Homework 3 Problem 1

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A tree diagram showing the sample space is given in Figure 1. We see that the total number of possible outcomes is $2^4 = 16$, 6 of which result in exactly two heads. The corresponding probability of obtaining exactly two heads in 4 tosses of a single coin is then $6/16 = 37.5\%$.

Great explanation. Exactly what I was looking for.

You also reminded me of the existence of MS Paint.

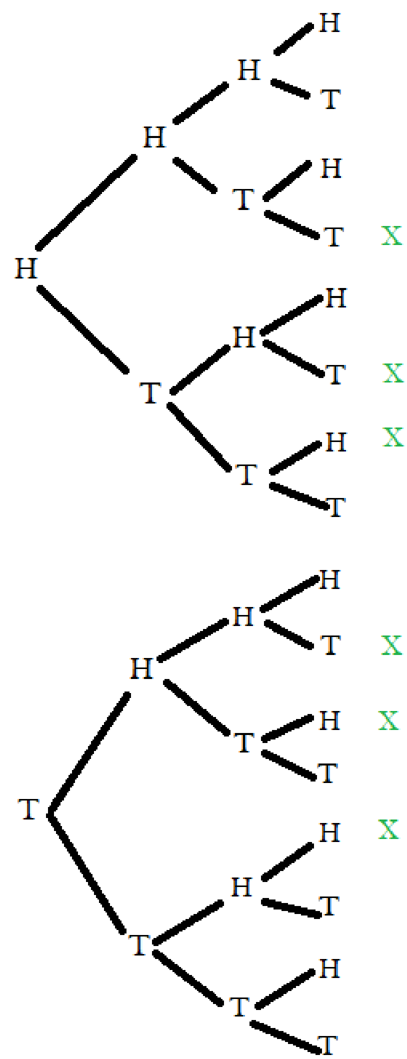

Figure 1: Tree diagram of flipping a single coin 4 times. A green X indicates an outcome of exactly two heads.

Next, consider the binomial distribution for a N coin tosses with k heads:

$$P(k) = \binom{N}{k}(1-\theta)^{N-k}\theta^k$$

To gain insight into why this formula gives the probability of k heads in N tosses, let us consider a simplified case where $\theta = 0.5$. This corresponds to flipping a 'fair' coin:

$$P(k) = \binom{N}{k}\left(\frac{1}{2}\right)^N$$

Notice that k cancels out in the terms containing θ. Ignoring the N choose k term for a moment, we easily see that this formula tells us the probability of flipping N heads in a row for a fair coin. Including the N choose k term, this formula tells us the probability of obtaining k heads from N tosses for a fair coin. Thus, we see that the term $\binom{N}{k}$ accounts for how many possible ways there are to "arrange" the desired outcome.

For example, if I wanted to flip a coin 10 times and calculate the probability of obtaining 10 heads, this formula would just be

$$P(10) = 1 * \frac{1}{2^{10}} = \frac{1}{1024} \approx 0.000977$$

Alternatively, if I wanted the probability of obtaining exactly 1 head in 10 tosses, I would just have

$$P(1) = 10 * \frac{1}{2^{10}} = \frac{10}{1024} \approx 0.00977$$

The above assumes we have a fair coin, but what if we have a weighted coin? Looking at the original distribution we started out with, we can easily see how a weighted coin can be worked into the calculation. As before, the term $\binom{N}{k}$ tells us how many ways we can order the desired outcome. The term θ^k is related to the probability of obtaining k heads, assuming heads has a probability of θ. The term $(1-\theta)^{N-k}$ simply tells us the probability that the other tosses are not heads. If we only have two possible outcomes–heads and tails–then the probability of tails must be $(1-\theta)$ and occur $(N-k)$ times.

As a last example, suppose we had a weighted coin with a 75% probability to land on heads. Then the probability of flipping the coin 10 times and obtaining 10 heads is

$$P(10) = 1 * (1 - 0.75)^{10-10}\, 0.75^{1}0 \approx 0.056$$

and the probability of obtaining exactly one head is

$$P(1) = 10 * (1 - 0.75)^{10-1}\, 0.75^{1} \approx 2.86 \times 10^{-5}$$

Hence, we see that

$$P(k) = \binom{N}{k}(1-\theta)^{N-k}\theta^{k}$$

simply corresponds to a generalized coin toss.