Homework 5 Problem 2

William Matzko

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In order to show the desired result, we must prove a few related arguments.

Lemma 1: $Var(\overline{X}) = \sigma^2/n$

Starting with the definition of variance and applying simple properties of the variance, we have:

$$\begin{aligned}
Var(\overline{X}) &= Var\left(\frac{X_1 + X_2 + \dots}{n}\right) \\
&= Var(X_1/n + X_2/n + \dots) \\
&= 1/n^2 * Var(X_1) + 1/n^2 * Var(X_2) + \dots \\
&= n\sigma^2/n^2 \\
&= \sigma^2/n
\end{aligned}$$

Lemma 2: $Var(x) = E(x^2) - E(x)^2$

Applying the definition of the variance and μ and using properties of the expectation value, we have

$$\begin{aligned}
Var(x) &= E[(x-\mu)^2] \\
&= E[(x - E(x))^2] \\
&= E[x^2 + E(x)^2 - 2xE(x)] \\
&= E(x^2) + E(E(x)^2) - 2E(x)E(x) \\
&= E(x^2) + E(x)^2 - 2E(x)^2 \\
&= E(x^2) - E(x)^2
\end{aligned}$$

Lemma 3: $E(\overline{X}^2) = \sigma^2/n + \mu^2$

Applying Lemma 2 and Lemma 1, as well as the definition of $E(x)$, we easily see this result holds.

$$\begin{aligned}
E(\overline{X}^2) &= Var(\overline{X}) + E(\overline{X})^2 \\
&= \sigma^2/n + \mu^2
\end{aligned}$$

Lemma 4: $E[(\overline{x} - \mu)^2] = \sigma^2/n$

Expanding the square and applying Lemma 3, we see

$$\begin{aligned}
E[(\overline{x} - \mu)^2] &= E(\overline{x}^2 + \mu^2 - 2\mu\overline{x}) \\
&= E(\overline{x}^2) + E(\mu^2) - 2\mu E(\overline{x}) \\
&= \sigma^2/n + \mu^2 + \mu^2 - 2\mu^2 \\
&= \sigma^2/n
\end{aligned}$$

Proof $E(S_b^2) = \sigma^2(n-1)/n$

We begin with the definition of S_b^2, rewrite and expand the squared term, then apply Lemma 4 as necessary and the definition of σ^2 given in the assignment.

$$
\begin{aligned}
E(S_b^2) &= E\left[\frac{1}{n}\sum_i^n (x_i - \overline{x})\right] \\
&= \frac{1}{n}E\left[\sum_i^n \left((x_i-\mu) - (\overline{x}-\mu)\right)^2\right] \\
&= \frac{1}{n}E\left[\sum_i^n \left((x_i-\mu)^2 + (\overline{x}-\mu)^2 - 2(\overline{x}-\mu)(x_i-\mu)\right)\right] \\
&= \frac{1}{n}E\left[\sum_i^n (x_i-\mu)^2 + n(\overline{x}-\mu)^2 - 2(\overline{x}-\mu)\sum_i^n (x_i-\mu)\right] \\
&= \frac{1}{n}E\left[\sum_i^n (x_u-\mu)^2 + n(\overline{x}-\mu)^2 - 2n(\overline{x}-\mu)^2\right] \\
&= \frac{1}{n}E\left[\sum_i^n (x_i-\mu)^2 - n(\overline{x}-\mu)^2\right] \\
&= \frac{1}{n}\left(E\left[\sum_i^n (x_i-\mu)^2\right] - nE\left[(\overline{x}-\mu)^2\right]\right) \\
&= \frac{1}{n}\left(n\sigma^2 - \sigma^2\right) \\
&= \sigma^2\frac{n-1}{n}
\end{aligned}
$$

which is the desired result.