Regression

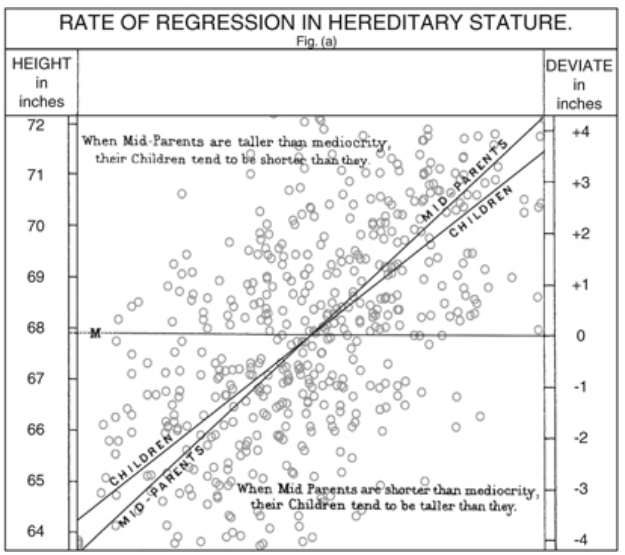
Schwartz

December 18, 2016

The Sophomore Slump

...or sophomore jinx or sophomore jitters refers to an instance in which a second, or sophomore, effort fails to live up to the standards of the first effort. It is commonly used to refer to the apathy of students (second year of high school, college or university), the performance of athletes (second season of play), singers/bands (second album), television shows (second seasons) and films (sequels/prequels). In the United Kingdom, the "sophomore slump" is more commonly referred to as "second year blues", particularly when describing university students. In Australia, it is known as "second year syndrome", and is particularly common when referring to professional athletes who have a mediocre second season following a stellar debut. The phenomenon of a "sophomore slump" can be explained psychologically, where earlier success has a reducing effect on the subsequent effort, but it can also be explained statistically, as an effect of the regression towards the mean.

The concept of "regression" comes from genetics and was popularized by Sir Francis Galton's late 19th century publication of "Regression towards mediocrity in hereditary stature." Galton observed that extreme characteristics (e.g., height) in parents are not completely passed on to offspring, but rather the characteristics in the offspring "regress" towards a mediocre point. By measuring the heights of hundreds of people Galton was able to quantify this "regression" and in so doing invented linear regression analysis, thus laying the groundwork for much of modern statistical modeling. The term "regression" stuck.



Objectives

- Terminology
- Least squares fit
- Normal distribution theory
- Coefficient testing
- Linear models and Multiple variables and Alternatives
- Model fit and Model selection
- Model diagnostics and Model evaluation

Linear Models and Regression Terminology

- $Y_i = \beta_0 + x_i\beta_1 + \epsilon_i, \epsilon_i \overset{i.i.d.}{\sim} Normal(0, \sigma^2)$

Linear Models and Regression Terminology

Outcome / Response / Label / Dependent/Endogenous Var.

- $Y_i = \beta_0 + x_i\beta_1 + \epsilon_i, \epsilon_i \overset{i.i.d.}{\sim} Normal(0, \sigma^2)$

Feature / Covariate / Independent/Exogenous Var.

Linear Models and Regression Terminology

- $Y_i = \beta_0 + x_i\beta_1 + \epsilon_i, \epsilon_i \overset{i.i.d.}{\sim} Normal(0, \sigma^2)$

Coefficient: β_1

Intercept: β_0

Error: ϵ_i

Noise: $\epsilon_i \overset{i.i.d.}{\sim} Normal(0, \sigma^2)$

Linear Models and Regression Terminology

Coefficient

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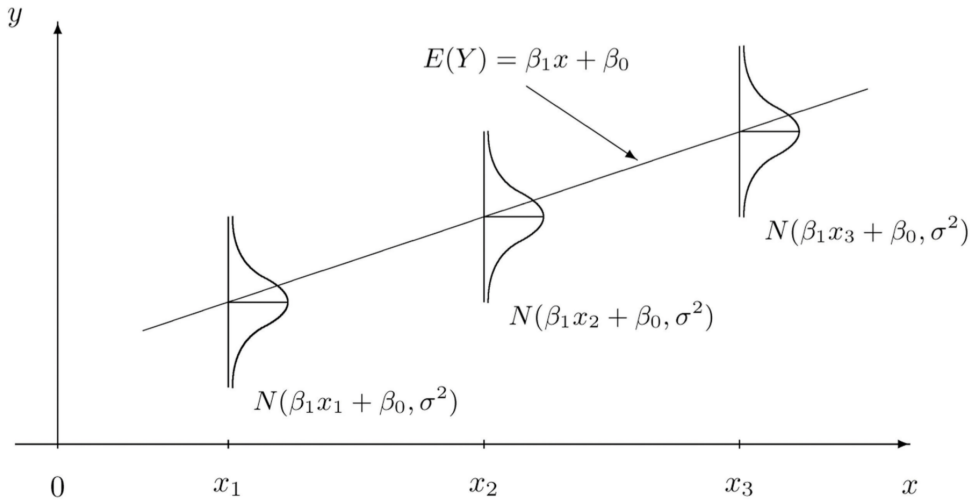
Intercept Error Noise



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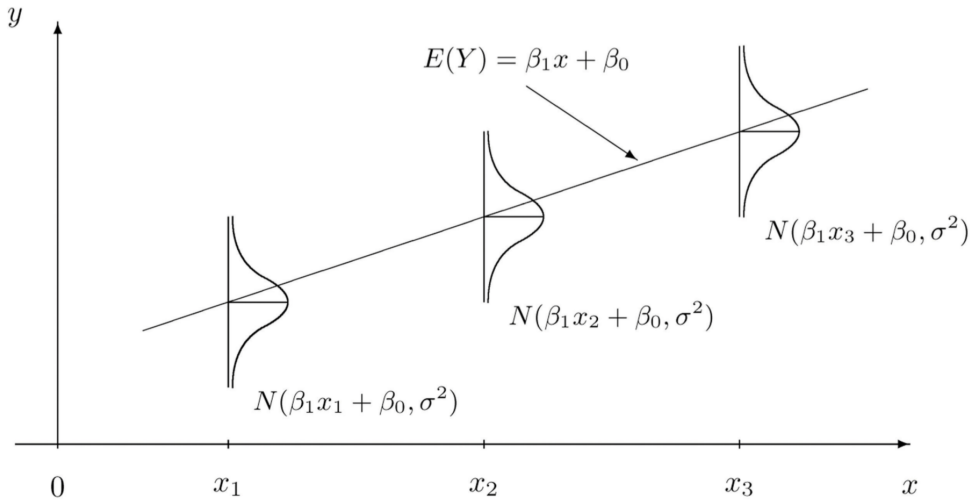

- $Y_i = \hat{\beta}_0 + x_i\hat{\beta}_1 + \hat{\epsilon}_i, \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} \hat{\epsilon}_i^2}{n-p-1}$ $(p = \#of\ coefficients)$

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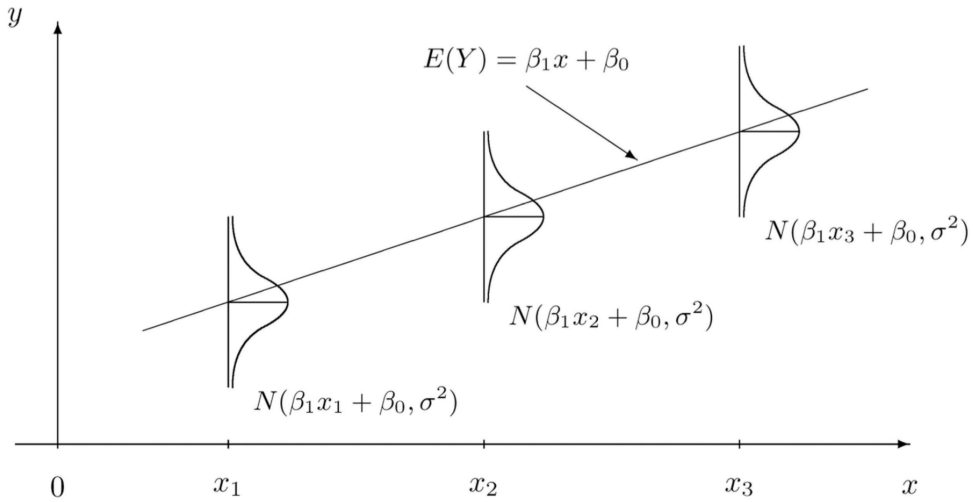

Fitted/Predicted value Residual Standard Error (RSE)

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Residual

Linear Models and Regression Terminology

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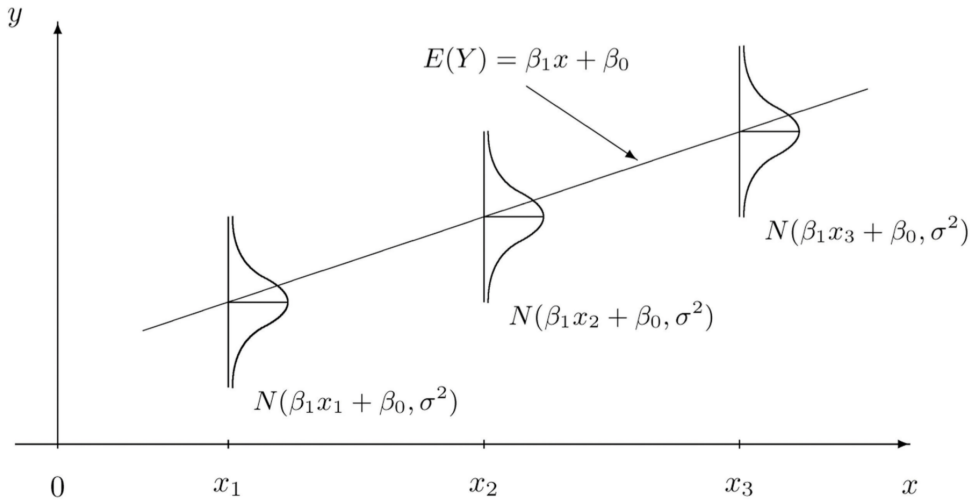
Intercept Error Noise



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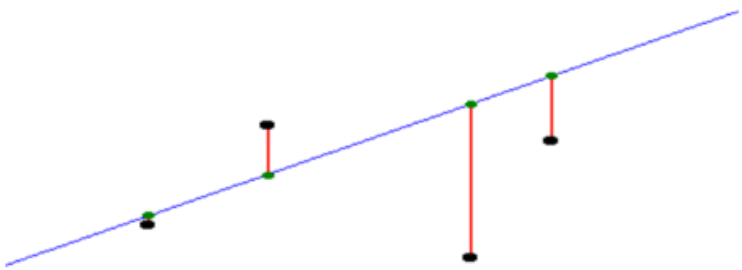
Residual



Least Squares Fit

- $Y_i = \hat{\beta}_0 + x_i \hat{\beta}_1 + \hat{\epsilon}_i$

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$$[\hat{\beta}_0, \hat{\beta}_1] = \underset{[\beta_0, \beta_1]}{argmin} \sum_{i=1}^{n} \hat{\epsilon}_i^2 = \underset{\beta}{argmin} \sum_{i=1}^{n} (Y_i - \mathbf{x}_i^T \beta)^2$$

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$$\sum_{i=1}^{n} (Y_i - \mathbf{x}_i^T\beta)^2 = (\mathbf{Y} - \mathbf{x}\beta)^T(\mathbf{Y} - \mathbf{x}\beta)$$

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$$\begin{aligned} &\nabla_\beta \beta^T(\mathbf{x}^T\mathbf{x})\beta - 2\mathbf{Y}^T\mathbf{x}\beta + \mathbf{Y}^T\mathbf{Y} \\ &= 2(\mathbf{x}^T\mathbf{x})\beta - 2\mathbf{Y}^T\mathbf{x} \qquad \text{(set to } \mathbf{0} \text{ to minimize)} \\ \Longrightarrow\ &\hat{\beta} = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{Y} \Longrightarrow \text{fitted values } \hat{\boldsymbol{Y}} = \mathbf{x}\hat{\beta} \end{aligned}$$

Least Squares Fit *bonus*

1. In simple linear regression the $\underset{\beta}{argmin}(\mathbf{Y} - \mathbf{x}\beta)^T(\mathbf{Y} - \mathbf{x}\beta)$ is

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{R_{xY} S_Y}{S_x}$$

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2. Maximum likelihood estimation (MLE) is equivalent

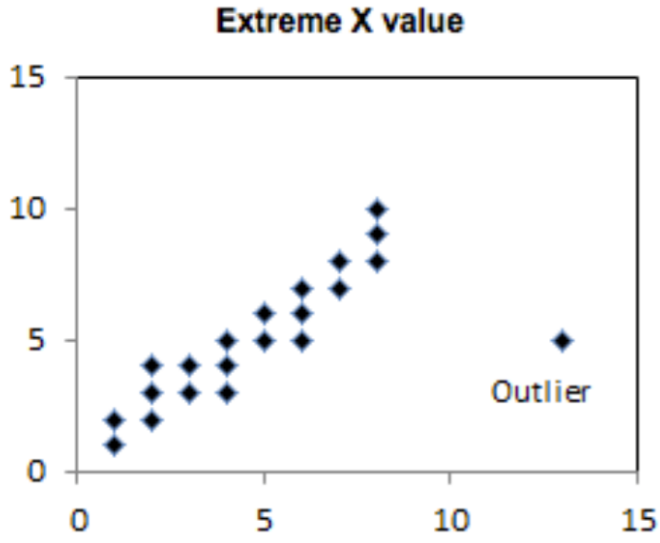
$$\underset{\beta}{argmax} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(Y_i - \mathbf{x}_i^T\beta)^2}$$

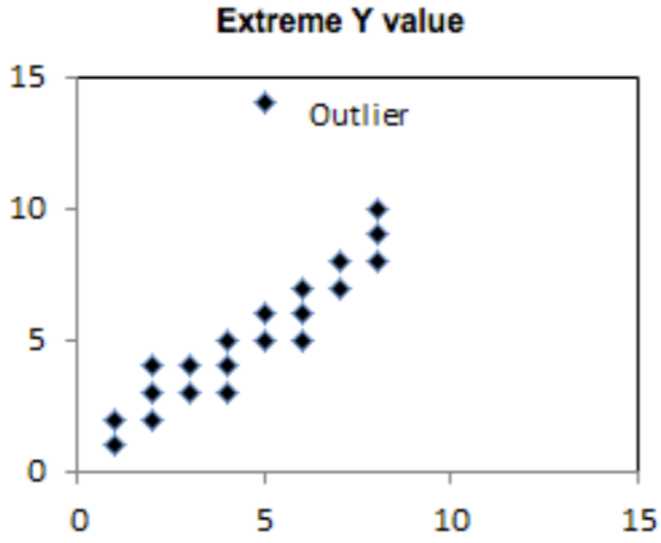
$$= \underset{\beta}{argmax} (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}(\mathbf{Y}-\mathbf{x}\beta)^T(\mathbf{Y}-\mathbf{x}\beta)}$$

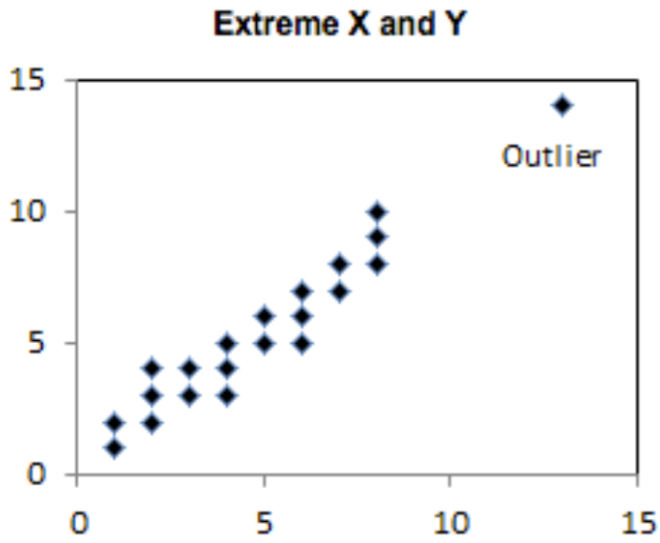
$$= \underset{\beta}{argmax} -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(\mathbf{Y} - \mathbf{x}\beta)^T(\mathbf{Y} - \mathbf{x}\beta)$$

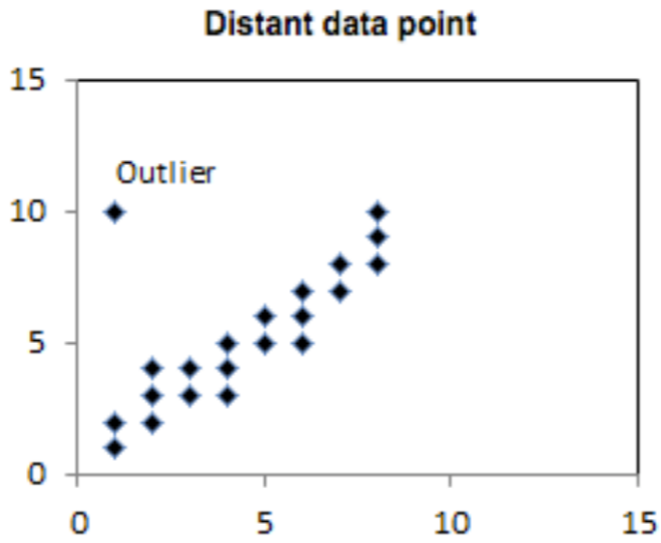
$$= \underset{\beta}{argmin} (\mathbf{Y} - \mathbf{x}_i\beta)^T(\mathbf{Y} - \mathbf{x}\beta)$$

Outliers


Extreme X value
15
10
5
0
0
5
10
15
Outlier


Extreme Y value
15
10
5
0
0
5
10
15
Outlier


Extreme X and Y
15
10
5
0
0
5
10
15
Outlier


Distant data point
15
10
5
0
0
5
10
15
Outlier

Leverage

The *hat* matrix H "puts the hat on $\boldsymbol{Y}$": H projects $\boldsymbol{Y}$ onto $\hat{\boldsymbol{Y}}$ – the (least squares) closest vector (to $\boldsymbol{Y}$) in the column space of x

$$
\begin{aligned}
H &= \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T \\
\hat{\boldsymbol{Y}} &= \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{Y} \\
&= H\mathbf{Y}
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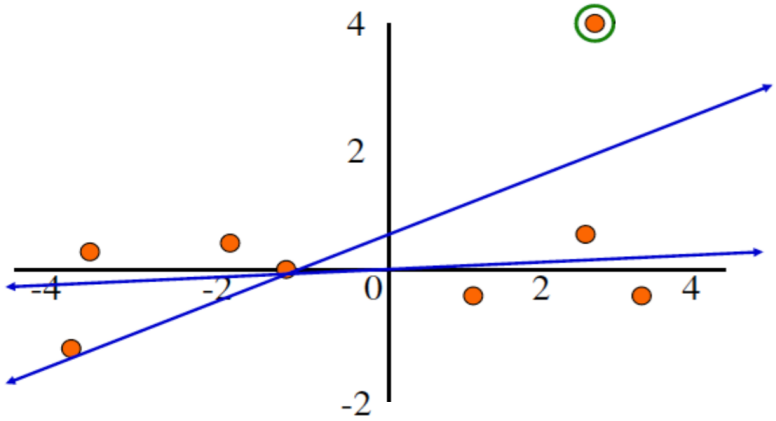
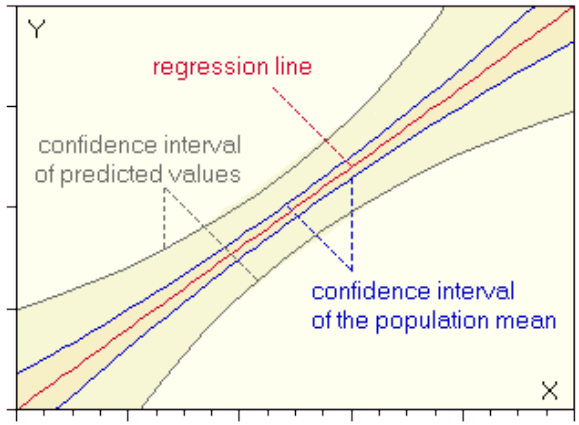


- Diagonal element $H_{ii} \in [0, 1]$, and $\sum_{i=1}^{n} H_{ii} = rank(\mathbf{x})$
- H_{ii} shows much $\hat{Y}_i$ depends on Y_i, however
  H_{ii} actually measures "extremeness" of x_i
- Relative comparison of H_{ii}'s id.'s "high leverage observations"
  H_{ii} is called the *leverage* of observation i

Influential Data Points

Studentized Residuals have a t-distribution

$$r_i = \frac{\hat{\epsilon}_i}{\hat{\sigma}\sqrt{1 - h_{ii}}}$$



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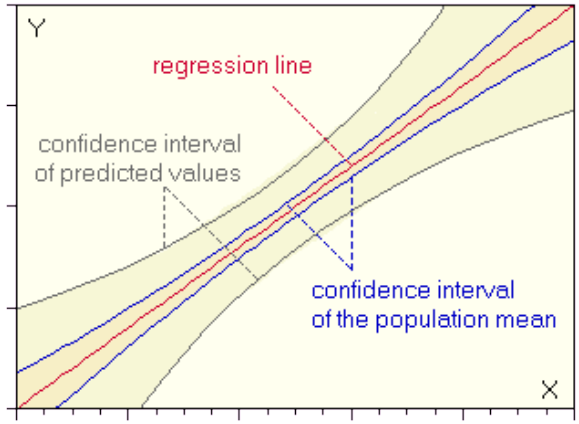
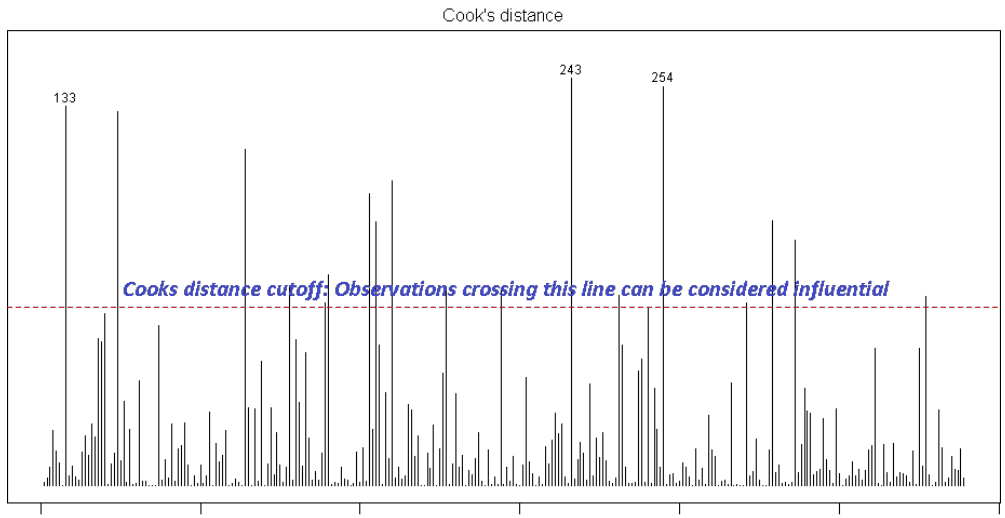


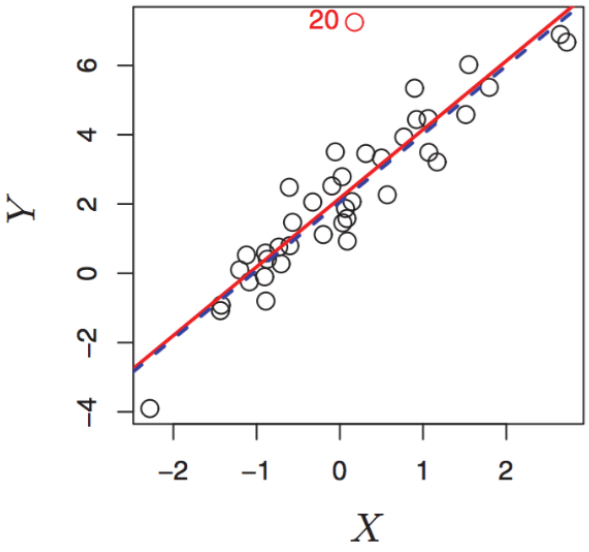
Influential data points i may have $D_i > \{3 \times \bar{D}, 1, 4/n, F^{1-\alpha}_{p,n-p}\}$

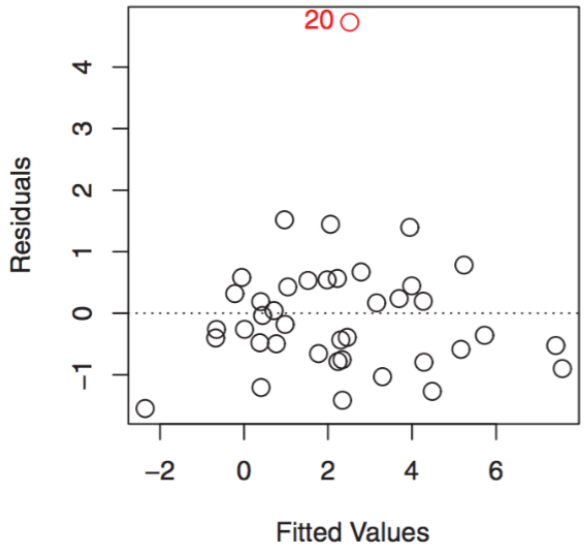
$$D_i = \frac{\sum_{j=1}^{n}\left(\hat{y}_j - \hat{y}_{j(-i)}\right)^2}{\hat{\sigma}^2 p}$$
$$= \frac{\hat{\epsilon}_i}{\hat{\sigma}^2 p}\frac{h_{ii}}{(1-h_{ii})^2}$$

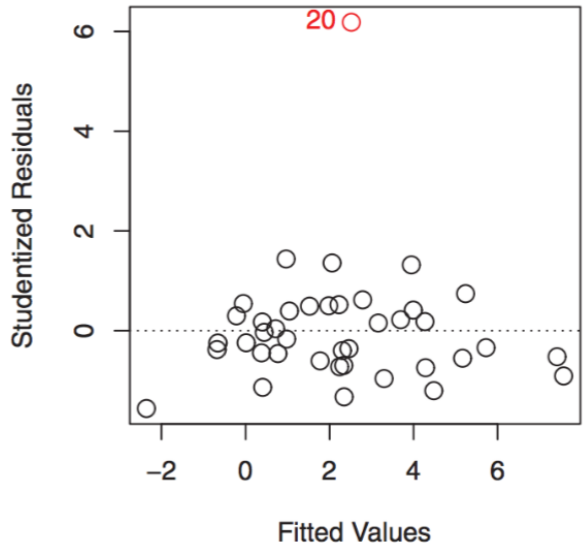


Regression Diagnostics

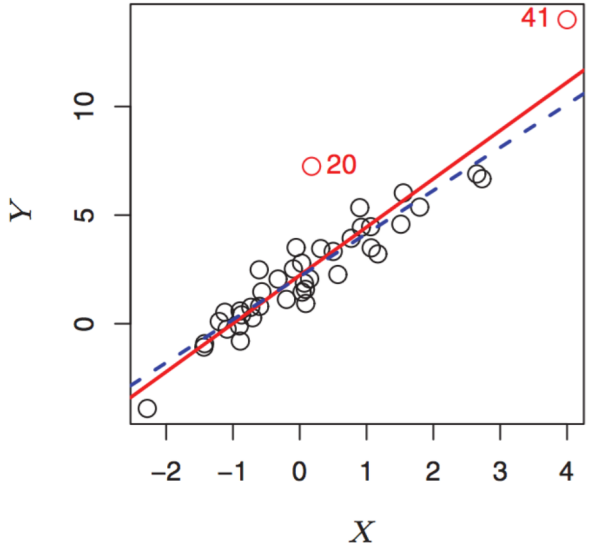
Outliers :

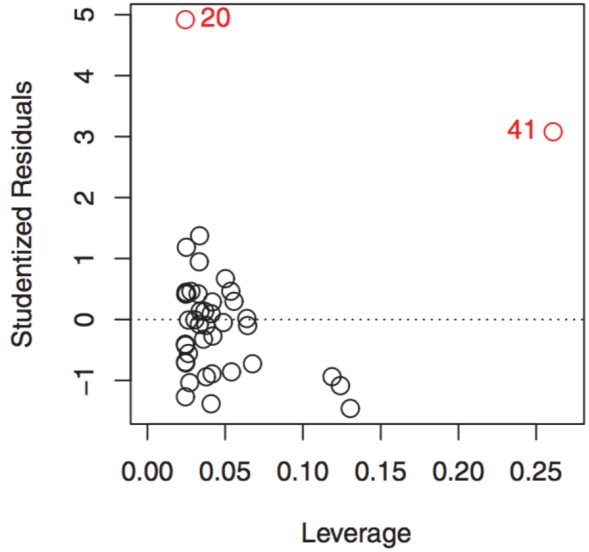






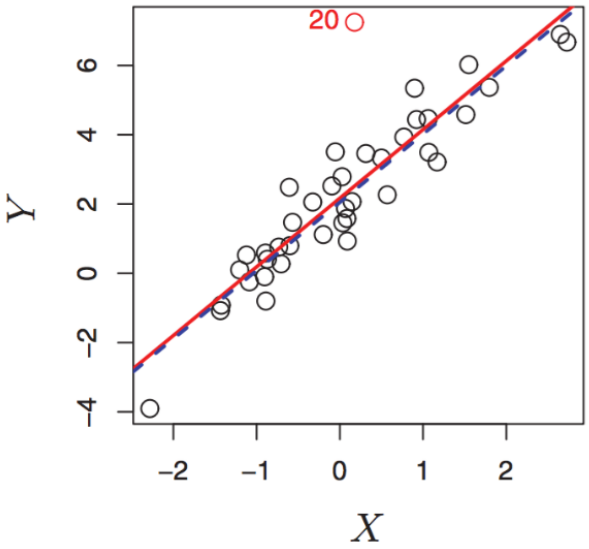
High Leverage Points :

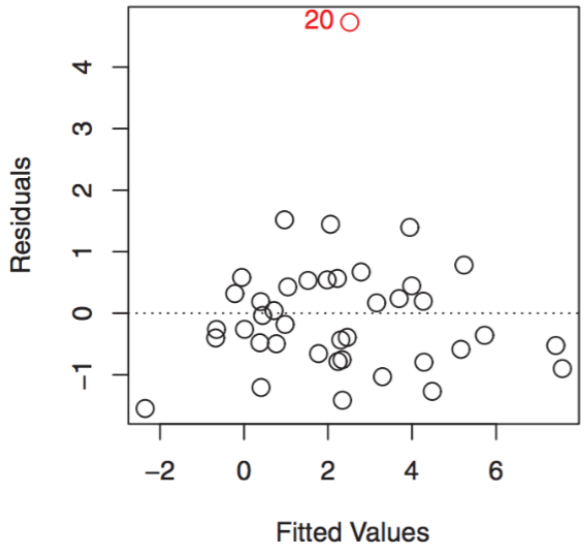


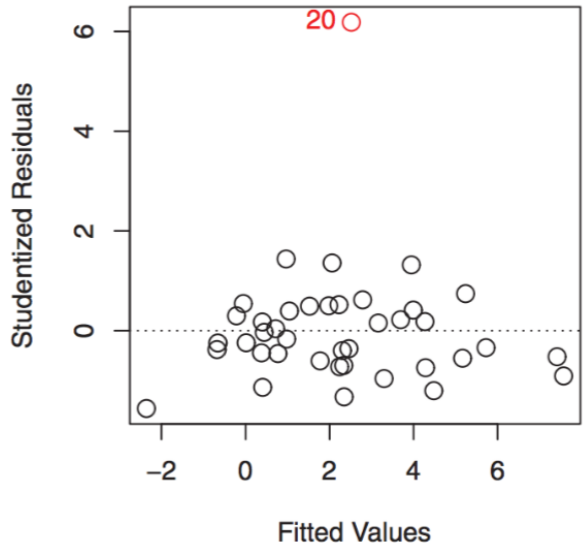


Regression Diagnostics

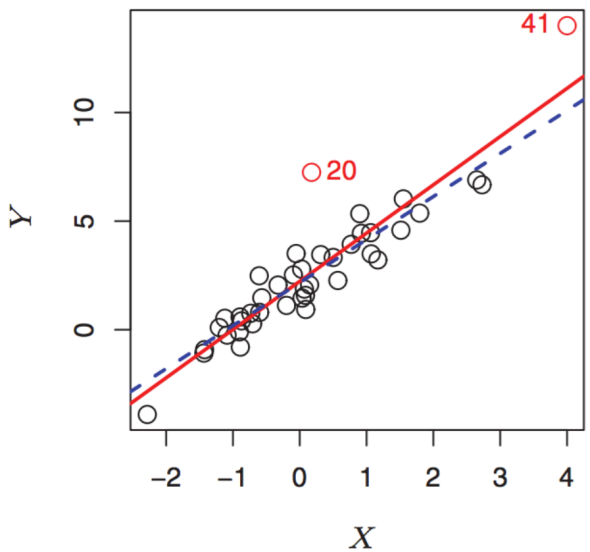
Outliers affecting estimates of the residual variance:

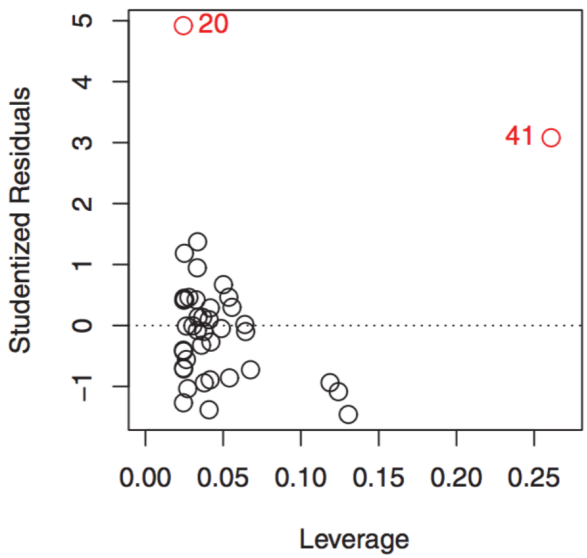






High Leverage Points affecting the fitted value predictions:





Regression is Multivariate Normal (MVN)

(inference in regression is based on this normality)

$$f(\mathbf{Y}|\mathbf{x}, \beta, \sigma^2) = \prod_{i=1}^{n} f(Y_i|\mathbf{x}_i, \beta, \sigma^2)$$

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f(\mathbf{Y}|\mathbf{x}, \beta, \sigma^2) &= \prod_{i=1}^{n} f(Y_i|\mathbf{x}_i, \beta, \sigma^2) \\
&= \prod_{i=1}^{n} N(\mathbf{x}_i^T \beta, \sigma^2) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(Y_i - \mathbf{x}_i^T \beta)^2} \\
&= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}(\mathbf{Y} - \mathbf{x}\beta)^T(\mathbf{Y} - \mathbf{x}\beta)} \quad = MVN(\mathbf{x}\beta, \sigma^2 I)
\end{aligned}
$$

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&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(Y_i-\mathbf{x}_i^T\beta)^2} \\
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\end{aligned}
$$

$$
\begin{aligned}
&\propto e^{-\frac{1}{2\sigma^2}(\beta-(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{Y})^T\mathbf{x}^T\mathbf{x}(\beta-(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{Y})} \\
&= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}(\beta-\hat{\beta})^T\mathbf{x}^T\mathbf{x}(\beta-\hat{\beta})} \\
\Longrightarrow\ & f(\hat{\beta}|\mathbf{x},\beta,\sigma^2) = MVN\left(\beta,\sigma^2(\mathbf{x}^T\mathbf{x})^{-1}\right)
\end{aligned}
$$

A SERIOUSLY MAJOR TRANSITION JUST HAPPENED

I just added more feature variables without telling you...

Multivariate Regression

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i, \ \epsilon_i \overset{i.i.d.}{\sim} \mathrm{N}(0, \sigma^2)$$

$$\boldsymbol{Y} \sim \mathrm{MVN}(\mathbf{x}\boldsymbol{\beta}, \sigma^2 I)$$

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Assumptions, violations, and remedial measures

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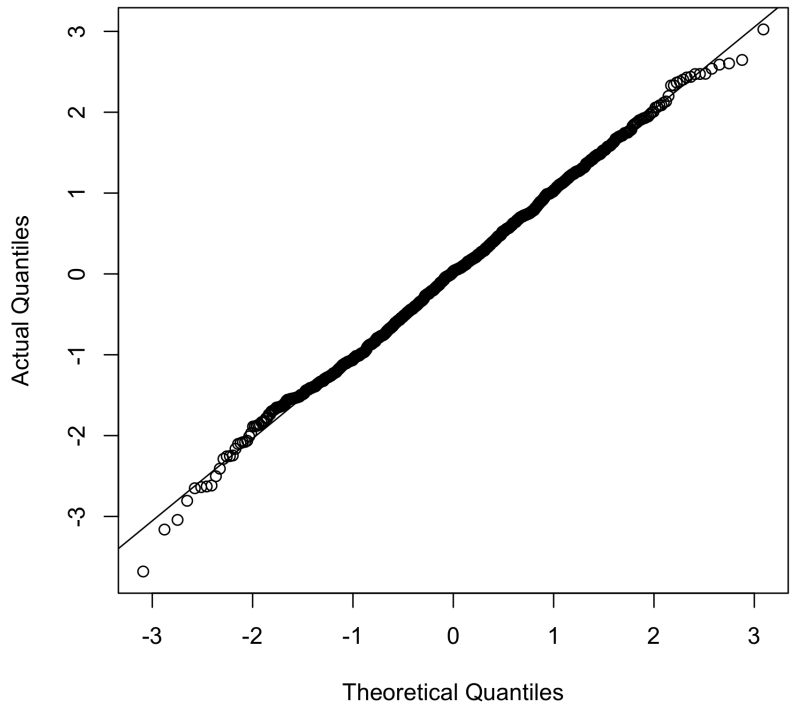
- Normality



Q-Q Plot

Hypothesis testing depends on distributional assumptions

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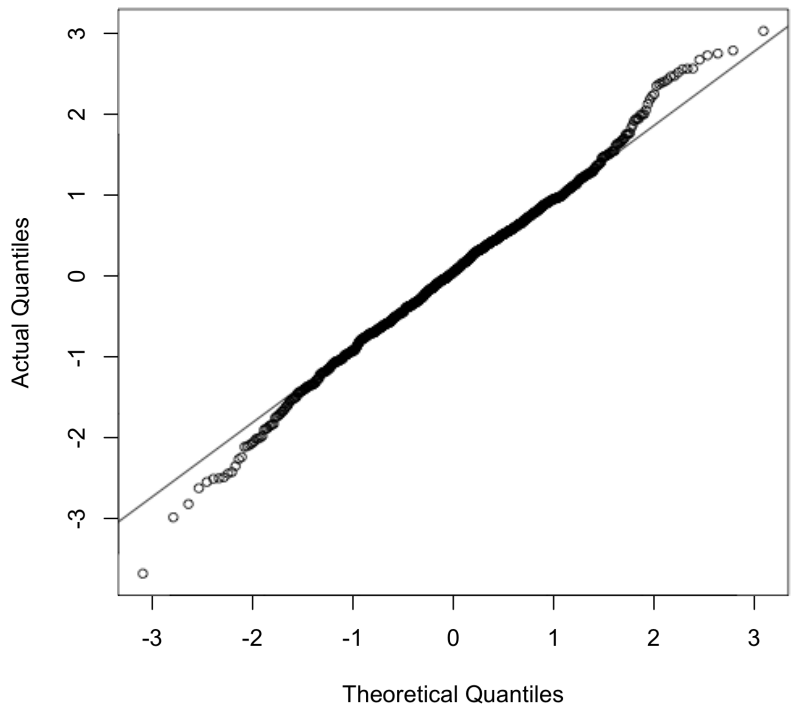

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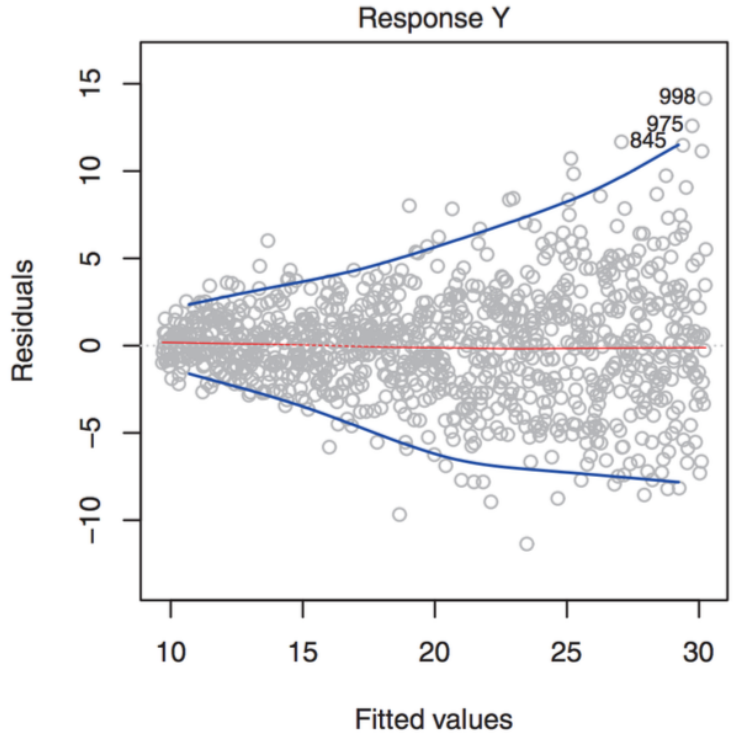

Residuals versus Fitted Values

Box-Cox transformations $\frac{Y^\lambda - 1}{\lambda}$ can help

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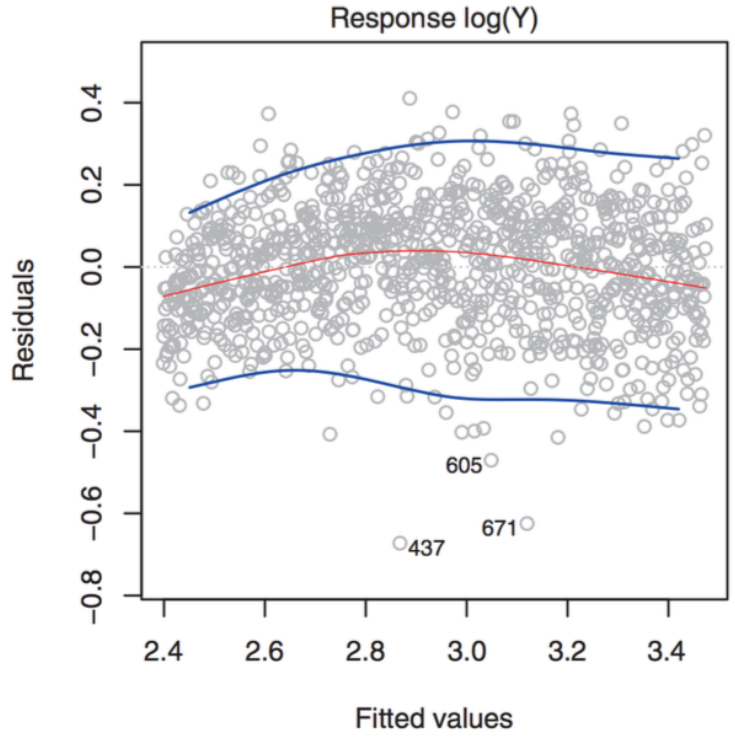

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- Independence

$$\mathrm{Cov}[\boldsymbol{Y} - \boldsymbol{X}\beta] \approx \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

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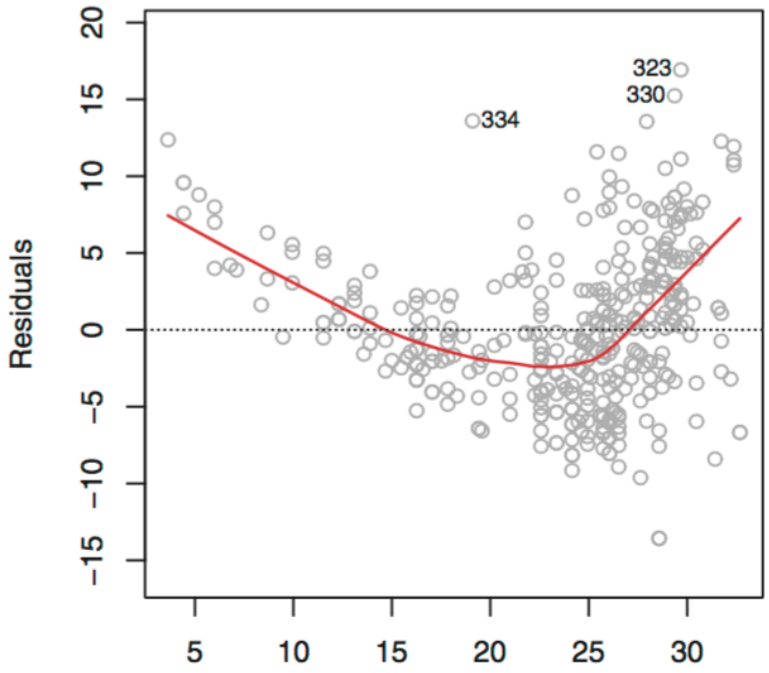
- Linear form



Residuals versus Feature Values

"All models are wrong, some are useful"
– George Box

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Multicollinearity and the Variance Inflation Factor (VIF)

And when you have any number of covariates (features)...

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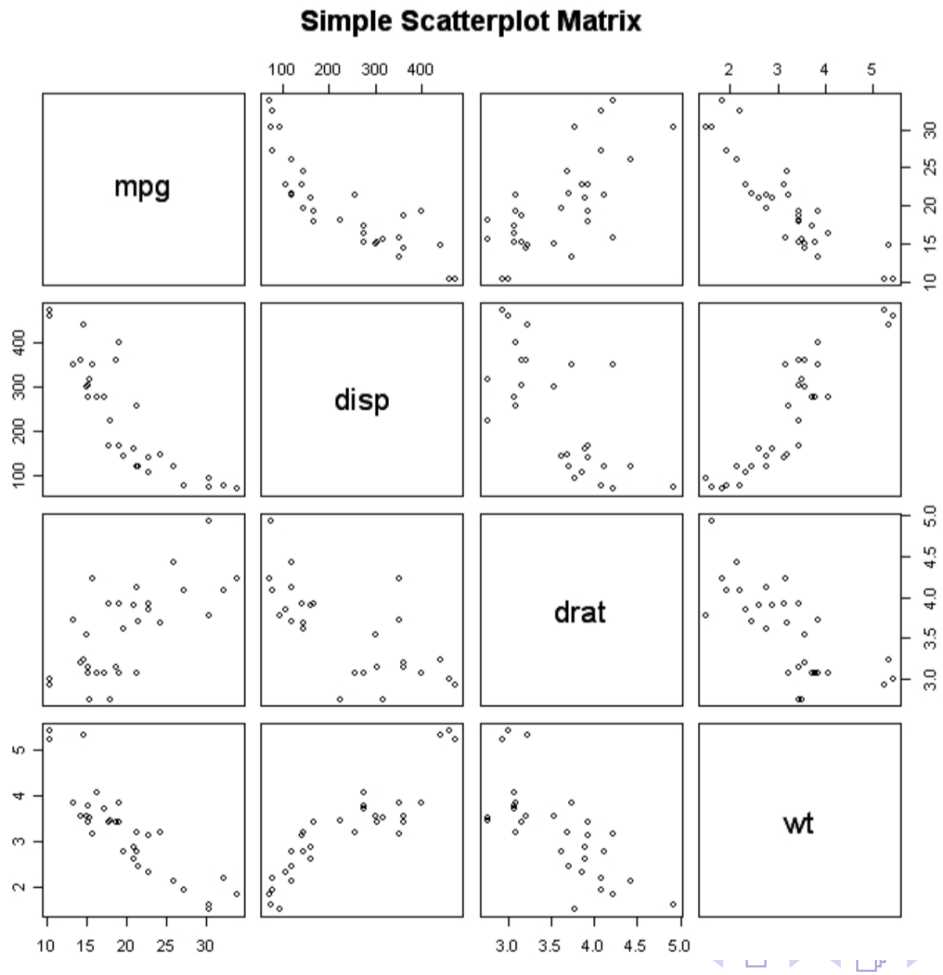

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	DJIA	S&P 500	Nasdaq	Canada	Mexico	Brazil	Stoxx 50	FTSE 100	CAC 40	DAX	IBEX	Italy	Netherlands	Sweden	Switzerland	Nikkei	Hang Seng	Australia
DJIA		0.97	0.85	0.57	0.56	0.52	0.52	0.48	0.51	0.56	0.49	0.50	0.50	0.42	0.42	0.09	0.11	0.07
S&P 500	0.97		0.91	0.62	0.58	0.55	0.50	0.47	0.50	0.55	0.48	0.50	0.49	0.41	0.41	0.09	0.11	0.05
Nasdaq	0.85	0.91		0.58	0.56	0.52	0.48	0.43	0.48	0.54	0.47	0.48	0.48	0.42	0.38	0.14	0.16	0.07
Canada	0.57	0.62	0.58		0.53	0.53	0.42	0.45	0.41	0.41	0.42	0.42	0.39	0.37	0.35	0.17	0.22	0.17
Mexico	0.56	0.58	0.56	0.53		0.56	0.42	0.42	0.44	0.43	0.43	0.44	0.39	0.38	0.38	0.17	0.25	0.17
Brazil	0.52	0.55	0.52	0.53	0.56		0.33	0.35	0.32	0.34	0.34	0.34	0.29	0.30	0.28	0.17	0.22	0.15
Stoxx 50	0.52	0.50	0.48	0.42	0.42	0.33		0.92	0.94	0.89	0.87	0.88	0.92	0.78	0.86	0.26	0.30	0.24
FTSE 100	0.48	0.47	0.43	0.45	0.42	0.35	0.92		0.86	0.80	0.80	0.82	0.84	0.73	0.78	0.26	0.30	0.26
CAC 40	0.51	0.50	0.48	0.41	0.44	0.32	0.94	0.86		0.89	0.88	0.89	0.92	0.78	0.84	0.28	0.32	0.25
DAX	0.56	0.55	0.54	0.41	0.43	0.34	0.89	0.80	0.89		0.83	0.84	0.86	0.75	0.77	0.26	0.29	0.21
IBEX	0.49	0.48	0.47	0.42	0.43	0.34	0.87	0.80	0.88	0.83		0.84	0.83	0.75	0.77	0.27	0.32	0.26
Italy	0.50	0.50	0.48	0.42	0.44	0.34	0.88	0.82	0.89	0.84	0.84		0.85	0.74	0.78	0.24	0.29	0.23
Netherlands	0.50	0.49	0.48	0.39	0.39	0.29	0.92	0.84	0.92	0.86	0.83	0.85		0.75	0.82	0.27	0.30	0.23
Sweden	0.42	0.41	0.42	0.37	0.38	0.30	0.78	0.73	0.78	0.75	0.75	0.74	0.75		0.75	0.29	0.33	0.27
Switzerland	0.42	0.41	0.38	0.35	0.38	0.28	0.86	0.78	0.84	0.77	0.77	0.78	0.82	0.75		0.29	0.32	0.29
Nikkei	0.09	0.09	0.14	0.17	0.17	0.17	0.26	0.26	0.28	0.26	0.27	0.24	0.27	0.29	0.29		0.52	0.49
Hang Seng	0.11	0.11	0.16	0.22	0.25	0.22	0.30	0.30	0.32	0.29	0.32	0.29	0.30	0.33	0.32	0.52		0.48
Australia	0.07	0.05	0.07	0.17	0.17	0.15	0.24	0.26	0.25	0.21	0.26	0.23	0.23	0.27	0.29	0.49	0.48	

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where R_j^2 is the R^2 of X_j regressed on all the other X's

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Scaling X's (putting X's on the same scale) helps numerically

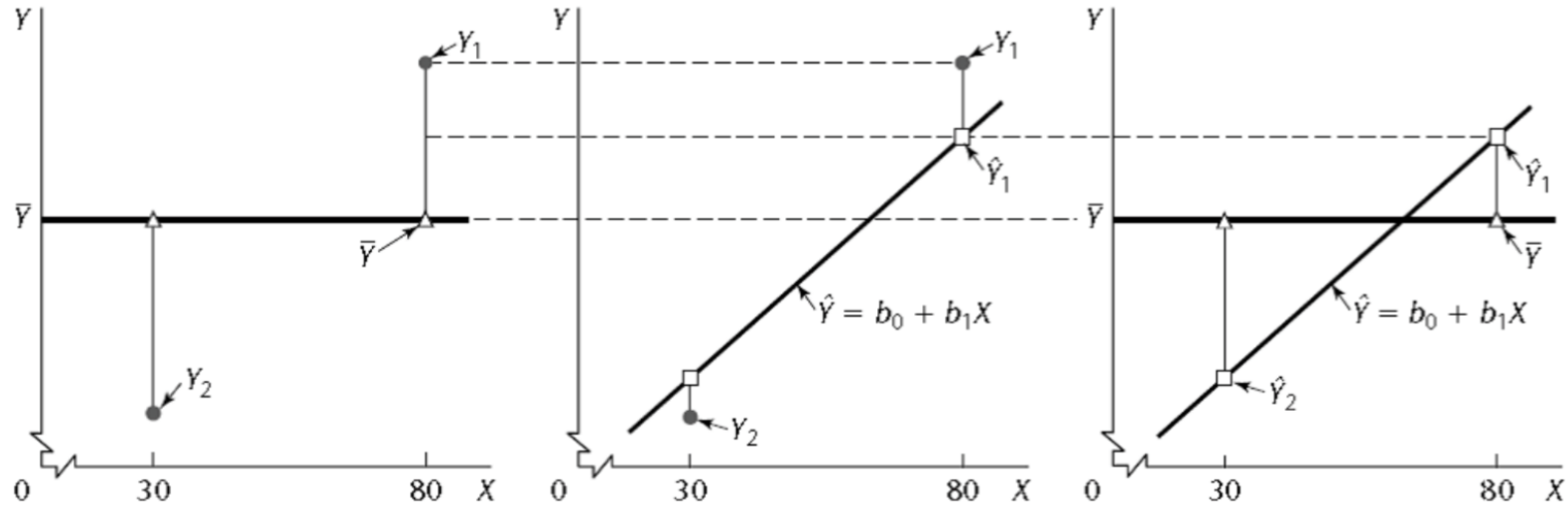
Model Fit

Residual Variation	Total Variation
$RSS = \sum(Y_i - \hat{Y}_i)^2 = \sum \hat{\epsilon}_i^2$	$TSS = \sum(Y_i - \bar{Y})^2 = RSS + \sum(\hat{Y}_i - \bar{Y})^2$
Residual Standard Error	**Proportion of Variance Explained**
$RSE = \sqrt{\frac{1}{n-p-1} RSS} = \sqrt{\frac{\sum(Y_i - \hat{Y}_i)^2}{n-p-1}}$	$R^2 = \frac{TSS-RSS}{TSS} = 1 - \frac{RSS}{TSS}$

F-test

$$F = \frac{(TSS-RSS)/p}{RSS/(n-p-1)} = \frac{\sum(\hat{Y}_i - \bar{Y})^2/p}{\sum(Y_i - \hat{Y})/(n-p-1)}$$

Decomposition of Total Variation



$$
\begin{aligned}
TSS &= \sum (Y_i - \bar{Y})^2 = \sum (Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2 \\
&= \sum (Y_i - \hat{Y}_i)^2 + 2\sum (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) + \sum (\hat{Y}_i - \bar{Y})^2 \\
&= \sum (Y_i - \hat{Y}_i)^2 + 2\sum \hat{\epsilon}_i (\hat{Y}_i - \bar{Y}) + \sum (\hat{Y}_i - \bar{Y})^2 \\
&\qquad \sum \hat{\epsilon}_i = 0 \uparrow \quad \uparrow \sum \hat{\epsilon}_i \hat{Y}_i = 0 \\
&= \sum (Y_i - \hat{Y}_i)^2 + \sum (\hat{Y}_i - \bar{Y})^2 = RSS + \sum (\hat{Y}_i - \bar{Y})^2
\end{aligned}
$$

Model Selection

- R^2 (model fit) is insufficient – more features means larger R^2

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- Classical Model Selection Criterion

Mallow's C_p	$\frac{1}{n}(RSS + 2p\hat{\sigma}^2)$	
AIC	$-2\log L + 2p$	$D_M = -2\log f(Y\|\hat{\theta}^{M_p}) + 2\log f(Y\|Y)$
BIC	$-2\log L + p\log n$	$D_M \overset{approx.}{\sim} \chi^2_{n-p-1}$
Adjusted R^2	$1 - \frac{RSS/(n-p-1)}{TSS/(n-1)}$	

Testing: an example with simple linear regression

$$f(\mathbf{Y}|\mathbf{x}, \beta, \sigma^2) = MVN(\mathbf{x}\beta, \sigma^2 I)$$

$$\implies f(\hat{\beta}|\mathbf{x}, \beta, \sigma^2) = MVN\left(\beta, \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}\right)$$

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where

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{x} \quad \hat{\beta}_1 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{R_{xY}S_Y}{S_x}$$

$$\text{Var}(\hat{\beta}_1) = \sigma^2 \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad \text{Var}(\hat{\beta}_0) = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]$$

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$$\implies f(\hat{\beta}|\mathbf{x}, \beta, \sigma^2) = MVN\left(\beta, \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}\right)$$

For simple linear regression then

$$\hat{\beta} \sim MVN\left(\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \frac{\sigma^2}{n\sum(x_i - \bar{x})^2}\begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}\right)$$

where

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{x} \quad \hat{\beta}_1 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{R_{xY}S_Y}{S_x}$$

$$\text{Var}(\hat{\beta}_1) = \sigma^2 \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad \text{Var}(\hat{\beta}_0) = \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]$$

$$\text{SE}(\hat{\beta}_0) = \sqrt{\text{Var}(\hat{\beta}_0)} \quad \text{SE}(\hat{\beta}_1) = \sqrt{\text{Var}(\hat{\beta}_1)}$$

Testing *Extra Credit*

- What is $\text{Var}(\hat{Y}_0)$? (Suppose we know σ^2)

  Hint: $\hat{Y}_0 = \hat{\beta}_0 + x_0\hat{\beta}_1$

  Hint: $Var[aX + bY] = ?$

- $\text{Var}(Y_0)$? For a *new observation* Y_0 according to our model? (Suppose we know σ^2)

- *Hint:* $Y_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0 + \epsilon$

Coefficient Testing *in General*

For $\hat{\beta} = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{Y}$, since (under H_0)

$$f(\hat{\beta}|\beta, \sigma^2) = MVN\left(\beta, \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}\right)$$

we have that

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and if we estimate

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(where p is the number of coefficients) then we have that

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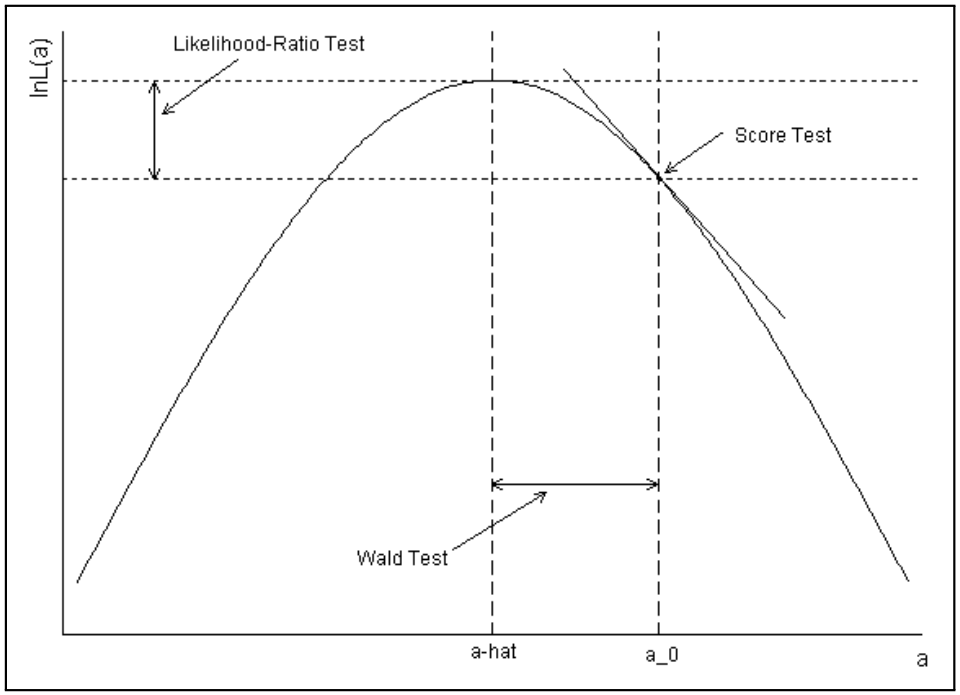
And this works for any number of feature variables...

Coefficient Testing *in even more Generality*



Wald test

$$\frac{(\hat{\theta}-\theta_0)^2}{Var(\hat{\theta})} \overset{approx.}{\sim} N(0,1) \text{ under } H_0$$

Likelihood-Ratio (LR) test

$$-2ln\left(\frac{L(\theta_0|x)}{L(\theta|x)}\right) \overset{approx.}{\sim} \chi^2_k \text{ under } H_0$$

Score test

$$\frac{\left(\frac{\partial}{\partial\theta}\log L(\theta_0|x)\right)^2}{-\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\log L(\theta_0|x)\right]} \overset{approx.}{\sim} \chi^2_1 \text{ under } H_0$$

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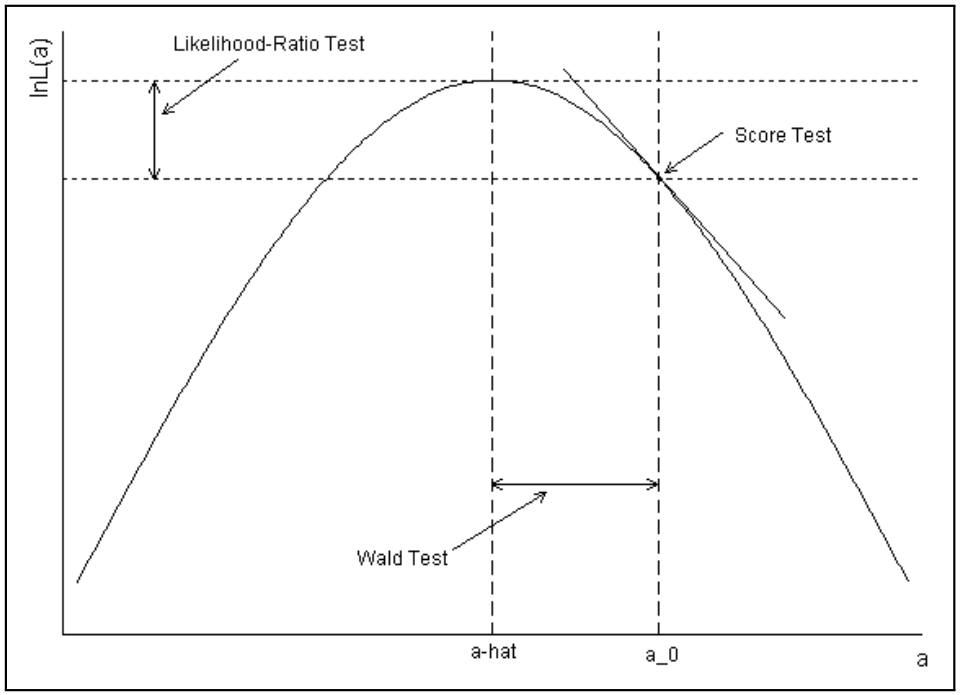

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Hypothesis Testing for Feature Selection

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OLS Regression Results

Dep. Variable:	y	R-squared:	0.933
Model:	OLS	Adj. R-squared:	0.928
Method:	Least Squares	F-statistic:	211.8
Date:	Mon, 03 Nov 2014	Prob (F-statistic):	6.30e-27 ←
Time:	14:45:06	Log-Likelihood:	-34.438
No. Observations:	50	AIC:	76.88
Df Residuals:	46	BIC:	84.52
Df Model:	3		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[95.0% Conf.	Int.]
x1	0.4687	0.026	17.751	0.000	0.416	0.522
x2	0.4836	0.104	4.659	0.000	0.275	0.693
x3	-0.0174	0.002	-7.507	0.000	-0.022	-0.013
const	5.2058	0.171	30.405	0.000	4.861	5.550

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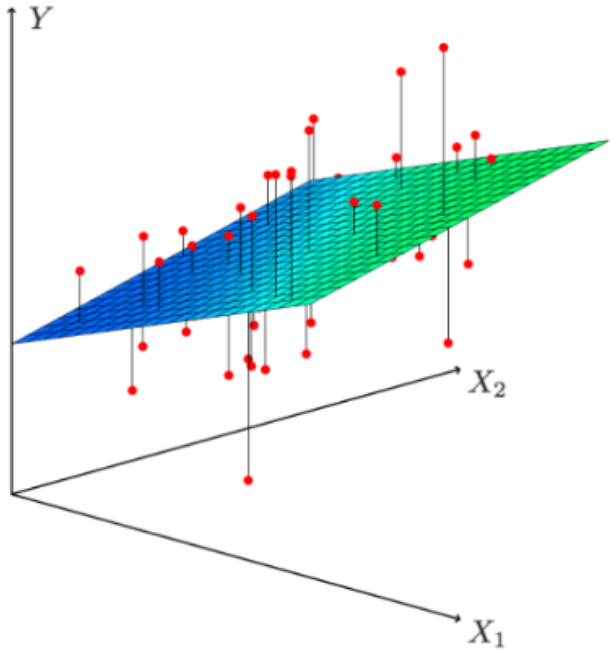

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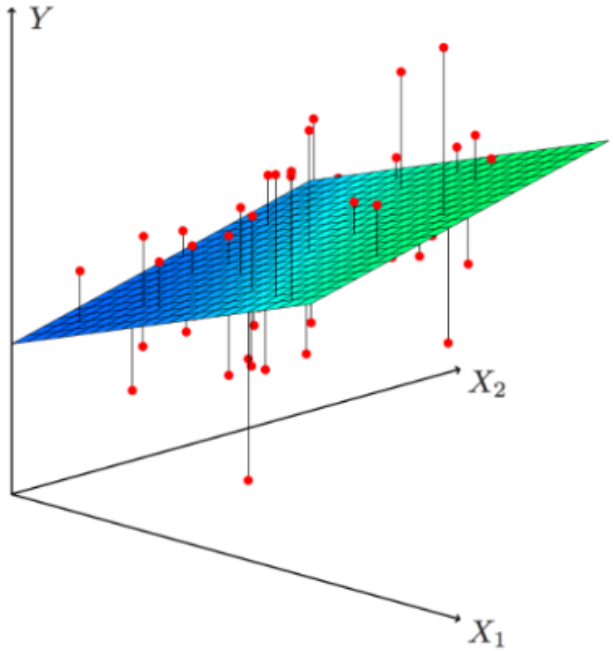

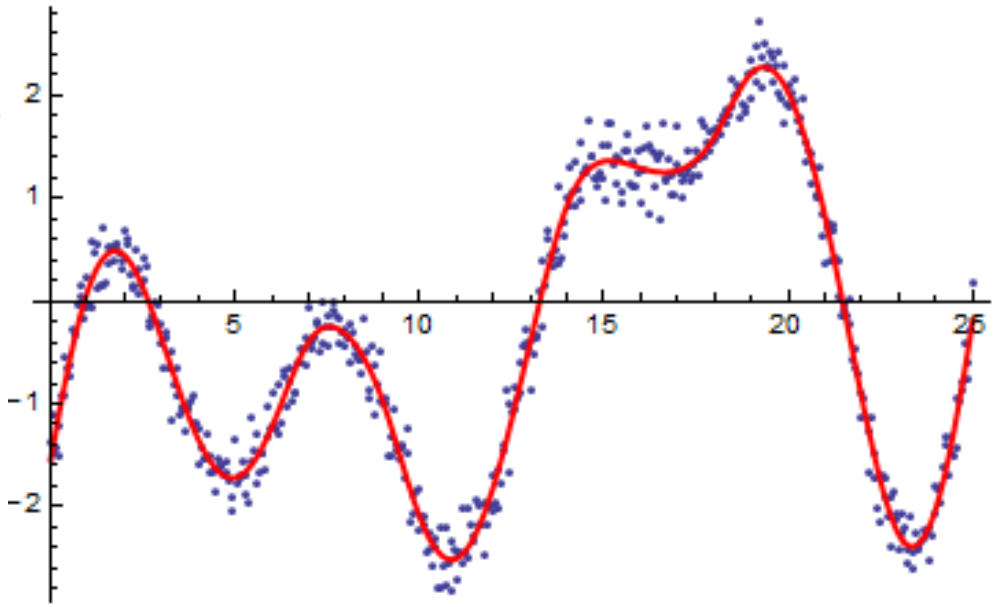

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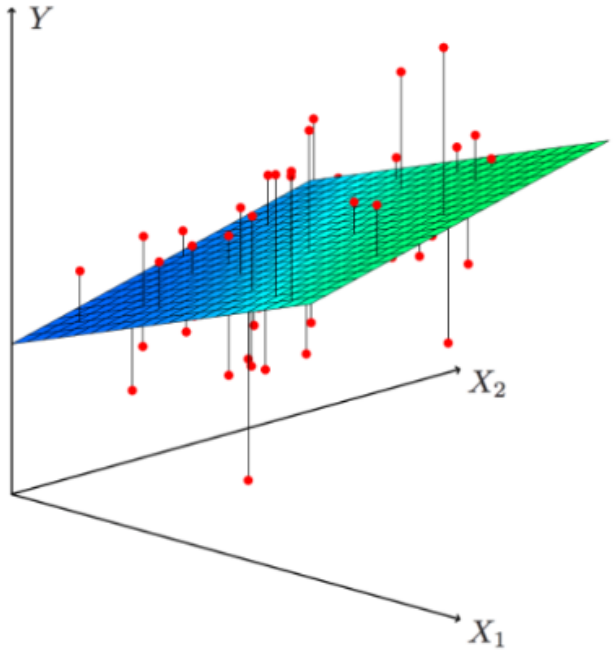

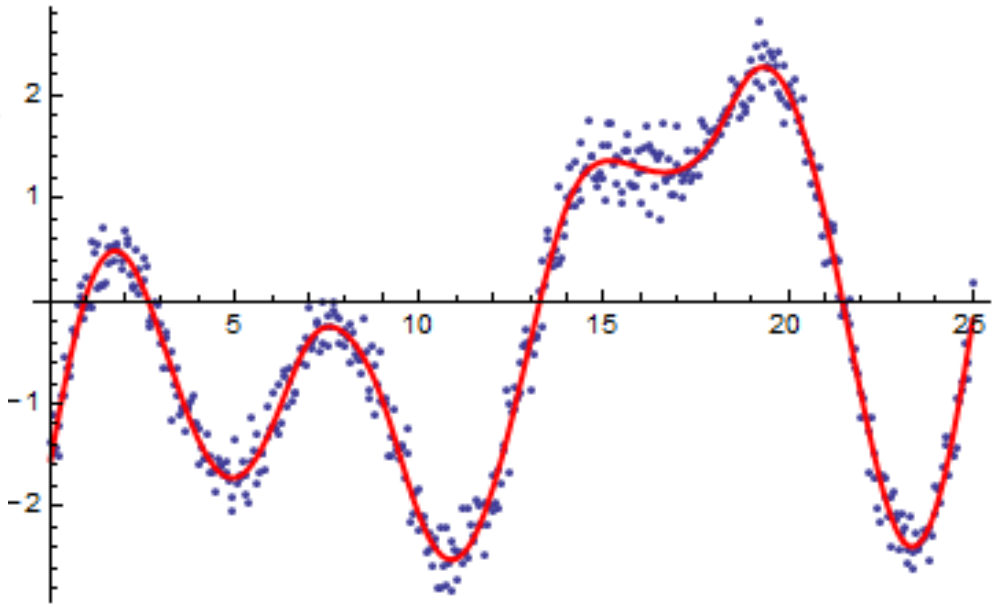

- "Linear" models are only linear in the coefficients

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i$$

- The x's can be pretty wild...

Linear models that produce “non-linear” response surfaces

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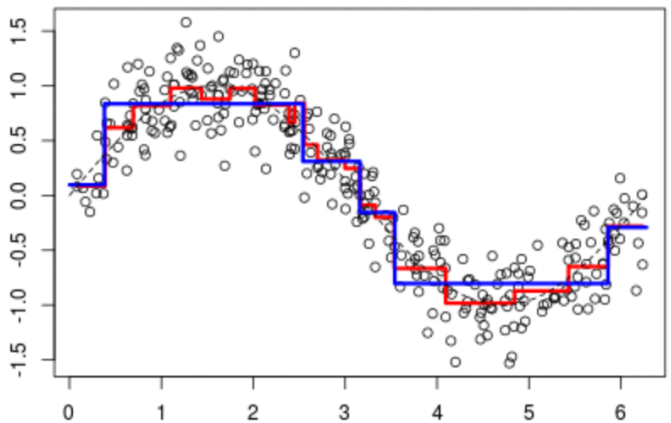
Step functions

$Y_i = \beta_j : if\ a_j \leq X_i < b_j$



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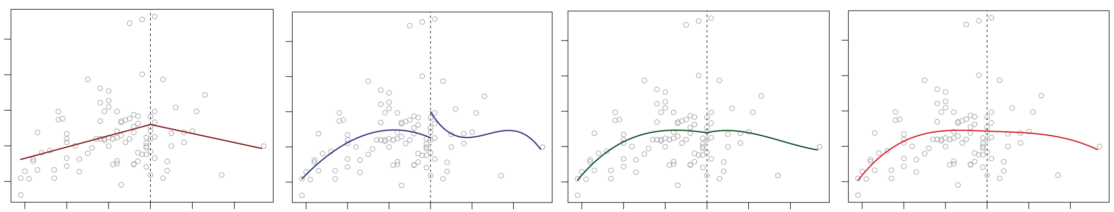
Step functions
Regression Splines



$$Y_i = \begin{cases} \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \epsilon_i & : if\ X_i \leq c \\ \beta_0^* + \beta_1 X_i + \beta_2^* X_i^2 + \beta_3^* X_i^3 + \epsilon_i & : if\ X_i > c \end{cases}$$

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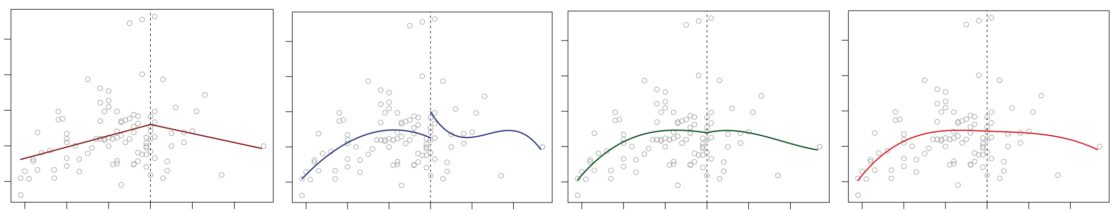

$$h(X_i, \xi) = \begin{cases} (x - \xi)^3 & : if\ X_i > \xi \\ 0 & : if\ X_i \leq \xi \end{cases} \qquad Y_i = \begin{array}{l} \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 \\ +\beta_{s1} h(X_i, \xi_1) + \cdots + \epsilon_i \end{array}$$

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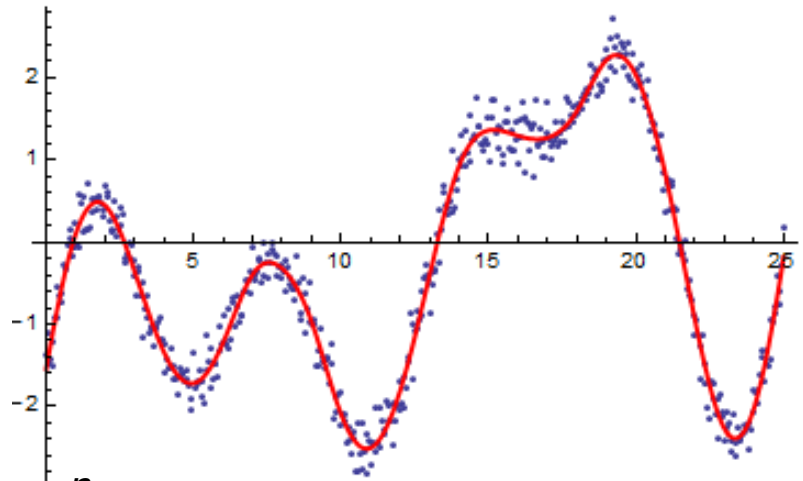
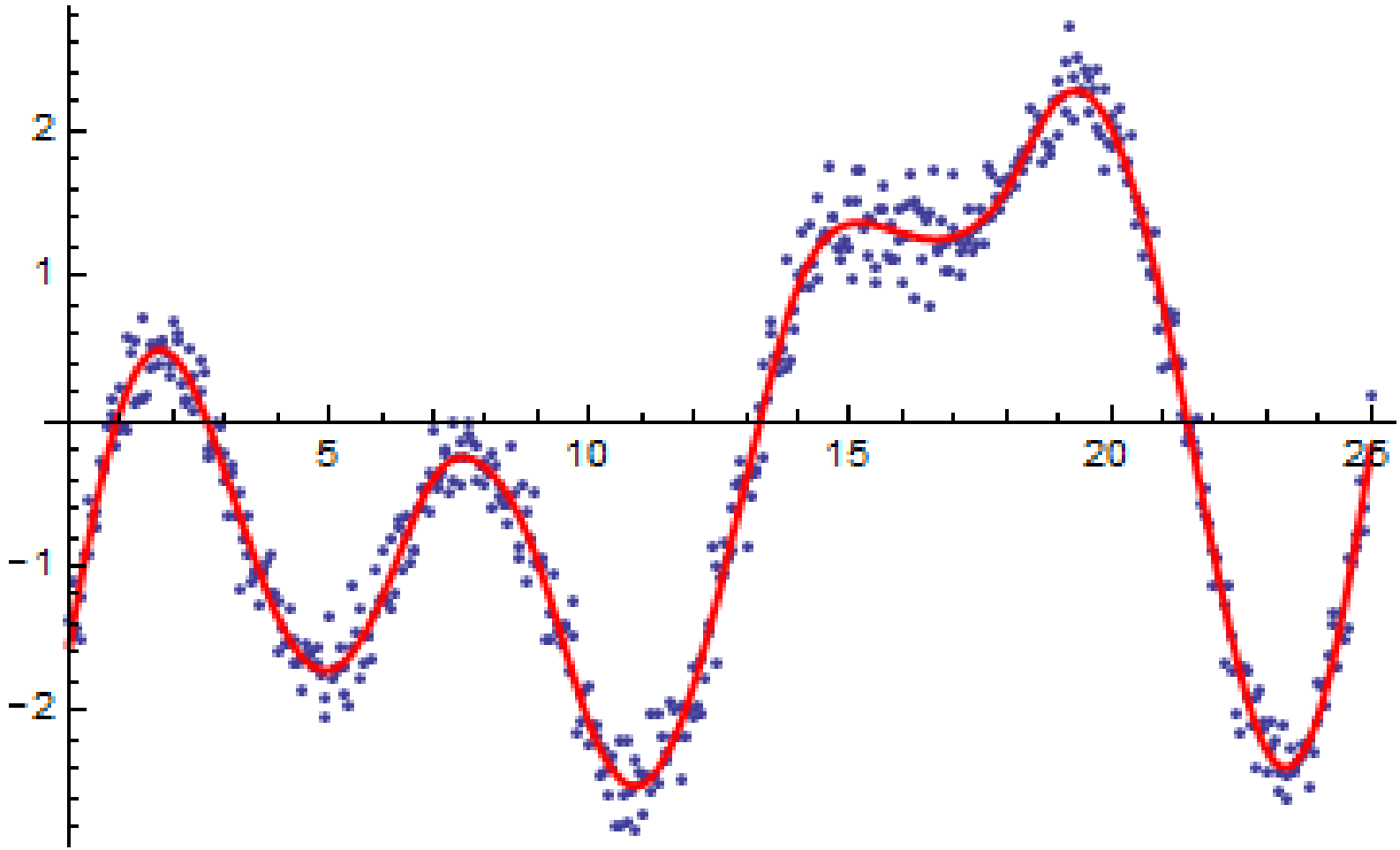
Step functions
Regression Splines
Smoothing Splines



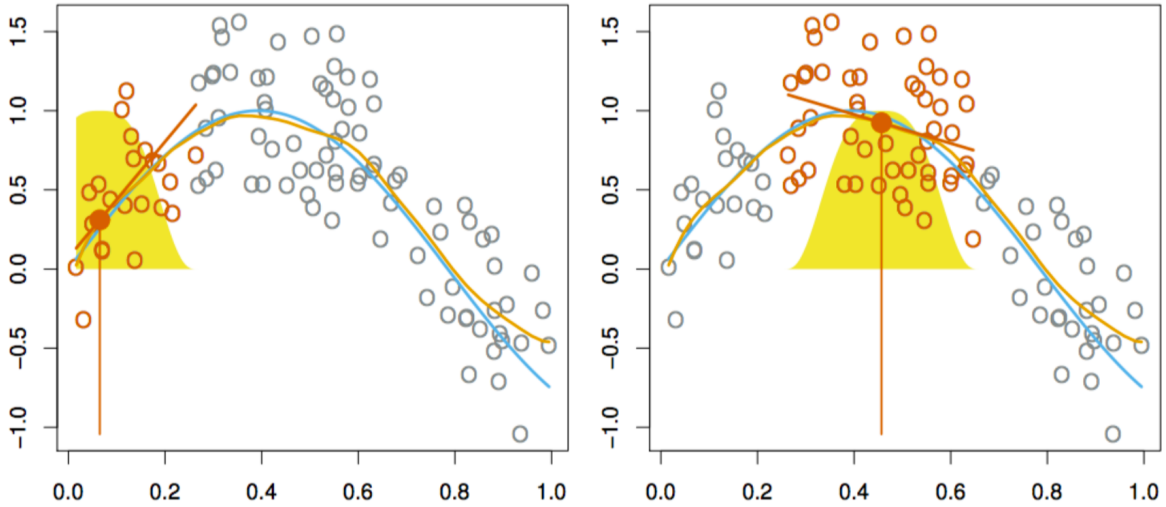
$$\min_g \sum_{i=1}^{n} (Y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$

Linear models aren't really so "linear"



Other ways to get “non linear” response surfaces

- Local Regression (LOESS)



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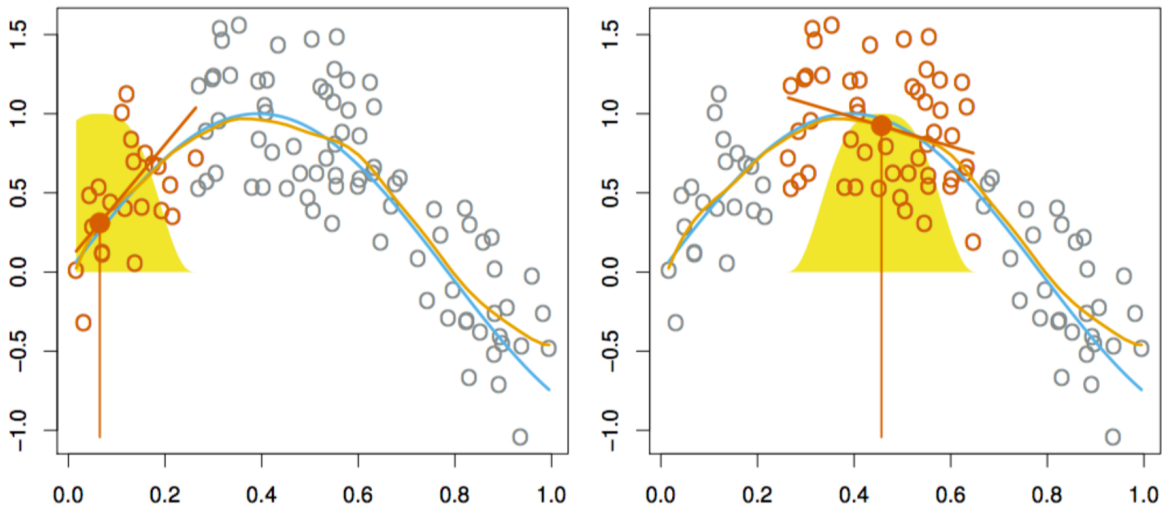

- Generalized Additive Models

$$y_i = \beta_0 + f_1(x_{i1}) + f_2(x_{i2}) + \cdots + f_p(x_{ip}) + \epsilon_i.$$