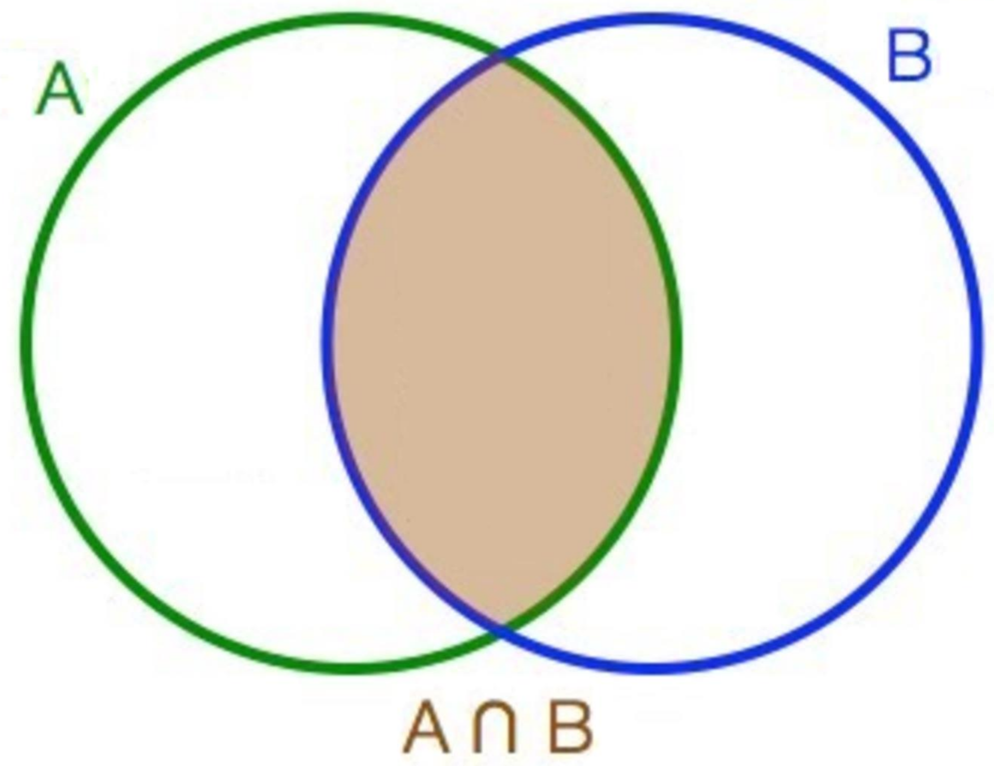
- Review conditional probability
- Bayes’ rule
- Bayesian inference: define posterior, prior, likelihood, evidence
- Solve a discrete Bayes problem by hand
- Pair programming: Bayesian analysis, verification by simulation

Bayes’ Rule: Motivation

- How to relate conditional probabilities between two events
- How to incorporate prior knowledge and belief into interpretation of data

Conditional Probability Review

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



Poll: What's the probability of rolling a dice with a value less than 4 knowing that the value is odd

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

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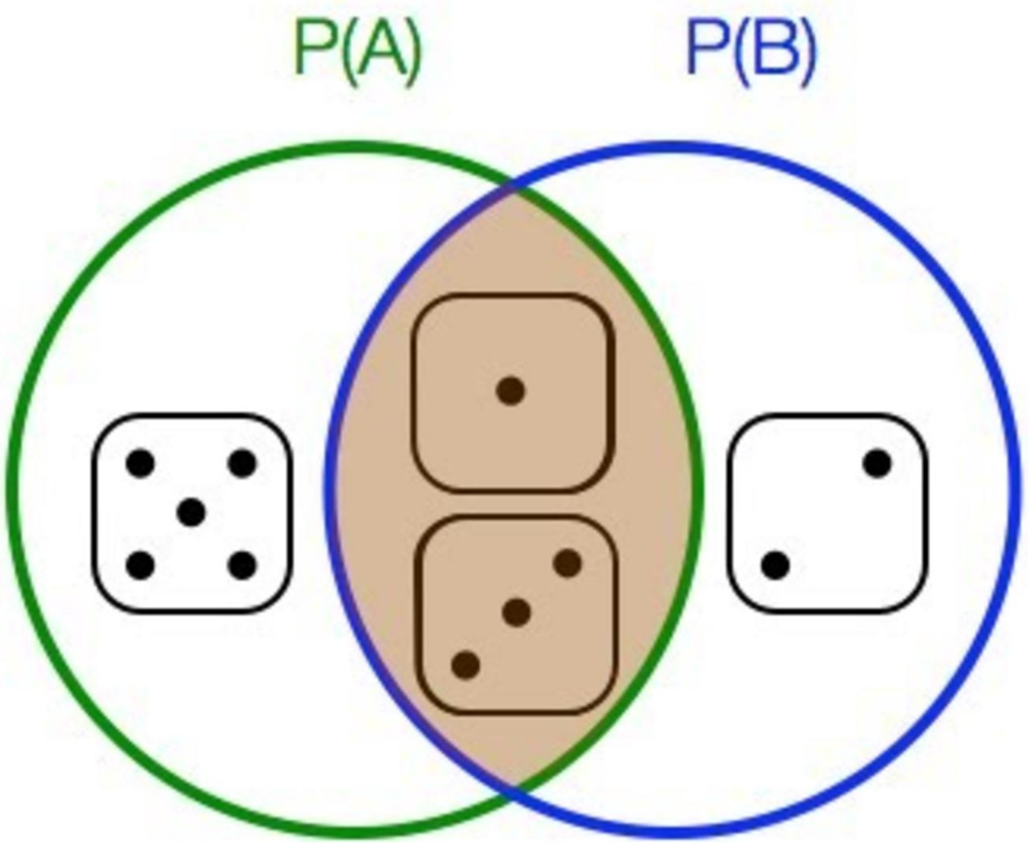
Poll: What's the probability of rolling a dice with a value less than 4 knowing that the value is odd

B = dice with a value less than 4

A = dice with an odd number

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2}{3}$$



Bayes' Rule

likelihood of
the evidence
"E" if the
hypothesis "H"
is true

posterior
probability of
"H" given the
evidence

prior probability
of "H"

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

prior probability that the
evidence itself is true
(but also a normalizing
constant)

Bayes' Rule after expanding $P(E)$

$$P(H|E) = \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|H^c)P(H^c)}$$

- $E = E \cap (H \cup H^c) = (E \cap H) \cup (E \cap H^c)$
- $P(E) = P(E \cap H) + P(E \cap H^c)$ (independent events)
- $P(E) = P(E|H)P(H) + P(E|H^c)P(H^c)$

Bayes' Rule after considering $P(E)$ as a normalizing constant

$$P(H|E) \propto P(E|H)P(H)$$

- $\propto$ means proportional: Calculate all your $P(H|E)$, then normalize them using their sum so their normalized sum is 1

Poll: You are planning a picnic today

You are planning a picnic today, but the morning is cloudy. What is the chance that it will rain during the day knowing that:

- 50% of all rainy days start off cloudy
- Cloudy mornings are common (40% of days start cloudy)
- This month is usually a dry month (only 3 of 30 days tend to be rainy)

Poll: You are planning a picnic today

$$\mathrm{P}(\text{rainy day} \mid \text{cloudy morning}) =$$

$$\frac{\mathrm{P}(\text{cloudy morning} \mid \text{rainy day})\mathrm{P}(\text{rainy day})}{\mathrm{P}(\text{cloudy morning})} =$$

$$\frac{.5 \ * 3/30}{.4} = .125$$

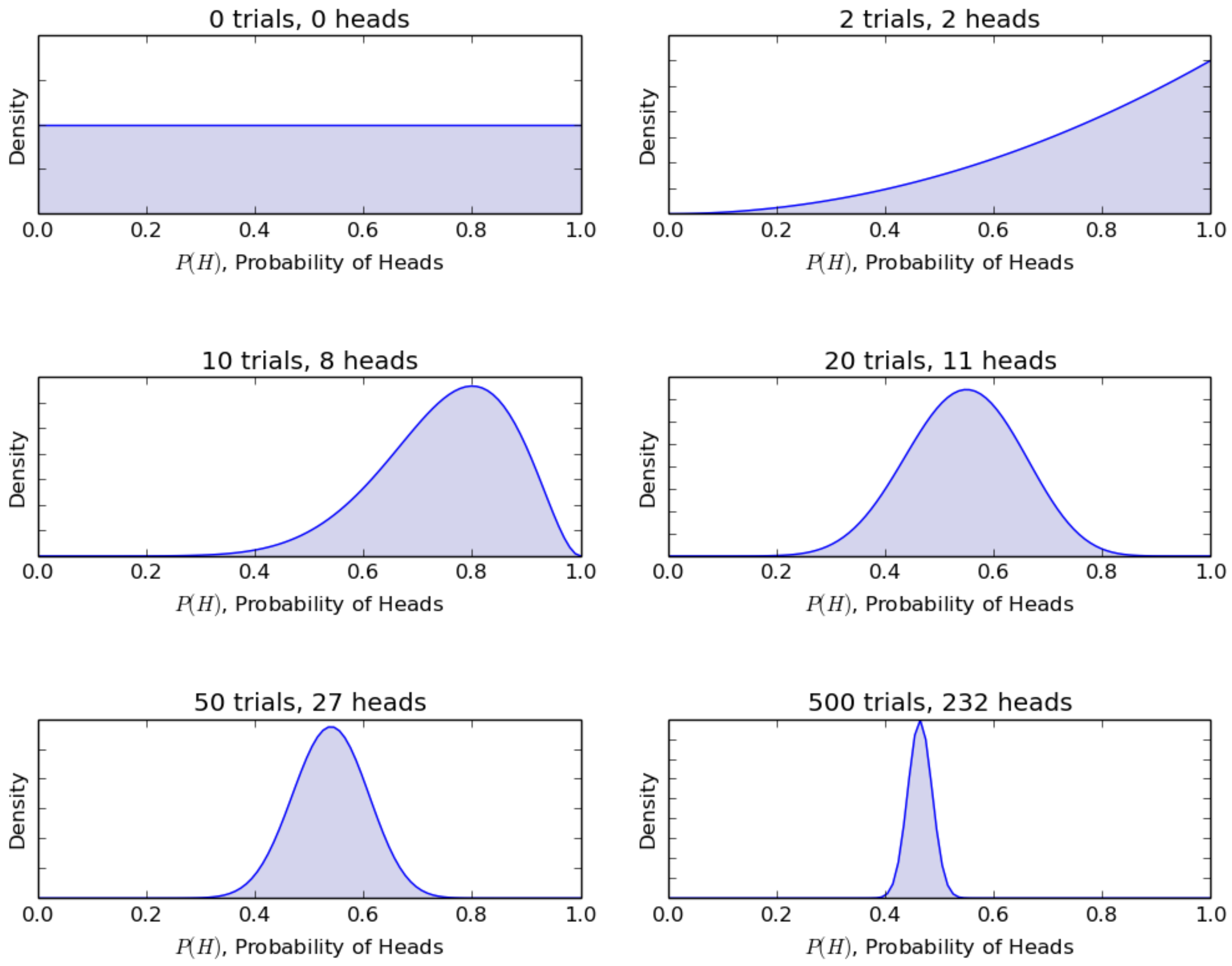
12.5%; not too bad compared with 50%..., let's have a picnic!

Bayesian Inference

- Bayesian updates his or her beliefs after seeing evidence
  - John Maynard Keynes, a great economist and thinker, said “When the facts change, I change my mind. What do you do, sir?”

- Probability is seen as a measure of believability in events

Bayesian Updates (pair programming)



Relating Prior Knowledge/Belief to Data

You have a drawer of 100 coins, 10 of which are biased

$P(\text{head} \mid \text{fair}) = .5$

$P(\text{head} \mid \text{biased}) = .25$

You randomly choose a coin and flip it once. It comes up heads

1. What is $P(\text{fair} \mid \text{head})$?
2. What if you flip it a second time and it comes up heads again?

Relating Prior Knowledge/Belief to Data

After the first coin flip, E = [head]

- H = fair coin

$$P(fair \mid head) =$$

$$\frac{P(head \mid fair)P(fair)}{P(head \mid fair)P(fair) + P(head \mid biased)P(biased)} = \frac{.5 \times .9}{.5 \times .9 + .25 \times .1} = .947$$

- H = biased coin

$$P(biased \mid head) =$$

$$\frac{P(head \mid biased)P(biased)}{P(head \mid fair)P(fair) + P(head \mid biased)P(biased)} = \frac{.25 \times .1}{.5 \times .9 + .25 \times .1} = .053$$

Relating Prior Knowledge/Belief to Data

After the second coin flip, E = [head, head]

- H = fair coin

$$P(fair \mid head) =$$

$$\frac{P(head \mid fair)P(fair)}{P(head \mid fair)P(fair) + P(head \mid biased)P(biased)} = \frac{.5 \times .947}{.5 \times .947 + .25 \times .053} = .973$$

- H = biased coin

$$P(biased \mid head) =$$

$$\frac{P(head \mid biased)P(biased)}{P(head \mid fair)P(fair) + P(head \mid biased)P(biased)} = \frac{.25 \times .053}{.5 \times .947 + .25 \times .053} = .027$$

This last example again but now using $P(E)$ as a normalizing constant

(you'll use this for the pair assignment…)

After the second coin flip, E = [head, head]

- H = fair coin

$$P(fair \mid head) \propto P(head \mid fair)P(fair) = .5 \times .947 = .474$$

- H = biased coin

$$P(biased \mid head) \propto P(head \mid biased)P(biased) = .25 \times .053 = .0133$$

- Normalizing constant is

$$.474 + .0133 = .487$$

- H = fair coin

$$P(fair \mid head) = {}^{.474}/_{.487} = .973$$

- H = biased coin

$$P(biased \mid head) = {}^{.0133}/_{.487} = .027$$

Afternoon pairing