Statistical Power and Bayesian Inference

Today’s objectives and plan

Morning:

- Review the hypothesis testing and type I and type II errors; which error is more serious?
- Compute power; understand relationship and trade-offs against other factors (significance level, effect size, variance, sample size) that influence power
- Morning assignment: Compute power and sample size

Afternoon:

- Review conditional probability
- Bayes' rule
- Bayesian inference
- Pair programming: Bayesian analysis, verification by simulation

Morning: Statistical Power

Hypothesis Testing: Possible Outcomes

	H_0 is true	H_0 is false
Fail to reject H_0	Correct Decision $(1 - \alpha)$	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision $(1 - \beta = \pi)$

Type I and Type II Errors

Type I Error

$= P(\text{Reject } H_0 \mid H_0 \text{ is true})$

$= \alpha$ (alpha, the significance level)

Type II Error

$= P(\text{Fail to reject } H_0 \mid H_0 \text{ is false})$

$= \beta$ (beta) $= 1 - \pi$ (one minus power)

Which Type of Errors is More Serious?

Example 1: You have been put on trial for murder

H_0: You are innocent (presumption of innocence)

H_A: You are guilty

The question for the jury is whether it finds enough evidence that you are guilty

Example 1: You have been put on trial for murder (cont.)

	H_0 is true You are **truly** innocent	H_0 is false You are **truly** guilty
Fail to reject H_0 Did not found enough evidence that you are guilty: You are **declared** innocent	Correct	Type II Error You are guilty but set free
Reject H_0 Found enough evidence that you are guilty: You are **declared** guilty	Type I Error You are found guilty of a murder that you did not commit (and the murderer is still free…) **The American justice system puts a lot of emphasis on avoiding type I errors**	Correct

Example 2: You are being screened for a disease

H_0: You don’t have the disease

H_A: You have the disease

The question for the test is whether it finds enough evidence (e.g., markers) that you have the disease

Example 2: You are being screened for a disease (cont.)

	H_0 is true You **truly** are disease-free	H_0 is false You **truly** have the disease
Fail to reject H_0 Did not found enough evidence that you are sick: You are **declared** to be disease-free	Correct	Type II Error You'll have the incorrect assurance that you don't have the disease **As a result, you won't be treated for your disease**
Reject H_0 Found enough evidence that you are sick: You are **declared** to have the disease	Type I Error You'll be anxious for a while but this will lead to other testing procedures. Eventually, you'll discover that the initial test was incorrect	Correct

Example 3: You are working on a new drug

H_0: The new drug is not more effective than current drugs

H_A: The new drug more effective than current drugs

Underpowered study:

- With a type II error, you would fail to reject that the new drug is not more effective than current drugs while said drug being really more effective

Minimizing the type II error (β)

Is equivalent to maximize π

$= 1 - \beta$

$= 1 -$ P(Fail to reject H_0 | H_0 is false)

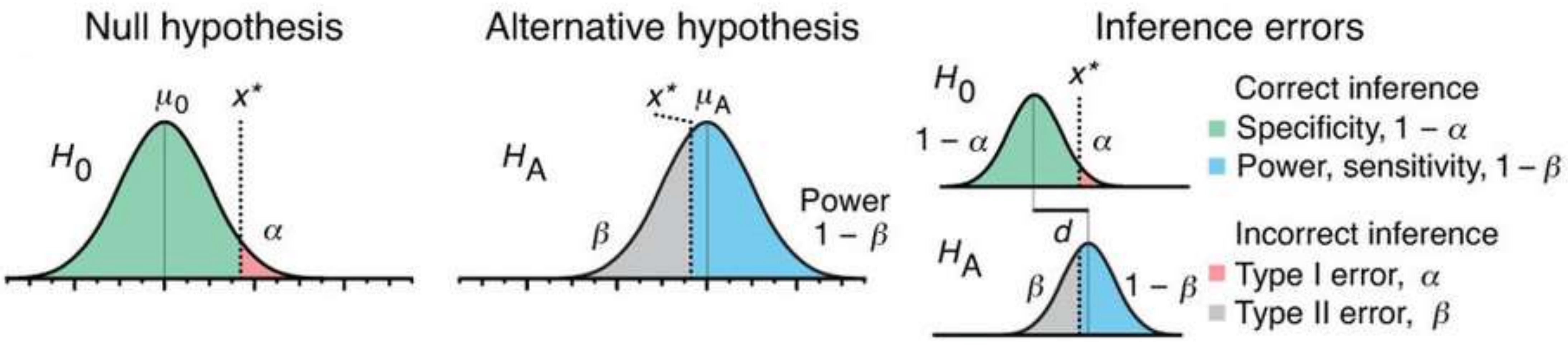
= P(Reject H_0 | H_0 is false)

= P(Reject H_0 | H_A is true)

Hypothesis Testing: Possible Outcomes

	H_0 is true	H_0 is false
Fail to reject H_0	Correct Decision $(1 - \alpha)$	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision $(1 - \beta = \pi)$

Hypothesis Testing: Outcomes and Regions



Compute Statistical Power of a Test

Example: One-sample Test of Mean

H_0: $\mu = \mu_0$ (e.g., the test finds no evidence of the better effectiveness of the new drug)

H_A: $\mu = \mu_1$ ($> \mu_0$) (e.g., the test finds evidence of the better effectiveness of the new drug)

Compute Statistical Power of a Test (cont.)

First, we reject H_0 when:

$$Z_\alpha \leq Z = \frac{X - \mu_0}{s/\sqrt{n}} \ or \ \mu_0 + Z_\alpha \frac{s}{\sqrt{n}} = X^* \leq X$$

Then, we find the corresponding cut-off of this value under H_A is:

$$X^* = \mu_1 + Z_{1-\beta} \frac{s}{\sqrt{n}} = \mu_1 - Z_\beta \frac{s}{\sqrt{n}}$$

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

Statistical Power and its relationship with other Factors

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

The equation above links the following variables:

- α (type I error; significance level)
- β (type II error; $\pi = 1 - \beta$, the statistical power)
- $\mu_1 - \mu_0$ (effect size)
- s (standard deviation)
- n (sample size)

Define 4 of these variables and solve for the remaining one

A. Type I Error and Type II Error Trade-Off

Assuming all other variables are fixed

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{{s}/{\sqrt{n}}}$$

If $\alpha \downarrow$ (Type I Error increases):

?

A. Type I Error and Type II Error Trade-Off (cont.)

Assuming all other variables are fixed

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

If $\alpha \downarrow$ (Type I Error increases):

- $Z_\alpha \uparrow$, $Z_\beta \downarrow$
- $\beta \uparrow$ (Type II error increases)
- $1 - \beta \downarrow$ (Power decreases)

B. Power and Effect Size Trade-Off

Assuming all other variables are fixed

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

If 1 - β ↑ (Power increases):

?

B. Power and Effect Size Trade-Off (cont.)

Assuming all other variables are fixed

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

If 1 - $\beta \uparrow$ (Power increases):

- $\beta \downarrow$
- $Z_\beta \uparrow$
- μ_1 - $\mu_0 \uparrow$ (Effect size increases)

C. Power and Sample Size Trade-Off

Assuming all other variables are fixed

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

If 1 - β↑ (Power increases):

?

C. Power and Sample Size Trade-Off (cont.)

Assuming all other variables are fixed

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

If 1 - β ↑ (Power increases):

- β ↓
- Z_β ↑
- $\frac{\mu_1 - \mu_0}{s/\sqrt{n}}$ ↑
- $s/\sqrt{n}$ ↓
- n ↑ (Sample size increases)

(as your sample size increases, the power of your test increases)

D. Sample Size and Effect Size Trade-Off

Assuming all other variables are fixed

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

If n $\uparrow$ (Sample size increases):

?

D. Sample Size and Effect Size Trade-Off (cont.)

Assuming all other variables are fixed

$$Z_\alpha + Z_\beta = \frac{\mu_1 - \mu_0}{s/\sqrt{n}}$$

If n ↑ (Sample size increases):

- $s/\sqrt{n}$ ↓
- $\mu_1 - \mu_0$ ↓ (Effect size decreases)

Power analysis allows you to determine the sample size needed to detect a particular effect (with more samples, you can detect smaller size effects)

Morning assignment

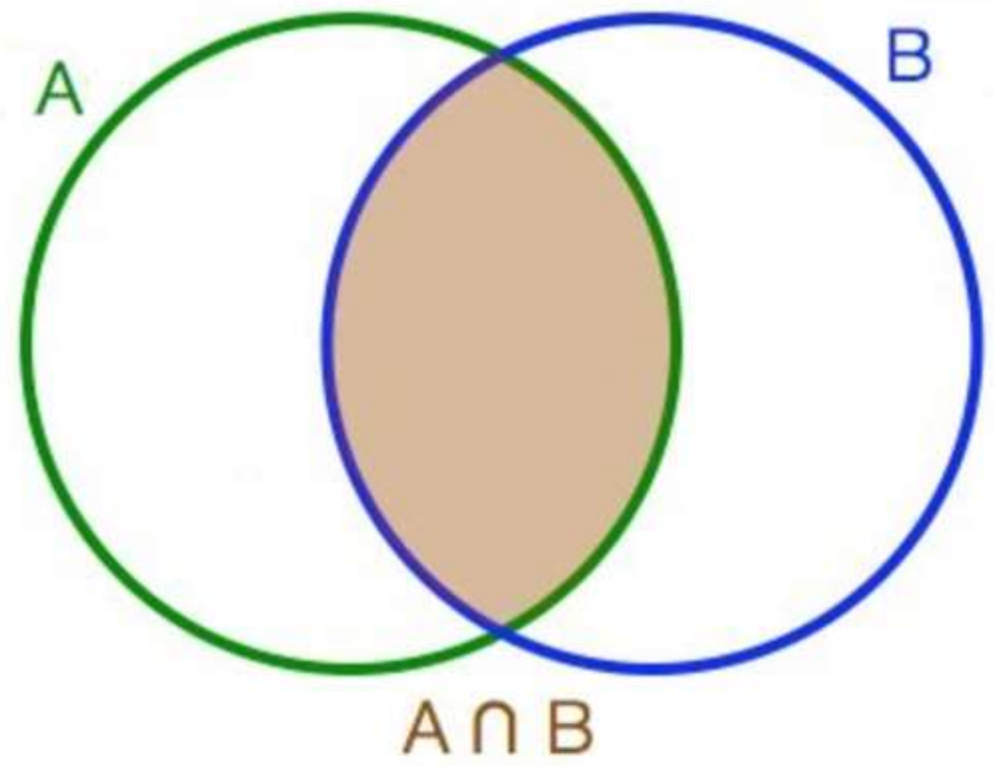
Afternoon: Bayesian Inference

Bayes' Rule: Motivation

- How to relate conditional probabilities between two events
- How to incorporate prior knowledge and belief into interpretation of data

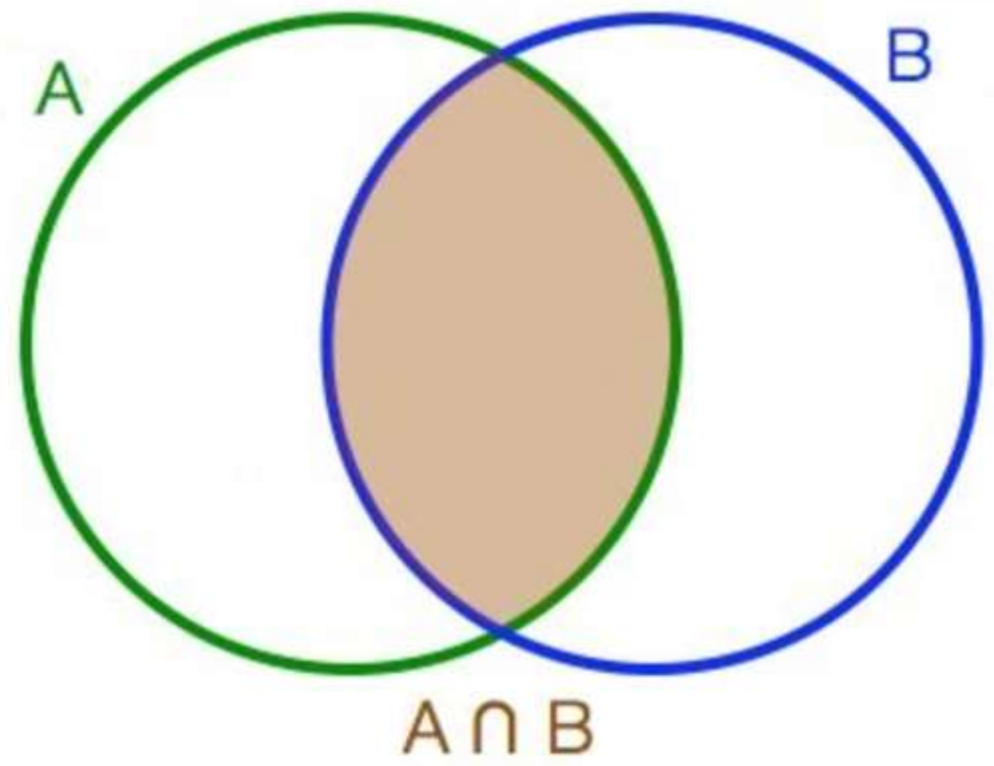
Conditional Probability Review

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



Poll: What's the probability of rolling a dice with a value less than 4 knowing that the value is odd

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

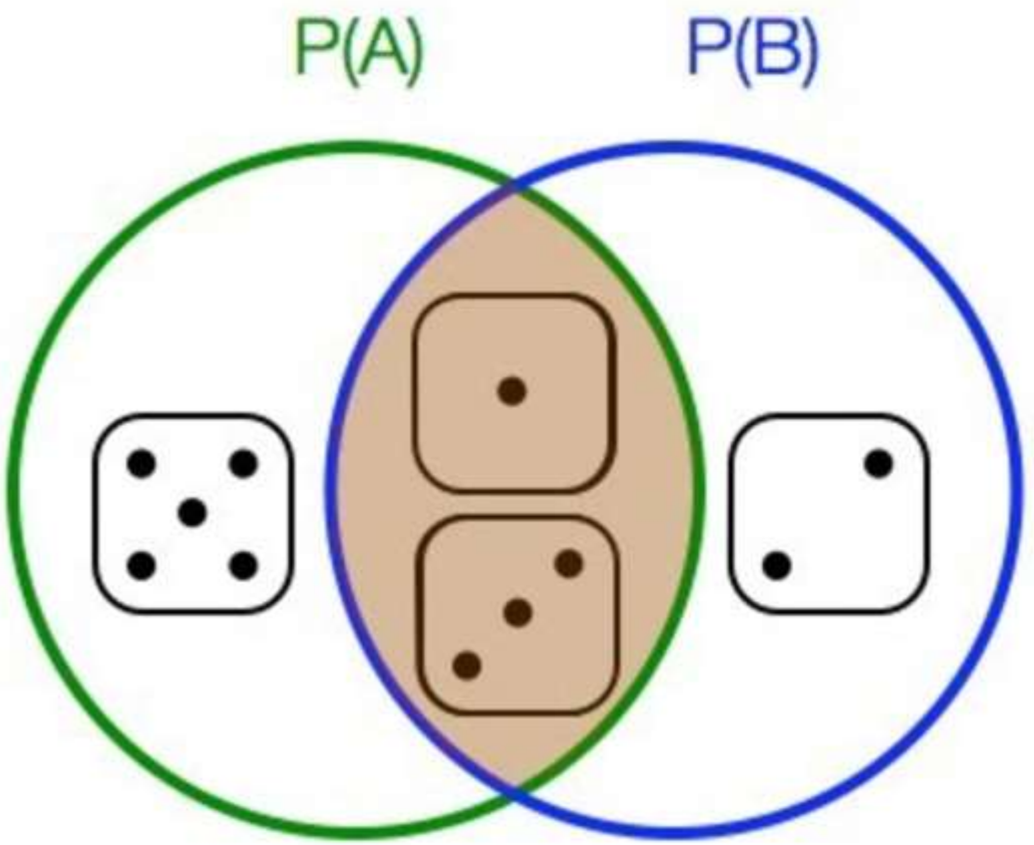


Poll: What's the probability of rolling a dice with a value less than 4 knowing that the value is odd

B = dice with a value less than 4

A = dice with an odd number

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2}{3}$$



Bayes' Rule

likelihood of the evidence “E” if the hypothesis “H” is true

prior probability of “H”

posterior probability of “H” given the evidence

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

prior probability that the evidence itself is true (but also a normalizing constant)

Bayes' Rule after expanding $P(E)$

$$P(H|E) = \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|H^c)P(H^c)}$$

- $E = E \cap (H \cup H^c) = (E \cap H) \cup (E \cap H^c)$
- $P(E) = P(E \cap H) + P(E \cap H^c)$ (independent events)
- $P(E) = P(E|H)P(H) + P(E|H^c)P(H^c)$

Bayes' Rule after considering $P(E)$ as a normalizing constant

$$P(H|E) \propto P(E|H)P(H)$$

- $\propto$ means proportional: Calculate all your $P(H|E)$, then normalize them using their sum so their normalized sum is 1

Poll: You are planning a picnic today

You are planning a picnic today, but the morning is cloudy. What is the chance that it will rain during the day knowing that:

- 50% of all rainy days start off cloudy
- Cloudy mornings are common (40% of days start cloudy)
- This month is usually a dry month (only 3 of 30 days tend to be rainy)

Poll: You are planning a picnic today

$$P(\text{rainy day} \mid \text{cloudy morning}) =$$

$$\frac{P(\text{cloudy morning} \mid \text{rainy day})P(\text{rainy day})}{P(\text{cloudy morning})} =$$

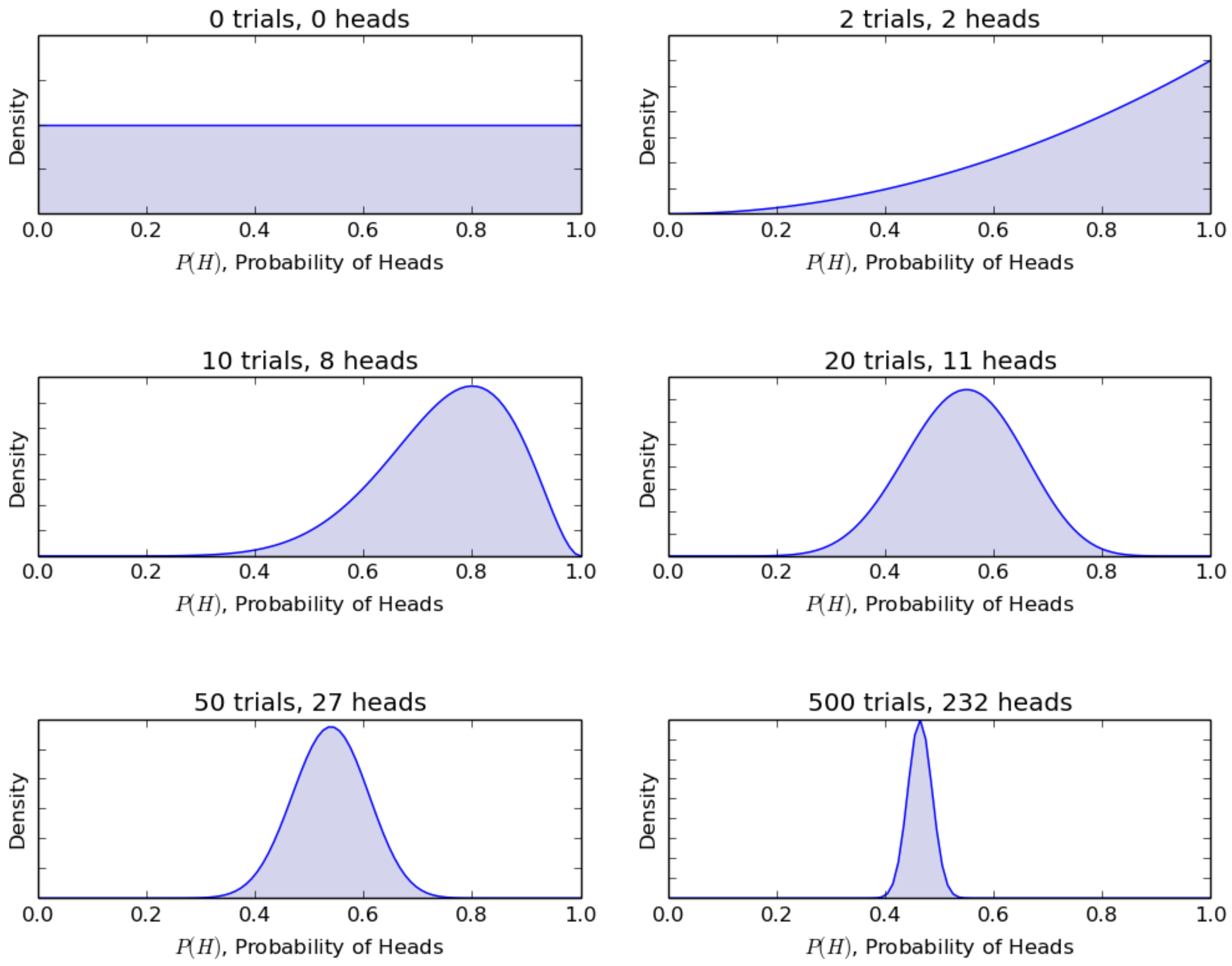
$$\frac{.5 \; * 3/30}{.4} = .125$$

12.5%; not too bad compared with 50%..., let's have a picnic!

Bayesian Inference

- Bayesian updates his or her beliefs after seeing evidence
  - John Maynard Keynes, a great economist and thinker, said “When the facts change, I change my mind. What do you do, sir?”
- Probability is seen as a measure of believability in events

Bayesian Updates (pair programming)



Relating Prior Knowledge/Belief to Data

You have a drawer of 100 coins, 10 of which are biased

P(head | fair) = .5

P(head | biased) = .25

You randomly choose a coin and flip it once. It comes up heads

1. What is P(fair | head)?
2. What if you flip it a second time and it comes up heads again?

Relating Prior Knowledge/Belief to Data

After the first coin flip, E = [head]

- H = fair coin

$$P(fair \mid head) =$$

$$\frac{P(head \mid fair)P(fair)}{P(head \mid fair)P(fair) + P(head \mid biased)P(biased)} = \frac{.5 \times .9}{.5 \times .9 + .25 \times .1} = .947$$

- H = biased coin

$$P(biased \mid head) =$$

$$\frac{P(head \mid biased)P(biased)}{P(head \mid fair)P(fair) + P(head \mid biased)P(biased)} = \frac{.25 \times .1}{.5 \times .9 + .25 \times .1} = .053$$

Relating Prior Knowledge/Belief to Data

After the second coin flip, E = [head, head]

- H = fair coin

$$P(fair \mid head) =$$

$$\frac{P(head \mid fair)P(fair)}{P(head \mid fair)P(fair) + P(head \mid biased)P(biased)} = \frac{.5 \times .947}{.5 \times .947 + .25 \times .053} = .973$$

- H = biased coin

$$P(biased \mid head) =$$

$$\frac{P(head \mid biased)P(biased)}{P(head \mid fair)P(fair) + P(head \mid biased)P(biased)} = \frac{.25 \times .053}{.5 \times .947 + .25 \times .053} = .027$$

This last example again but now using $P(E)$ as a normalizing constant

(you'll use this for the pair assignment…)

After the second coin flip, E = [head, head]

- H = fair coin

$$P(fair \mid head) \propto P(head \mid fair)P(fair) = .5 \times .947 = .474$$

- H = biased coin

$$P(biased \mid head) \propto P(head \mid biased)P(biased) = .25 \times .053 = .0133$$

- Normalizing constant is

$$.474 + .0133 = .487$$

- H = fair coin

$$P(fair \mid head) = {}^{.474}/_{.487} = .973$$

- H = biased coin

$$P(biased \mid head) = {}^{.0133}/_{.487} = .027$$

Afternoon pairing