The Rank Deficiency of Certain Sized *Lights Out* Boards

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1 Intro Conjectures

Let $f(n,x)$ be the Chebyshev polynomial over $GF(2)$ that we defined previously. Recall that the rank deficiency of an $n \times n$ *Lights Out* board, $d(n)$, is the degree of $\gcd(f(n,x), f(n,x+1))$. Let $g : \mathbb{N} \to \mathbb{N}$ where $g(k) = 2^{k+1} + 2^{k-1} - 1$.

Conjecture 1. *Let $k \in \mathbb{N}$. Then*

$$f(g(k),x) = x^{2^{k-1}-1}\left(x^{2^{k+1}} + x^{2^k} + 1\right).$$

Conjecture 2. *Let $k \in \mathbb{N}$. Then*

$$f(g(k),x+1) = \left(x^{2^{k+1}} + x^{2^k} + 1\right)\left(x^{2^{k-1}-1} + \cdots + 1\right).$$

2 Main Result

The following relies on 1 and 2 being true.

Theorem 1. *Let $k \in \mathbb{N}$. Then*

$$\gcd\left(f(g(k),x), f(g(k),x+1)\right) = x^{2^{k+1}} + x^{2^k} + 1.$$

Proof. We can see from conjectures 1 and 2 that the desired gcd is a common factor of both polynomials. So, we just need to show that over $GF(2)$ that the remaining factors of both polynomials have no common factors. This result is easily verified by a computer.

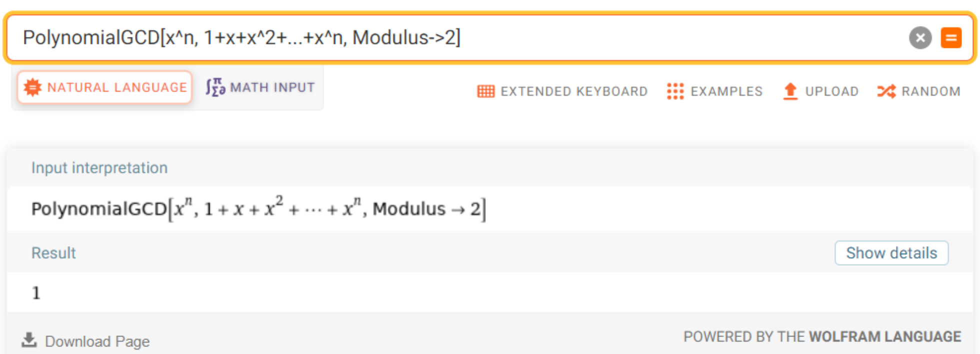

Figure 1: Wolfram|Alpha: PolynomialGCD[$x^n, 1 + x + x^2 + ... + x^n$, Modulus→2]

$\square$

Corollary 1. *A Lights Out board of size $g(k) \times g(k)$ will have nullity 2^{k+1}.*

3 Concluding Conjectures

Conjecture 3. *Let* $h : \mathbb{N} \to \mathbb{N}$ *where*

$$h(n) = \max\{g(m) \mid m \in \mathbb{N}, g(m) \leq n\}.$$

Then for all $n \in \mathbb{N}$,

$$\max\{d(m) \mid 1 \leq m \leq n\} = d(h(n)).$$

Corollary 2. $d(n) = n$ *only for* $n = 4$. *Otherwise,* $d(n) < n$.

Proof. Observe that $d(1) = d(2) = d(3) = 0$, and $d(g(1)) = d(4) = 4$.

Assume for contradiction that there exists some $n > 4$ such that $d(n) \geq n$.
Notice that for $k > 1$,

$$d(g(k)) = 2^{k+1} < g(k) = 2^{k+1} + 2^{k-1} - 1.$$

So,

$$d(h(n)) < h(n) \leq n = d(n).$$

However,

$$\max\{d(m) \mid 1 \leq m \leq n\} = d(n) > d(h(n)),$$

which contradicts conjecture 3. □