

# Homework 5 Problem 2

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In order to show the desired result, we must prove a few related arguments.

**Lemma 1:**  $Var(\bar{X}) = \sigma^2/n$

Starting with the definition of variance and applying simple properties of the variance, we have:

$$\begin{aligned} Var(\bar{X}) &= Var\left(\frac{X_1 + X_2 + \dots}{n}\right) \\ &= Var(X_1/n + X_2/n + \dots) \\ &= 1/n^2 * Var(X_1) + 1/n^2 * Var(X_2) + \dots \\ &= n\sigma^2/n^2 \\ &= \sigma^2/n \end{aligned}$$

If you evaluated  $E[Var(\bar{X})]$ , then this is OK b/c  $E[Var(X_i)] = \sigma^2$ .

**Lemma 2:**  $Var(x) = E(x^2) - E(x)^2$

Applying the definition of the variance and  $\mu$  and using properties of the expectation value, we have

$$\begin{aligned} Var(x) &= E[(x - \mu)^2] \\ &= E[(x - E(x))^2] \\ &= E[x^2 + E(x)^2 - 2xE(x)] \\ &= E(x^2) + E(E(x)^2) - 2E(x)E(x) \\ &= E(x^2) + E(x)^2 - 2E(x)^2 \\ &= E(x^2) - E(x)^2 \end{aligned}$$

**Lemma 3:**  $E(\bar{X}^2) = \sigma^2/n + \mu^2$

Applying Lemma 2 and Lemma 1, as well as the definition of  $E(x)$ , we easily see this result holds.

$$\begin{aligned} E(\bar{X}^2) &= Var(\bar{X}) + E(\bar{X})^2 \\ &= \sigma^2/n + \mu^2 \end{aligned}$$

**Lemma 4:**  $E[(\bar{x} - \mu)^2] = \sigma^2/n$

Expanding the square and applying Lemma 3, we see

$$\begin{aligned} E[(\bar{x} - \mu)^2] &= E(\bar{x}^2 + \mu^2 - 2\mu\bar{x}) \\ &= E(\bar{x}^2) + E(\mu^2) - 2\mu E(\bar{x}) \\ &= \sigma^2/n + \mu^2 + \mu^2 - 2\mu^2 \\ &= \sigma^2/n \end{aligned}$$

**Proof**  $E(S_b^2) = \sigma^2(n-1)/n$

We begin with the definition of  $S_b^2$ , rewrite and expand the squared term, then apply Lemma 4 as necessary and the definition of  $\sigma^2$  given in the assignment.

$$\begin{aligned} E(S_b^2) &= E \left[ \frac{1}{n} \sum_i^n (x_i - \bar{x})^2 \right] \\ &= \frac{1}{n} E \left[ \sum_i^n ((x_i - \mu) - (\bar{x} - \mu))^2 \right] \\ &= \frac{1}{n} E \left[ \sum_i^n ((x_i - \mu)^2 + (\bar{x} - \mu)^2 - 2(\bar{x} - \mu)(x_i - \mu)) \right] \\ &= \frac{1}{n} E \left[ \sum_i^n (x_i - \mu)^2 + n(\bar{x} - \mu)^2 - 2(\bar{x} - \mu) \sum_i^n (x_i - \mu) \right] \\ &= \frac{1}{n} E \left[ \sum_i^n (x_i - \mu)^2 + n(\bar{x} - \mu)^2 - 2n(\bar{x} - \mu)^2 \right] \\ &= \frac{1}{n} E \left[ \sum_i^n (x_i - \mu)^2 - n(\bar{x} - \mu)^2 \right] \\ &= \frac{1}{n} \left( E \left[ \sum_i^n (x_i - \mu)^2 \right] - nE[(\bar{x} - \mu)^2] \right) \\ &= \frac{1}{n} (n\sigma^2 - \sigma^2) \\ &= \sigma^2 \frac{n-1}{n} \end{aligned}$$

which is the desired result.