Intro to Linear Algebra: A Summary

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Updated December 29, 2021

Contents

Chapter 0

Background & Review

Everything mentioned in this chapter should already be familiar to you from other math classes. These topics span two main areas: algebra/pre-calculus and single variable calculus. Ideas from algebra and pre-calculus will be used either implicitly or with only a passing reference. Topics from single variable calculus will mostly be used in setting up problems.
If you are unfamiliar with anything mentioned, you can use many of the great only resources like Khan Academy, to familiarize yourself before moving forward.

0.1 Algebra and Pre-Calculus

0.1.1 Sets

Definition. A set A is a collection of distinct elements. Those elements can be anything, like numbers, functions, and even other sets.

We can define a set by giving its elements, like $A = \{-2, 5, 3\}$ or by describing its properties, like $A = \{x \mid x > 0\}$ where the vertical bar means "such that". If an object x is a member of the set A, we write $x \in A$.

A set A is called a subset of a set B if every element of A is also an element of B. We can write this as $A \subseteq B$. For example, $\{7, 10, 16\} \subseteq \{5, 6, 7, 9, 10, 11, 16\}$. Note that this relation can be strict if there exists at least one element in B that is not also an element of A. Some common sets and their informal definitions are given below:

Set Name	Symbol	Informal Definition
Natural numbers	$\mathbb{N}$	$\{1, 2, 3, \dots\}$
Integers	$\mathbb{Z}$	$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
Rational numbers	$\mathbb{Q}$	$\{\frac{m}{n} \mid m, n \in \mathbb{Z}$ and $n \neq 0\}$
Real numbers	$\mathbb{R}$	Any number on the number line[1]

This means that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

There are several common operations that can be performed on sets. The union $A \cup B$ of two sets A and B is the set of all elements that are elements of A or of B. Similarly, the intersection $A \cap B$ of two sets A and B is the set of all elements that are also elements of both A and B.

Example. If $A = \{\sqrt{2}, 2, 5, 8\}$ and $B = \{-9, 8, 2.3\}$, what are $A \cup B$ and $A \cap B$?

To find the union, we combine the sets, making sure to include any repeated element only once:

$$A \cup B = \{-9, \sqrt{2}, 2, 2.3, 5, 8\}.$$

Then, since the only element both sets share is 8, we also have

$$A \cap B = \{8\}.$$

0.1.2 Intervals

Definition. We call a subset I of $\mathbb{R}$ an interval if, for any $a, b \in I$ and $x \in \mathbb{R}$ such that $a \leq x \leq b$, then $x \in I$.

We can write an interval more simply using the notation $[a, b]$, which is equivalent to $\{x \in \mathbb{R} \mid a \leq x \leq b\}$. This is called a closed interval, and to make the inequalities strict, we can also define an open interval by using parantheses instead of square brackets.

In addition, we can mix the two to create half-open intervals, where one inequality is strict and the other isn't. For instance, $(2, 5]$ refers to the set $\{x \in \mathbb{R} \mid x < 2 \leq 5\}$ Finally, if the interval is unbounded in either direction, we use the notations $-\infty$ and ∞ to indicate that there is no minimum or maximum, respectively.

Example. Is $8 \in (-\infty, 4) \cup [8, 100)$?

Since $8 \leq 8 < 100$ is a true statement, $8 \in [8, 100)$. Since we are taking the union with another set, all of the members of the right interval will also be members of the union of intervals. Therefore, the statement is true.

0.1.3 Functions

Definition. A function f is a rule between a pair of sets, denoted $f : D \to C$, that assigns values from the first set, the domain D, to the second set, the codomain C.

We call the subset of the codomain C that constitutes all values f can actually attain the range $R \subseteq C$. Note that when we draw a graph of a function, all we are doing is drawing all ordered pairs $\{(x, f(x)) \mid x \in D\}$.

Example. Find the domain of the following function:

$$f(x) = \frac{1}{(1-x)\sqrt{5-x^2}}$$

We know that $\frac{n}{0}$ is undefined for all $n \in \mathbb{R}$ and $\sqrt{x}$ is only defined for $x \geq 0$. The first condition applies to the first term in the denominator and both conditions apply to the second, giving us

$$(1-x) \neq 0 \text{ and } 5 - x^2 > 0$$

The first condition implies $x \neq 1$ while the second implies $|x| < \sqrt{5}$. Putting these together, we find that the domain is

$$\{x \mid x \neq 1, |x| < \sqrt{5}\} \text{ or } (-\sqrt{5}, 1) \cup (1, \sqrt{5})$$

We can also compose two functions, such that the ouput of one function is the input of another:

$$(f \circ g)(x) = f(g(x)).$$

Definition. A function g is called an inverse function of f if $f(g(x)) = x$ for all x in the domain of g and $g(f(x))$ for all x in the domain of f. We write this as $g = f^{-1}$.

One common algorithm for finding an inverse function is to set $y = f(x)$, substitute all x's for y's, and then solve for y.

Example. Find the inverse function of

$$f(x) = \frac{5x+2}{4x-3}.$$

We first make the substitutions to set up the algorithm:

$$y = \frac{5x+2}{4x-3} \text{ followed by } x = \frac{5y+2}{4y-3}$$

After multiplying both sides by the denominator and simplifying, we have

$$\implies 4xy - 3x = -5y - 2 \implies y = f^{-1}(x) = \frac{3x-2}{4x+5}.$$

We say that a function f is even if it satisfies $f(-x) = f(x)$ for all $x \in D$. Likewise, we say that a function f is odd if it satisfies $f(-x) = -f(x)$ for all $x \in D$. Geometrically, we can see that the graph of an even function is symmetric with respect to the y-axis, while the graph of an odd function is symmetric with respect to the origin.

Example. Is $f(x) = 2x - x^2$ even, odd, or neither?

$$f(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, the function is neither even nor odd.

0.1.4 Complex Numbers

Definition. i is called the imaginary unit. It's defined by $i^2 = -1$.

The set of complex numbers ($\mathbb{C}$) is an extension of the real numbers. Complex numbers have the form $z = \alpha + \beta i$, where α and β are real numbers. The α part of z is called the real part, so $\Re(z) = \alpha$. The β part of z is called the imaginary part, so $\Im(z) = \beta i$.

Often, complex numbers are visualized as points or vectors in a 2D plane, called the complex plane, where α is the x-component, and β is the y-component. Thinking of complex numbers like points helps us define the magnitude of complex numbers and compare them. Since a point (x, y) has a distance $\sqrt{x^2 + y^2}$ from the origin, we can say the magnitude of z, $|z|$ is $\sqrt{\alpha^2 + \beta^2}$. Thinking of complex numbers like vectors helps us understand adding two complex numbers, since you just add the components like vectors.

A common operation on complex numbers is the complex conjugate. The complex conjugate of $z = \alpha + \beta i$ is $\overline{z} = \alpha - \beta i$. z and $\overline{z}$ are called a conjugate pair.

Conjugate pairs have the following properties. Let $z, w \in \mathbb{C}$ and $n \in \mathbb{Z}$.

$$\begin{aligned}
\overline{z \pm w} &= \overline{z} \pm \overline{w} \\
\overline{zw} &= \overline{z} \cdot \overline{w} \\
\overline{z} &= z \Leftrightarrow z \in \mathbb{R} \\
z\overline{z} &= |z|^2 = |\overline{z}|^2 \\
\overline{\overline{z}} &= z \\
\overline{z}^n &= \overline{z^n} \\
z^{-1} &= \frac{\overline{z}}{|z|^2}
\end{aligned}$$

0.1.5 Factoring Polynomials

We want to break up a polynomial like $f(x) = a_0 + a_1x^1 + \ldots a_nx^n$ into linear factors so that $f(x) = c(x - b_1) \cdot \ldots \cdot (x - b_n)$. This form makes it simple to see that the roots of f, solutions to $f(x) = 0$, are $x = b_1 \ldots b_n$.

For quadratics, $f(x) = ax^2 + bx + c$, there exists a simple formula that will give us both roots, the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We can see that when $b^2 - 4ac < 0$, like for $f(x) = x^2 + 5x + 10$, we will get complex roots $\alpha \pm \beta i$. For any polynomial, these roots come in pairs, so if $\alpha + \beta i$ is a root, then so is $\alpha - \beta i$. This means that every conjugate pair $\alpha \pm \beta i$ has a quadratic equation with those roots. Sometimes we will not factor quadratics with complex roots into linear terms.

Although there do exist explicit formulas for finding roots for cubic (degree 3) and quartic (degree 4) equations, they are too long and not useful enough to memorize. When working by hand, we instead use other tricks to find roots.

There are a few useful tricks that can help. If the polynomial doesn't have a constant term, then 0 is a root. If all the coefficients sum to 0, then 1 is a root. For certain polynomials with an even number of terms, like all cubics of the form $ax^3 + bx^2 + cax + cb$ we can factor out a term from the first two and last two terms to get $x^2(ax + b) + c(ax + b) = (ax + b)(x^2 + c)$. For other polynomials, we might just try guessing and checking values. However, we need a more efficient way that works in general.

Since we are looking to find linear factors $f(x) = (x - b_1) \cdot \ldots \cdot (x - b_n)$, we can see that the constant term in the polynomial is the product of the roots $b_1 \ldots b_n$. In fact, since the coefficients of polynomials are completely determined by the roots and the leading coefficient, all the coefficients are sums and products of roots. You might remember when factoring quadratics that the coefficient of x term is the sum of the two roots. These rules are called Vieta's formulas.

So, if we have the constant term, we can check all of its integer factors to see if any are roots. For each root, we can divide, using a technique like synthetic division, to continue finding the rest of the roots. This method is especially useful on tests because the roots tend to be integers.

Example. Factor the polynomial $x^5 + x^4 - 2x^3 + 4x^2 - 24x$.

We can immediately see that there is no constant term, so $x = 0$ is a root. Now we need to work on factoring $x^4 + x^3 - 2x^2 + 4x - 24$.
The factors of -24 are: -24, -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12, and 24. Starting from roots close to 0 and working outwards, we find that $x = 2$ is a root. So, we synthetic divide like so

$$\begin{array}{r|ccccc}
x=2 & 1 & 1 & -2 & 4 & -24 \\
 & \downarrow & 2 & 6 & 8 & 24 \\
\hline
 & 1 & 3 & 4 & 12 & |\,0
\end{array}$$

to see that now we need to work on factoring $x^3 + 3x^2 + 4x + 12$. $x^3 + 3x^2 + 4x + 12 = x^2(x + 3) + 4(x + 3) = (x + 3)(x^2 + 4)$, so $x = -3$ is a root, and we need to work on factoring $x^2 + 4$. $x^2 + 4$ has two complex roots $\pm 2i$, so we'll leave it as a quadratic.

$$x^5 + x^4 - 2x^3 + 4x^2 - 24x = x(x - 2)(x - 3)(x^2 + 4)$$

0.1.6 Trig Functions & The Unit Circle

Imagine a circle of radius 1 centered at the origin that we'll call the unit circle. The x and y coordinates of a point on the unit circle are completely determined by the angle θ in radians between the x-axis and a line from the origin to the point.

The function $\cos\theta$ tells us x-coordinate of the point, while $\sin\theta$ tells us the y-coordinate of the point. The function $\tan\theta = \frac{\sin\theta}{\cos\theta}$ tells us the slope of the line from the origin to the point. Most of the trig functions have geometric interpretations as shown below. The most used ones are sin, cos, $\tan = \frac{\sin}{\cos}$, $\cot = \frac{\cos}{\sin}$, $\csc = \frac{1}{\sin}$, and $\sec = \frac{1}{\cos}$.

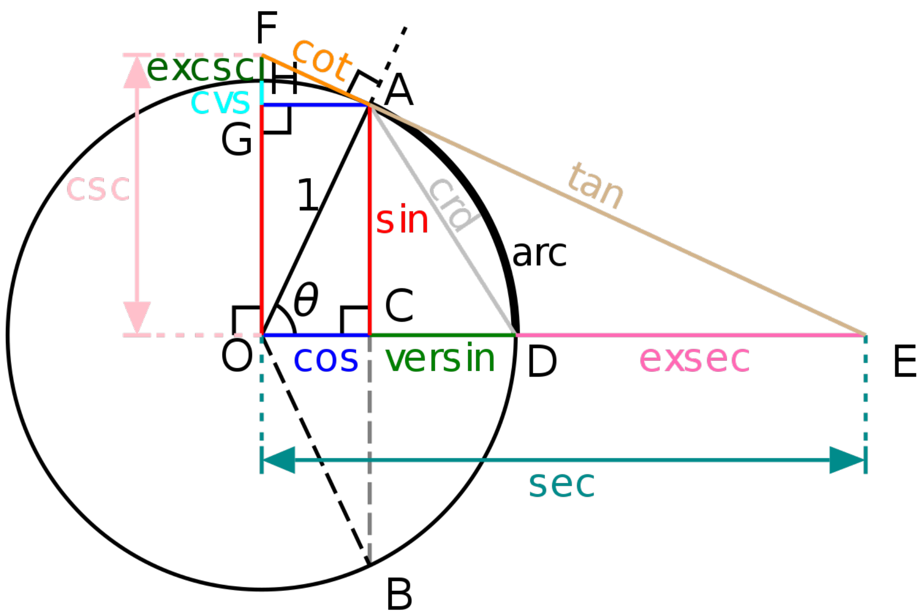

Figure 1: Wikipedia - Unit circle

We can also think about the inverses of these trig functions. These are either notated with a -1 exponent on the function, or the prefix arc in front of the function name. Many of these functions are only defined on a part of the domain $[0, 2\pi]$. Below is a table of the inverse trig functions and their domains.

Function	Domain
arcsin	$[-1, 1]$
arccos	$[-1, 1]$
arctan	$(-\infty, \infty)$
arccot	$(-\infty, \infty)$
arccsc	$(-\infty, -1] \cup [1, \infty)$
arcsec	$(-\infty, -1] \cup [1, \infty)$

0.1.7 Trig Identities

As we could see in Figure 0.1.6, sin and cos form a right triangle with hypotenuse 1. So, using the Pythagorean Theorem,

$$\sin^2\theta + \cos^2\theta = 1.$$

By dividing by $\sin^2$ or $\cos^2$, we can also get

$$1 + \cot^2\theta = \csc^2\theta \text{ and } \tan^2\theta + 1 = \sec^2\theta.$$

Together, these 3 identities are called the Pythagorean Identities.

We can also relate functions and co-functions.

$$\text{xxx}(\theta) = \text{coxxx}\left(\frac{\pi}{2} - \theta\right).$$

Some of the most useful and used identities are the sum and difference.

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \\
\sin\alpha \pm \sin\beta &= 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \\
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \\
\cos\alpha - \cos\beta &= -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)
\end{aligned}$$

0.1.8 Exponentials & Logarithms

Definition. e is the base of the natural logarithm. It's defined by the limit

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

$\exp x = e^x$ and $\ln x$ are inverse functions of each other such that

$$e^{\ln x} = x \text{ and } \ln e^x = x.$$

Just like other exponentials, the normal rules for adding, subtracting, and multiplying exponents apply:

$$e^x e^y = e^{x+y},\ \frac{e^x}{e^y} = e^{x-y}, \text{ and } (e^x)^k = e^{xk}.$$

Similar rules apply for logarithms:

$$\ln x + \ln y = \ln xy,\ \ln x - \ln y = \ln\left(\frac{x}{y}\right), \text{ and } \ln\left(a^b\right) = b\ln a.$$

We can also write a logarithm of any base using natural logarithms:

$$\log_b a = \frac{\ln a}{\ln b}.$$

e is also unique in that it is the only real number a satisfying the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x,$$

meaning e^x is its own derivative.

0.1.9 Partial Fractions

If we have a function of two polynomials $f(x) = \frac{P(x)}{Q(x)}$, it's often easier to break this quotient into a sum of parts where the denominator is a linear or quadratic factor and the numerator is always a smaller degree than the denominator.

Example.

$$\frac{2x-1}{x^3-6x^2+11x-6} = \frac{1/2}{x-1} + \frac{-3}{x-2} + \frac{5/2}{x-3}.$$

One natural way to find these small denominators comes from the linear factors of the denominator where we keep quadratics with complex roots. This way, when making a common denominator, we get back the original big denominator. However, there are a few special cases we have to take care of.

Linear Factors

This is the the most basic type where the degree of the numerator is less than the degree of the denominator and the denominator factors into all linear factors with no repeated roots. In this case we can write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \ldots + \frac{A_n}{(x-a_n)}.$$

Multiplying each side by $Q(x)$,

$$P(x) = A_1(x-a_2)\ldots(x-a_n) + \ldots + A_n(x-a_1)\ldots(x-a_{n-1}).$$

We can then find each A_i by evaluating both sides at $x = a_i$, since every term except the ith has an $(x-a_i)$ factor that will go to 0. So,

$$A_i = \frac{P(a_i)}{(x-a_i)\ldots(x-a_{i-1})(x-a_{i+1})\ldots(x-a_n)}.$$

Example. Find the partial fraction decomposition of the following expression:

$$\frac{2x-1}{x^3-6x^2+11x-6}.$$

Factoring,

$$x^3-6x^2+11x-6=(x-1)(x-2)(x-3).$$

So,

$$\frac{2x-1}{x^3-6x^2+11x-6}=\frac{A_1}{x-1}+\frac{A_2}{x-2}+\frac{A_3}{x-3}.$$

Multiplying each side by the denominator,

$$2x-1=A_1(x-2)(x-3)+A_2(x-1)(x-3)+A_3(x-1)(x-2).$$

At $x=1$,

$$1=A_1(1-2)(1-3) \implies A_1=\frac{1}{2}.$$

At $x=2$,

$$3=A_2(2-1)(2-3) \implies A_2=-3.$$

At $x=3$,

$$5=A_3(3-1)(3-2) \implies A_3=\frac{5}{2}.$$

So,

$$\frac{2x-1}{x^3-6x^2+11x-6}=\frac{1/2}{x-1}+\frac{-3}{x-2}+\frac{5/2}{x-3},$$

just as was shown in the previous example.

Repeated Linear Factors

If $Q(x)$ has repeated roots, it factors into

$$Q(x)=R(x)(x-a)^k,\ k\geq 2 \text{ and } R(a)\neq 0.$$

When making the common denominator for each repeated root of multiplicity k, we do

$$\frac{P(x)}{R(x)(x-a)^k}=(\text{Decomposition of } R(x))+\frac{A_1}{x-a}+\ldots+\frac{A_k}{(x-a)^k}.$$

You would then multiply each side by the denominator like in the linear factors case and solve for the coefficients. The only additional difficulty is that you might have to use previous results or solve a system of linear equations to get some of the constants.

Example. Find the partial fraction of the following expression:

$$\frac{x^2+5x-6}{x^3-7x^2+16x-12}.$$

Factoring,

$$x^3 - 7x^2 + 16x - 12 = (x-3)(x-2)^2.$$

So,

$$\frac{x^2+5x-6}{x^3-7x^2+16x-12} = \frac{A_1}{x-3} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2}.$$

Multiplying each side by the denominator,

$$x^2+5x-6 = A_1(x-2)^2 + A_2(x-2)(x-3) + A_3(x-3).$$

At $x=2$,

$$8 = A_3(2-3) \implies A_3 = -8.$$

At $x=3$,

$$18 = A_1(3-2)^2 \implies A_1 = 18.$$

Now we'll use our results for A_1 and A_3 to find A_2 using a value for x that isn't 2 or 3 so the A_2 term doesn't become 0. A good choice is $x=0$.
At $x=0$,

$$-6 = 18(0-2)^2 + A_2(0-2)(0-3) + -8(0-3) \implies A_2 = -17.$$

So,

$$\frac{x^2+5x-6}{x^3-7x^2+16x-12} = \frac{18}{x-3} - \frac{17}{x-2} - \frac{8}{(x-2)^2}.$$

Quadratic Factors

If a quadratic doesn't have real roots, then we have a quadratic factor. Here, we'll assume that the quadratic factor isn't repeated. So, $Q(x) = R(x)(ax^2+bx+c)$, $b^2-4ac < 0$, and $R(x)$ is not evenly divisible by ax^2+bx+c. In this case, we say

$$\frac{P(x)}{R(x)(ax^2+bx+c)} = (\text{Decomposition of } R(x)) + \frac{A_1x+B_1}{ax^2+bx+c}.$$

We then solve for the constants in the numerator, possibly having to solve a system of equations or using previous results and less convenient values for x.

Example. Find the partial fraction decomposition of the following expression:

$$\frac{6x^2+21x+11}{x^3+5x^2+3x+15}.$$

Factoring,

$$x^2 + 5x^2 + 3x + 15 = (x+5)(x^2+3).$$

So,

$$\frac{6x^2+21x+11}{x^3+5x^2+3x+15} = \frac{A_1}{x+5} + \frac{A_2x+B_2}{x^2+3}.$$

Multiplying each side by the denominator,

$$6x^2 + 21x + 11 = A_1(x^2 + 3) + (A_2x + B_2)(x + 5).$$

At $x = -5$,

$$56 = 28A_1 \implies A_1 = 2.$$

Now we'll use the previous result and another value for x. We can use $x = 0$ to not have to worry about the A_2 term. At $x = 0$,

$$11 = 2(3) + (B_2)(5) \implies B_2 = 1.$$

Now we'll use the previous 2 results to find A_2. $x = 1$ is a good choice to keep the numbers small. At $x = 1$,

$$38 = 2(1 + 3) + (A_2 + 1)(6) \implies A_2 = 4.$$

So,

$$\frac{6x^2 + 21x + 11}{x^3 + 5x^2 + 3x + 15} = \frac{2}{x + 5} + \frac{4x + 1}{x^2 + 3}.$$

Repeated Quadratic Factors

If a quadratic factor that can't be broken into linear factors is repeated, then we can write $Q(x) = R(x)(ax^2 + bx + c)^k$, $k \geq 0$, and $R(x)$ is not divisible by $(ax^2 + bx + c)^k$. Now we have to do a combination of what we did for repeated linear factors and quadratic factors. We say

$$\frac{P(x)}{R(x)(ax^2 + bx + c)^k} = (\text{Decomposition of } R(x)) + \frac{A_1x + B_1}{ax^2 + bx + c} + \ldots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}.$$

We then solve for the coefficients in the numerator.

Example. Find the partial fraction decomposition of $\frac{3x^4 - 2x^3 + 6x^2 - 3x + 3}{x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12}$.

Factoring,

$$x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12 = (x + 3)(x^2 + 2)^2.$$

So,

$$\frac{3x^4 - 2x^3 + 6x^2 - 3x + 3}{x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12} = \frac{A_1}{x + 3} + \frac{A_2x + B_2}{x^2 + 2} + \frac{A_3x + B_3}{(x^2 + 2)^2}.$$

Multiplying each side by the denominator,

$$3x^4 - 2x^3 + 6x^2 - 3x + 3 = A_1(x^2 + 2)^2 + (A_2x + B_2)(x^2 + 2)(x + 3) + (A_3x + B_3)(x + 3).$$

At $x = -3$,

$$363 = 121A_1 \implies A_1 = 3.$$

Now, we'll use our result for A_1 and pick a value for x that minimizes the number of things we need to solve for. We'll have to solve a linear system with 4 unknowns, so we'll need up to 4 values. At $x = 0$,

$$3 = 3(2)^2 + B_2(2)(3) + B_3(3) \implies 2B_2 + B_3 = -3.$$

At $x = 1$,

$$7 = 3(3)^2 + (A_2 + B_2)(3)(4) + (A_3 + B_3)(4) \implies 3A_2 + A_3 + 3B_2 + B_3 = -5.$$

At $x = -1$,

$$17 = 3(3)^2 + (-A_2 + B_2)(3)(2) + (-A_3 + B_3)(2) \implies -3A_2 - A_3 + 3B_2 + B_3 = -5.$$

At $x = 2$,

$$53 = 3(6)^2 + (2A_2 + B_2)(6)(5) + (2A_3 + B_3)(5) \implies 12A_2 + 2A_3 + 6B_2 + B_3 = -11.$$

Now we have the following system of equations:

$$\begin{cases} 0A_2 + 0A_3 + 2B_2 + B_3 & = -3 \\ 3A_2 + A_3 + 3B_2 + B_3 & = -5 \\ -3A_2 - A_3 + 3B_2 + B_3 & = -5 \\ 12A_2 + 2A_3 + 6B_2 + B_3 & = -11 \end{cases}.$$

Solving,

$$A_2 = 0, A_3 = 0, B_2 = -2, \text{ and } B_3 = 1.$$

So,

$$\frac{3x^4 - 2x^3 + 6x^2 - 3x + 3}{x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12} = \frac{3}{x+3} - \frac{2}{x^2+2} + \frac{1}{(x^2+2)^2}.$$

Improper Fractions

If the degree of the numerator is greater than or equal to the degree of the denominator, we have a case of improper fractions. In this case, we have to do polynomial long division to get a quotient and remainder and then decompose the remainder if necessary. So,

$$\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}.$$

Example. Find the partial fraction decomposition of the following expression:

$$\frac{x^3 + 3}{x^2 - 2x - 3}.$$

First we do polynomial long division to find that

$$\frac{x^3 + 3}{x^2 - 2x - 3} = x + 2 + \frac{7x + 9}{x^2 - 2x - 3}.$$

Now that the numerator is of a lesser degree than the denominator, we can decompose it normally.

$$x^2 - 2x - 3 = (x - 3)(x + 1).$$

So,

$$\frac{7x+9}{x^2-2x-3} = \frac{A_1}{x-3} + \frac{A_2}{x+1}.$$

Multiplying each side by the denominator,

$$7x+9 = A_1(x+1) + A_2(x-3).$$

At $x = -1$,

$$2 = -4A_2 \implies A_2 = \frac{-1}{2}.$$

At $x = 3$,

$$30 = 4A_1 \implies A_1 = \frac{15}{2}.$$

So,

$$\frac{x^3+3}{x^2-2x-3} = x+2+\frac{15/2}{x-3} + \frac{-1/2}{x+1}.$$

0.2 Single Variable Calculus

0.2.1 Derivatives and Integrals

Derivatives

The derivative of a function $y = f(x)$, notated $f'(x)$, gives the slope of the tangent line to f at x.

Definition.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Below are some properties of the derivative. Let f and g be functions of x and p a scalar.

Linearity

$$(pf \pm g)' = pf' \pm g'$$

Product Rule

$$(fg)' = f'g + fg'$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Chain Rule

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Power Rule

$$\frac{\mathrm{d}x^x}{\mathrm{d}p} = px^{p-1}, p \neq 0$$

Exponent Rule

$$\frac{\mathrm{d}x^x}{\mathrm{d}p} = p^x \ln p, p > 0$$

The Power Rule and Exponent Rule are two cases of the same rule

$$\frac{\mathrm{d}}{\mathrm{d}x} f^g = g f^{g-1} f' + f^g \ln f g'.$$

Using the definition of the derivative and these rules, we can find the derivatives to some common functions.

$$\begin{array}{r|c}
\frac{\mathrm{d}}{\mathrm{d}x} p = 0 & \frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x \\
\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x} & \frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \\
\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x & \frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x
\end{array}$$

Integrals

The definite integral of a function $f(x)$ from $x = a$ to $x = b$ where $a \leq b$ is the area between $f(x)$ and the x-axis bounded by the lines $x = a$ and $x = b$ where area above the x-axis is positive, and area below the x-axis is negative.

Definition.

$$\int_a^b f(x)\mathrm{d}x = \lim_{h \to 0} \sum_{n=1}^{\frac{b-a}{h}} f(a + (n-1)h) \cdot h.$$

We also define an indefinite integral, or antiderivative of $f(x)$, notated $F(x)$ where

$$F'(x) = f(x) \implies \int f(x)\mathrm{d}x = F(x).$$

Note that there are infinitely many such functions F, since adding a constant to F does not affect its derivative. To notate this, we add a constant C to the indefinite integral. Given an initial condition for f, we can solve for C.

Below are some properties of the integral. Let f and g be functions of x and p, a, b, and c where $a < b < c$, and f and g are continuous on the closed interval $[a, c]$.

Linearity

$$\int (pf \pm g)\mathrm{d}x = p \int f\mathrm{d}x \pm \int g\mathrm{d}x$$

Flipped Bounds

$$\int_a^b f\mathrm{d}x = -\int_b^a f\mathrm{d}x$$

Union of Intervals

$$\int_a^b f\mathrm{d}x + \int_b^c f\mathrm{d}x = \int_a^c f\mathrm{d}x$$

Power Rule

$$\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

U-Substitution

$$\int (f' \circ g)\, g'\mathrm{d}x = f \circ g + C$$

Integration by Parts

$$\int f'g\mathrm{d}x = fg - \int fg'\mathrm{d}x$$

Fundamental Theorem of Calculus

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(s)\mathrm{d}s = f(x)$$

Using the definition of the integral and the above rules, we can find the indefinite integral of some common functions.

$$\int \frac{1}{x}\mathrm{d}x = \ln\big|x\big| + C$$
$$\int \sin x\mathrm{d}x = -\cos x + C$$
$$\int \cos x\mathrm{d}x = \sin x + C$$
$$\int \tan x\mathrm{d}x = -\ln\big|\cos x\big| + C$$

0.2.2 Taylor Series

A Taylor series as a way of approximating a function about a point $x = a$ using polynomials. The first approximation just keeps the same value at $x = a$, the second approximation keeps the same value and first derivative at $x = a$, etc.

Definition.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

If we approximate a function about $x = 0$, we call this a Maclaurin series. Below are some common Maclaurin series, and their radii of convergence if applicable.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
\frac{1}{1+x} &= 1 - x + x^2 - \dots, \text{ where } \left|x\right| < 1 \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \text{ where } \left|x\right| < 1
\end{aligned}$$

Euler's Identity

Let's see what happens when we look at the Maclaurin series for e^{ix}.

$$\begin{aligned}
e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \dots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \dots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right).
\end{aligned}$$

The two expressions in parenthesis are exactly the Maclaurin series for $\cos x$ and $\sin x$. So,

$$e^{ix} = \cos x + i \sin x.$$

In the case that $x = \pi$,

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0.$$

So,

$$e^{i\pi} + 1 = 0.$$