Eigenvalues of Square *Lights Out* Games

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1 Definitions

Definition 1. *The nth Lights Out matrix with diagonal k is the $n^2 \times n^2$ block matrix of the form*

$$M_{n,k} = \begin{bmatrix} D_{n,k} & I_n & O_n & O_n & O_n & \dots & O_n \\ I_n & D_{n,k} & I_n & O_n & O_n & \dots & O_n \\ O_n & I_n & D_{n,k} & I_n & O_n & \dots & O_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ O_n & \dots & O_n & I_n & D_{n,k} & I_n & O_n \\ O_n & \dots & O_n & O_n & I_n & D_{n,k} & I_n \\ O_n & \dots & O_n & O_n & O_n & I_n & D_{n,k} \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix, O_n is the $n \times n$ zero matrix, and $D_{n,k}$ is the tridiagonal $n \times n$ matrix with k along the main diagonal, 1 above and below the main diagonal, and 0 elsewhere.

$$D_{n,k} = \begin{bmatrix} k & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & k & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & k & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & k & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & k & 1 \\ 0 & \dots & 0 & 0 & 0 & 1 & k \end{bmatrix}.$$

$k \in \mathbb{N}$.

Definition 2. *The inner product of two vectors in $\mathbb{Z}_2^{n^2}$ is the result of their standard inner product in $\mathbb{R}^{n^2} \mod 2$.*

Definition 3. *A vector $\vec{v}$ in $\mathbb{R}^{n^2}$ is convertible to a vector in $\mathbb{Z}_2^{n^2}$ if there exists a real, non-zero constant c such that $c\vec{v}$ has all integer components. The converted vector is $\vec{v_2} = c\vec{v} \mod 2$.*

2 Some Lemmas

Below are some lemmas that will be useful in proving our main theorems and their corollaries. They are in this section rather than immediately before where they are first applied because they are applicable outside the specific context of *Lights Out*.

Lemma 1. *All eigenvectors with rational eigenvalues of a matrix with all rational entries are convertible.*

Proof. Let A be a square matrix with all rational entries. Let λ be an eigenvalue of A. Consider the process of finding eigenvectors from a rational eigenvalue. One would do elementary row operations on the homogeneous system $(A - \lambda I)\vec{v} = \vec{0}$. Since λ is rational, all of the entries in A are also rational, and elementary row operations do not introduce irrational numbers, then the resultant vector has all rational entries. Let

$$\vec{v} = \left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_{n^2}}{q_{n^2}}\right)^{\mathrm{T}}$$

be the resultant eigenvector, where each p_i/q_i is a fraction of integers in least terms. Then multiplying $\vec{v}$ by $k = \operatorname{lcm}(q_1, q_2, \ldots, q_{n^2})$ gives a vector with all integer components, so $\vec{v}$ is convertible. $\square$

Within the context of *Lights Out*, this means all eigenvectors of $M_{n,k}$ with rational eigenvalues are convertible. Eigenvectors with irrational eigenvalues may still be convertible if the ratio of any two components of the eigenvector is rational.

Lemma 2. *If λ is an eigenvalue of a matrix A, then $\lambda - c$ is an eigenvalue of $B = A - cI$.*

Proof. Let λ be an eigenvalue of A with corresponding eigenvector $\vec{v}$. Then

$$B\vec{v} = (A - cI)\vec{v} = A\vec{v} - cI\vec{v} = \lambda\vec{v} - c\vec{v} = (\lambda - c)\,\vec{v}.$$

Thus, $\lambda - c$ is an eigenvalue of B with eigenvector $\vec{v}$. $\square$

Lemma 3. *Let p and q be rational numbers where $0 < p < q < 1$. Then there are no solutions to*

$$\cos(\pi p) + cos(\pi q) = 1.$$

Proof. Professor Jason Bell at the University of Waterloo proved in this Reddit thread that for any rational number C where

$$\cos(\pi p) + cos(\pi q) = C,$$

that N, the common denominator of p and q, is such that $\phi(N) < 8/|C|$ where ϕ is Euler's totient function. Taking this bound, I have checked via computer program that the only solution for $C = 1$ is

$$\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) = 1.$$

Since this would make $p = q = 1/3$, and $p < q$, so there are indeed no solutions. $\square$

We can apply the same techniques to show that for $C = -1$, the only solutions (allowing $p = q$) are $(p, q) = (2/3, 2/3)$; for $C = 1/2$, the only solutions are $(p, q) = (1/3, 1/2)$ and $(1/5, 3/5)$; for $C = -1/2$, the only solutions are $(p, q) = (1/2, 2/3)$ and $(2/5, 4/5)$. What's curious is that these solutions and the one given in lemma 3 appear to be the only solutions, as conjectured in conjecture 4.

3 Normal *Lights Out*: $k = 1$

We will start by looking at the traditional *Lights Out* game ($k = 1$) and then generalize the results to all $k \in \mathbb{N}$. $M_{n,k}$ models the effect of clicking some of the n^2 buttons on an $n \times n$ *Lights Out* board. In the on or off context of *Lights Out* lights, it is a endomorphism over $\mathbb{Z}_2^{n^2}$, but we can still analyze it as a endomorphism over $\mathbb{R}^{n^2}$. For example, since M_n is real and symmetric, all of its n^2 eigenvalues and the components of its eigenvectors are real. Since the components of all vectors in $\mathbb{Z}_2^{n^2}$ are either 0 or 1, any eigenvectors of $M_{n,k}$ in $\mathbb{Z}_2^{n^2}$ would have an eigenvalue of 0 or 1. These are the most important eigenpairs in the context of *Lights Out*.

Lemma 4. *The eigenvalues of* $D_{n,1}$ *are*

$$\lambda_i = 1 + 2\cos\left(\frac{i}{n+1}\pi\right), \quad 1 \le i \le n.$$

Proof. p_n, the characteristic polynomial of $D_{n,1}$ satisfies the recurrence relation

$$p_n = \begin{cases} 1 & n = 0 \\ 1 - \lambda & n = 1 \\ (1-\lambda)p_{n-1} - p_{n-2} & n \ge 2 \end{cases}.$$

Letting $2x = 1 - \lambda$, our recurrence relation becomes the definition of a Chebyshev polynomial of the second kind, $U_n(x)$. It's well-know that the roots of $U_n(x)$ are

$$x_i = \cos\left(\frac{i}{n+1}\pi\right), \quad 1 \le i \le n.$$

Since $\lambda = 1 - 2x$, we see that the roots of the characteristic polynomial of D_n are

$$\lambda_i = 1 - 2\cos\left(\frac{i\pi}{n+1}\right), \quad 1 \le i \le n.$$

Since

$$1 - 2\cos\left(\frac{i\pi}{n+1}\right) = 1 + 2\cos\left(\frac{n-i+1}{n+1}\pi\right),$$

and $n - i + 1$ has the same bounds as i, so we can write the eigenvalues as

$$\lambda_i = 1 + 2\cos\left(\frac{i}{n+1}\pi\right), \quad 1 \le i \le n.$$

□

Theorem 1. *The eigenvalues of* $M_{n,1}$ *are are given by*

$$\lambda_{i,j} = \left(1 + 2\cos\left(\frac{i}{n+1}\pi\right)\right) + \left(2\cos\left(\frac{j}{n+1}\pi\right)\right), \quad 1 \leq i, j \leq n.$$

Proof. Since $M_{n,1}$ is a block matrix with similar structure to $D_{n,1}$, we can write it as a Kronecker product.

$$M_{n,1} = D_{n,1} \otimes I_n + I_n \otimes (D_{n,1} - I_n).$$

In this case, Kronecker products have the property that the eigenvalues of M_n are the sums of pairs of eigenvalues of $D_{n,1}$ and $D_{n,1} - I_n$. By lemma 2 and lemma 4, we get that the eigenvalues of $M_{n,1}$ are

$$\lambda_{i,j} = \left(1 + 2\cos\left(\frac{i}{n+1}\pi\right)\right) + \left(2\cos\left(\frac{j}{n+1}\pi\right)\right), \quad 1 \leq i, j \leq n.$$

□

4 Generalization: $k \in \mathbb{N}$

We are now ready to generalize theorem 1 and make some interesting corollaries.

Theorem 2. *The eigenvalues of* $M_{n,k}$ *are given by*

$$\lambda_{i,j} = \left(k + 2\cos\left(\frac{i}{n+1}\pi\right)\right) + \left(2\cos\left(\frac{j}{n+1}\pi\right)\right), \quad 1 \leq i, j \leq n.$$

Proof.

$$M_{n,k} = M_{n,1} + (k-1)I.$$

Applying lemma 2 to the results of theorem 1, we get the eigenvalues as

$$\lambda_{i,j} = \left(k + 2\cos\left(\frac{i}{n+1}\pi\right)\right) + \left(2\cos\left(\frac{j}{n+1}\pi\right)\right), \quad 1 \leq i, j \leq n. \tag{1}$$

□

We will focus on integer eigenvalues. Our goal as we work through these corollaries is to understand the structure and symmetries of the eigenvalues to the point where we can give a full description of where and under what conditions integer eigenvalues occur.

Corollary 1.

$$\lambda_{i,j} = \lambda_{j,i}.$$

Proof.

$$\begin{aligned}\lambda_{i,j} &= k + 2\cos\left(\frac{i}{n+1}\pi\right) + 2\cos\left(\frac{j}{n+1}\pi\right)\\ &= k + 2\cos\left(\frac{j}{n+1}\pi\right) + 2\cos\left(\frac{i}{n+1}\pi\right)\\ &= \lambda_{j,i}.\end{aligned}$$

□

Although this corollary is certainly the simplest, it describes an important symmetry in the eigenvalues of $M_{n,k}$ that we will use repeatedly to prove more interesting results. If the eigenvalues of $M_{n,k}$ were laid out on an $n \times n$ matrix according to their i and j values – $(i, j) = (1, 1)$ in the upper left and (n, n) in the bottom right – then this result tells us that the matrix is real and symmetric about the major diagonal.

Corollary 2. *The characteristic polynomial of $M_{n,k}$ has a root of multiplicity n at $\lambda = k$.*

Proof. Rearranging equation (1) to use multiplication instead of addition,

$$\cos\left(\frac{i+j}{n+1}\cdot\frac{\pi}{2}\right)\cos\left(\frac{i-j}{n+1}\cdot\frac{\pi}{2}\right) = \frac{k-\lambda_{i,j}}{4}, \quad 1 \le i, j \le n. \tag{2}$$

Substituting $\lambda_{i,j} = k$,

$$\begin{aligned}\cos\left(\frac{i+j}{n+1}\cdot\frac{\pi}{2}\right)\cos\left(\frac{i-j}{n+1}\cdot\frac{\pi}{2}\right) &= 0, \quad 1 \le i, j \le n\\ \implies \cos\left(\frac{i+j}{n+1}\cdot\frac{\pi}{2}\right) &= 0, \quad 1 \le i, j \le n\end{aligned}$$

because the second cos term is never 0 for our bounds on i and j.

$$\implies i + j = n + 1, \quad 1 \le i, j \le n$$

giving exactly n solutions. □

We've now added some more detail to our matrix of eigenvalues. Along the minor diagonal, from bottom left to top right, all eigenvalues are k. This hints at some sort of symmetry about the minor diagonal, which we will address next.

Corollary 3.

$$\lambda_{i,j} + \lambda_{n-i+1,n-j+1} = 2k.$$

Proof.

$$\begin{aligned}
&\lambda_{i,j} + \lambda_{n-i+1,n-j+1} \\
&= k + 2\cos\left(\frac{i}{n+1}\pi\right) + 2\cos\left(\frac{j}{n+1}\pi\right) + k + 2\cos\left(\frac{n-i+1}{n+1}\pi\right) + 2\cos\left(\frac{n-j+1}{n+1}\pi\right) \\
&= 2k + 2\cos\left(\frac{i}{n+1}\pi\right) + 2\cos\left(\frac{n-i+1}{n+1}\pi\right) + 2\cos\left(\frac{j}{n+1}\pi\right) + 2\cos\left(\frac{n-j+1}{n+1}\pi\right) \\
&= 2k + 2\cos\left(\frac{i}{n+1}\pi\right) + 2\cos\left(\pi - \frac{i}{n+1}\pi\right) + 2\cos\left(\frac{j}{n+1}\pi\right) + 2\cos\left(\pi - \frac{j}{n+1}\pi\right) \\
&= 2k + 2\cos\left(\frac{i}{n+1}\pi\right) - 2\cos\left(\frac{i}{n+1}\pi\right) + 2\cos\left(\frac{j}{n+1}\pi\right) - 2\cos\left(\frac{j}{n+1}\pi\right) \\
&= 2k
\end{aligned}$$

□

Note how this result is compatible with what we know about $M_{n,k}$ and the sum of eigenvalues for any real matrix: $\mathrm{tr}(M_{n,k}) = kn^2$, and the sum of all eigenvalues as given by the above result is also kn^2. This result also is compatible with corollary 2 because when $i + j = n + 1$, $\lambda_{i,j} = \lambda_{n-i+1,n-j+1} = k$.

This result also solidifies the symmetry we first got a glimpse of in corollary 2. If we know the value of an eigenvalue on the upper left side of the minor diagonal, then we can easily find the corresponding eigenvalue on the lower right side of the minor diagonal. These two corresponding eigenvalues have much in common. For example, if one is an integer of a certain parity (odd or even), then the other is also an integer with the same parity.

Next, we'll bound from above and below the possible eigenvalues of $M_{n,k}$, which will also bound the possible number of integer eigenvalues that could possibly occur in our eigenvalue matrix.

Corollary 4.

$$k - 4 < \lambda_{i,j} < k + 4.$$

Proof. Looking at equation (2), we can see that $\lambda_{i,j}$ is maximized when i and j are minimized. The smallest values for i and j are $i = j = 1$. Thus,

$$\max \lambda_{i,j} = k + 4\cos\left(\frac{\pi}{n+1}\right).$$

This value approaches $k + 4$ as n grows large, but is always less for any finite n. Corollary 3 tells us this eigenvalue must have a corresponding one that sums to $2k$, so

$$\min \lambda_{i,j} = 2k - \max \lambda_{i,j},$$

which approaches $k - 4$ as n grows large, but is always greater for any finite n. □

We've now shown that $M_{n,k}$ can have at most 7 distinct integer eigenvalues: $k-3$, $k-2$, $k-1$, k, $k+1$, $k+2$, and $k+3$. We've already given a description of when eigenvalue k occurs. All of the others occur in pairs that sum to $2k$, meaning one member of the pair is on the upper left side of the minor diagonal, and the other member of the pair is on the lower-right side of the diagonal. Next, we'll bound the number of integer eigenvalues other than k that can occur.

Corollary 5. *Counting multiplicity, $M_{n,k}$ has at most 10 integer eigenvalues other than k.*

Proof. If $\lambda_{i,j}$ is an integer not equal to k, then corollary 4 tells us that

$$\lambda_{i,j} \in \{k-3, k-2, k-1, k+1, k+2, k+3\}$$
$$\cos\left(\frac{i}{n+1}\pi\right) + \cos\left(\frac{j}{n+1}\pi\right) \in \left\{-\frac{3}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}\right\}.$$

Using the method and notation described in lemma 3 on each of these values, we get the the only solutions are

$$(p,q,C) \in \left\{\left(\frac{2}{3},\frac{2}{3},-1\right), \left(\frac{1}{2},\frac{2}{3},-\frac{1}{2}\right), \left(\frac{2}{5},\frac{4}{5},-\frac{1}{2}\right), \left(\frac{1}{3},\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{5},\frac{3}{5},\frac{1}{2}\right), \left(\frac{1}{3},\frac{1}{3},1\right)\right\},$$

where $0 < p \leq q < 1$. This list already counts pairs occurring across the minor diagonal of the eigenvalues matrix as described in corollary 3, but it does not consider that swapping p and q also gives a solution, which corresponds to swapping i and j in equation (1). If $p \neq q$, each solution corresponds to two distinct solutions, and if $p = q$, then each solution corresponds to only one solution. Thus, we have $1+2+2+2+2+1 = 10$ solutions that could result in an integer eigenvalue. □

Not only does this result show that there can never be more than 10 integer eigenvalues in $M_{n,k}$, no matter n, it also shows that certain integer eigenvalues, although they are in the range of possibility given in corollary 4, never occur. Namely, $k-3$ and $k+3$ are never integer eigenvalues of $M_{n,k}$.

The next few corollaries will describe for what n certain integer eigenvalues occur. This will culminate in a complete description of for which n certain integer eigenvalues occur as well as describing the relationship between the eigenvalues for different n's.

Corollary 6. *$M_{n,k}$ has two eigenvalues equivalent to $k \mod 2$ other than k if and only if $n \equiv 2 \mod 3$.*

Proof. We'll start by looking for which i, j, and n $\lambda_{i,j} = k+2$ and $i+j < n+1$. Finding one eigenvalue with these constraints implies the existence of another where $i+j > n+1$, as shown in corollary 3. As we touched on in proving corollary 4, $\lambda_{i,j}$ decrease as i and j increase. In fact,

$$\lambda_{i,j} > k \Leftrightarrow i+j < n+1, \;\; \lambda_{i,j} < k \Leftrightarrow i+j > n+1.$$

So, $k+2$ is the only possible eigenvalue equivalent to $k \mod 2$ that could appear within our constraints.

$$k + 2\cos\left(\frac{i}{n+1}\pi\right) + 2\cos\left(\frac{j}{n+1}\pi\right) = k+2, \quad 1 \leq i, j \leq n$$
$$\cos\left(\frac{i}{n+1}\pi\right) + \cos\left(\frac{j}{n+1}\pi\right) = 1, \quad 1 \leq i, j \leq n.$$

As shown in the proof of lemma 3 and restated in the proof of corollary 5, the only solution is

$$\frac{i}{n+1} = \frac{j}{n+1} = \frac{1}{3} \implies i = j = \frac{n+1}{3}.$$

Since i and j are integers, $n \equiv 2 \mod 3$. □

Thinking back to what we found in corollaries 4 and 2, it makes sense that $i = j$. Since entries across from each other on the minor diagonal have the same parity, there must be exactly one eigenvalue in the upper left half of the eigenvalue matrix and another in the lower right half. Since values are the same across the major diagonal, the only way for there to be one such value in each half is for it to occur on the major diagonal. As we will see all other non-k integer eigenvalues of the same parity occur in fours: one in each "quadrant" with the major and minor diagonals as "axes".

Corollary 7. *If $n \equiv 4 \mod 5$, then $M_{n,k}$ has four integer eigenvalues equivalent to $k+1 \mod 2$.*

Proof. Let $n = 5m - 1$ where m is an integer greater than or equal to 1.

$$\begin{aligned}
\lambda_{3m,m} &= k + 2\cos\left(\frac{3m}{(5m-1)+1}\pi\right) + 2\cos\left(\frac{3m}{(5m-1)+1}\pi\right) \\
&= k + 2\cos\left(\frac{3}{5}\pi\right) + 2\cos\left(\frac{1}{5}\pi\right) \\
&= k+1.
\end{aligned}$$

Also,

$$\lambda_{3m,m} = \lambda_{m,3m} = k+1, \;\; \lambda_{2m,4m} = \lambda_{4m,2m} = k-1$$

as demonstrated in corollary 3. So, $\lambda_{3m,m}$, $\lambda_{m,3m}$, $\lambda_{2m,4m}$, and $\lambda_{4m,2m}$ are the four integer eigenvalues equivalent to $k+1 \mod 2$ when $n \equiv 4 \mod 5$. Since m is always greater than or equal to 1, these eigenvalues are distinct. □

This result describes one of the two possible ways that $M_{n,k}$ can have integer eigenvalues equivalent to $k+1 \mod 2$. Note that since it is possible for $n \equiv 4 \mod 5$ and $n \not\equiv 2 \mod 3$, $M_{n,k}$ can have exactly four integer eigenvalues other than k, all equivalent to $k+1 \mod 2$.

Corollary 8. *If $n \equiv 5 \mod 6$, then $M_{n,k}$ has four integer eigenvalues equivalent to $k+1 \mod 2$.*

Proof. Let $n = 6m - 1$ where m is an integer greater than or equal to 1.

$$\begin{aligned}\lambda_{2m,3m} &= k + 2\cos\left(\frac{2m}{(6m-1)+1}\pi\right) + 2\cos\left(\frac{3m}{(6m-1)+1}\pi\right)\\ &= k + 2\cos\left(\frac{1}{3}\pi\right) + 2\cos\left(\frac{1}{2}\pi\right)\\ &= k+1.\end{aligned}$$

Also,

$$\lambda_{2m,3m} = \lambda_{3m,2m} = k+1, \;\; \lambda_{3m,4m} = \lambda_{4m,3m} = k-1$$

as demonstrated in corollary 3. So, $\lambda_{2m,3m}$, $\lambda_{3m,2m}$, $\lambda_{3m,4m}$, $\lambda_{4m,3m}$ are the four integer eigenvalues equivalent to $k+1 \mod 2$ when $n \equiv 5 \mod 6$. Since m is always greater than or equal to 1, these eigenvalues are distinct. $\square$

Note that if $n \equiv 5 \mod 6$, then $n \equiv 2 \mod 3$, meaning there are at least six integer eigenvalues other than k. These eigenvalues are also distinct from those also equivalent to $k+1 \mod 2$ in corollary 7.

With all these corollaries complete, we can now fully describe where and under what conditions all integer eigenvalues of $M_{n,k}$ occur.

Corollary 9. *k is an eigenvalue of $M_{n,k}$ with multiplicity n. Let h be the number of integer eigenvalues of $M_{n,k}$ other than k.*

$$h = \begin{cases} 0 \Leftrightarrow & n \not\equiv 2 \mod 3 \textit{ and } n \not\equiv 4 \mod 5 \textit{ and } n \not\equiv 5 \mod 6\\ 2 \Leftrightarrow & n \equiv 2 \mod 3 \textit{ and } n \not\equiv 4 \mod 5 \textit{ and } n \not\equiv 5 \mod 6\\ 4 \Leftrightarrow & n \not\equiv 2 \mod 3 \textit{ and } n \equiv 4 \mod 5 \textit{ and } n \not\equiv 5 \mod 6\\ 6 \Leftrightarrow & n \equiv 2 \mod 3 \textit{ and } n \equiv 4 \mod 5 \textit{ and } n \not\equiv 5 \mod 6\\ \quad \textit{OR} & n \equiv 2 \mod 3 \textit{ and } n \not\equiv 4 \mod 5 \textit{ and } n \equiv 5 \mod 6\\ 10 \Leftrightarrow & n \equiv 2 \mod 3 \textit{ and } n \equiv 4 \mod 5 \textit{ and } n \equiv 5 \mod 6 \end{cases}$$

Proof. We've already proven that k is an eigenvalue of $M_{n,k}$ with multiplicity n in corollary 2. As we've already noted, some of the conditions are redundant. We've also already proven the $h = 2$ and $h = 10$ cases in both directions in corollaries 6 and 5 respectively. Applying the method of lemma 3 exactly as we did in corollary 5, we can see that corollaries 6, 7, and 8 establish strict bounds, meaning these corollaries describe all possible solutions for integer eigenvalues. $\square$

Using the results and symmetries we found working toward describing integer eigenvalues, we can also describe the relationship between the eigenvalues of different n's.

Corollary 10. *If the dimension of some eigenspace for $M_{n,k}$, V_λ, is d, then the dimension of the same eigenspace for $M_{2n+1,k}$ is $\geq d$.*

Proof. Let λ be an eigenvalue of $M_{n,k}$. If V_λ has dimension d for $M_{n,k}$, then there exists at least d (i,j) pairs such that

$$\cos\left(\frac{i+j}{n+1}\cdot\frac{\pi}{2}\right)\cos\left(\frac{i-j}{n+1}\cdot\frac{\pi}{2}\right)=\frac{k-\lambda}{4},\quad 1\le i,j\le n$$
$$\cos\left(\frac{2i+2j}{(2n+1)+1}\cdot\frac{\pi}{2}\right)\cos\left(\frac{2i-2j}{(2n+1)+1}\cdot\frac{\pi}{2}\right)=\frac{k-\lambda}{4},\quad 1\le i,j\le n$$

So, for each such (i,j) pair for $M_{n,k}$, $(2i,2j)$ is such a pair for $M_{2n+1,k}$. □

Note $V_{\lambda=0}$ is the null space. It's also true by the same reasoning that if $M_{2n+1,k}$ has nullity (dimension of null space) d then, $M_{n,k}$ has nullity at most d. I think this corollary applies to Sutner's conjecture in *Linear Cellular Automata and the Garden of Eden* on page 52. However, Sutner himself may have given a similar result in *On σ Automaton* as proposition 3.1.

We can even combine these two areas to find the integer eigenvalues of $M_{2n+1,k}$ given information about the integer eigenvalues of $M_{n,k}$.

MAYBE TALK ABOUT HOW NICE PROPERTIES SEEM TO HOLD BETWEEN N AND 2N+1? SHOW PRETTY PICTURES OF FIRST ROW / LAST ROW SEE / DO MATRICES?

Corollary 11. *If $M_{n,k}$ has no integer eigenvalues other than k, then $M_{2n+1,k}$ also has no integer eigenvalues other than k.*

Proof. Let $M_{n,k}$ have no integer eigenvalues other than k.

Corollary 9 tells us that $n\not\equiv 2 \mod 3$. If instead $n\equiv 0 \mod 3$, then $2n+1\equiv 1 \mod 3$. If $n\equiv 1 \mod 3$, then $2n+1\equiv 0 \mod 3$. In either case, $n\not\equiv 2 \mod 3$.

Corollary 9 also tells us that $n\not\equiv 4 \mod 5$. If instead $n\equiv 0 \mod 5$, then $2n+1\equiv 1 \mod 5$. If $n\equiv 1 \mod 5$, then $2n+1\equiv 3 \mod 5$. If $n\equiv 2 \mod 5$, then $2n+1\equiv 0 \mod 5$. If $n\equiv 3 \mod 5$, then $2n+1\equiv 2 \mod 5$. If $n\equiv 0 \mod 5$, then $2n+1\equiv 1 \mod 5$. In any case, $2n+1 \not equiv 4 \mod 5$.

Corollary 9 also tells us that $n\not\equiv 5 \mod 6$. If instead $n\equiv 0 \mod 6$, then $2n+1\equiv 1 \mod 6$. If $n\equiv 1 \mod 6$, then $2n+1\equiv 3 \mod 6$. If $n\equiv 2 \mod 6$, then $2n+1\equiv 5 \mod 6$. However, $n\equiv 2 \mod 6 \implies n\equiv 2 \mod 3$, which we know can't happen and $M_{n,k}$ have on integer eigenvalues. If $n\equiv 3 \mod 6$, then $2n+1\equiv 1 \mod 6$. If $n\equiv 4 \mod 6$, then $2n+1\equiv 3 \mod 6$. In any possible case, $2n+1\not\equiv 5 \mod 6$.

So, if n satisfies all of the criteria given in corollary 9 for $h=0$, then $2n+1$ also satisfies thse criteria, and $M_{2n+1,k}$ has no integer eigenvalues other than k. □

Corollary 12. *If $M_{n,k}$ has exactly two integer eigenvalues other than k, then $M_{2n+1,k}$ has exactly six integer eigenvalues other than k.*

Proof. Let $M_{n,k}$ has exactly two integer eigenvalues other than k.

Corollary 9 tells us that $n\equiv 2 \mod 3$, meaning $n=3m-1$ for some positive integer m. $2n+1=6m-1$, so $2n+1\equiv 2 \mod 3\equiv 5 \mod 6$.

Corollary 9 also tells us that $n \not\equiv 4 \mod 5$. As we already showed in the proof of corollary 11, $n \not\equiv 4 \mod 5 \implies 2n+1 \not\equiv 4 \mod 5$.

$2n+1$ satisfies the second requirement for $h = 6$, so $M_{2n+1,k}$ has exactly six integer eigenvalues other than k. $\square$

Corollary 13. *If $M_{n,k}$ has exactly four integer eigenvalues other than k, then $M_{2n+1,k}$ also has exactly four integer eigenvalues other than k.*

Proof. Let $M_{n,k}$ have exactly four integer eigenvalues other than k.

Corollary 9 tells us that $n \not\equiv 2 \mod 3$. As we already showed in the proof of corollary 11, $n \not\equiv 2 \mod 3 \implies 2n+1 \not\equiv 2 \mod 3$.

Corollary 9 also tells us that $n \equiv 4 \mod 5$, meaning $n = 5m - 1$ for some positive integer m. $2n+1 = 10m - 1$, so $2n+1 \equiv 4 \mod 5$.

Corollary 9 also tells us that $n \not\equiv 5 mod 6$. As we already showed in the proof of corollary 11, $n \not\equiv 5 \mod 6$ implies $2n+1 \not\equiv 5 \mod 6$ unless $n \equiv 2 \mod 3$. So, $2n+1 \not\equiv 5 \mod 6$.

$2n+1$ satisfies the requirement for $h = 4$, so $M_{2n+1,k}$ has exactly four integer eigenvalues other than k. $\square$

Corollary 14. *If $M_{n,k}$ has exactly six integer eigenvalues other than k due to satisfying the first requirement for $h = 6$ given in corollary 9, then M_{2n+1} has exactly ten integer eigenvalues other than k.*

Proof. Let $M_{n,k}$ have exactly six integer eigenvalues other than k due to satisfying the first requirement for $h = 6$ given in corollary 9.

Corollary 9 tells us that $n \equiv 2 \mod 3$. As we already already showed in corollary 12, $n \equiv 2 \mod 3 \implies 2n+1 \equiv 2 \mod 3 \equiv 5 \mod 6$.

Corollary 9 also tells us that $n \equiv 4 \mod 5$. As we already showed in corollary 13, $n \equiv 4 \mod 5 \implies 2n+1 \equiv 4 \mod 5$.

$2n+1$ satisfies the requirement for $h = 10$, so $M_{2n+1,k}$ has exactly ten integer eigenvalues other than k. $\square$

Corollary 15. *If $M_{n,k}$ has exactly six integer eigenvalues other than k due satisfying the second requirement for $h = 6$ given in corollary 9, then M_{2n+1} also has exactly six integer eigenvalues other than k due to satisfying the same requirement.*

Proof. Let $M_{n,k}$ have exactly six integer eigenvalues other than k due to satisfying the second requirement for $h = 6$ given in corollary 9.

Corollary 9 tells us that $n \equiv 2 \mod 3$. This is a redundant condition as we also see that $n \equiv 5 \mod 6$. As we already already showed in corollary 12, $n \equiv 2 \mod 3 \implies 2n+1 \equiv 2 \mod 3 \equiv 5 \mod 6$.

Corollary 9 also tells us that $n \not\equiv 4 \mod 5$. As we already showed in corollary 11, $n \not\equiv 4 \mod 5 \implies 2n+1 \not\equiv 4 \mod 5$.

$2n+1$ satisfies the second requirement for $h = 6$, so $M_{2n+1,k}$ has exactly six integer eigenvalues other than k. $\square$

Corollary 16. *If $M_{n,k}$ has exactly ten integer eigenvalues other than k, then $M_{2n+1,k}$ also has exactly ten integer eigenvalues other than k.*

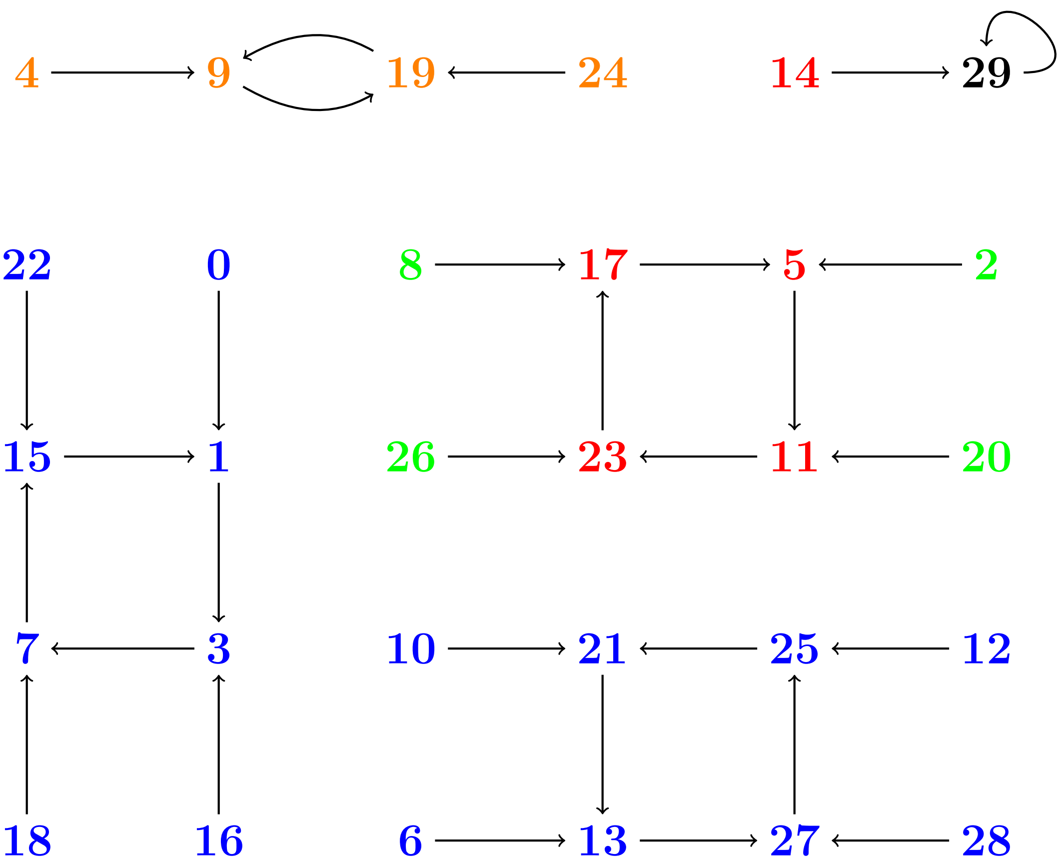
Proof. Let $M_{n,k}$ have exactly ten integer eigenvalues other than k.

Corollary 9 tells us that $n \equiv 2 \mod 3$. As we already showed in corollary 12, $n \equiv 2 \mod 3 \implies 2n+1 \equiv 2 \mod 3 \equiv 5 \mod 6$.

Corollary 9 also tells us that $n \equiv 4 \mod 5$. As we already showed in corollary 13, $n \equiv 4 \mod 5 \implies 2n+1 \equiv 4 \mod 5$.

$2n+1$ satisfies the requirement for $h = 10$, so $M_{2n+1,k}$ has exactly ten integer eigenvalues other than k. $\square$

Since corollary 9 tells us that the number of integer eigenvalues other than k occur regularly $\mod \text{lcm}(3,5,6) = 30$, we can visualize the relationships described in corollaries 11, 12, 13, 15, 14, and 16 as a colored directed graph where the nodes are the values of $n \mod 30$ and are colored according to their h value. Two nodes a and b are connected if $2a+1 \equiv b \mod 30$.



Blue = 0 Green = 2 Orange = 4 Red = 6 Black = 10

I think that corollaries 10, 11, 12, 13. 15, 14, and 16 prove, or at the very least contribute to finding a solution for Sutner's conjecture in *Linear Cellular Automata and the Garden of Eden*. We may still need a proof of something similar to conjecture 2 to prove it in $\mathbb{Z}_2$ rather than $\mathbb{R}$.

5 Conjectures

Conjecture 1. *There exist n linearly independent eigenvectors in $\mathbb{R}^{n^2}$ with eigenvalue $\lambda = k$ whose converted vectors are linearly independent in $\mathbb{Z}_2^{n^2}$.*

This proof seems to be relevant, but I don't know if it applies to finite fields like $\mathbb{Z}_2$.

Conjecture 2. *For $M_{n,k}$ the following hold.*

1. *All eigenvectors with integer eigenvalues are convertible to a vector in $\mathbb{Z}_2^{n^2}$*
2. *If $\vec{u} \in \mathbb{R}^{n^2}$ is an eigenvector of $M_{n,k}$ with eigenvalue λ, then $\vec{u} \mod 2$ is an eigenvector with eigenvalue $\lambda \mod 2$.*

The first statement is proven by lemma 1. I haven't had any luck in proving the second statement. Below is an example for a small value of n that demonstrates the idea.

Example 1. *For $n = 4$, the eigenvalues with multiplicity are*

$$\lambda_4 = \left\{2+\sqrt{5}, 1+\sqrt{5}, 1+\sqrt{5}, \sqrt{5}, 2, 2, 1, 1, 1, 1, 0, 0, 2-\sqrt{5}, 1-\sqrt{5}, 1-\sqrt{5}, -\sqrt{5}\right\}.$$

If an element of λ_4 is an integer, then there is a corresponding eigenvector with all integer components. I won't list all 16 eigenvectors here, but I'll give an example of an integer and non-integer eigenvalue.

Take the eigenvalue $\lambda = 2$. One of the two corresponding eigenvectors is

$$\vec{v}_{\lambda=2} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The components have been reshaped into a grid for space concerns and to better illustrate the relationship between these vectors and a Lights Out board. We can see that $\vec{v}_{\lambda=2} \in \mathbb{Z}^{25}$. We can also check that

$$\vec{v}_{\lambda=2} \mod 2 \equiv \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \equiv \vec{v}_{\lambda=0}$$

meaning that the converted version of $\vec{v}_{\lambda=2}$ is an eigenvector in $\mathbb{Z}_2^{16}$ with eigenvalue 0.

Take the eigenvalue $\lambda = \sqrt{5}$. The only corresponding eigenvector is

$$\vec{v}_{\lambda=\sqrt{5}} = \begin{bmatrix} 1 & \phi-1 & 1-\phi & -1 \\ \phi-1 & 2-\phi & \phi-2 & 1-\phi \\ 1-\phi & \phi-2 & 2-\phi & \phi-1 \\ -1 & 1-\phi & \phi-1 & 1 \end{bmatrix}$$

where ϕ is the golden ratio. Once again, the vector has been reshaped into a grid. We can see that since some components are rational and others irrational, that $\vec{v}_{\lambda=\sqrt{5}}$ is not convertible to a vector in $\mathbb{Z}_2^{16}$. However, if we go ahead and find $\vec{v}_{\lambda=\sqrt{5}} \mod 2$, we see that

$$\vec{v}_{\lambda=\sqrt{5}} \mod 2 \equiv \begin{bmatrix} 1 & 1-\phi & 1-\phi & 1 \\ 1-\phi & 2-\phi & 2-\phi & 1-\phi \\ 1-\phi & 2-\phi & 2-\phi & 1-\phi \\ 1 & 1-\phi & 1-\phi & 1 \end{bmatrix} \equiv \vec{v}_{\lambda=2-\sqrt{5}}.$$

This hints at a deeper relationship between eigenvectors $\mod 2$ whose corresponding eigenvalues sum to 2 (likely to $2k$ in general), likely relating to corollary 3.

Even if true, this conjecture doesn't always give a way to find a basis for all null patterns (kernel of $M_{n,k}$ in $\mathbb{Z}_2^{n^2}$). For example, the the nullity of $M_{19,1}$ in $\mathbb{Z}_2^{361}$ is 16, which is greater than the possible number of eigenvalues other than 1 (we showed it was at most 10). This should imply that certain irrational eigenvalues give convertible eigenvectors.

Conjecture 3. *The nullity of $M_{n,1}$ is always even.*

I couldn't think of a way to extend this to all k. We may have already proved this. We've already proved that the algebraic multiplicity of $\lambda_{i,j} = 0$ is even. The concern is that the results don't apply in finite fields like $\mathbb{Z}_2$. Since $M_{n,1}$ is real and symmetric, it is always diagonalizable, meaning for every eigenvalue, the geometric multiplicity equals the algebraic multiplicity. It's shown here that when working in $\mathbb{Z}_2$, the nullity is the degree of $\gcd\left(U_n(\frac{x}{2}), U_n(\frac{x+1}{2})\right)$, where $U_n(x)$ is the nth Chebyshev polynomial as described in lemma 4. Here is another site that makes this claim and has done some other relevant mathematical analysis of *Lights Out*. In question 2.3, it asks about a relationship in $\mathbb{Z}_2$ between what we call $M_{n,k}$ and $M_{2n+1,k}$ that we have answered in $\mathbb{R}$.

Conjecture 4. *Let $0 < p < q < 1$ be rational numbers. Let $-2 < C < 2$ be a non-zero rational number. The only solutions to*

$$\cos(p\pi) + \cos(q\pi) = C$$

are

$$(p, q, C) = \left(\frac{1}{3}, \frac{1}{3}, 1\right), \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{5}, \frac{3}{5}, \frac{1}{2}\right), \left(\frac{2}{3}, \frac{2}{3}, -1\right), \left(\frac{1}{2}, \frac{2}{3}, -\frac{1}{2}\right), \textit{and} \left(\frac{2}{5}, \frac{4}{5}, -\frac{1}{2}\right).$$

I have used the method described in lemma 3 to check that if any other such solutions do exist, then the denominator of C will be greater than 200. In my discussion with Professor Bell, who proved lemma 3, he wondered if we could use some of the properties taken advantage of in his proof to provably show that many values of C cannot have a solution. If we could bound the denominator of C to something computationally reasonable, we could then prove this conjecture by checking the remaining denominators. Conway and Jones also seem to describe the problem more generally and may have even solved something equivalent to or even stronger than this conjecture here as theorem 7.