

# Regression

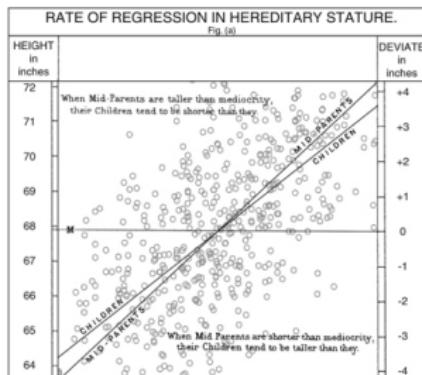
Schwartz

September 8, 2016

# The Sophomore Slump

...or sophomore jinx or sophomore jitters refers to an instance in which a second, or sophomore, effort fails to live up to the standards of the first effort. It is commonly used to refer to the apathy of students (second year of high school, college or university), the performance of athletes (second season of play), singers/bands (second album), television shows (second seasons) and films (sequels/prequels). In the United Kingdom, the “sophomore slump” is more commonly referred to as “second year blues”, particularly when describing university students. In Australia, it is known as “second year syndrome”, and is particularly common when referring to professional athletes who have a mediocre second season following a stellar debut. The phenomenon of a “sophomore slump” can be explained psychologically, where earlier success has a reducing effect on the subsequent effort, but it can also be explained statistically, as an effect of the regression towards the mean.

The concept of “regression” comes from genetics and was popularized by Sir Francis Galton's late 19th century publication of “Regression towards mediocrity in hereditary stature.” Galton observed that extreme characteristics (e.g., height) in parents are not completely passed on to offspring, but rather the characteristics in the offspring “regress” towards a mediocre point. By measuring the heights of hundreds of people Galton was able to quantify this “regression” and in so doing invented linear regression analysis, thus laying the groundwork for much of modern statistical modeling. The term “regression” stuck.



# Objectives

- ▶ Terminology
- ▶ Least squares fit
- ▶ Normal distribution theory
- ▶ Coefficient testing
- ▶ Linear models and Multiple variables and Alternatives
- ▶ Model fit and Model selection
- ▶ Model diagnostics and Model evaluation

# Terminology

- ▶  $Y_i = \beta_0 + x_i\beta_1 + \epsilon_i, \epsilon_i \stackrel{i.i.d.}{\sim} Normal(0, \sigma^2)$

## Terminology

Coefficient

$$\blacktriangleright Y_i = \beta_0 + x_i \beta_1 + \epsilon_i, \epsilon_i \stackrel{i.i.d.}{\sim} Normal(0, \sigma^2)$$

Intercept

Error

Noise

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$$\blacktriangleright Y_i = \hat{\beta}_0 + x_i \hat{\beta}_1 + \hat{\epsilon}_i, \hat{\sigma}^2 = \frac{\sum_{i=1}^n \hat{\epsilon}_i^2}{n-p-1} \quad (p = \# of coefficients)$$

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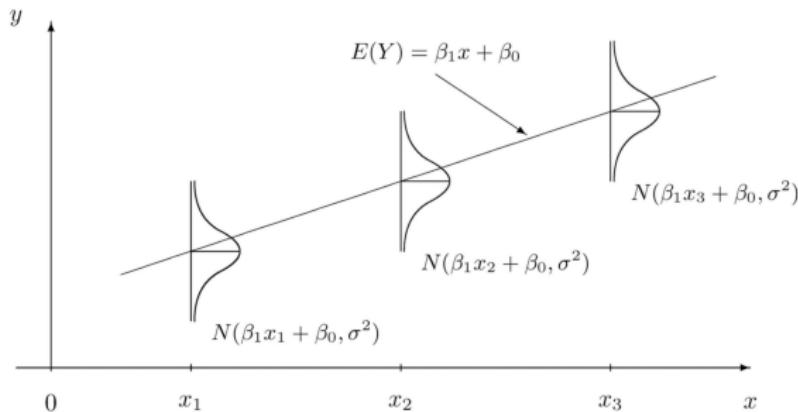
Fitted value Residual

- ▶  $Y_i = \hat{\beta}_0 + x_i\hat{\beta}_1 + \hat{\epsilon}_i, \hat{\sigma}^2 = \frac{\sum_{i=1}^n \hat{\epsilon}_i^2}{n-p-1} \quad (p = \# of coefficients)$

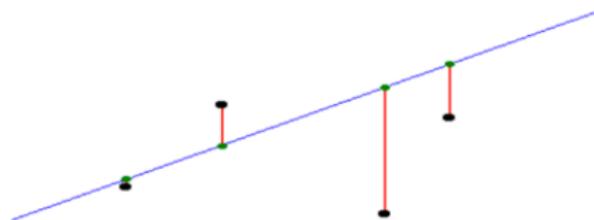
Residual Standard Error (RSE)

# Terminology

- $E[Y_i] = \beta_0 + x_i\beta_1 + \epsilon_i, \epsilon_i \stackrel{i.i.d.}{\sim} Normal(0, \sigma^2)$



- $Y_i = \hat{\beta}_0 + x_i\hat{\beta}_1 + \hat{\epsilon}_i, \hat{\sigma}^2 = \frac{\sum_{i=1}^n \hat{\epsilon}_i^2}{n-p-1}$  ( $p = \# of coefficients$ )



# Least Squares Fit

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$$[\hat{\beta}_0, \hat{\beta}_1] = \underset{[\beta_0, \beta_1]}{\operatorname{argmin}} \sum_{i=1}^n \hat{\epsilon}_i^2 = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^n (Y_i - \mathbf{x}_i^T \beta)^2$$

where  $\mathbf{x}_i^T = [1, x_i]$  and  $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$

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$$\sum_{i=1}^n (Y_i - \mathbf{x}_i^T \beta)^2 = (\mathbf{Y} - \mathbf{x}\beta)^T (\mathbf{Y} - \mathbf{x}\beta)$$

where  $\mathbf{Y}^T = [Y_1, Y_2, \dots, Y_n]$  and  $\mathbf{x}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}$

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$$\begin{aligned} & \nabla_{\beta} \beta^T (\mathbf{x}^T \mathbf{x}) \beta - 2\beta^T \mathbf{x}^T \mathbf{Y} + \mathbf{Y}^T \mathbf{Y} \\ &= 2(\mathbf{x}^T \mathbf{x}) \beta - 2\mathbf{x}^T \mathbf{Y} \quad (\text{set to } \mathbf{0} \text{ to minimize}) \\ \implies & \hat{\beta} = (\mathbf{x}^T \mathbf{x})^{-1} \mathbf{x}^T \mathbf{Y} \implies \text{fitted values } \hat{\mathbf{Y}} = \mathbf{x} \hat{\beta} \end{aligned}$$

## Least Squares Fit *bonus*

1. In simple linear regression the  $\underset{\beta}{\operatorname{argmin}}(\mathbf{Y} - \mathbf{x}\beta)^T(\mathbf{Y} - \mathbf{x}\beta)$  is

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{R_{xY} S_Y}{S_x}$$

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2. Maximum likelihood estimation (MLE) is equivalent

$$\underset{\beta}{\operatorname{argmax}} \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(Y_i - \mathbf{x}_i^T \beta)^2}$$

$$= \underset{\beta}{\operatorname{argmax}} (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}(\mathbf{Y} - \mathbf{x}\beta)^T(\mathbf{Y} - \mathbf{x}\beta)}$$

$$= \underset{\beta}{\operatorname{argmax}} -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(\mathbf{Y} - \mathbf{x}\beta)^T(\mathbf{Y} - \mathbf{x}\beta)$$

$$= \underset{\beta}{\operatorname{argmin}} (\mathbf{Y} - \mathbf{x}\beta)^T(\mathbf{Y} - \mathbf{x}\beta)$$

# The Multivariate Normal Distribution (MVN)

$$f(\mathbf{Y}|\mathbf{x}, \boldsymbol{\beta}, \sigma^2) = \prod_{i=1}^n f(Y_i|\mathbf{x}_i, \boldsymbol{\beta}, \sigma^2)$$

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## Testing

$$f(\mathbf{Y}|\mathbf{x}, \boldsymbol{\beta}, \sigma^2) = MVN(\mathbf{x}\boldsymbol{\beta}, \sigma^2 I)$$
$$\implies f(\hat{\boldsymbol{\beta}}|\mathbf{x}, \boldsymbol{\beta}, \sigma^2) = MVN\left(\boldsymbol{\beta}, \sigma^2(\mathbf{x}^T \mathbf{x})^{-1}\right)$$

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For simple linear regression then

$$\hat{\boldsymbol{\beta}} \sim MVN\left(\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \frac{\sigma^2}{n \sum (x_i - \bar{x})^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}\right)$$

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where

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x} \quad \hat{\beta}_1 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{R_{xY} S_Y}{S_x}$$

$$\text{Var}(\hat{\beta}_1) = \sigma^2 \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad \text{Var}(\hat{\beta}_0) = \sigma^2 \left[ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right]$$

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$$\text{Var}(\hat{\beta}_1) = \sigma^2 \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad \text{Var}(\hat{\beta}_0) = \sigma^2 \left[ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right]$$

$$\text{SE}(\hat{\beta}_0) = \sqrt{\text{Var}(\hat{\beta}_0)} \quad \text{SE}(\hat{\beta}_1) = \sqrt{\text{Var}(\hat{\beta}_1)}$$

## Testing Extra Credit I

- ▶ What is  $\text{Var}(\hat{Y}_0)$ ? (Suppose we know  $\sigma^2$ )

- ▶ Hint:  $\hat{Y}_0 = \hat{\beta}_0 + x_0 \hat{\beta}_1$
- ▶ Hint:  $\text{Var}[aX + bY] = ?$

## Testing Extra Credit II

- ▶  $\text{Var}(Y_0)$ ? For a *new observation*  $Y_0$  according to our model?  
(Suppose we know  $\sigma^2$ )
- ▶ Hint:  $Y_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0 + \epsilon$

# A SERIOUSLY MAJOR TRANSITION JUST HAPPENED

And you didn't even notice

## Coefficient Testing

For  $\hat{\beta} = (\mathbf{x}^T \mathbf{x})^{-1} \mathbf{x}^T \mathbf{Y}$ , since (under  $H_0$ )

$$f(\hat{\beta} | \beta, \sigma^2) = MVN \left( \beta, \sigma^2 (\mathbf{x}^T \mathbf{x})^{-1} \right)$$

we have that

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we have that

$$\frac{\hat{\beta}_i - \beta_i}{\text{SE}(\hat{\beta}_i)} \sim N(0, 1)$$

and if we estimate

$$\hat{\sigma}^2 = \frac{(\mathbf{Y} - \mathbf{x}\hat{\beta})^T (\mathbf{Y} - \mathbf{x}\hat{\beta})}{n - p - 1}$$

(where  $p$  is the number of coefficients) then we have that

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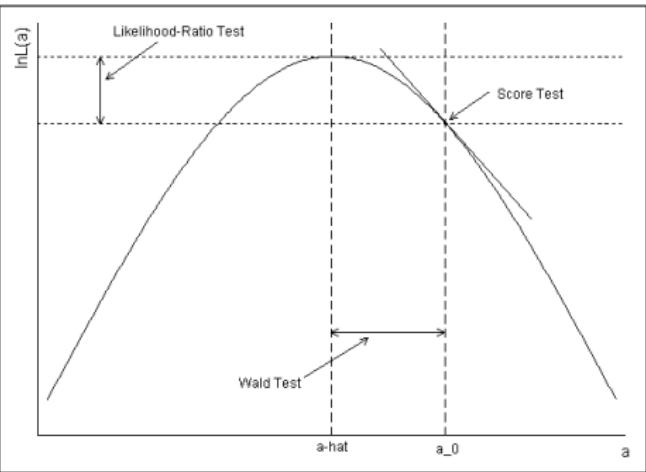
$$\hat{\sigma}^2 = \frac{(\mathbf{Y} - \mathbf{x}\hat{\beta})^T (\mathbf{Y} - \mathbf{x}\hat{\beta})}{n - p - 1}$$

(where  $p$  is the number of coefficients) then we have that

$$\frac{\hat{\beta}_i - \beta_i}{\text{SE}(\hat{\beta}_i)} \sim t_{n-p-1}$$

And this works for any number of covariates..

# More Coefficient Testing



Wald test  
$$\frac{(\hat{\theta} - \theta_0)^2}{Var(\hat{\theta})} \stackrel{\text{approx.}}{\sim} N(0, 1)$$
 under  $H_0$

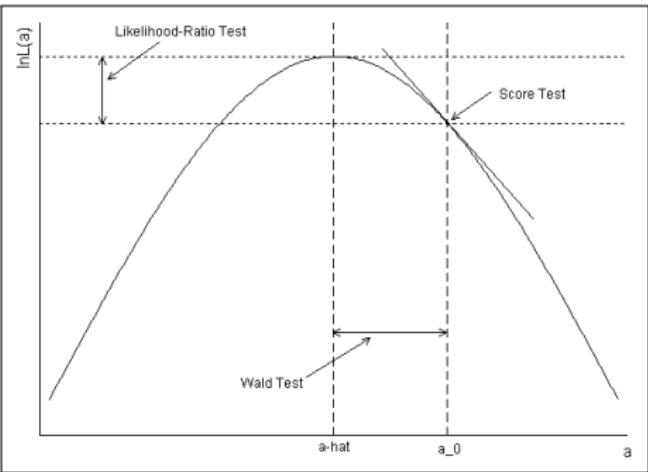
Likelihood-Ratio (LR) test

$$-2\ln \left( \frac{L(\theta_0|x)}{L(\hat{\theta}|x)} \right) \stackrel{\text{approx.}}{\sim} \chi_k^2 \text{ under } H_0$$

Score test

$$\frac{\left( \frac{\partial}{\partial \theta} \log L(\theta_0|x) \right)^2}{-E \left[ \frac{\partial^2}{\partial \theta^2} \log L(\theta_0|x) \right]} \stackrel{\text{approx.}}{\sim} \chi_1^2 \text{ under } H_0$$

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# Multicollinearity and the Variance Inflation Factor (VIF)

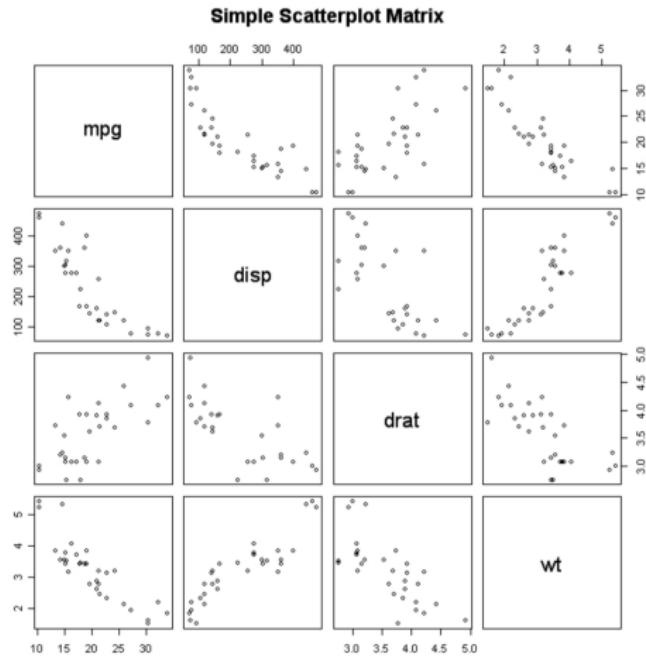
And when you have any number of covariates (features)...

$$\hat{\beta} \sim MVN\left(\beta, \sigma^2(\mathbf{x}^T \mathbf{x})^{-1}\right)$$

# Multicollinearity and the Variance Inflation Factor (VIF)

And when you have any number of covariates (features)...

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$$\hat{\beta} \sim MVN \left( \beta, \sigma^2 (\mathbf{x}^T \mathbf{x})^{-1} \right)$$

	DJIA	S&P 500	Nasdaq	Canada	Mexico	Brazil	Stoxx 50	FTSE 100	CAC 40	DAX	IBEX	Italy	Netherlands	Sweden	Switzerland	Nikkei	Hang Seng	Australia
DJIA	0.97	0.85	0.57	0.56	0.52	0.52	0.48	0.51	0.56	0.49	0.50	0.50	0.50	0.42	0.42	0.09	0.11	0.07
S&P 500	0.97	0.91	0.62	0.58	0.55	0.50	0.47	0.50	0.55	0.48	0.50	0.49	0.41	0.41	0.09	0.11	0.05	
Nasdaq	0.85	0.91	0.58	0.56	0.52	0.48	0.43	0.48	0.54	0.47	0.48	0.48	0.42	0.38	0.14	0.16	0.07	
Canada	0.57	0.62	0.58	0.53	0.53	0.42	0.45	0.41	0.41	0.42	0.42	0.39	0.37	0.35	0.17	0.22	0.17	
Mexico	0.56	0.58	0.56	0.53	0.56	0.42	0.42	0.44	0.43	0.43	0.44	0.39	0.38	0.38	0.17	0.25	0.17	
Brazil	0.52	0.55	0.52	0.53	0.56	0.33	0.35	0.32	0.34	0.34	0.34	0.29	0.30	0.28	0.17	0.22	0.15	
Stoxx 50	0.52	0.50	0.48	0.42	0.42	0.33	0.92	0.94	0.89	0.87	0.88	0.92	0.78	0.86	0.26	0.30	0.24	
FTSE 100	0.48	0.47	0.43	0.45	0.42	0.35	0.92	0.86	0.80	0.80	0.82	0.84	0.73	0.78	0.26	0.30	0.26	
CAC 40	0.51	0.50	0.48	0.41	0.44	0.32	0.94	0.86	0.89	0.88	0.89	0.92	0.78	0.84	0.28	0.32	0.25	
DAX	0.56	0.55	0.54	0.41	0.43	0.34	0.89	0.80	0.89	0.83	0.84	0.86	0.75	0.77	0.26	0.29	0.21	
IBEX	0.49	0.48	0.47	0.42	0.43	0.34	0.87	0.80	0.88	0.83	0.84	0.83	0.75	0.77	0.27	0.32	0.26	
Italy	0.50	0.50	0.48	0.42	0.44	0.34	0.88	0.82	0.89	0.84	0.84	0.85	0.74	0.78	0.24	0.29	0.23	
Netherlands	0.50	0.49	0.48	0.39	0.39	0.29	0.92	0.84	0.92	0.86	0.83	0.85	0.75	0.82	0.27	0.30	0.23	
Sweden	0.42	0.41	0.42	0.37	0.38	0.30	0.78	0.73	0.78	0.75	0.75	0.74	0.75	0.75	0.29	0.33	0.27	
Switzerland	0.42	0.41	0.38	0.35	0.38	0.28	0.86	0.78	0.84	0.77	0.77	0.78	0.82	0.75	0.29	0.32	0.29	
Nikkei	0.09	0.09	0.14	0.17	0.17	0.17	0.26	0.26	0.28	0.26	0.27	0.24	0.27	0.29	0.29	0.52	0.49	
Hang Seng	0.11	0.11	0.16	0.22	0.25	0.22	0.30	0.30	0.32	0.29	0.32	0.29	0.30	0.33	0.32	0.52	0.48	
Australia	0.07	0.05	0.07	0.17	0.17	0.15	0.24	0.26	0.25	0.21	0.26	0.23	0.23	0.27	0.29	0.49	0.48	

# Multicollinearity and the Variance Inflation Factor (VIF)

And when you have any number of covariates (features)...

$$\hat{\beta} \sim MVN\left(\beta, \sigma^2(\mathbf{x}^T \mathbf{x})^{-1}\right)$$

$$\widehat{\text{Var}}[\hat{\beta}_j] = \frac{\hat{\sigma}^2}{(n - 1)\widehat{\text{Var}}[X_j]} \cdot \frac{1}{1 - R_j^2} \quad [\text{VIF}]$$

where  $R_j^2$  is the  $R^2$  of  $X_j$  regressed on all the other  $X$ 's

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Scaling  $X$ 's (putting  $X$ 's on the same scale) helps numerically

# Multivariate Regression

That seriously major transition that just happened that you didn't even notice?

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i, \quad \epsilon_i \stackrel{i.i.d.}{\sim} N(0, \sigma^2)$$

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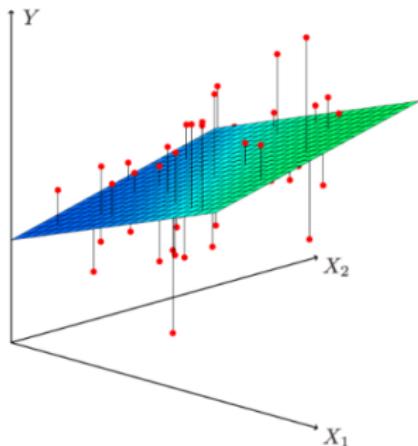
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# Linear Models

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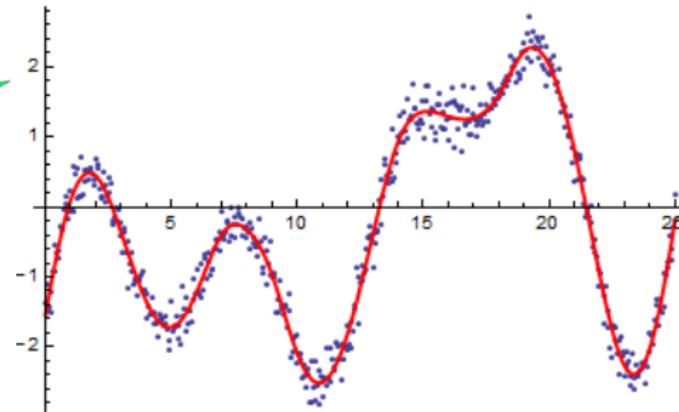
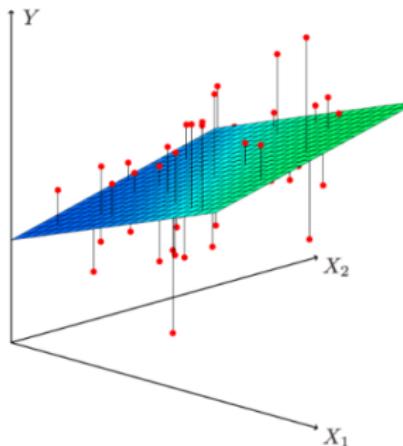
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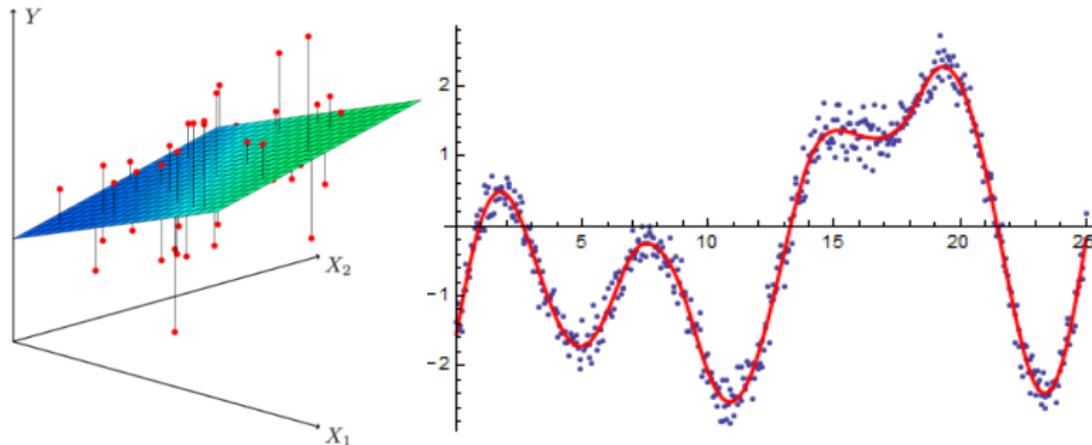
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- ▶ “Linear” models are only linear in the coefficients

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i$$

- ▶ The \$x\$'s can be pretty wild...

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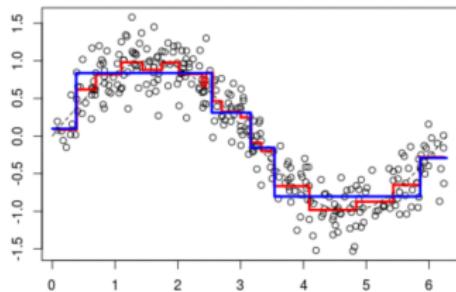
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Step functions

$$Y_i = \beta_j : \text{if } a_j \leq X_i < b_j$$



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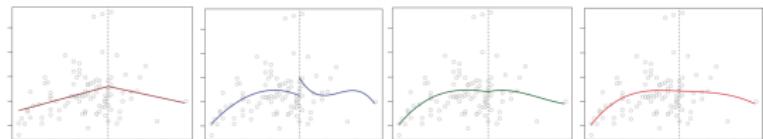
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Step functions



Regression Splines

$$Y_i = \begin{cases} \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \epsilon_i & : \text{if } X_i \leq c \\ \beta_0^* + \beta_1 X_i + \beta_2^* X_i^2 + \beta_3^* X_i^3 + \epsilon_i & : \text{if } X_i > c \end{cases}$$

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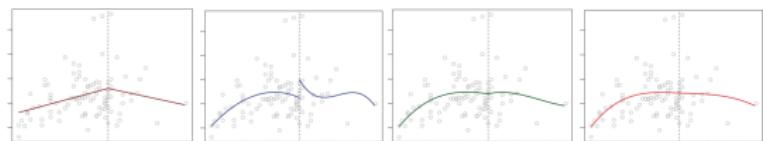
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Is it possible to “vary one  $X$  and hold all others constant”?

Step functions



Regression Splines

$$h(X_i, \xi) = \begin{cases} (x - \xi)^3 & : \text{if } X_i > \xi \\ 0 & : \text{if } X_i \leq \xi \end{cases} \quad Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \beta_{s1} h(X_i, \xi_1) + \dots + \epsilon_i$$

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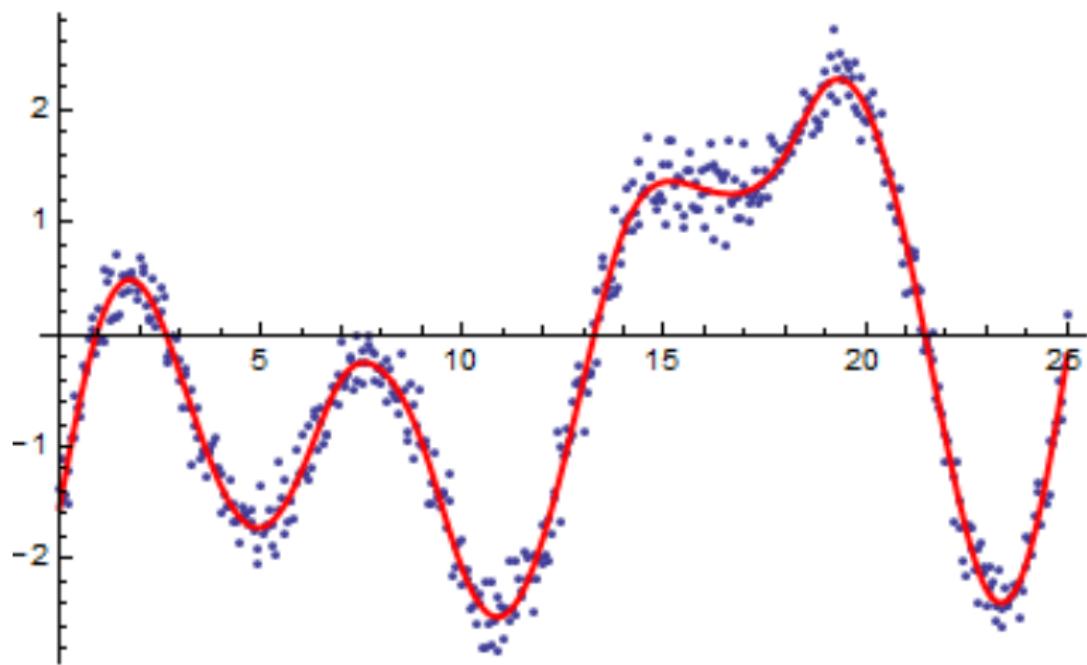
Step functions

Regression Splines

Smoothing Splines

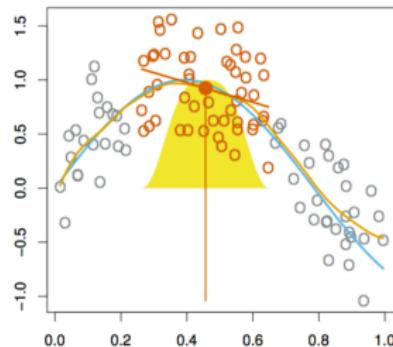
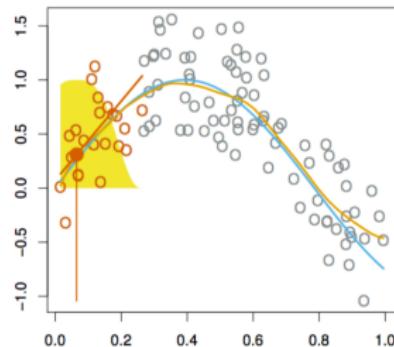
$$\min_g \sum_{i=1}^n (Y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$

## Linear models aren't really so “linear”



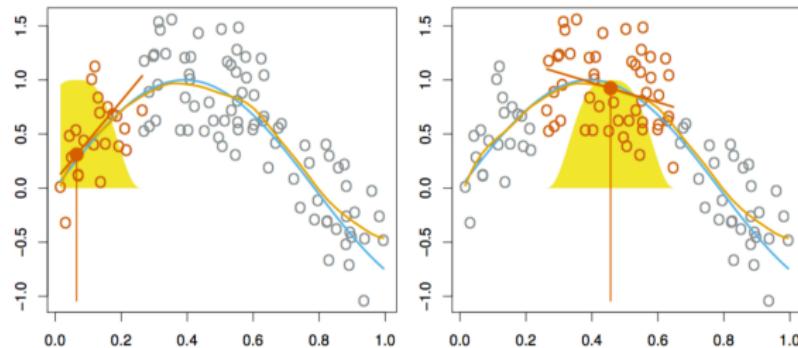
# Other ways to get “non linear” response surfaces

- ▶ Local Regression (LOESS)



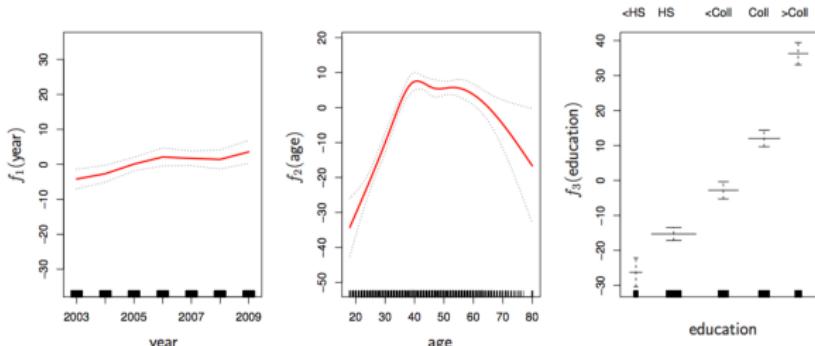
# Other ways to get “non linear” response surfaces

- ▶ Local Regression (LOESS)



- ▶ Generalized Additive Models

$$y_i = \beta_0 + f_1(x_{i1}) + f_2(x_{i2}) + \cdots + f_p(x_{ip}) + \epsilon_i.$$



# Model Fit

Residual Variation	Total Variation
$RSS = \sum(Y_i - \hat{Y}_i)^2 = \sum \hat{\epsilon}_i^2$	$TSS = \sum(Y_i - \bar{Y})^2$ $= RSS + \sum(\hat{Y}_i - \bar{Y})^2$
Residual Standard Error	Proportion of Variance Explained

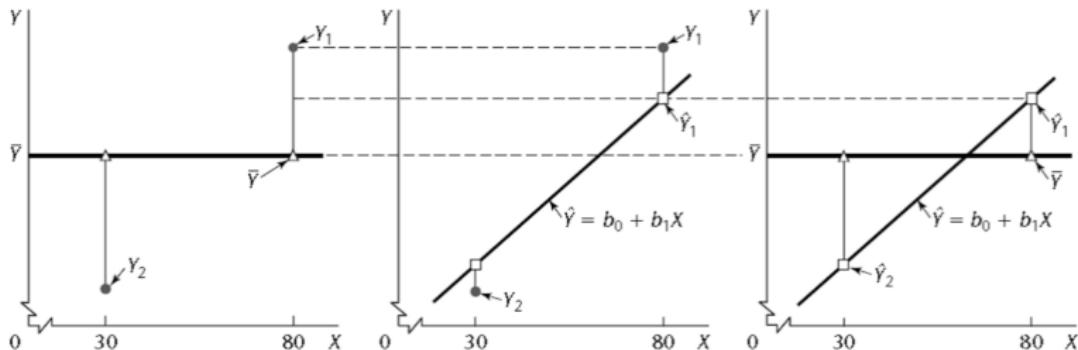
$$RSE = \sqrt{\frac{1}{n-p-1} RSS}$$
$$= \sqrt{\frac{\sum(Y_i - \hat{Y}_i)^2}{n-p-1}}$$

$$R^2 = \frac{TSS - RSS}{TSS}$$
$$= 1 - \frac{RSS}{TSS}$$

## F-test

$$F = \frac{(TSS - RSS)/p}{RSS/(n-p-1)} = \frac{\sum(\hat{Y}_i - \bar{Y})^2/p}{\sum(Y_i - \hat{Y})/(n-p-1)}$$

# Decomposition of Total Variation



$$\begin{aligned} TSS &= \sum(Y_i - \bar{Y})^2 = \sum(Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2 \\ &= \sum(Y_i - \hat{Y}_i)^2 + 2\sum(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) + \sum(\hat{Y}_i - \bar{Y})^2 \\ &= \sum(Y_i - \hat{Y}_i)^2 + 2\sum\hat{\epsilon}_i(\hat{Y}_i - \bar{Y}) + \sum(\hat{Y}_i - \bar{Y})^2 \\ &\quad \sum\hat{\epsilon}_i = 0 \uparrow \uparrow \sum\hat{\epsilon}_i \hat{Y}_i = 0 \\ &= \sum(Y_i - \hat{Y}_i)^2 + \sum(\hat{Y}_i - \bar{Y})^2 = RSS + \sum(\hat{Y}_i - \bar{Y})^2 \end{aligned}$$

## Hypothesis Testing for Feature Selection

$$F = \frac{(TSS - RSS)/p}{RSS/(n - p - 1)} = \frac{\sum(\hat{Y}_i - \bar{Y})^2/p}{\sum(Y_i - \hat{Y})/(n - p - 1)}$$

$F \sim F_{p,n-p-1}$  (tests if any coefficient is *non-zero*)

$\frac{\hat{\beta}_i - \beta_i}{SE(\hat{\beta}_i)} \sim t_{n-p-1}$  (tests if a *specific* coefficient is non-zero\*)

\*in the presence of all the others (this is a “last-in” test)

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OLS Regression Results

Dep. Variable:	y	R-squared:	0.933
Model:	OLS	Adj. R-squared:	0.928
Method:	Least Squares	F-statistic:	211.8
Date:	Mon, 03 Nov 2014	Prob (F-statistic):	6.30e-27 ←
Time:	14:45:06	Log-Likelihood:	-34.438
No. Observations:	50	AIC:	76.88
Df Residuals:	46	BIC:	84.52
Df Model:	3		
Covariance Type:	nonrobust		
coef	std err	t	P> t
x1	0.4687	0.026	17.751
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- ▶ Classical Model Selection Criterion

$$\text{Mallow's } C_p = \frac{1}{n}(RSS + 2p\hat{\sigma}^2)$$

$$AIC = -2 \log L + 2p$$

$$BIC = -2 \log L + p \log n$$

$$\text{Adjusted } R^2 = 1 - \frac{RSS/(n-p-1)}{TSS/(n-1)}$$

$$D_M = -2 \log f(Y|\hat{\theta}^{M_p}) + 2 \log f(Y|Y)$$

$$D_M \stackrel{\text{approx.}}{\sim} \chi^2_{n-p-1}$$

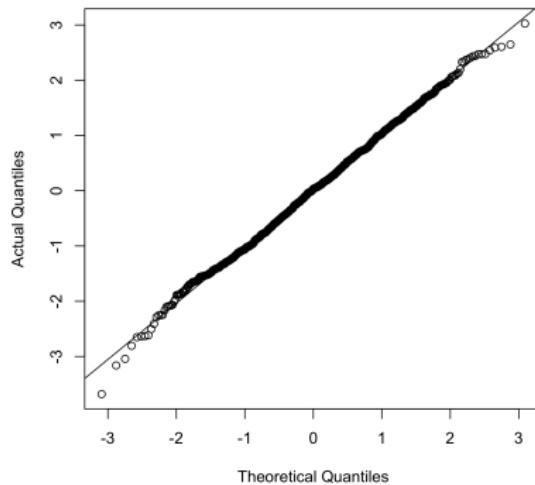
## Assumptions, violations, and remedial measures

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i, \quad \epsilon_i \stackrel{i.i.d.}{\sim} N(0, \sigma^2)$$

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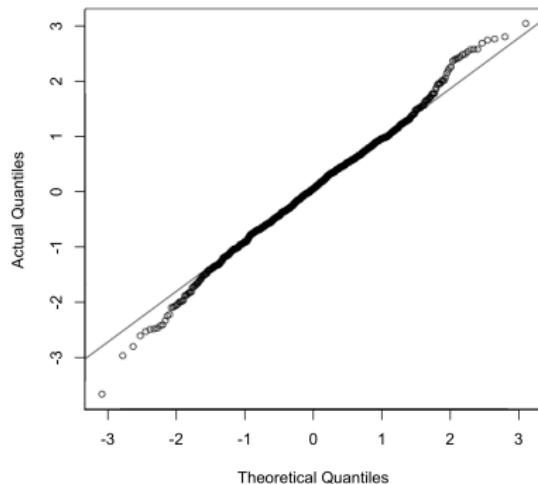
Q-Q Plot

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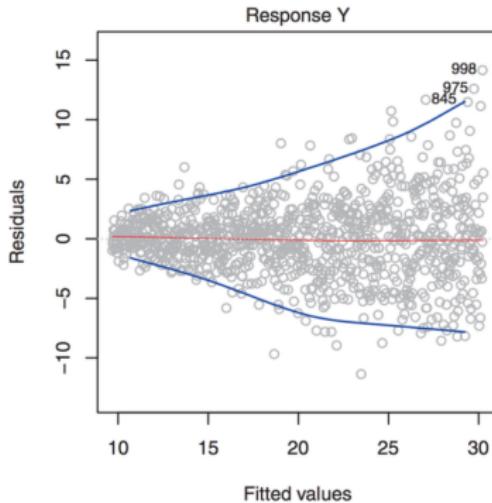
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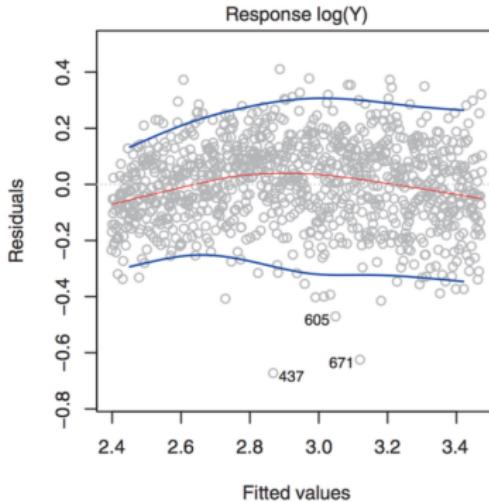
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- ▶ Homoskedasticity
- ▶ Independence

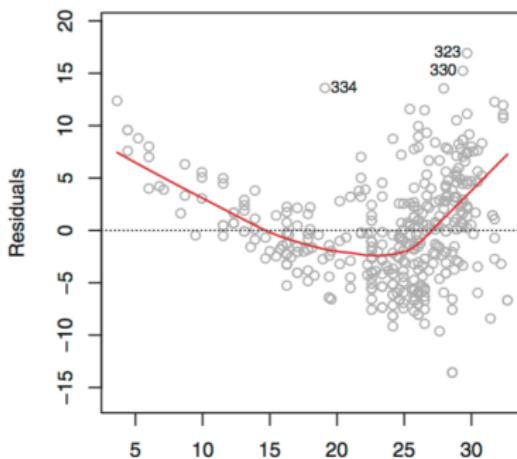
$$\text{Cov}[\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}] = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^T(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})$$

$$\approx \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

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Residuals versus Feature Values

“All models are wrong, some are useful”  
– George Box

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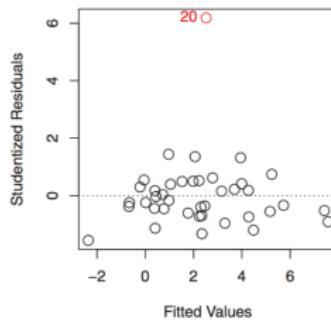
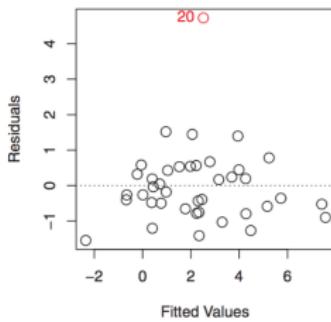
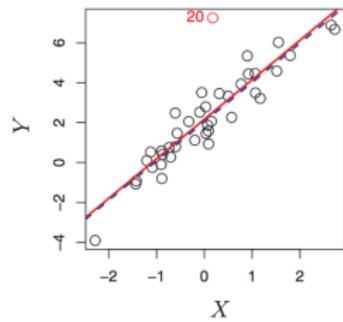
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- ▶ Linear form
- ▶ Fixed  $x$ 's

# Concerns: Outliers

Studentized Residuals (as in “Student’s t-distribution” residuals)

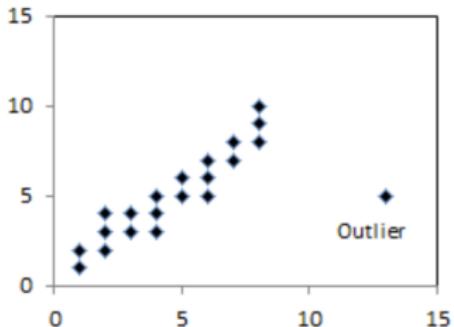
$$r_i = \frac{\hat{\epsilon}_i}{\hat{\sigma} \sqrt{1 - h_{ii}}}$$

have a t-distribution

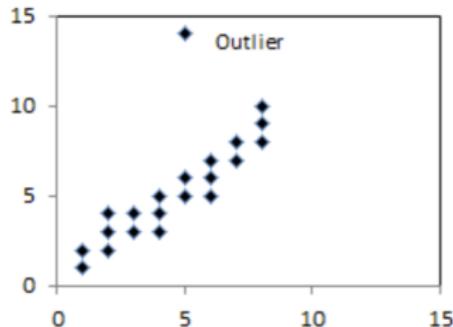


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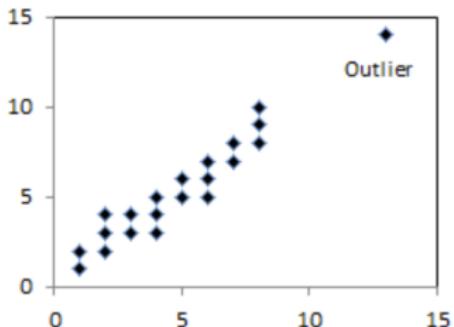
Extreme X value



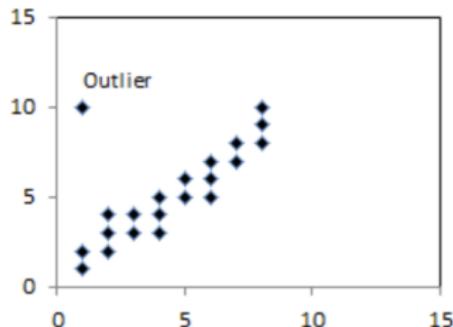
Extreme Y value



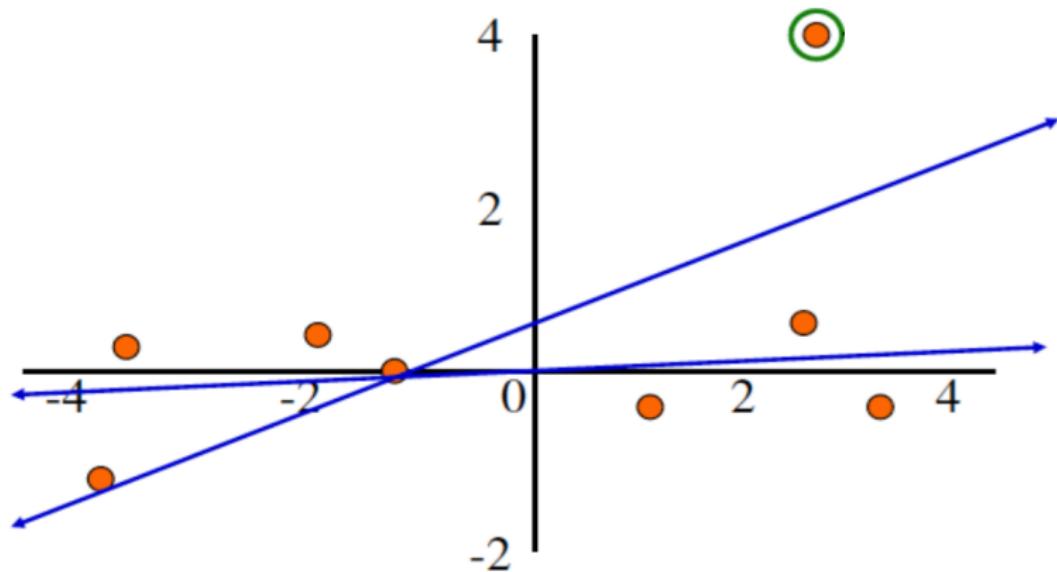
Extreme X and Y



Distant data point



## Concerns: Outliers



## Concerns: High Leverage Points

The *hat* matrix  $H$  “puts the hat on  $\mathbf{Y}$ ”:  $H$  projects  $\mathbf{Y}$  onto  $\hat{\mathbf{Y}}$  – the (least squares) closest vector (to  $\mathbf{Y}$ ) in the column space of  $\mathbf{x}$

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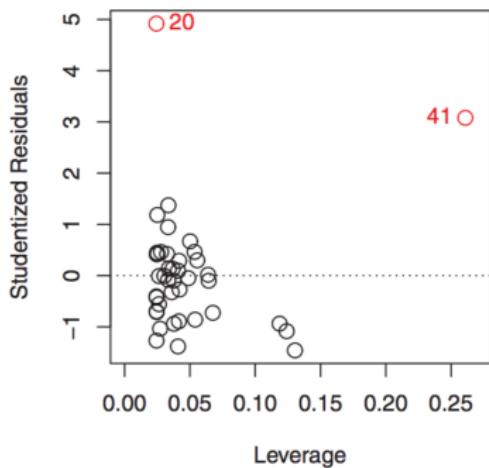
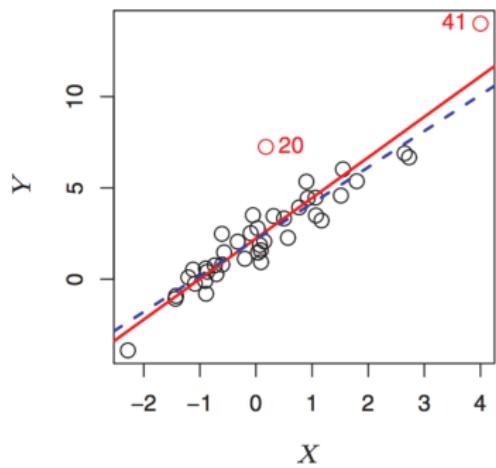
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 $H_{ii}$  is called the *leverage* of observation  $i$

# Concerns: High Leverage Points



## Concerns: Highly Influential Points – *Cook's distance*

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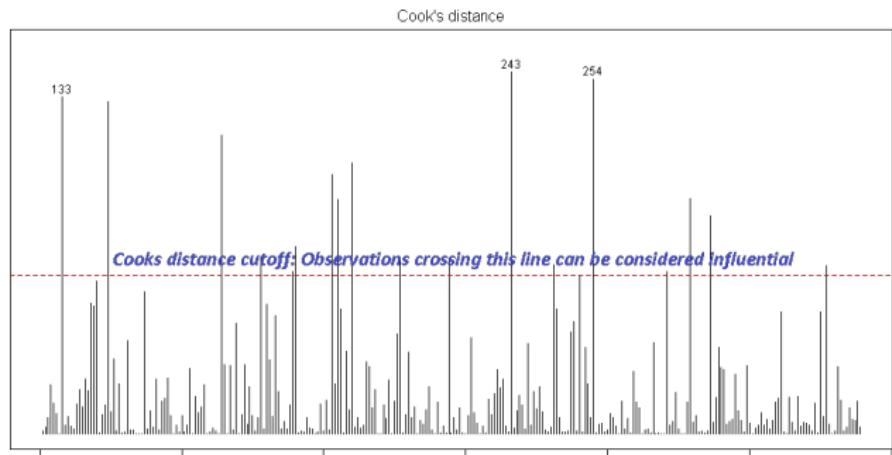
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