what is:
$A, P(A)$

has anyone studied ~~...~~ formal logic?

we have propositions:
statements about the world
can be true or false

we want to follow deductive reasoning

$A \Rightarrow B$ (if A is true then B must be true)

observe A is true
then B is true

or its converse
$A \Rightarrow B$
observe B is false
then A is false

however, we more frequently have
$A \Rightarrow B$
observe B is true
or observe A is false

how does this affect our knowledge of A, B, their relationships

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First, some notation:

$AB \equiv A$ and B (in sets $A \cap B$)

$A+B \equiv A$ or B (in sets $A \cup B$)

These are for propositions (which are not numbers)

$\bar{A} \equiv A$ is false

~~A ... is false~~

$\overline{AB}$ AB is false

$\bar{A}\bar{B}$ A is false and B is false

AND, OR, NOT form a complete set of logical operations (in fact NAND or NOR one is sufficient)

Some rules of logic operations

$AA = A$

$A+A = A$

$AB = BA$

$A+B = B+A$

$A(BC) = (AB)C = ABC$

$A+(B+C) = (A+B)+C = A+B+C$

$A(B+C) = AB + AC$

$A+(BC) = (A+B)(A+C)$

$C = AB$

$\bar{C} = \bar{A} + \bar{B}$

$D = A+B$

$\bar{D} = \bar{A}\bar{B}$

Deductive reasoning does not allow for the ~~propositions we want~~ inference we want to do!

instead we want to reason about the plausability of propositions (or groupings of them)

we can derive (not shown) based on requirements of basic reasoning that we represent these with functions of propositions

$p(A)$ will be some real number

$0 \leq p(A) \leq 1$ where 0 is impossibility and 1 is certainty

$p(A|B)$ [$A|B$ is plausability of A understanding B to be true (or A given B)]

$$p(AB) = p(A)\, p(B|A) = p(B)\, p(A|B)$$

* [note: some people refuse to write un conditional probabilities (they are correct, but it is annoying)]

$$p(A) + p(\bar{A}) = 1$$

these three rules allow us to calculate most (if not all) probabilities.

~~But what about numbers:~~

example:

$$P(A+B) = 1 - P(\bar{A}\bar{B})$$
$$= 1 - p(\bar{A})\, p(\bar{B}|\bar{A})$$
$$= 1 - p(\bar{A})[1 - p(B|\bar{A})]$$
$$= 1 - p(\bar{A}) + p(\bar{A})\, p(B|\bar{A})$$
$$= p(A) + p(\bar{A}B)$$
$$= p(A) + p(B)\, p(\bar{A}|B)$$
$$= p(A) + p(B)[1 - p(A|B)]$$
$$= p(A) + p(B) - p(B)\, p(A|B)$$
$$= p(A) + p(B) - P(AB)$$

what if A, B are mutually exclusive i.e. $P(AB) = 0$

what if we have a number of mutually exclusive events $P(A_i A_j) = p(A_i)\,\delta_{ij}$

the $P(\sum_i A_i) = \sum_i P(A_i)$

if $P(A_i) = P(A_j)$ for all i, j

then $P(A_i) = \frac{1}{n}$ for ~~all~~ $1 \leq i \leq n$

~~"principle of indifference"~~

So if we have some set of exhaustive, mutually exclusive events and they are all equally likely then we have a starting point and rules to determine results.

die

roll dice once what is probability of 3 or 4

A: roll of 3

B: roll of 4

$$P(A+B) = P(A) + P(B) + \cancel{P(AB)} \to 0$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

what if we roll twice?

A: 3 roll 1

B: 3 roll 2

C: 4 roll 1

D: 4 roll 2

~~only~~

~~$P(A+B+C+D) = P(A) + P(B) + P(C) + P(D) -$~~

$$P(A+B+C+D) = P(A+B+C) + P(D) - \underbrace{P((A+B+C)D)}_{P(AD+BD+CD)}$$

$$P(A+B+C) = P(A+B) + P(C) - P(AC+BC)$$

$$P(AD+BD+CD) = P(AD+BD) + P(CD) - P(ADC+BDC)$$

$$\cancel{P(A) + P(B) + P(C) + P(D) - P(AB)}$$

$$P(A+B) + P(C) + P(D) - P(AC+BC) - P(AD+BD) - P(CD) + P(ADC+BDC)$$

$$P(A+B) = P(A) + P(B) - P(AB)$$

$$P(AC+BC) = P(AC) + P(BC) - P(ABC)$$

$$P(AD+BD) = P(AD) + P(BD) - P(ABD)$$

$$P(ADC+BDC) = P(ADC) + P(BDC) - P(ABCD)$$

$$P(A) + P(B) + P(C) + P(D) - P(AB) - \cancel{P(AC)} - P(BC) + \cancel{P(ABC)} - P(AD) - \cancel{P(BD)} + \cancel{P(ABD)} + \cancel{P(ADC)} + \cancel{P(BDC)} - \cancel{P(ABCD)}$$

$$= P(A) + P(B) + P(C) + P(D) - P(AB) - \cancel{P(AC)}^{\,0} - P(BC) - P(AD)$$

$$P(AB) = P(BC) = P(AD) = P(CD) = \frac{1}{36}$$

$$= \frac{4}{6} - \frac{3}{36} = \frac{24}{36} - \frac{4}{36} = \frac{20}{36}$$

that was annoying

counting is a thing

- this gets us to combinatorics:

$n!$ number of ways to arrange n items

permutations $\frac{n!}{(n-k)!}$ (choose k from n items) order matters

combinations $\frac{n!}{(n-k)!\,k!}$ (choose k from n items order doesn't matter

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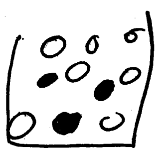
for example
urn with M red balls and N balls total
non-red balls are white



if we draw a ball
what is probability we will draw a red ball.
- [wait for answer]

what if we draw multiple balls

$$P(R_1 R_2) = \frac{M}{N}\frac{(M-1)}{(N-1)}$$

what is $P(R_2|R_1) = ??$ maybe
[wait for answer]

$$P(R_1 R_2 \ldots R_r) = \frac{M}{N} \cdot \frac{(M-1)}{(N-1)} \cdot \ldots \frac{(M-r+1)}{(N-r+1)}$$

$$= \frac{M!}{(M-r)!} \cdot \frac{(N-r)!}{N!}$$

~~$P(R_1 R_2 W_1 R_3 R_4) = P(R_1 R_2 R_3 R_4 W_1)$~~
~~(why??)~~

$P(R_1 R_2 W_3) = ??$

well remember $P(AB) = P(A)P(B|A)$

call $R_1 R_2 = A$

then we have $P(R_1 R_2) P(W_3 | R_1 R_2)$

$= P(R_1) P(R_2 | R_1) P(W_3 | R_1 R_2)$

this works for arbitrary statements

in general:

$$P\left(\bigcap_{k=1}^{n} A_k\right) = \prod_{k=1}^{n} P\left(A_k \,\middle|\, \bigcap_{j=1}^{k-1} A_j\right)$$

"chain rule of probability"

So if we look at

$P(R_1 R_2 W_3) = P(W_1 R_2 R_3)$

why?? (discuss??)

work it out if necessary

~~then~~ back to counting

if we want to know probability of drawing 5 balls and having 3 be red (without caring about order)

then we just need to calculate

$$P(R_1 R_2 R_3 W_4 W_5) = \frac{M!(N-M)!\,(N-5)!}{(M-3)!\,(N-M-2)!\,N!}$$

and multiply by $\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$

in general

P(r red balls in n draws)

$$= \frac{\binom{M}{r}\binom{N-M}{n-r}}{\binom{N}{n}}$$

this is a probability distribution (called the hypergeometric) will talk more about these this afternoon.

~~$P(+) = 0.9$.~~ $P(+ | \text{disease})$ is ~~specificity~~ sensitivity

$P(+ | \text{no disease}) = ?$

~~[illegible]~~ $P(+ | \text{no disease}) + P(\bar{+} | \text{no disease}) = 1$

$P(+ | \text{no disease}) = 1 - \text{specificity}$

$$P(+) = 0.9 \cdot 0.03 + (1 - 0.99) \cdot (1 - 0.03)$$
$$= \cancel{0.0377}\ 0.0367$$

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what about $P(\text{disease} | +)$??

for this we can use Bayes Rulle

remember $P(A|B)P(B) = P(B|A)P(A)$

$$\therefore\ P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

if we have $P(+ | \text{disease})$

can get $P(\text{disease} | +)$

$$P(\text{disease} | +) = \frac{P(+ | \text{disease})\, P(\text{disease})}{P(+)}$$

$$= \frac{0.9 \cdot 0.03}{0.0367}$$

$$\cong \cancel{0.7}\ \cancel{[illegible]}\ 0.736$$

additional useful ideas:

independence:

$$P(A|B) = P(A)$$

[– discuss] i.e. probability it will rain tomorrow given I have a sister.

its really important, will talk more about later.

law of total probability:

if B_i is a partition of some event space.

i.e. set of disjoint events that cover every outcome

then $P(A) = \sum_i P(A|B_i)P(B_i)$

~~I tell you i rolled a ... or ...~~

test for a disease has a sensitivity of 0.90 and specificity of 0.01

sensitivity is probability of a positive result given that they have disease.

specificity is probability of a negative result given they do not have the disease

if 3% of population has the disease what is chance of testing positive

$P(+) = P(+|\text{disease})P(\text{disease}) + P(+|\text{no-disease})P(\text{no-disease})$