Essential Probability

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Objectives

Today's objectives:

- Review probability definitions and concepts
- Review combinatorics definitions and concepts
- Review properties of distributions
- Understand which distributions to use to model which types of processes

Agenda

Today's plan:

1. Combinatorics
2. Probability
3. Random variables and probability distributions

References

A couple helpful references:

- Mathematical Statistics: A Unified Introduction provides an accessible introduction with many examples
- Statistical Inference provides a graduate level introduction
- All of Statistics: A Concise Course in Statistical Inference
- Probability and Measure provides a rigorous mathematical foundation for those with 'extreme math' skills

Review: sets

Some definitions:

- A *set* S consists of all possible outcomes or events and is called the *sample space*
- *Union*: $A \cup B = \{x : x \in A \,\texttt{or}\, x \in B\}$
- *Intersection*: $A \cap B = \{x : x \in \texttt{ and}\, x \in B\}$
- *Complement*: $A^c = \{x : x \notin A\}$
- *Disjoint*: $A \cap B = \emptyset$
- *Partition*: a set of pairwise disjoint sets, $\{A_j\}$, such that $\overset{\infty}{\underset{j=1}{\cup}} A_j = S$
- Plus the commutative, associative, distributive, and DeMorgan's laws

Combinatorics

Example: tea

R. A. Fischer is invited to tea with a lady who claims she can tell whether tea or milk is added to the cup first. Fisher is incredulous and proposes the following experiment:

- He will prepare three cups with tea added first and milk second and three cups prepared in the opposite order
- He will order the cups randomly
- The lady will guess which are which
- What is the probability she guesses all three correctly by chance?

Factorial

Factorial counts the number of ways of ordering or picking something when order matters:

- We write $n! = n \times (n-1) \times ... \times 1$
- $0! = 1$ by convention
- Example: how many ways can we shuffle a deck of cards?

Combination

Combination counts the number of ways of picking something when order doesn't matter:

- $\binom{n}{k} = \frac{n!}{(n-k)!k!}$
- We say '*n choose k*'
- This is the number of ways of choosing k items from n total items
- Typically, the items are identical
- Urns and balls are the classic example:
  - If I draw k balls from an urn with n balls, how many different sets are possible?
  - If I draw W white balls and B black balls from an urn, how many different orderings are possible?

Example: tea revisited

What if we prepare eight cups with four cups tea first and four milk first:

- What is the probability she can guess at least three out of four cups correctly?
- Will R. A. Fisher be impressed?

Multinomial

The number of ways of assigning $(n_1, n_2, ..., n_k)$ objects to k different categories:

- $\binom{n!}{n_1 n_2 \cdots n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$
- Example: an urn contains red, white, and blue balls. . .

Probability

Introduction

Probability provides the mathematical tools we use to model randomness:

- Probability tells us how likely an event (Frequentist) is or how likely our beliefs are to be correct (Bayesian)
- Provides the foundation for statistics and machine learning
- Often our intuitions about randomness are incorrect because we live only one realization
- Enumerating all possible outcomes (using combinatorics) can help us compute the probability of an event

Definition of probability

Given a sample space, S, a *probability function*, P, has three properties:

- $P(A) \geq 0, \forall A \in S$
- $P(S) = 1$
- For a set of pairwise disjoint sets $\{A_j\}$, $P(\bigcup\limits_j A_j) = \sum\limits_j P(A_j)$

Note: for those who really care about the details, you need to use measure theory and sigma algebras

Example: tossing a coin

Consider a coin toss:

- $S = \{H, T\}$
- $P(H) = P(T) = \frac{1}{2} > 0$
- $P(S) = 1$

Note: this means $P(A) = 1 - P(A^c)$

Independence

Two events A and B are said to be *independent* if

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$$

or, equivalently, if

$$\Pr[B|A] = \Pr[B],$$

i.e., knowledge of A provides no information about B

- $A \perp B$ means A and B are independent
- To compute the probability that any one of a set of independents events, $\{A_n\}$, occurs:

$$\Pr[\bigcup_k A_k] = \sum \Pr[A_k],$$

where $A_i \perp A_j, \forall i \neq j$

Multiplication rule

To compute the probability that two independent events occur, multiply their probabilities:

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$$

Example:

- What is the probability that A and B happen?

Example: coin tosses

Take a moment to solve this question:

- Three types of fair coins are in an urn: HH, HT, and TT
- You pull a coin out of the urn, flip it, and it comes up H
- **Q**: what is the probability it comes up H if you flip it a second time?

Conditional probability

We often care about whether one event provides information about another event. The *conditional probability* of B given A is:

$$\Pr[B|A] = \frac{\Pr[A \cap B]}{\Pr[A]}$$

- We say this is the '*probability of B conditional on A*'
- I.e., if A has occurred, what is the probability B will occur?
- For a pdf of two random variables,

$$f(y|x) = \frac{f(x, y)}{f(x)}$$

Probability chain rule

Can condition on an arbitrary number of variables:

- Simple example:

$$\Pr[A_3, A_2, A_1] = \Pr[A_3|A_2, A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_1]$$

- General case:

$$\Pr[A_n, ..., A_1] = \prod_j \Pr[A_j|A_{j-1}, ..., A_1]$$

or

$$\Pr[\bigcap_j^n A_j] = \prod_j^n \Pr[A_j| \bigcap_k^{j-1} A_k]$$

Law of total probability

If $\{B_n\}$ is a partition of the sample space, the *Law of total probability* states:

$$\Pr[A] = \sum_j \Pr[A \cap B_j]$$

or

$$\Pr[A] = \sum_j \Pr[A|B_j] \cdot \Pr[B_j]$$

$\Pr[A]$ is said to be a *marginal distribution* of $\Pr[A, B]$

Bayes's Rule

Use Bayes's Rule when you need to compute conditional probability for $B|A$ but only have probability for $A|B$:

$$\Pr[B|A] = \frac{\Pr[A|B] \cdot \Pr[B]}{\Pr[A]}$$

- Proof: use the definition of conditional probability
- For an arbitrary partition of event space, $\{A_j\}$, use the general form of Bayes's rule:

$$\Pr[A_k|B] = \frac{\Pr[A_k|B] \cdot \Pr[B]}{\sum_j \Pr[B|A_j] \cdot \Pr[A_j]}$$

Example: drug testing

A test for EPO has the following properties:

Variable	Value
$\Pr[+ \mid doped]$	0.99
$\Pr[+ \mid clean]$	0.05
$\Pr[doped]$	0.005

Q: What is the probability the cyclist is using EPO if the test is positive? I.e., what is $\Pr[doped \mid +]$?

Solution: drug testing

1. Compute probability of being clean:

$$\Pr[\mathit{clean}] = 1 - \Pr[\mathit{doped}]$$

2. Use Bayes's Rule:

$$\begin{aligned}
\Pr[\mathit{doped}|+] &= \frac{\Pr[+|\mathit{doped}] \cdot \Pr[\mathit{doped}]}{\Pr[+|\mathit{doped}] \cdot \Pr[\mathit{doped}] + \Pr[+|\mathit{clean}] \cdot \Pr[\mathit{clean}]} \\
&= \frac{0.99 \cdot 0.005}{0.99 \cdot 0.005 + 0.05 \cdot (1 - 0.005)} \\
&= 0.090
\end{aligned}$$

Based on this example

Random variables and probability distributions

Definition: random variable

Given a sample space S, a *random variable*, X, is a function such that $X(s) : S \mapsto \mathbb{R}$:

- Use capital letters to refer to a random variable, e.g., X
- Use lower case to refer to a specific realization, x, or $X = x$
- Consequently, $\Pr[X = x] = \Pr[\{s \in S : X(s) = x\}]$
- We write $X \sim \texttt{XYZ}(\alpha, \beta, ...)$ to mean X is distributed like the *XYZ* distribution with parameters α, β, ...
- We say a series of random variables are *i.i.d.* if they are '*independent and identically distributed*'
- Example: $X \sim \texttt{N}(\mu, \sigma^2)$ or $X \sim \texttt{U}(0, 1)$

Cumulative distribution function (CDF)

Definition: the cumulative distribution function $F_X(x) = \Pr[X \leq x]$:

- Properties:
  - $0 \leq F_X(x) \leq 1$
  - $\lim_{x \to -\infty} F_X(x) = 0$
  - $\lim_{x \to \infty} F_X(x) = 1$
  - $F_X(x)$ is monotonically increasing
- Applies to discrete and continuous random variables
- Note: $\Pr[a \leq X \leq b] = F_X(b) - F_X(a)$

Discrete: probability mass function (PMF)

For a random variable, X, which takes discrete values $\{x_i\}$, use a PMF to determine the probability of an individual event:

- $f_X(x) = P(X = x), \forall x$
- We say there is *probability mass* p_i on x_i, where $p_i = \Pr[X = x_i]$
- Example: tossing coins
  - $X \in \{H, T\}$
  - $p_H = p_T = \frac{1}{2}$

Continuous probability density function (PDF)

For a continuous random variable, X, use a PDF:

- $f_X(x)dx = \Pr[x < X < x + dx]$
- $f_X(x) = \frac{dF_x(x)}{dx}$, assuming some regularity conditions
- $F_X(x) = \int_{-\infty}^{x} f_X(s)ds$
- Example: survival time, T, of uranium before decay
  - $T \sim \texttt{Exp}(\lambda)$
  - PDF: $f_T(t) = \lambda \cdot \exp(-\lambda \cdot t)$
  - CDF: $F_T(t) = 1 - \exp(-\lambda \cdot t)$ if $t \geq 0$
  - What fraction survives longer than t?

Properties of distributions

Use these properties to characterize a distribution:

- Expectation/mean
- Variance/standard deviation
- Skew
- Kurtosis
- Correlation

We often compute sample analogs of these properties to compare the empirical distribution of our data to standard distributions

Expectation/mean

The *expectation*, *mean*, or *expected value* is a measure of what is a likely value of a random variable:

- $\mu_{g(X)} = \mathbb{E}_X[g(x)]$:
  - Continuous: $\mathbb{E}_X[g(x)] = \int_{-\infty}^{\infty} g(s) f_X(s) ds$
  - Discrete: $\mathbb{E}_X[g(x)] = \sum_{s \in \{x_i\}} g(s) f_X(s)$
- Expectation is a linear operator
- The mean is $\mathbb{E}_X[x] = \int_{-\infty}^{\infty} s f(s) ds$
- The sample mean is $\overline{x} = \frac{1}{n} \sum_{j=1}^{n} x_j$

Variance

Variance measures the spread of a distribution:

- $\texttt{Var}[x] = \mathbb{E}_X[(x - \mu_x)^2]$
- Sometimes variance is written as $\sigma^2(x) = \texttt{Var}(x)$
- Often, we use *standard deviation*, $\sigma(X) = \sqrt{\texttt{Var}[x]}$ which has the same dimensions as X
- Note: the sample variance is $s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \overline{x})^2$

Skew and kurtosis

Skew and kurtosis are higher order moments:

- Skewness:
  - $\gamma_1 = \mathbb{E}[\left(\frac{X-\mu}{\sigma}\right)^3]$
  - Measures asymmetry of a distribution
  - Sign of skewness tells whether distribution is left or right skewed
- Kurtosis:
  - $\kappa = \mathbb{E}[\left(\frac{X-\mu}{\sigma}\right)^4]$
  - Measures the 'fatness' of the tails of the distribution

Variance of the mean

Statistics like the sample mean are random variables:

- Thus, they have a distribution
- Can compute their variance:

$$\mathtt{Var}(\overline{x}) = \frac{\mathtt{Var}(x)}{N}$$

- Hence, the standard deviation is:

$$\sigma(\overline{x}) = \sqrt{\frac{\mathtt{Var}(x)}{N}}$$

or

$$\sigma(\overline{x}) = \frac{\sigma(x)}{\sqrt{N}}$$

Quantiles (percentiles)

Quantiles are another way to characterize the distribution of data:

- The *quantile function* of X is

$$Q_\alpha(x) = \min_x \{x : \Pr(X \leq x) \geq \alpha\}$$
$$Q_\alpha(x) = \min_x \{x : F(x) \geq \alpha\},$$

where $\alpha \in (0, 1)$

- Given regularity conditions, $Q_\alpha[x] = F^{-1}(\alpha)$
- If $U \sim \texttt{Uniform}(0, 1)$ then $X = Q(U)$
- *percentiles* are just the quantile $\times 100$

Common quantiles

During EDA, it is often helpful to examine:

- Median: $Q_{0.5}[x]$
- Upper quartile: $Q_{0.75}[x]$
- Lower quartile: $Q_{0.25}[x]$
- Note: the median usually does not equal the mean, especially for data with a long tail

Pro tip: compute a box plot

Multivariate distributions

Model relationships between multiple random variables with a multivariate (joint) distribution:

- Let $X(s) : S \mapsto \mathbb{R}^k$, i.e., X is a vector of random variables, $X(s) = (x_1(s), x_2(s), ..., x_k(s))^T$
- CDF:

$$F(x_1, x_2, ..., x_k) = \Pr[X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k]$$

- PDF:

$$F(x_1, x_2, ..., x_k) = \int_{-\infty \cdots -\infty}^{x_1 x_2 \cdots x_k} f(s_1, s_2, ..., s_k) ds_1 ds_2 \cdots ds_k$$

Multivariate moments

Can compute vector analogs of all moments we have discussed:

- Mean: $\mu_x = \mathbb{E}[x]$
- Variance: $\texttt{Var}[x] = \mathbb{E}[(x - \mu_x) \cdot (x - \mu_x)^T]$
- Covariance: $\texttt{Cov}[x, y] = \mathbb{E}[(x - \mu_x) \cdot (y - \mu_y)^T]$
- Correlation: $\rho_{XY}(x, y) = \dfrac{\texttt{Cov}[x, y]}{\sigma(x) \cdot \sigma(y)}$

Marginal and conditional distributions

To compute the marginal distribution from the joint (multivariate) distribution, just integrate (sum) over the other variable(s):

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, s) ds$$

For a bivariate distribution, conditional pdf is:

$$f(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Covariance and correlation

To explore the relationship between variables compute:

- *Covariance*:
  - $\texttt{Cov}(x, y) = \mathbb{E}[(x - \mu_x) \cdot (y - \mu_y)]$
  - Size changes with scaling of variables
  - For random variables which are vectors, use $\texttt{Cov}[x, y] = \mathbb{E}[(x - \mu_x) \cdot (y - \mu_y)^T]$
- *Correlation* (Pearson):
  - Dimensionless measure relationship
  - $\rho_{XY}(x, y) = \dfrac{\texttt{Cov}(x, y)}{\sigma(x) \cdot \sigma(y)}$
  - Thus, $\rho_{XY} \in [-1, 1]$
  - Other correlation coefficients, such as Spearman, use rank and are more robust
- Correlation is not causation!

Correlation and linearity

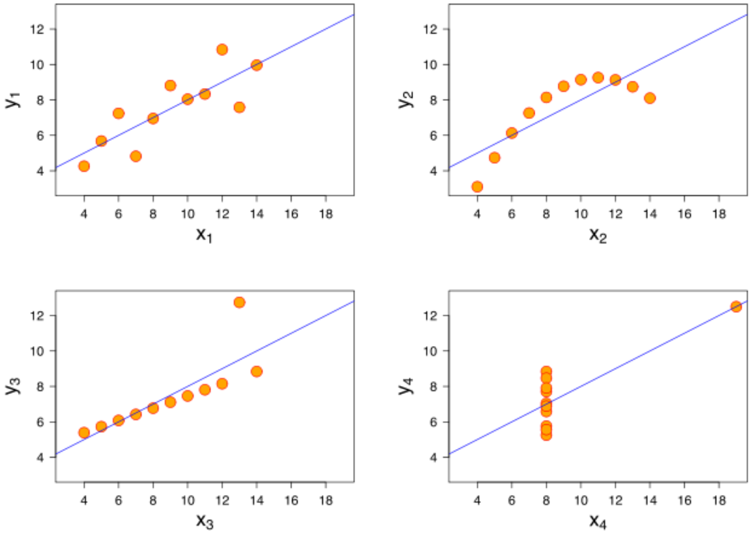

Figure 1: Correlation and linearity: $r = 0.816$. From Wikipedia.

Correlation captures noisiness and direction

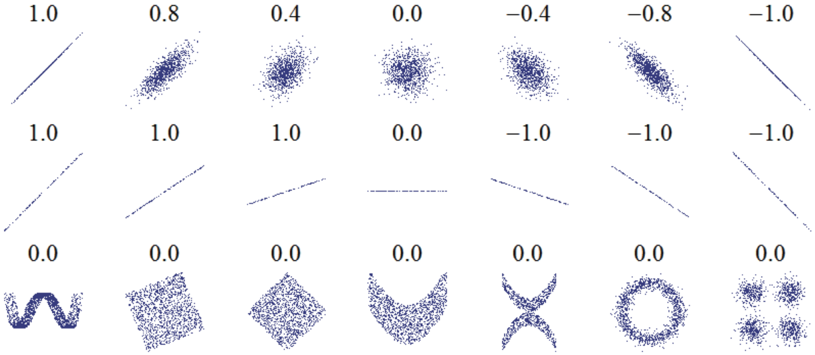

Figure 2: Correlation and non-linearity. From Wikipedia.

The weak law of large numbers and the analog principle

The *weak law of large numbers* states that, given some regularity conditions,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = \mathbb{E}[x]$$

This motivates the *analog principle*: when creating sample estimators, replace expectations, $\mathbb{E}$, with sums, $\frac{1}{n}\sum_{i=1}^{n}$

Common distributions

Overview

We now review the properties of some common distributions:

- Discrete
  - Bernoulli
  - Binomial
  - Geometric
  - Poisson
- Continuous
  - Uniform
  - Exponential
  - Gaussian a.k.a. Normal
  - χ^2
  - Student's t
  - F distribution

Bernoulli

Models a toss of an unfair coin or clicking on a website:

- $X \sim \texttt{Bernoulli}(p)$
- PMF: $\Pr[H] = p$ and $\Pr[T] = 1 - p$
- Mean: $\mathbb{E}[x] = p$
- Variance: $\texttt{Var}[x] = p \cdot (1 - p)$

Example: click through rate

Given N visitors of whom n click on the 'Buy' button:

- What is click through rate (CTR)?
- What is the variance of the click through rate?

Binomial

Models repeated tosses of a coin:

- $X \sim \texttt{Binomial}(n, p)$ for n tosses of a coin where $\Pr[H] = p$
- PMF: $\Pr[X = k] = \binom{n}{k} p^k \cdot (1 - p)^{(n-k)}, \forall\, 0 \leq k \leq n$
- Mean: $n \cdot p$
- Variance: $n \cdot p \cdot (1 - p)$
- Approaches Gaussian for limit of large n

Geometric

Models probability succeeding on the k-th try:

- $X \sim \texttt{Geometric}(p, k)$
- PMF: $\Pr[X = k] = p \cdot (1 - p)^{(k-1)}$
- Mean: $\dfrac{1}{p}$
- Variance: $\dfrac{1 - p}{p^2}$

Poisson

Models number of events in a period of time, such as number of visitors to website:

- $X \sim \texttt{Poisson}(\lambda)$
- PMF: $\Pr[X = k] = \exp(-\lambda) \cdot \frac{\lambda^k}{k!}, \forall\, k = 1, 2, \ldots$
- Mean $=$ variance $= \lambda$
- λ is the number of events during the interval of interest
- Note: $\Pr[X = k]$ is just one term in the Taylor's series expansion of $\exp(x)$ when suitably normalized

Remark: the assumption that mean $=$ variance is very strong. In practice, better to fit a model with *overdispersion* such as the negative binomial distribution, and test whether the assumption holds

Uniform

Models a process where all values in an interval are equally likely:

- $X \sim \mathtt{U}(a, b)$
- PDF: $f(x) = \dfrac{1}{b-a}, \forall\, x \in [a, b]$ and 0 otherwise
- Mean: $\dfrac{a+b}{2}$
- Variance: $\dfrac{(b-a)^2}{12}$
- Note: any continuous random variable can be transformed into a uniformly distributed variable by letting $u = F_X(x)$

Exponential

Models survival, such as the fraction of uranium which has not decayed by time t or time until a bus arrives:

- $T \sim \texttt{Exp}(\lambda)$
- $1/\lambda$ is the half-life
- CDF: $\Pr[T \leq t] = 1 - \exp(\lambda \cdot t), x \geq 0, \lambda \geq 0$
- Mean: $1/\lambda$
- Variance: $1/\lambda^2$
- 'Memory-less'

Gaussian a.k.a. Normal

A benchmark distribution:

- $X \sim (N)(\mu, \sigma^2)$
- PDF: $f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$
- Often, compute the 'z-statistic':
  - $z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$
  - Perform a 'z-test' to check probability of observed value
- 'Standard normal' is $\mathtt{N}(0, 1)$:
  - PDF is $\phi(x)$
  - CDF is $\Phi(x)$
- Will discuss Central Limit Theorem tomorrow

This is the famous 'Bell-curve' distribution and is associated with many processes, such as white noise, Brownian motion, etc.

Other distributions

Some other distributions:

- χ^2:
  - Models sum of k squared, independent, normally-distributed random variables
  - Use for goodness of fit tests
- Student's t: distribution of the *t-statistic*:
  - t-statistic: $t = \frac{\overline{x} - \mu}{s/\sqrt{n}}$, where s is the standard error
  - Perform a 't-test' to check probability of observed value
  - Has fatter tails than normal distribution
- F-distribution:
  - Distribution of the ratio of two χ^2 random variables
  - Use to test restrictions and ANOVA

Digression: random numbers

Bad news: the computer generates *pseudo*-random numbers:

- Not truly random
- Generated using a variety of algorithms so that they satisfy statistical tests
- Most proofs use true random numbers ... so be careful they may not hold with pseudo-random numbers
- To generate pseudo-random numbers for an arbitrary distribution use the trick that $u = F_X(x) \sim U(0, 1)$

Summary

Summary

Q: When do you use factorial vs. combination?

Q: What is independence?

Q: What is conditional probability? How do I use Bayes's rule?

Q: What are the PDF and CDF?

Q: What are moments should you use to characterize a distribution? How do you calculate them?

Q: What is a quantile?

Q: What are some common distributions? What type of processes do they model?