TODO: [Nullity 2 *Lights Out* Boards]

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1 Intro

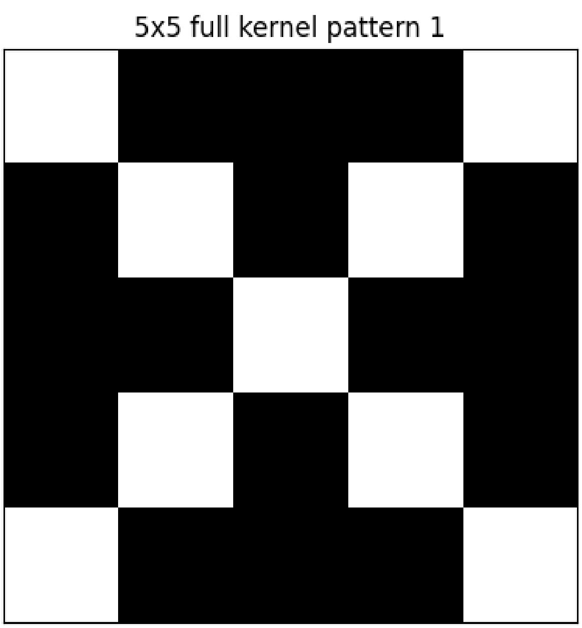
- What *Lights Out* is
- What quiet patterns are
- Certain board sizes require n^2 clicks because they have no quiet patterns
- Result: For boards where the space of quiet patterns has dimension 2, we can solve this problem exactly.

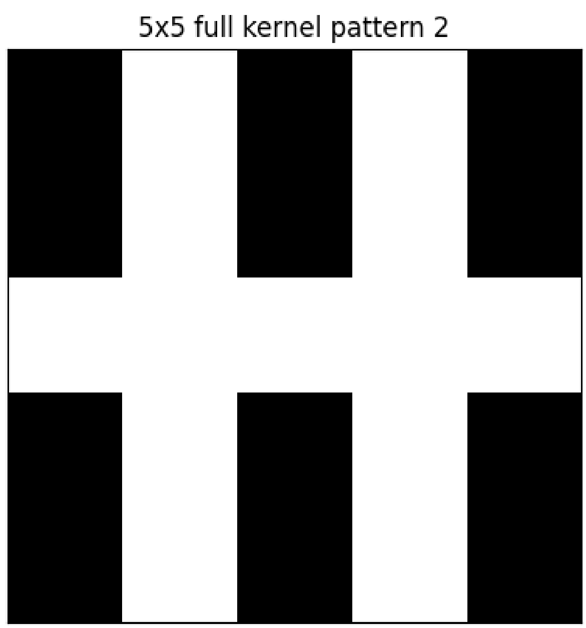
2 Solving a 5 x 5

The 5×5 board is the first example with nullity 2.

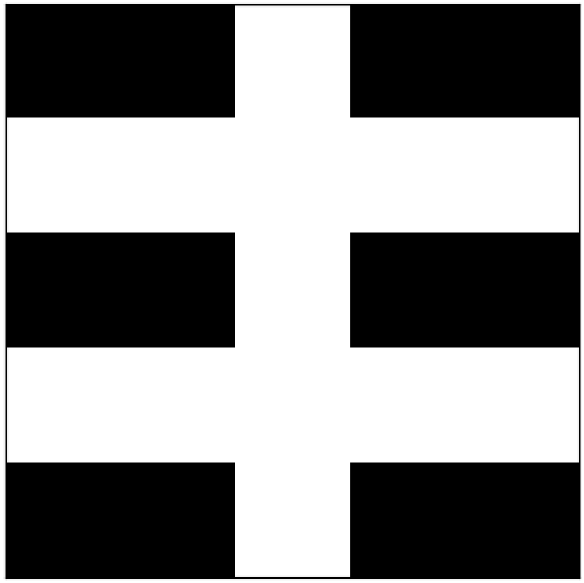
Lemma 1. 5×5 *board that is solvable can be done so in no more than 15 clicks.*

Proof. Below are the three non-trivial quiet patterns for the 5×5 board.



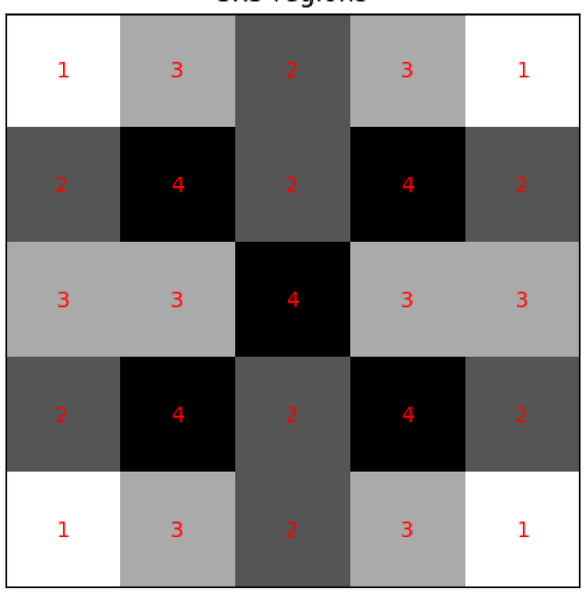


5x5 full kernel pattern 3



We can divide the squares of the 5×5 board into four regions based on which quiet patterns they are a part of.

5x5 regions



Region 1 is the intersection of quiet patterns 2 and 3; region 2 is the intersection of quiet patterns 1 and 2; region 3 is the intersection of quiet patterns 1 and 3; region 4 is in non of the quiet patterns.

Assume that we have a solvable board that requires the maximum number of clicks needed to solve optimally. Let A be the number of clicks in the solution in region 1, B in region 2, C in region 3, and D in region 4. Then the solution uses $A + B + C + D$ clicks.

If we apply quiet pattern 1, we will get an equivalent solution that uses $A + (8 - B) + (8 - C) + D$ clicks.

Since we assumed the solution we had to begin with was minimal,

$$A + B + C + D \leq A + (8 - B) + (8 - C) + D.$$

Rearranging,

$$B + C \leq 8. \tag{1}$$

If we apply quiet pattern 2, we will get an equivalent solution that uses $(4-A)+(8-B)+C+D$ clicks. Since we assumed the solution we had to begin with was minimal,

$$A + B + C + D \leq (4 - A) + (8 - B) + C + D.$$

Rearranging,

$$A + B \leq 6. \tag{2}$$

If we apply quiet pattern 3, we will get an equivalent solution that uses $(4-A)+B+(8-C)+D$ clicks. Since we assumed the solution we had to begin with was minimal,

$$A + B + C + D \leq (4 - A) + B + (8 - C) + D.$$

Rearranging,

$$A + C \leq 6. \tag{3}$$

In all these constraints we have derived, D is not contrained beyond region 4 containing five buttons. Thus, in any board that requires the maximum number of clicks to optimally solve, $D = 5$.

Putting all of these constraints together in matrix form for A, B, and C,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \leq \begin{bmatrix} 8 \\ 6 \\ 6 \end{bmatrix}. \tag{4}$$

Clearly, if these constraints were tight with A, B, and C as integers, than we have a maximal optimal solution. Row reducing, we can see that

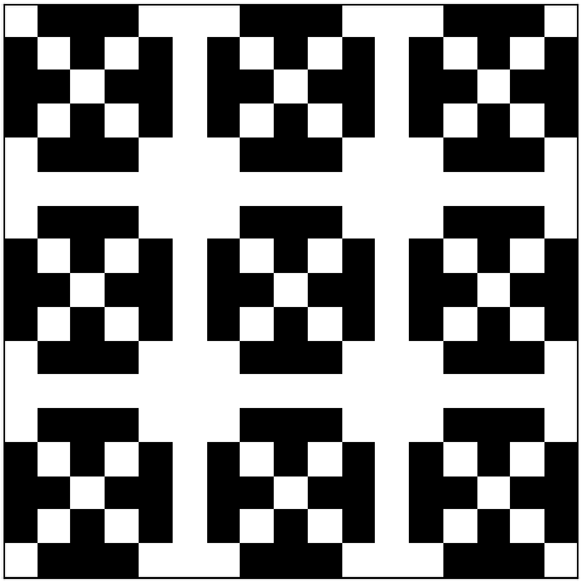
$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

is the solution to the tight constraints and has all variables as integers like required. Thus, it describes a maximal optimal solution. So, a maximal optimal solution uses no more than $A+B+C+D = 2+4+4+5 = 15$ clicks. $\square$

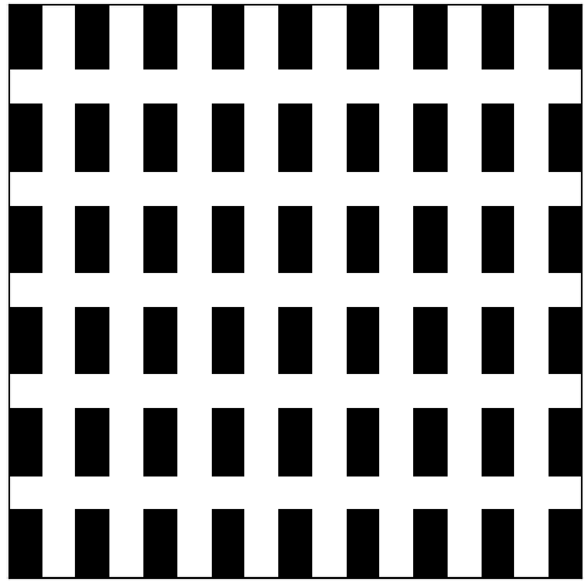
3 Extending to Larger Boards

The 5×5 kernel patterns have a nice property that because that the same buttons are pushed on the top/bottom and left/right edges, they tile to make kernel patterns on larger boards, specifically those of size $6k + 5 \times 6k + 5$, where k is an integer at least 0. For example, notice how the kernel patterns on the 17×17 board ($k = 2$) are the same patterns from the 5×5 board repeated 3 times in the vertical and horizontal directions.

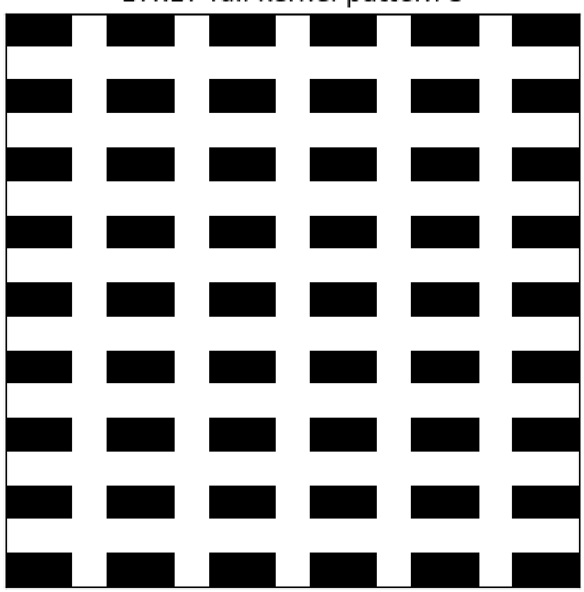
17x17 full kernel pattern 1



17x17 full kernel pattern 2



17x17 full kernel pattern 3



Thus, we can apply the same proof technique from the 5 board to get an upper bound of all boards of size $6k + 5 \times 6k + 5$.

Theorem 1. *Let k be an integer at least 0. Then a solvable $6k + 5 \times 6k + 5$ board can be solved in no more than $21k^2 + 40k + 15$ clicks.*

Proof. As shown above, the three non-trivial quiet patterns of a 5×5 board will tile a $6k + 5 \times 6k + 5$ board. Thus, we can divide the larger board into the same four regions where region 1 has $4(k + 1)^2$ lights, regions 2 and 3 both have $8(k + 1)^2$ lights, and region 4 has the remaining $5(k + 1)^2 + 11k^2 + 10k$ lights. Applying the three non-trivial quiet patterns and rearranging into matrix form, we get that D is unconstrained to

$5(k+1)^2 + 11k^2 + 10k$, and

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \le \begin{bmatrix} 8 \\ 6 \\ 6 \end{bmatrix} (k+1)^2. \tag{5}$$

So,

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} (k+1)^2.$$

Therefore, $A+B+C+D = 2(k+1)^2+4(k+1)^2+4(k+1)^2+5(k+1)^2+11k^2+10k = 15(k+1)^2+11k^2+10k = 26k^2+40k+15$. □

We can now establish some corollaries

Corollary 1. *All boards of size* $6k+5 \times 6k+5$ *have nullity at least 2.*

Proof. We've already shown that the three tiling quiet patterns much exist on all boards of size $6k+5\times 6k+5$. The third quiet pattern is the sum for the other two, so these quiet patterns form a space of dimension 2. □

Corollary 2. *This bound is tight for boards with nullity 2.*

Proof. If the board in question only has a null space of dimension 2, then the upper bound proof fully describes all possible equivalent solutions. □

The 5×5, 17×17, 41×41, 53×53, 77×77, and 113×113 boards are the boards of size $6k+5$ with nullity 2. Thus applying the corollary, they can be solved in no more than 15, 199, 1191, 1999, 4239, and 9159 clicks.

Corollary 3. *Let $d(n)$ be the nullity of an $n \times n$ board. Then for all integers $a \ge 2$ and $k \ge 0$,*

$$d(ak+a-1) \ge d(a-1).$$

Proof. Choose any quiet pattern on an $a-1 \times a-1$ board. Reflect the pattern k times horizontally and vertically, adding width one spaces between each reflected board. This will form a quiet pattern of size $ak+a-1 \times ak+a-1$. We can do this reflection operation with all $d(a-1)$ independent quiet patterns to form $d(a-1)$ independent quiet patterns on the larger boards. □

4 Conjectures

Conjecture 1. *If an $n \times n$ board has nullity 2, then $n \equiv 5 \mod 6$.*

Although we only examined the 5 modulo 6 case here, we can convert all board sizes into an integer programming problem. This does not bode well for being able to find solutions to arbitrary boards.

Conjecture 2. *The proof technique of assuming our integer program can be solved with all constraints tight will only work for boards with nullity 2, none higher.*

This isn't really a formal conjecture that can be proven correct or incorrect, but

Conjecture 3. *If we do manage to solve other nullity boards in a generalized (non brute-force) way like our approach for the 5x5, we'll be able to use a similar tiling argument to establish upper bounds.*