The Rank Deficiency of Certain Sized *Lights Out* Boards

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September 2, 2021

1 Intro Lemmas

Let $f(n,x)$ be the Chebyshev polynomial over $GF(2)$ that we defined previously. Recall that the rank deficiency of an $n \times n$ *Lights Out* board, $d(n)$, is the degree of $\gcd(f(n,x), f(n,x+1))$. Let $g : \mathbb{N} \to \mathbb{N}$ where $g(k) = 2^{k+1} + 2^{k-1} - 1$.

Lemma 1. *Let $k \in \mathbb{N}$. Then*

$$f(g(k),x) = x^{2^{k-1}-1}\left(x^{2^{k+1}} + x^{2^k} + 1\right).$$

Proof. Recall how we determine the coefficients of $f(n,x)$ using the parity of binomial coefficients. First, we need to find a value $K = 2^k - 1$ where k is the smallest integer $K \geq n$. Second, we find $S = 2(K-n)$. Finally, we find that the ith coefficient is 1 if and only if $\binom{2i+S}{S+i}$ is odd. Equivalently, it's 1 if and only if i `&` $S+i$ is zero. $K = 2^{k+2} - 1$ for $g(k) = 2^{k+1} + 2^{k-1} - 1$. So,

$$\begin{aligned}
S &= 2(K - g(k)) \\
&= 2\left((2^{k+2} - 1) - (2^{k+1} + 2^{k-1} - 1)\right) \\
&= 2\left(2^{k+2} - 2^{k+1} - 2^{k-1}\right) \\
&= 2\left(8 \cdot 2^{k-1} - 4 \cdot 2^{k-1} - 2^{k-1}\right) \\
&= 2\left(3 \cdot 2^{k-1}\right) \\
&= 3 \cdot 2^k.
\end{aligned}$$

In binary this is $S = 11\underbrace{0\ldots0}_{k}$.

Let's think about what happens when we do i `&` $S+i$ for various values of i.

- In $i = 0$, we see that the left side of the `&` is 0, so the entire result is 0. Therefore, the leading coefficient is a 1, as desired.
- In $i = 1 \ldots 2^k - 1$, the trailing k 0's in S will match whatever is in i's binary representation in $S+i$. So, the result of the `&` will be nonzero. Therefore, there will be $2^k - 1$ 0 coefficients following the leading 1, as desired.
- In $i = 2^k$, $S + i = 100\underbrace{0\ldots0}_{k}$. So, the `&` will result in a 0, meaning we get a coefficient of 1, as desired.
- In $i = 2^k + 1 \ldots 2^{k+1} - 1$, we see a combination of the previous two cases. We can write $i = 2^k + j$, where $1 \leq j \leq 2^k - 1$. As we saw in the previous case, $S + 2^k = 100\underbrace{0\ldots0}_{k}$ in binary. As in two cases ago, the trailing k 0's in $S + 2^k$ will match whatever is in j's binary representation in $S + 2^k + j$. So, the result of the `&` operation will be nonzero. Therefore, the second 1 coefficient will be followed by $2^k - 1$ 0 coefficients, as desired.

- In $i = 2^{k+1}$, $S + i = 101\underbrace{0\dots0}_{k}$. Since i does not have a 2^k or a 2^{k+2} in its binary representation, i `&` $S + i$ `==` 0. So, we get a coefficient of 1, as desired.
- In $i = 2^{k+1} + 1 \dots 2^{k+1} + 2^{k-1} - 1$, we again see a combination of previous cases. We can write $i = 2^{k+1} + j$, where $1 \leq j \leq 2^{k-1} - 1$. As we saw in the previous case, $S + 2^{k+1} = 101\underbrace{0\dots0}_{k}$. As in two cases ago, the trailing k 0's in $S + 2^{k+1}$ will match whatever is in j's binary representation in $S + 2^{k+1} + j$. So, the result of the `&` operation will be nonzero. Therefore, the third 1 coefficient will be floowed by $2^{k-1} - 1$ 0 coefficients, as desired.

$\square$

Lemma 2. *Let $k \in \mathbb{N}$. Then*

$$f(g(k), x+1) = \left(x^{2^{k+1}} + x^{2^k} + 1\right)\left(x^{2^{k-1}-1} + \cdots + 1\right).$$

Proof. In lemma 1, we showed that

$$f(g(k), x) = \left(x^{2^{k-1}-1}\right)\left(x^{2^{k+1}} + x^{2^k} + 1\right).$$

Substituting $x+1$ for x and using mod 2 arithmetic (i.e. $(a+b)^c = a^c + b^c$ where c is a power of 2; $1 = -1$),

$$\begin{aligned}
f(g(x), x+1) &= \left((x+1)^{2^{k-1}-1}\right)\left((x+1)^{2^{k+1}} + (x+1)^{2^k} + 1\right) \\
&= \left(x^{2^{k-1}} + 1\right)\frac{1}{x+1}\left(x^{2^{k+1}} + 1 + x^{2^k} + 1 + 1\right) \\
&= \left(x^{2^{k+1}} + x^{2^k} + 1\right)\frac{x^{2^{k-1}} - 1}{x - 1} \\
&= \left(x^{2^{k+1}} + x^{2^k} + 1\right)\left(x^{2^{k-1}-1} + x^{2^{k-1}-2} + \cdots + 1\right).
\end{aligned}$$

$\square$

2 Main Result

Now that we have lemmas 1 and 2, we can prove our main result.

Theorem 1. *Let $k \in \mathbb{N}$. Then*

$$\gcd\left(f(g(k), x), f(g(k), x+1)\right) = x^{2^{k+1}} + x^{2^k} + 1.$$

Proof. We can see from lemmas 1 and 2 that the desired gcd is a common factor of both polynomials. So, we just need to show that over $GF(2)$ that the remaining factors of both polynomials have no common factors. This result is easily verified by a computer.

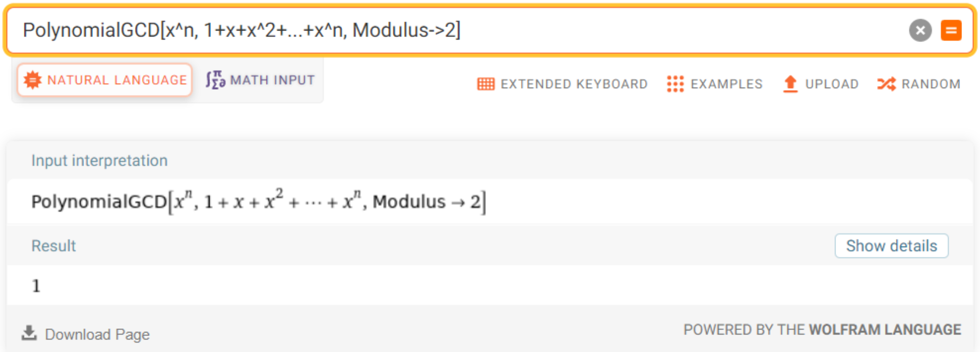

Figure 1: Wolfram|Alpha: PolynomialGCD[$x^n, 1 + x + x^2 + ... + x^n$, Modulus→2]

$\square$

Corollary 1. *A Lights Out board of size $g(k) \times g(k)$ will have nullity 2^{k+1}.*

3 Concluding Conjectures

Conjecture 1. *Let $h : \mathbb{N} \to \mathbb{N}$ where*

$$h(n) = \max\{g(m) \mid m \in \mathbb{N}, g(m) \leq n\}.$$

Then for all $n \in \mathbb{N}$,

$$\max\{d(m) \mid 1 \leq m \leq n\} = d(h(n)).$$

In other words, if we list the board sizes and their rank deficiencies, noting each time we encounter a rank deficiency higher than any other we've seen before, we will have noted exactly the board sizes described by g.

The following corollary relies on conjecture 1.

Corollary 2. *$d(n) = n$ only for $n = 4$. Otherwise, $d(n) < n$.*

Proof. Observe that $d(1) = d(2) = d(3) = 0$, and $d(g(1)) = d(4) = 4$.

Assume for contradiction that there exists some $n > 4$ such that $d(n) \geq n$.
Notice that for $k > 1$,

$$d(g(k)) = 2^{k+1} < g(k) = 2^{k+1} + 2^{k-1} - 1.$$

So,

$$d(h(n)) < h(n) \leq n = d(n).$$

However,

$$\max\{d(m) \mid 1 \leq m \leq n\} = d(n) > d(h(n)),$$

which contradicts conjecture 1. $\square$