ZIPFIAN ACADEMY

# WEBSITE OPTIMIZATION


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50%
50%
CAMPAIGN A
20% OFF
18% CONVERSION RATE
FREE SHIPPING
CAMPAIGN B
WINNER!
32% CONVERSION RATE

# BAYES THEOREM

"Diacronic"

If you see some new evidence

Update your belief in the hypothesis using this term

$$P(H \mid E) = \frac{P(H)P(E \mid H)}{P(E)}$$

H = Hypothesis
E = Evidence

# BAYES THEOREM

This is what you believed before you saw the evidence

Likelihood of seeing that evidence if your hypothesis is correct

Probability of my hypothesis given then I've seen some evidence

$$P(H \mid E) = \frac{P(H)P(E \mid H)}{P(E)}$$

Likelihood of seeing that evidence under any circumstances at all

$$P(E) = \sum_i P(H_i)P(E \mid H_i)$$

# SAMPLE PROBLEM

- Two bowls of cookies, Bowl 1 contains 30 vanilla and 10 chocolate cookies. Bowl 2 contains 20 of each.

Bowl 1
V=30
C=10

Bowl 2
V=20
C=20

- Suppose you choose a bowl at random, without looking, and pick a vanilla cookie. What's the probability it came from Bowl 1?

# BAYESIAN STATISTICS IN FOUR EASY STEPS

- Define the prior distribution that incorporates your subjective beliefs about a parameter (in your example the parameter of interest is the proportion of left-handers). The prior can be uninformative or informative.

- Gather data.

- Update your prior distribution with the data using Bayes' theorem to obtain a posterior distribution. The posterior distribution is a probability distribution that represents your updated beliefs about the parameter after having seen the data.

- Analyze the posterior distribution and summarize it (mean, median, sd, quantiles, ...).

The basic of all Bayesian statistics is Bayes' theorem, which is states:

$$\text{posterior} \propto \text{prior} \times \text{likelihood}$$

- In our case, the likelihood is binomial. If the prior and the posterior distribution are in the same family, the prior and posterior are called conjugate distributions.

- The beta distribution is a conjugate prior because the posterior is also a beta distribution.

$$\text{posterior} \propto \text{prior} \times \text{likelihood}$$

$$\text{beta} \propto \text{beta} \times \text{binomial}$$

- Conjugate analyses are convenient but rarely occur in real-world problems.
- In most cases, the posterior distribution has to be found numerically via MCMC .
- If the prior probability distribution does not integrate to 1, it is called an *improper prior*, if it does integrate to 1 it is called a *proper prior*.
- In most cases, an improper prior does not pose a major problem for Bayesian analyses.
- The posterior distribution must be proper though, i.e. the posterior must integrate to 1.

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# RULES OF THUMB

- If the prior is *uninformative*, the posterior is very much determined by the data (the posterior is data-driven)
- If the prior is *informative*, the posterior is a mixture of the prior and the data
- The more informative the prior, the more data you need to "change" your beliefs, so to speak because the posterior is very much driven by the prior information
- If you have a lot of data, the data will dominate the posterior distribution (they will overwhelm the prior)

# EXAMPLE

- In a group of students, there are 2 out of 18 that are left-handed.
- Find the posterior distribution of left-handed students in the population assuming uninformative prior. Summarize the results.
- According to the literature 5-20% of people are left-handed. Take this information into account in your prior and calculate new posterior.

# EXAMPLE

- Say your prior beta is Beta($\pi|\alpha,\beta$) where $\pi$ is the proportion of left-handers.

- To specify the prior parameters $\alpha$ and $\beta$, it is useful to know the mean and variance of the beta distribution (for example, if you want your prior to have a certain mean and variance).

- The mean of $\pi=\alpha/(\alpha+\beta)$. Thus, whenever $\alpha=\beta$, the mean is 0.5. The variance of the beta distribution is $\alpha\beta\ /(\alpha+\beta)^2(\alpha+\beta+1)$

# EXAMPLE

- Now, the convenient thing is that you can think of $\alpha$ and $\beta$ as previously observed (pseudo-)data, namely $\alpha$ left-handers and $\beta$ right-handers out of a (pseudo-)sample of size $n=\alpha+\beta$.
- The $\text{Beta}(\pi|\alpha=1,\beta=1)$ distribution is the uniform (all values of $\pi$ are equally probable) and is the equivalent of having observed two people out of which one is left-handed and one is right-handed.

# EXAMPLE

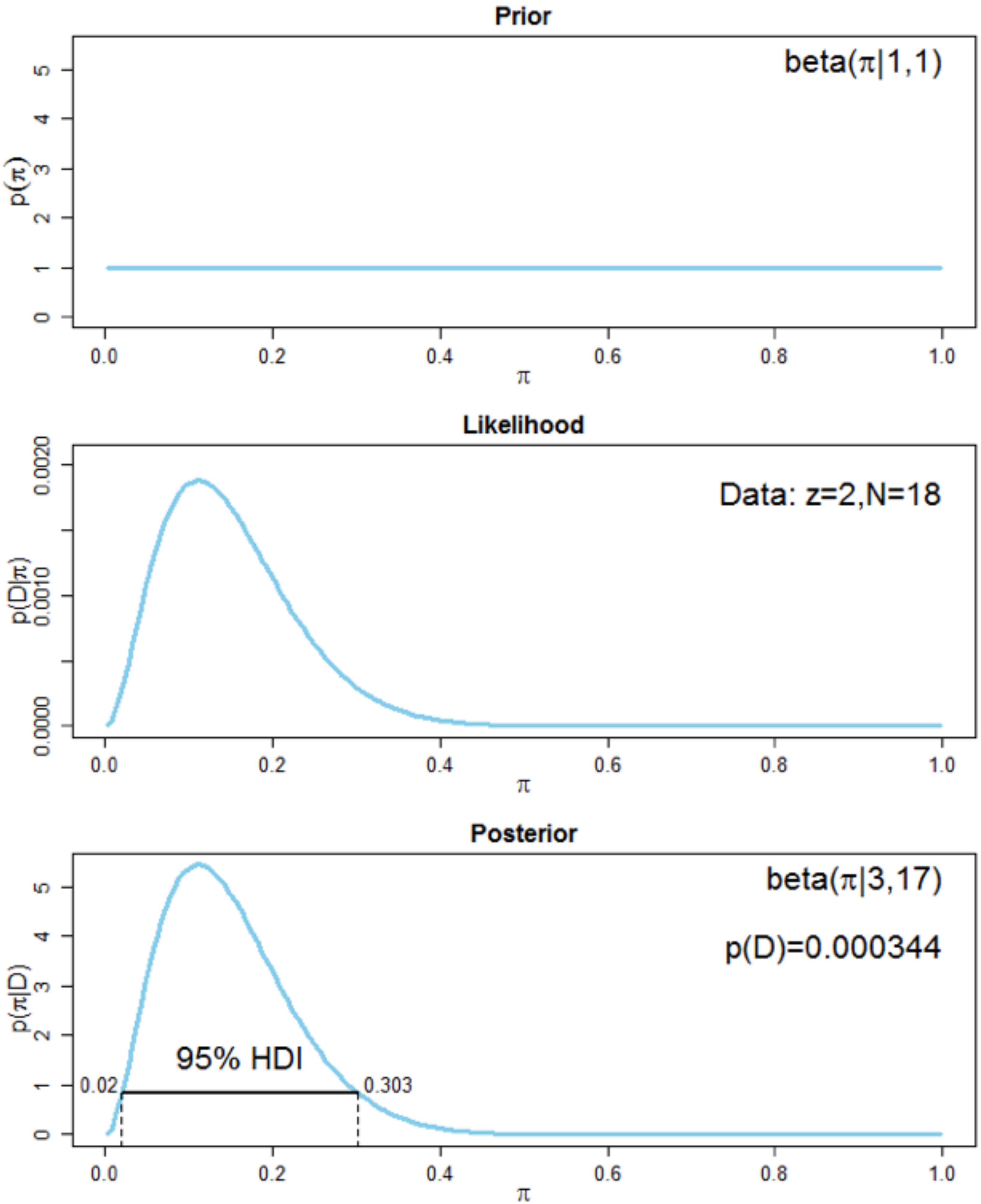
- The posterior beta distribution is simply Beta($z+\alpha$,$N-z+\beta$) where N is the size of the sample and z is the number of left-handers in the sample.

- The posterior mean of $\pi$LH is therefore $(z+\alpha)/(N+\alpha+\beta)$. So to find the parameters of the posterior beta distribution, we simply add z left-handers to $\alpha$ and $N-z$ right-handers to $\beta$.

- The posterior variance is $(z+\alpha)(N-z+\beta)/(N+\alpha+\beta)$^2$(N+\alpha+\beta+1)$. Note that a highly informative prior also leads to a smaller variance of the posterior distribution.

# EXAMPLE

- In this case, $z=2$ and $N=18$ and our prior is the uniform which is uninformative, so $\alpha=\beta=1$.
- The posterior distribution is therefore Beta(3,17). The posterior mean is $\pi=3/(3+17)=0.15$.


Prior
beta(π|1,1)
p(π)
π
Likelihood
Data: z=2,N=18
p(D|π)
π
Posterior
beta(π|3,17)
p(D)=0.000344
95% HDI
0.02
0.303
p(π|D)
π

# EXAMPLE

- For your second task, you're asked to incorporate the information that 5-20% of the population are left-handers into account.

- The easiest way is to say that the prior beta distribution should have a mean of 0.125 which is the mean of 0.05 and 0.2. But how to choose $\alpha$ and $\beta$ of the prior beta distribution?

# EXAMPLE

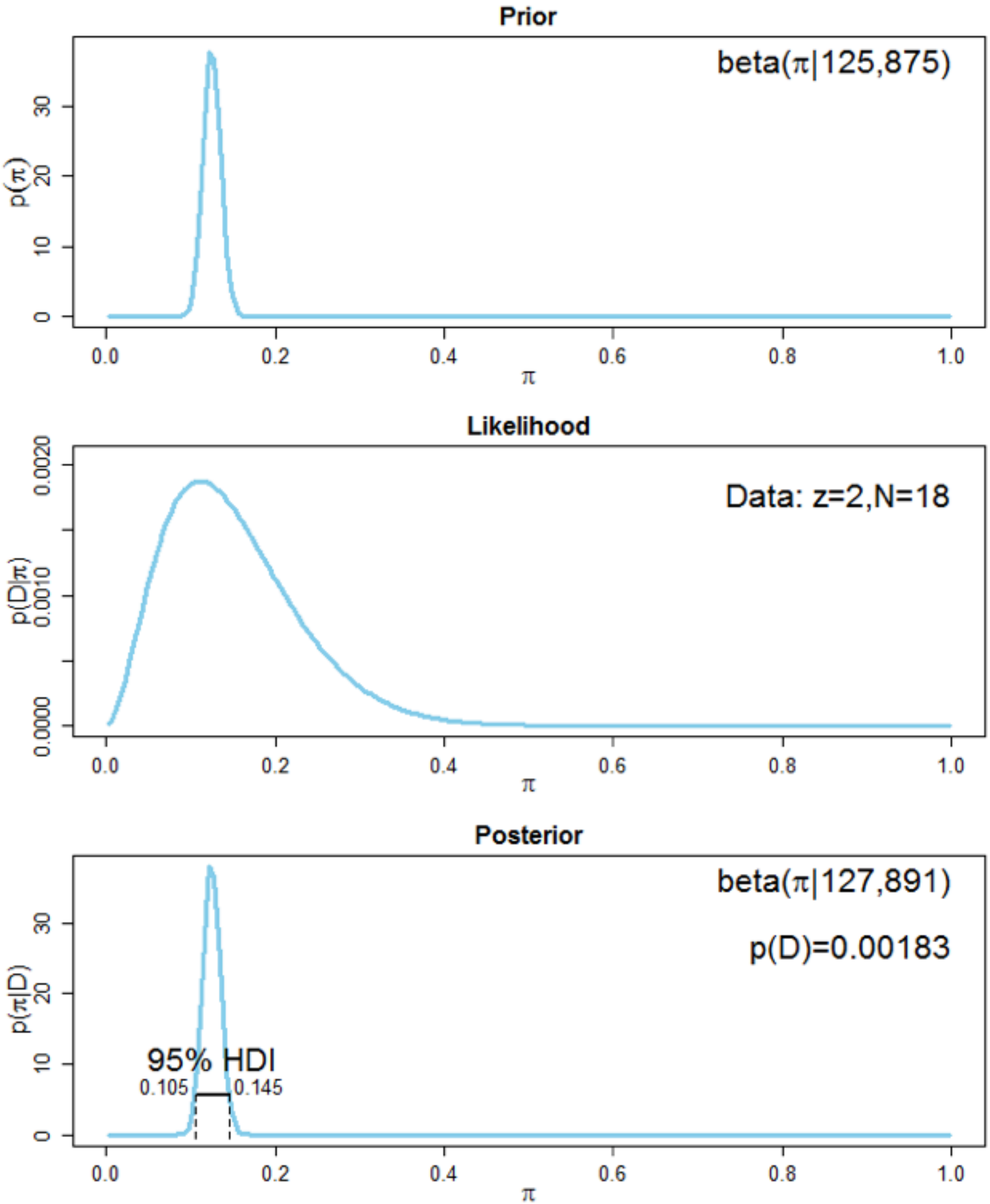
- First, you want your mean of the prior distribution to be 0.125 out of a pseudo-sample of equivalent sample size n_eq.
- More generally, if you want your prior to have a mean m with a pseudo-sample size n_eq, the corresponding $\alpha$ and $\beta$ values are: $\alpha = m * n_{eq}$ and $\beta = (1-m)n_{eq}$.

# EXAMPLE

- All you are left to do now is to choose the pseudo-sample size n_eq which determines how confident you are about your prior information.

- Let's say you are very sure about your prior information and set n_eq=1000. The parameters of your prior distribution are therefore $\alpha=0.125 \cdot 1000=125$ and $\beta=(1-0.125) \cdot 1000=875$.

- The posterior distribution is Beta(127,891) with a mean of about 0.125 which is practically the same as the prior mean of 0.125.

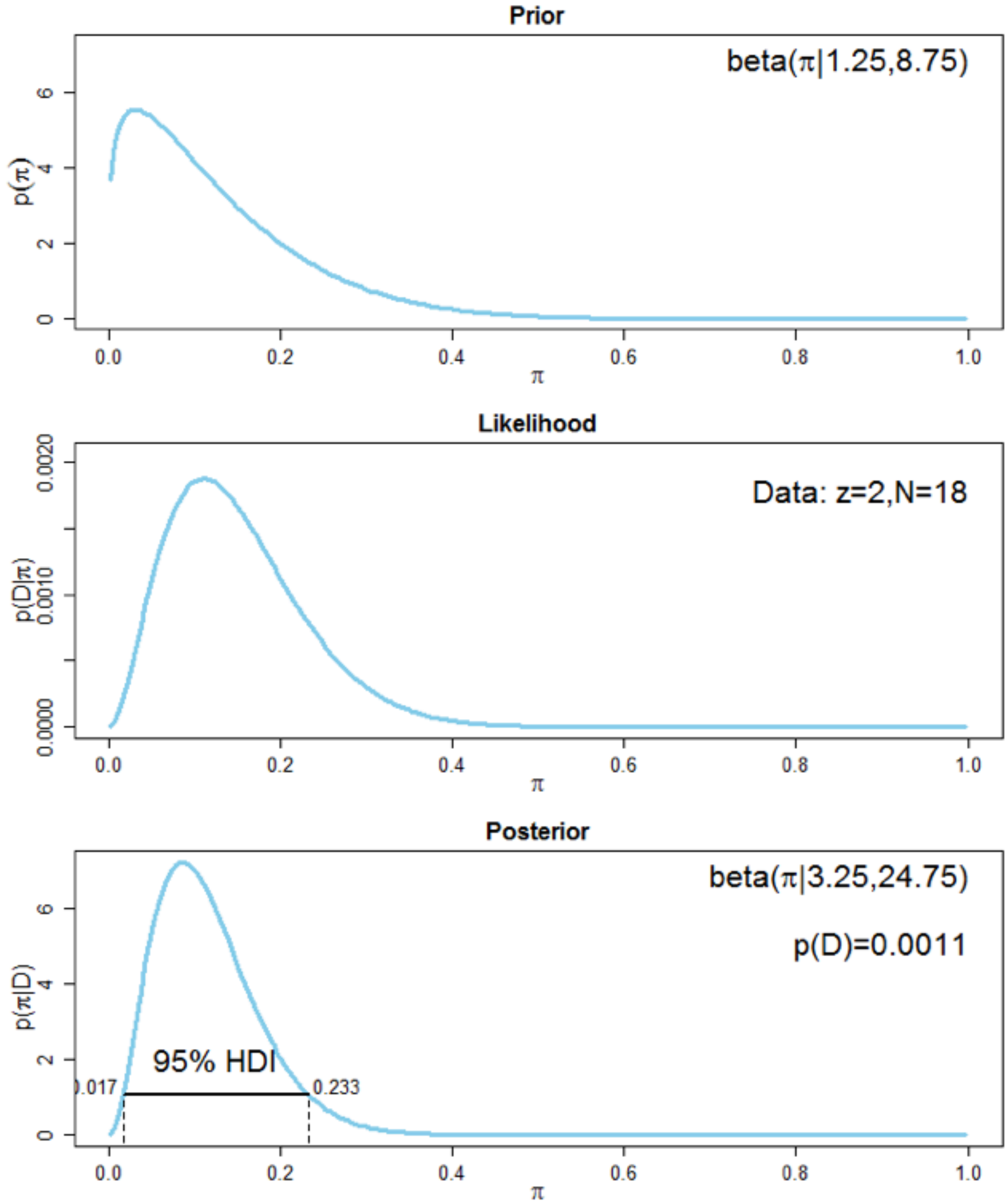


Prior
beta(π|125,875)
p(π)
π
Likelihood
Data: z=2,N=18
p(D|π)
π
Posterior
beta(π|127,891)
p(D)=0.00183
p(π|D)
95% HDI
0.105
0.145
π

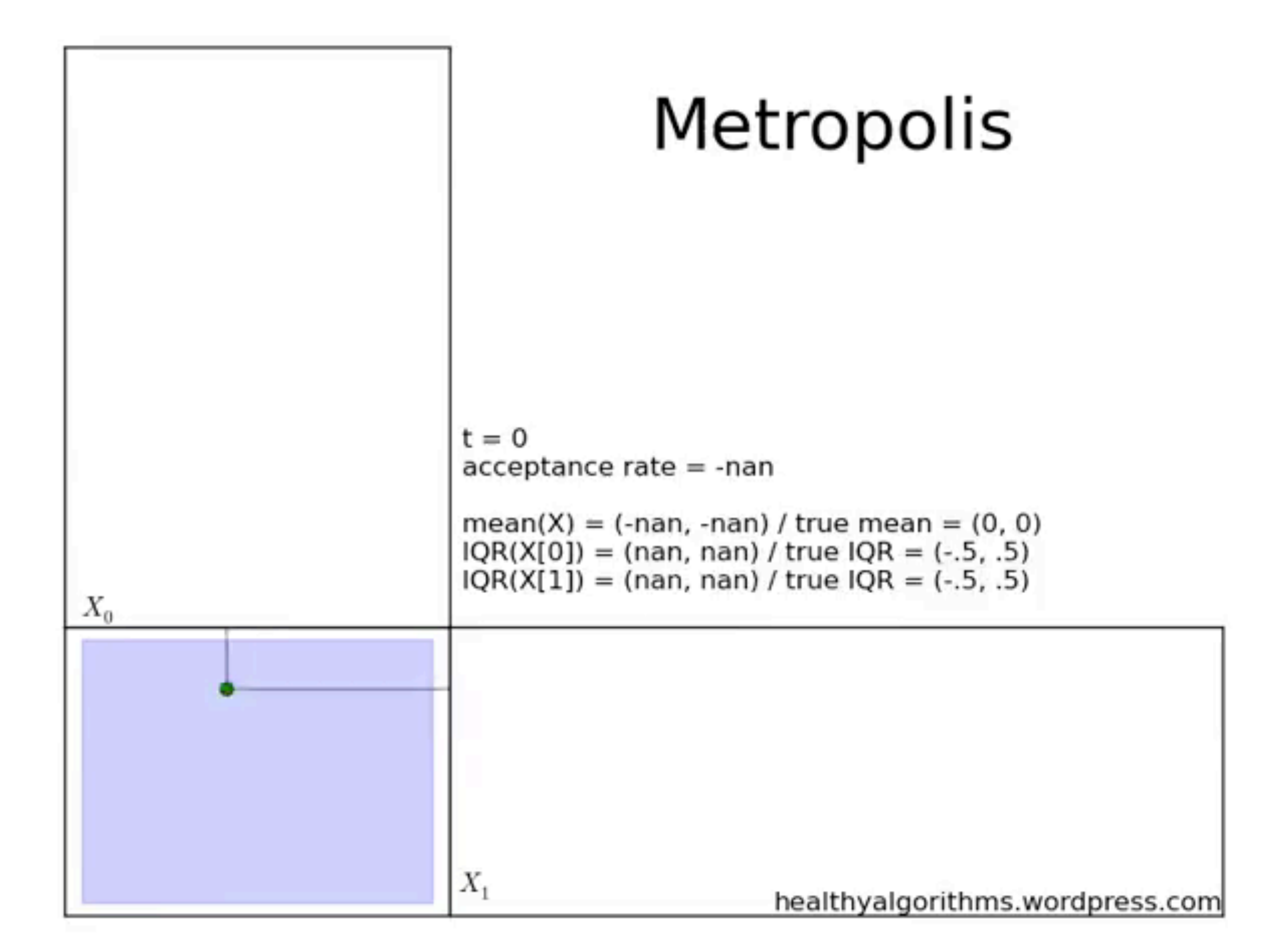
# EXAMPLE

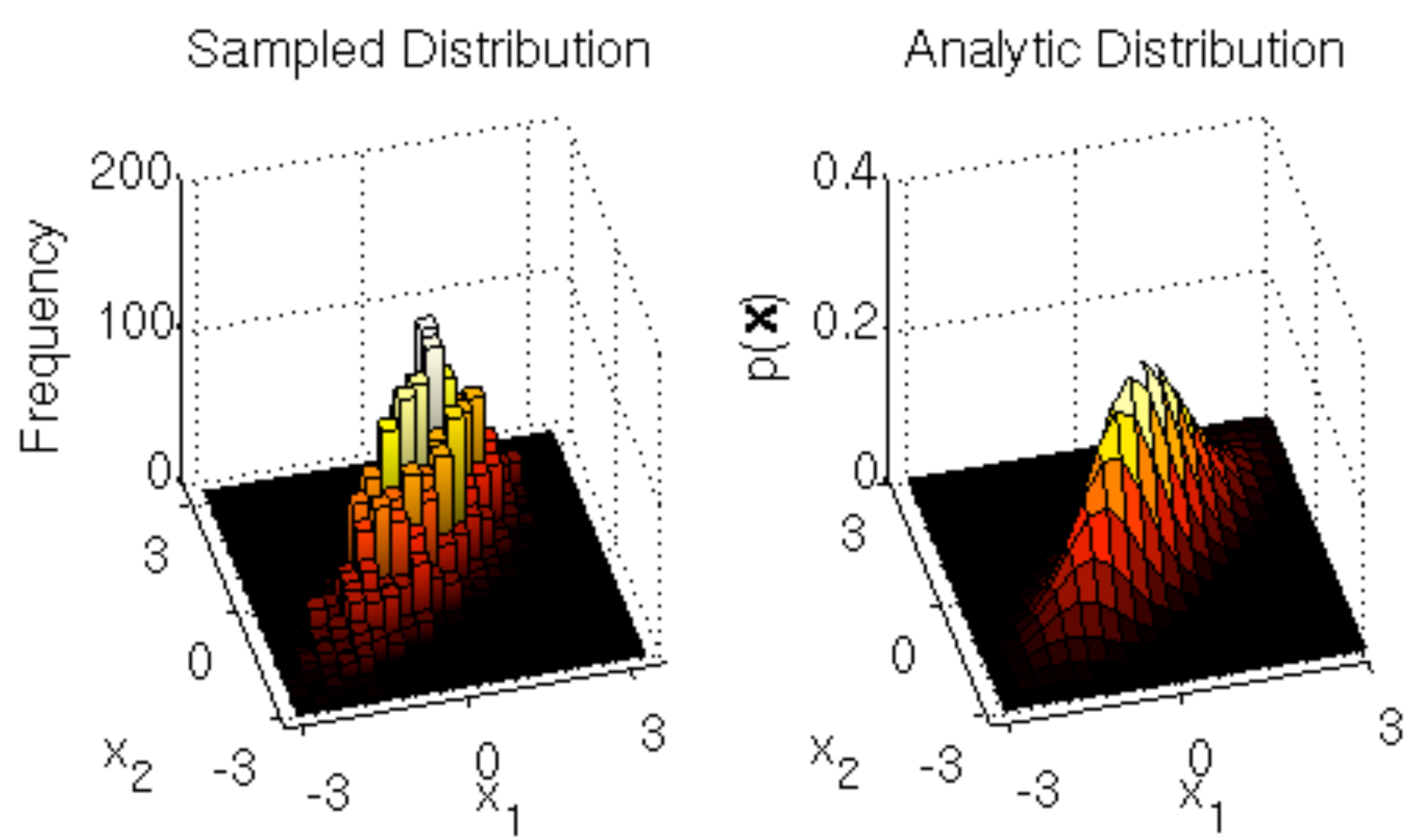
- If you are less sure about the prior information, you could set the n_eq of your pseudo-sample to, say, 10, which yields $\alpha$=1.25 and $\beta$=8.75 for your prior beta distribution.

- The posterior distribution is Beta(3.25,24.75) with a mean of about 0.116. The posterior mean is now near the mean of your data (0.111) because the data overwhelm the prior.



Prior
beta(π|1.25,8.75)
p(π)
π
Likelihood
Data: z=2,N=18
p(D|π)
π
Posterior
beta(π|3.25,24.75)
p(D)=0.0011
95% HDI
.017
0.233
p(π|D)
π


Metropolis
t = 0
acceptance rate = -nan
mean(X) = (-nan, -nan) / true mean = (0, 0)
IQR(X[0]) = (nan, nan) / true IQR = (-.5, .5)
IQR(X[1]) = (nan, nan) / true IQR = (-.5, .5)
$X_0$
$X_1$
healthyalgorithms.wordpress.com



Sampled Distribution
Frequency
200
100
0
3
0
-3
$x_2$
-3
0
3
$x_1$
Analytic Distribution
$p(\mathbf{x})$
0.4
0.2
0
3
0
-3
$x_2$
-3
0
3
$x_1$


EXPLORATION
vs.
EXPLOITATION

# TRADITIONAL A/B TESTING

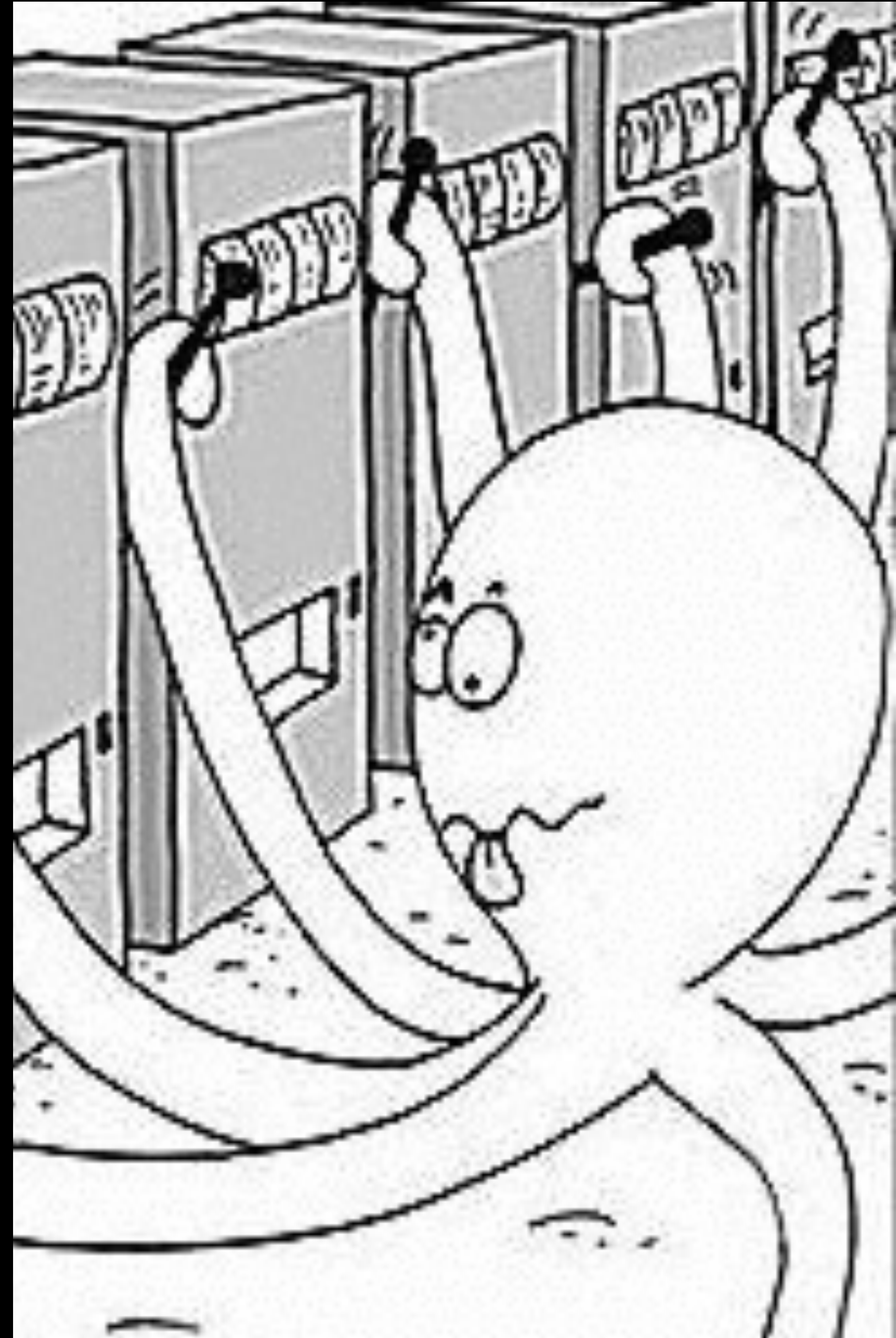
- A short period of *pure exploration*, in which you assign equal numbers of users to Group A and Group B
- A long period of *pure exploitation*, in which you send all of your users to the more successful version of your site and never come back to the option that seemed to be inferior

# QUIZ: WHY MIGHT THIS BE A BAD STRATEGY?

- A short period of *pure exploration*, in which you assign equal numbers of users to Group A and Group B
- A long period of *pure exploitation*, in which you send all of your users to the more successful version of your site and never come back to the option that seemed to be inferior

AN INTERESTING SOLUTION

# MULTI-ARM BANDIT ALGORITHMS



Arm A

Arm B

Arm C

Which one do you choose?

Arm A

Arm B

Arm C

Prior + Experience = Posterior

or in this case…

MEANS

```
np.argmax(self.wins / (self.trials + 1))
```

# EPSILON-GREEDY ALGORITHM

- Define some probability *epsilon* for which we will randomly explore. For example, if *epsilon* = 0.1, we will explore 10% of the time (fixed).

- How do we do this? Well, for each user, we flip a coin, simulated by drawing a random number between 0 and 1. If our draw is less than *epsilon*, explore randomly. Otherwise, pick the arm with the highest probability or expected value.

# DEMO

# EPSILON-GREEDY ALGORITHM HAS THE FOLLOWING WEAKNESSES:

- The algorithm's default choice is to select the arm that currently has the highest estimated value.

- The algorithm sometimes decides to explore and chooses an option that isn't the one that currently seems best:

    - The *epsilon-Greedy* algorithm explores by selecting from all of the arms completely at random. It makes one of these random exploratory decisions with probability *epsilon.*

# CAN WE DO BETTER?

# SOFTMAX

- Choose each arm in proportion to its estimated value
- Based on your past experiences, choose the arm with the highest probability based on proportions:
    - rA / (rA / rB)
    - rB / (rA / rB)

# SOFTMAX

- Choose each arm in proportion to its estimated value, scaled exponentially with an additional parameter, *tau*.
- Based on your past experiences, choose the arm with the highest probability based on:
  - exp(rA / tau) / (exp(rA / tau) + exp(rB / tau))
  - exp(rB / tau) / (exp(rA / tau) + exp(rB / tau))

# SOFTMAX

- What is *tau*?

- Analogous to a *temperature* of a system like in Physics, at low temperatures, atoms will behave orderly and produce solids, but at high temperatures, they behave randomly and will produce gases.

$$0 \xleftarrow{\text{deterministic}} \text{tau} \xrightarrow{\text{random}} \text{Inf}$$

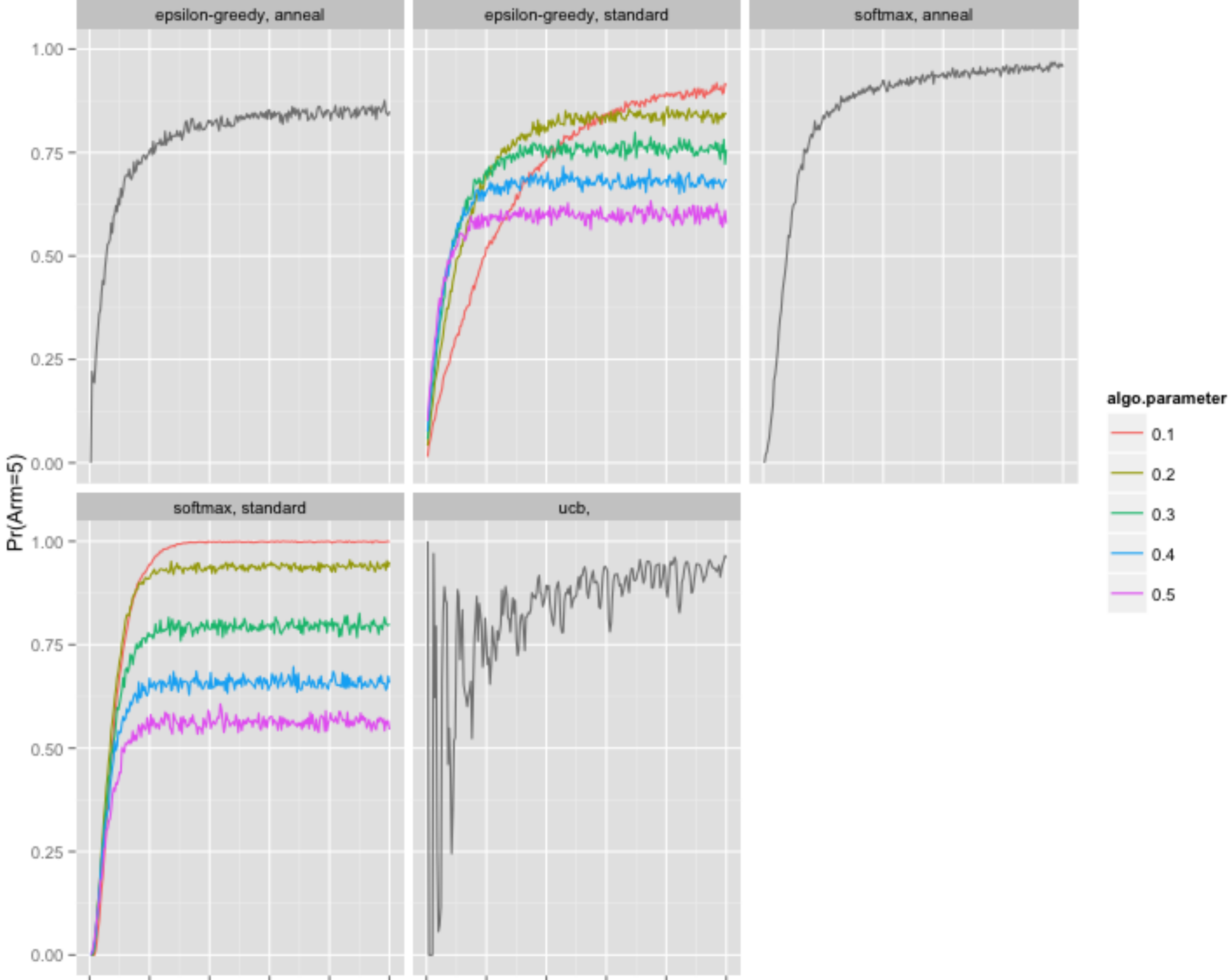
# UCB — STRENGTHS

- UCB doesn't use randomness at all. Unlike *epsilon-Greedy*, it's possible to know exactly how UCB will behave in any given situation. This can make it easier to reason about at times.

- UCB doesn't have free parameters that you need to configure before you can deploy it. This is a *major* improvement if you're interested in running in the wild, because it means that you can start UCB without having clear sense of what you expect the world to behave life.

# UPPER CREDIBLE BOUND (UCB)

- We compute an "upper credible bound", which follows from the Chernoff-Hoeffding bound (not covered here)
- Play machine j that maximizes $\bar{x_j} + \sqrt{\frac{2 \ln n}{n_j}}$ where x_j is the average reward obtained from machine j, n_j is the number of times machine j has been played so far, and n is the overall numbers of of total plays across all bandits.



epsilon-greedy, anneal
epsilon-greedy, standard
softmax, anneal
softmax, standard
ucb,
Pr(Arm=5)
1.00
0.75
0.50
0.25
0.00
algo.parameter
0.1
0.2
0.3
0.4
0.5

# UCB — HOW DOES IT WORK?

- The algorithms default choice is to select the arm that currently has the highest estimated value
- The algorithm decides to explore and chooses an option that isn’t the one that current seems best
    - The epsilion-Greedy algorithm explores by selecting from all other arms at random with probability *epsilon.*
- *Softmax* explores randomly by selecting the other arms proportional to the estimated value (reward) from each of the arms, but doesn’t take into account how many times it’s pulled an arm

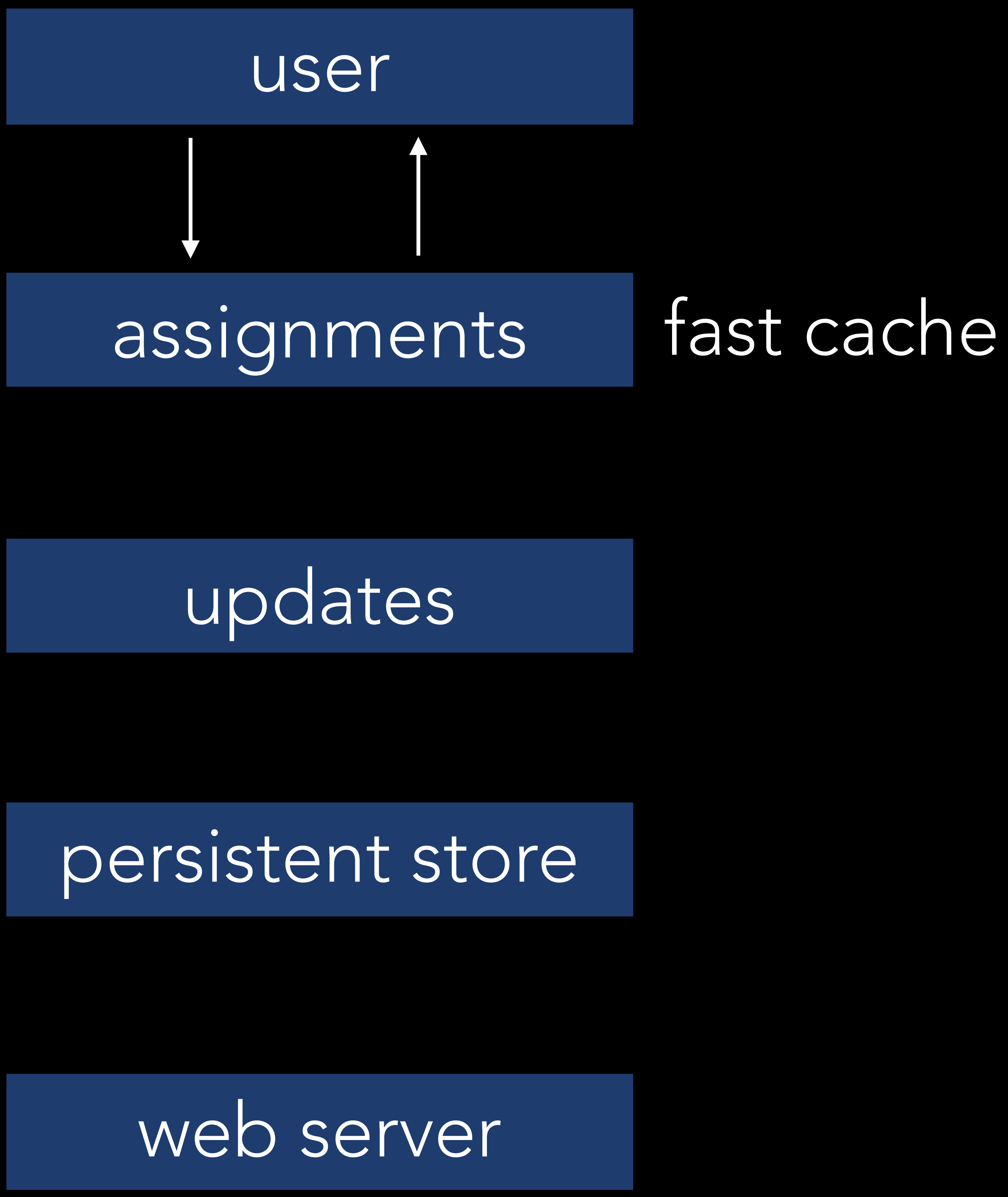
# BANDITS IN THE REAL WORLD

- A/A Testing.
- Concurrent Experiments
- Continuous Experimentation vs. Periodic Testing
- Bad Metrics of Success
- Scaling Problems
- Moving Worlds
- Contextual Bandits - *LinUCB* (Linear Regression) & *GLMUBC* (Generalized Linear Model)

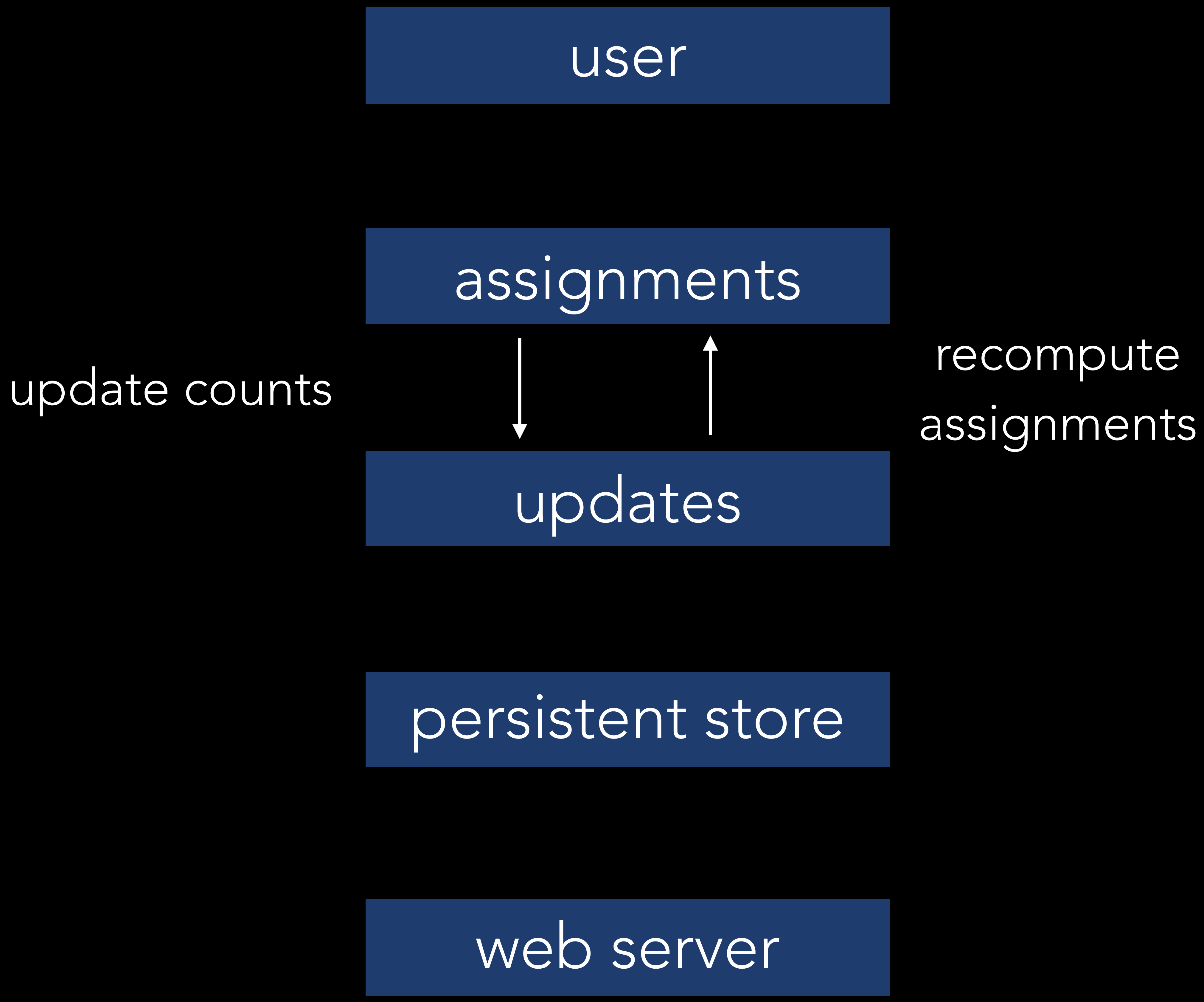
# BANDITS IN PRODUCTION

- Blocked Assignments
  - Assign incoming users to new arms in advance and draw this information from a fast cache when users actually arrive. Store their responses for batch processing later in another fast cache
- Blocked Updates
  - Update your estimates of arm values in batches on a regular interval and regenerate your blocked assignments. Because you work in batches, it will be easier to perform the kind of complex calculations you'll need to deal with correlated arms or contextual information

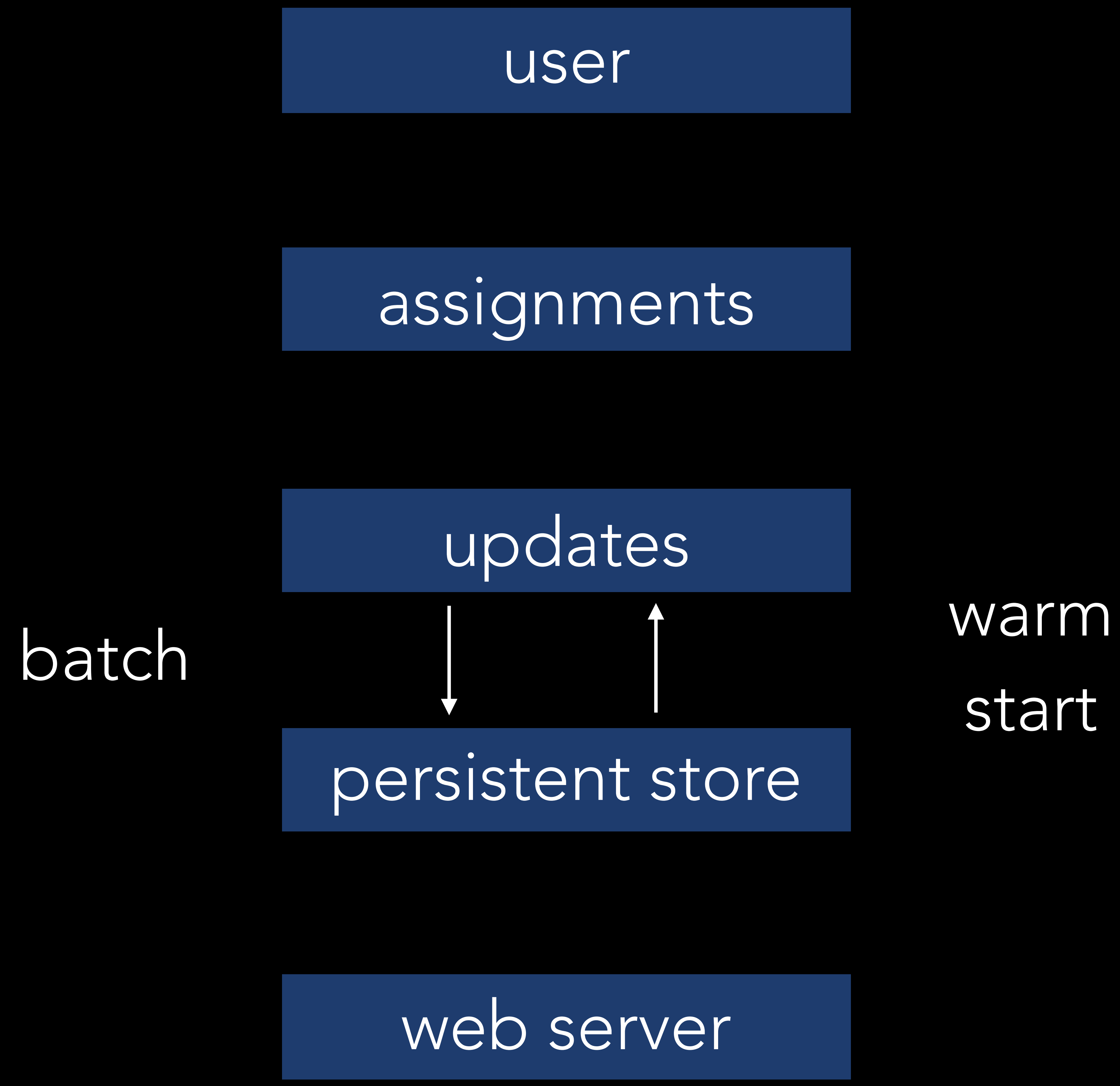
# BANDITS IN PRODUCTION



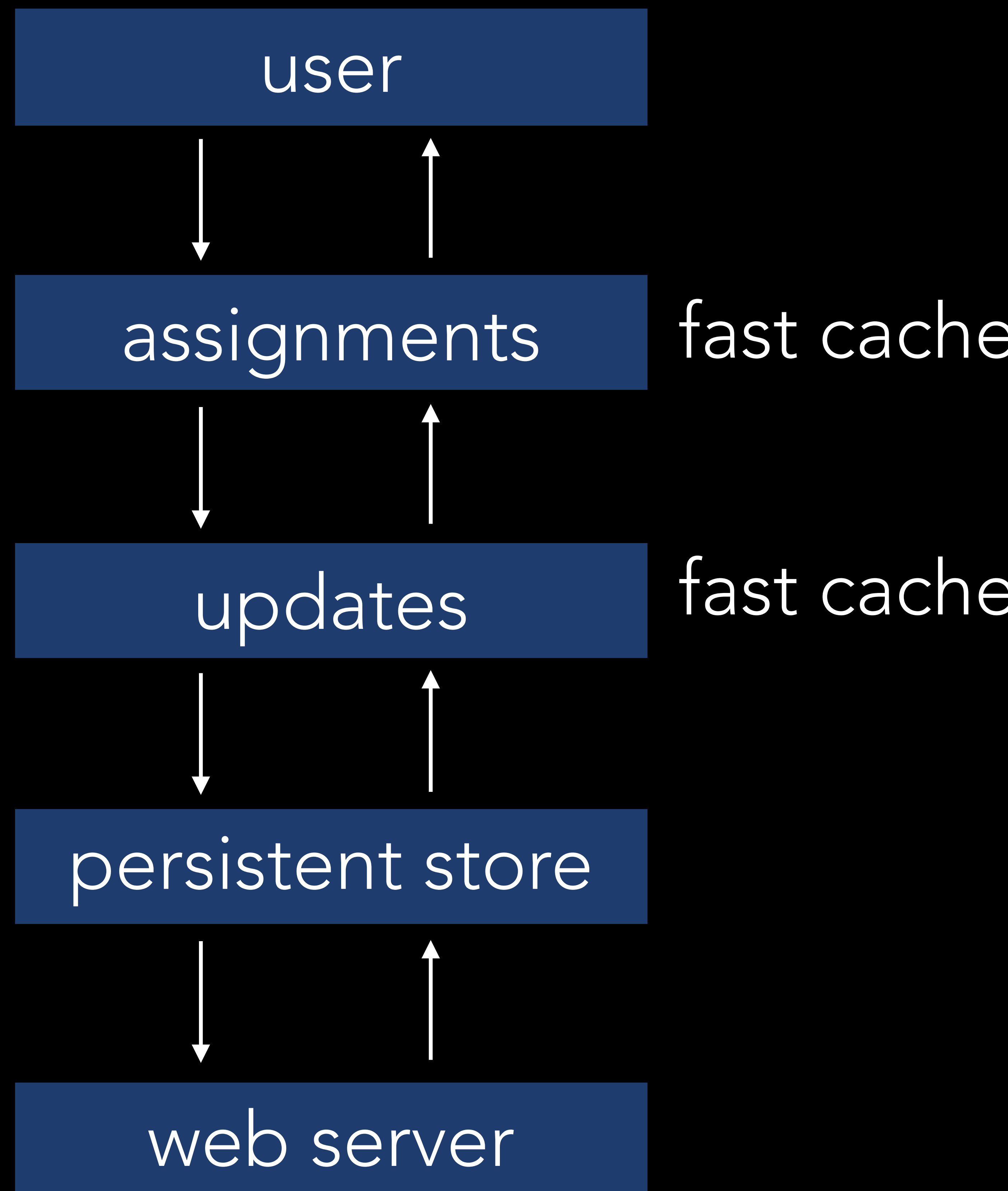
# BANDITS IN PRODUCTION



# BANDITS IN PRODUCTION



# BANDITS IN PRODUCTION



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# THANK YOU

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