

Most Moves Problem in *Lights Out*

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1 Introduction

Consider a game played on a simple graph $G = (V, E)$ where each vertex consists of a clickable light. A move consists of clicking any vertex v , which toggles the on/off state of v and its neighbors. Starting from an initial configuration of lights, one wins the game by finding a solution: a sequence of moves that turns off all the lights. This game is commercially available from Tiger Electronics as *Lights Out* for G a 5×5 grid.

Clicking a vertex twice is equivalent to not clicking it at all [7]. Thus, solutions can be sets of vertices rather than a sequences. Some initial configurations cannot be solved, meaning some lights will always remain on [1]. Further, Goldwasser and Klostermeyer show that determining if some initial configuration can be transformed through moves to a configuration with at least k lights off, which they call the *Maximizing Off Switches (MOS)* problem, is NP-complete. However, the case where $k = |V|$ is solvable in polynomial time, and the authors do not believe that MOS is NP-complete when G is restricted to grid graphs [3]. Thus, we restrict our attention only to solvable initial configurations. One natural question to ask about *Lights Out* is what we call the *Most Moves Problem (MMP)*:

How many moves does a worst-case initial configuration on G require to solve?

This question is analogous to finding the most turns of a Rubik's Cube needed to solve any scramble, which is known to be 20 [5].

Previous work determined when the answer to the MMP is the worst-case $|V|$. Eriksson, Eriksson, and Sjöstrand show that answer to the MMP is $|V|$ if and only if the number of perfect matchings in G is odd [2]. Sutner shows that the function $\Phi_G : P(V) \rightarrow P(V)$, that maps which vertices are clicked to which vertices turn on is linear with respect to symmetric difference. That is, for all $V_1, V_2 \subseteq V$,

$$\Phi_G(V_1 \Delta V_2) = \Phi_G(V_1) \Delta \Phi_G(V_2).$$

Further, Sutner shows that Φ_G is a bijection if and only if the answer to the MMP is $|V|$. Otherwise, the nullity of G , $\dim(\ker \Phi_G)$, is non-zero and even [7].

We review a technique from Scherphuis to solve the MMP for a 5×5 grid [6]. We generalize this approach to give an upper bound the MMP for all grids of size $(6k - 1) \times (6k - 1)$. This upper bound exactly solves the MMP for all such nullity 2 grids, which we conjecture solves the MMP for all nullity 2 grids.

2 Parity Covers

For any vertex $v \in V$, let $N(v)$, the neighborhood of v be all vertices that are affected when we click v . That is,

$$N(v) = \{u \in V \mid u = v \text{ or } (u, v) \in E\}.$$

If we have $V' \subseteq V$ such that for all $v \in V$, $|V' \cap N(v)|$ is even, then we say V' is an even parity cover of G . Similarly, if $|V' \cap N(v)|$ is always odd, then V' is an odd parity cover of G . Every graph has an odd parity cover, but not every graph has an even parity cover other than the trivial $V' = \emptyset$ [7].

Lemma 1. *Let $V' \subseteq V$. Then V' is an even parity cover of G if and only if $\Phi_G(V') = \emptyset$.*

Proof. The state of vertex v depends only on how often (even or odd) elements of $N(v)$ have been clicked [7]. So, $\Phi_G(V') = \emptyset$ if and only if every vertex in V contains an even number of elements from V' in its neighborhood, the definition of V' being an even parity cover. ■

Lemma 2. *Let V' be an even parity cover of G . If $V_1 \subseteq V$ is the solution to some initial configuration, then so is $V_1 \Delta V'$.*

Proof. Notice,

$$\begin{aligned}\Phi_G(V_1) &= \Phi_G(V_1) \Delta \emptyset \\ &= \Phi_G(V_1) \Delta \Phi_G(V') \\ &= \Phi_G(V_1 \Delta V').\end{aligned}$$

■

Lemma 3. *Let V_1 and V_2 be solutions to the same initial configuration. Then $V_1 \Delta V_2$ is an even parity cover of G .*

Proof. Since V_1 and V_2 are solutions to the same initial configuration, $\Phi_G(V_1) = \Phi_G(V_2)$. So,

$$\begin{aligned}\Phi_G(V_1 \Delta V_2) &= \Phi_G(V_1) \Delta \Phi_G(V_2) \\ &= \Phi_G(V_1) \Delta \Phi_G(V_1) \\ &= \emptyset.\end{aligned}$$

Thus, by Lemma 1, $V_1 \Delta V_2$ is an even parity cover of G . ■

These lemmas provide a way to the solution to a given initial configuration that uses the fewest moves. Once we have any solution to the initial configuration, apply every even parity cover to generate all equivalent solutions and take whichever contains the fewest elements.

3 MMP for 5×5 Grid

The 5×5 grid is smallest with $\dim(\ker \Phi_G) = 2$. Since we will apply nearly the same techniques in the next section, we will restate Scherphuis's proof of solving the MMP for a 5×5 grid in our own notation.

Theorem 1 (Scherphuis). *The answer to the Min Moves Problem for a 5×5 grid graph is 15.*

Proof. Since $\dim(\ker \Phi_G) = 2$ for G a 5×5 grid, there are $2^{\dim(\ker \Phi_G)} = 4$ even parity covers. One of these even parity covers is the trivial empty set. Below are the three non-trivial even parity covers. A black square means the corresponding vertex is an element of the even parity cover.

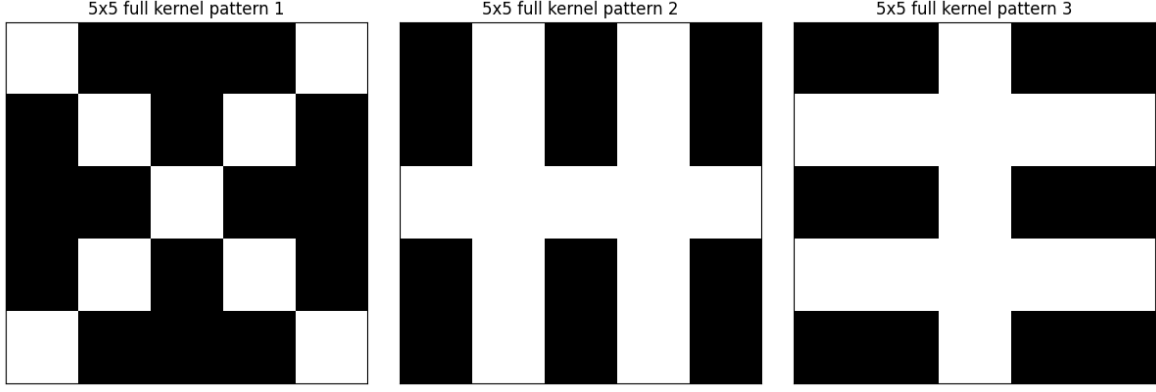


Figure 1: Non-trivial elements of $\ker \Phi_G$ for G a 5×5 grid

Partition the vertices of the 5×5 grid to one of four regions based on which even parity covers contain them, as shown below.

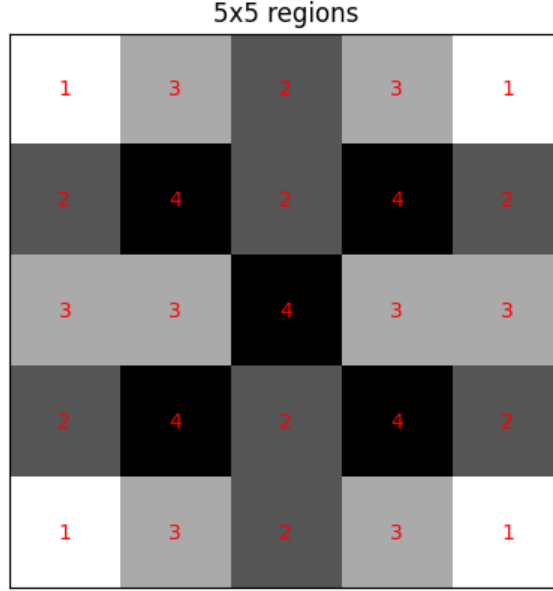


Figure 2: The four regions of a 5×5 grid

Region 1 is the intersection of covers 2 and 3; region 2 is the intersection of covers 1 and 2; region 3 is the intersection of covers 1 and 3; region 4 is the intersection of all the complements.

Let $Y \subseteq V$ be a worst-case initial configuration. Let $X \subseteq V$ be a solution to Y that uses the fewest moves among all solutions to Y . Let R_i be the number of vertices in X also contained in region i . Then the solution uses $|X| = R_1 + R_2 + R_3 + R_4$ moves.

If we take the symmetric difference of X and even parity cover 1, we obtain an equivalent solution that uses $R_1 + (8 - R_2) + (8 - R_3) + R_4$ moves. Since we assumed X was a minimal solution,

$$R_1 + R_2 + R_3 + R_4 \leq R_1 + (8 - R_2) + (8 - R_3) + R_4.$$

Rearranging,

$$R_2 + R_3 \leq 8. \tag{1}$$

If we take the symmetric difference of X and even parity cover 2, we obtain an equivalent solution that uses $(4 - R_1) + (8 - R_2) + R_3 + R_4$ moves. Since we assumed X was a minimal solution,

$$R_1 + R_2 + R_3 + R_4 \leq (4 - R_1) + (8 - R_2) + R_3 + R_4.$$

Rearranging,

$$R_1 + R_2 \leq 6. \quad (2)$$

If we take the symmetric difference of X and even parity cover 3, we obtain an equivalent solution that uses $(4 - R_1) + R_2 + (8 - R_3) + R_4$ moves. Since we assumed X was already a minimal solution,

$$R_1 + R_2 + R_3 + R_4 \leq (4 - R_1) + R_2 + (8 - R_3) + R_4.$$

Rearranging,

$$R_1 + R_3 \leq 6. \quad (3)$$

Since we have applied all even parity covers, we can be sure we considered all solutions to Y . None of (1), (2), or (3) involve R_4 . Thus, R_4 is not constrained beyond the fact that region 4 contains five vertices. Since we assumed Y was a worst-case initial configuration, $R_4 = 5$.

Combining (1), (2), and (3) into matrix form,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \leq \begin{bmatrix} 8 \\ 6 \\ 6 \end{bmatrix}. \quad (4)$$

We need to solve this integer linear program (ILP), maximizing the L^1 -norm $R_1 + R_2 + R_3$. Since all entries in the constraint matrix are non-negative, finding an integer solution where the constraints are tight (i.e. replace the \leq in (4) with $=$) would be the optimal solution to the ILP. Assuming the constraints are tight and performing row reduction, we find that

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

is the optimal solution to the ILP. So, the answer to the Min Moves Problem for a 5×5 grid is

$$\begin{aligned} |X| &= R_1 + R_2 + R_3 + R_4 \\ &= 2 + 4 + 4 + 5 \\ &= 15. \end{aligned}$$

■

4 Inequality for Larger Grids

For each even parity cover of a smaller grid, we can construct a unique even parity cover of a larger grid.

Theorem 2. *Let $d(n) = \dim(\ker \Phi_G)$ for G an $n \times n$ grid. Then for all $n, k \in \mathbb{N}$,*

$$d(nk - 1) \geq d(n - 1).$$

Proof. Since $nk - 1 = k(n - 1) + (k - 1)$, an $(nk - 1) \times (nk - 1)$ grid consists of a $k \times k$ grid of $(n - 1) \times (n - 1)$ grids, each separated by a horizontal or vertical strip of height or width 1.

Let Q be an even parity cover for an $(n - 1) \times (n - 1)$ grid. Let R be Q tiled k times horizontally and vertically onto an $(nk - 1) \times (nk - 1)$ grid, where each tile is a horizontal/vertical reflection of its horizontal/vertical neighboring tiles, as shown below.

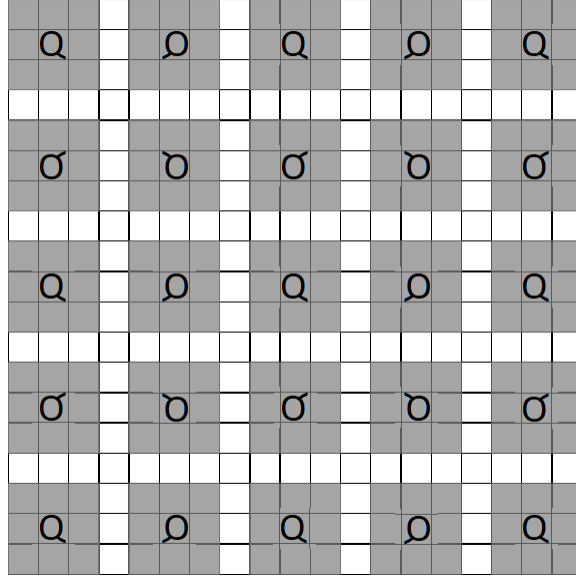


Figure 3: Pattern Q tiled 5 times.

Consider any vertex v in the $(nk - 1) \times (nk - 1)$ grid. If v is in one of the areas containing a reflection of Q , then $|N(v) \cap R|$ is even because Q is an even parity cover. Otherwise, v is on the border between reflections of Q . If v is in the intersection of a horizontal and vertical border, then $|N(v) \cap R| = 0$, which is even. Otherwise, $N(v)$ contains two vertices that are in neighboring reflections of Q . Since we reflected Q across this boarder, the two vertices are either both in R or neither are in R . In either case, $|N(v) \cap R|$ is even. So, R is an even parity cover.

So, each even parity cover of an $(n - 1) \times (n - 1)$ grid allows us to generate a unique even parity cover for an $(nk - 1) \times (nk - 1)$ grid. Since these even parity covers are independent on a $(n - 1) \times (n - 1)$ grid, they are also independent on the larger $(nk - 1) \times (nk - 1)$ grid. Thus,

$$d(nk - 1) \geq d(n - 1).$$

■

5 Establishing an Upper Bound

Applying Theorem 2 for $n = 6$, we get that for all $k \in \mathbb{N}$,

$$d(6k - 1) \geq d(5) = 2.$$

So, tiling the even parity covers from a 5×5 grid give even parity covers of $(6k - 1) \times (6k - 1)$ grids. For example, notice how the 17×17 even parity covers are simply the 5×5 even parity covers tiled $k = 3$ times.

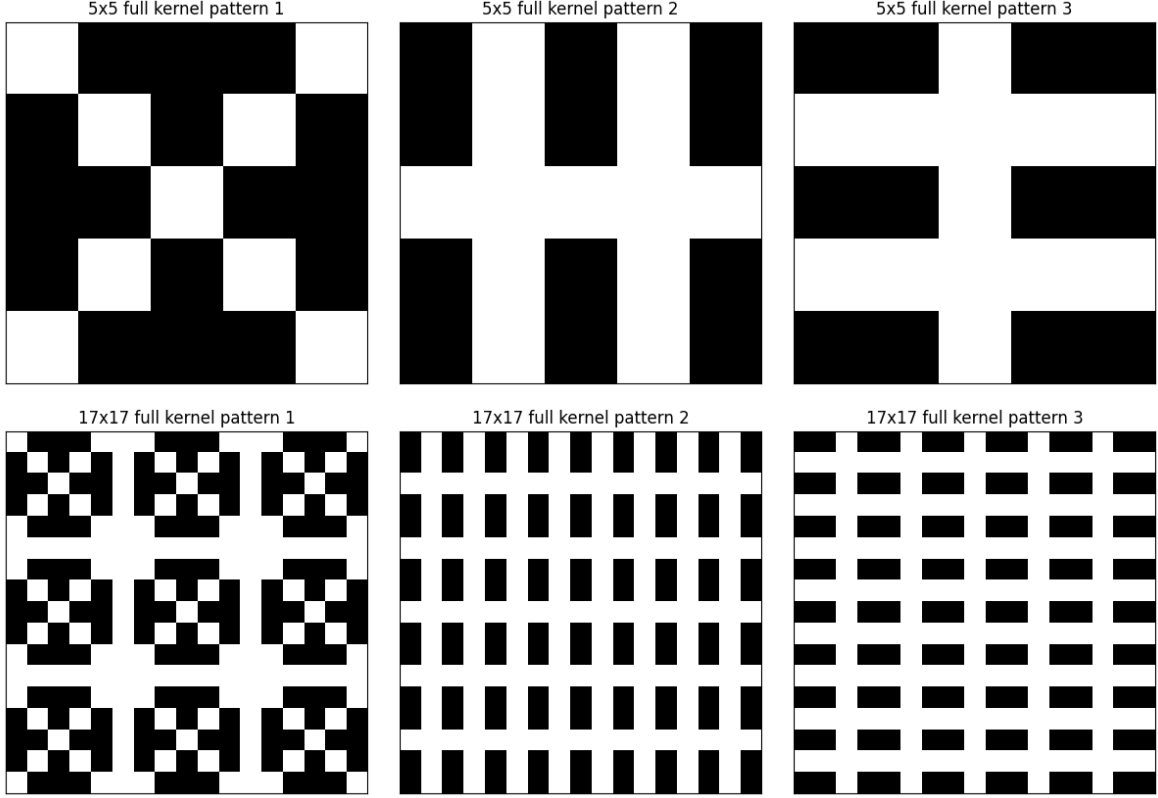


Figure 4: Non-trivial elements of $\ker \Phi_G$ for G a 5×5 and 17×17 grid

Thus, we can apply the same proof technique as in Theorem 1 to find an upper bound for the Min Moves Problem for such grid sizes. We'll also know that if $d(6k-1) = 2$, then we have considered all equivalent solutions, meaning this upper bound is exact, and we have solved the Min Moves Problem.

Theorem 3. *Let $k \in \mathbb{N}$. Then the answer to the MMP for a $(6k-1) \times (6k-1)$ grid is at most $26k^2 - 12k + 1$. This bound is exactly the answer to the MMP when $d(6k-1) = 2$.*

Proof. Applying Theorem 2 with $n = 6$, the even parity covers of the 5×5 grid will tile a $(6k-1) \times (6k-1)$ grid. We can partition the grid into four regions based on which of these even parity covers they are a part of. Using the same numbering for even parity covers as in Theorem 1, region 1 is the intersection of even parity covers 2 and 3; region 2 is the intersection of even parity covers 1 and 2; region 3 is the intersection of even parity covers 1 and 3; region 4 is the intersection of the complements. Region 1 contains $4k^2$ vertices, regions 2 and 3 contain $8k^2$ vertices, and region 4 contains the remaining $16k^2 - 12k + 1$ vertices.

Let $Y \subseteq V$ be a worst-case initial configuration. Let $X \subseteq V$ be a solution to Y that uses the fewest moves among all solutions to Y . Let R_i be the number of vertices in X also contained in region i . Then the solution uses $|X| = R_1 + R_2 + R_3 + R_4$ moves.

Applying the three even parity covers and deriving the constraints like we did in proving Theorem 1, we find that R_4 is unconstrained and can thus be set to $16k^2 - 12k + 1$, while R_1 , R_2 , and R_3 are constrained by the following ILP:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \leq \begin{bmatrix} 8 \\ 6 \\ 6 \end{bmatrix} k^2. \quad (5)$$

This is the exact same set of constraints as in Theorem 1, just with a factor of k^2 . Thus,

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} k^2.$$

is the solution to the ILP. Thus,

$$\begin{aligned} |X| &= R_1 + R_2 + R_3 + R_4 \\ &= 2k^2 + 4k^2 + 4k^2 + (16k^2 - 12k + 1) \\ &= 26k^2 - 12k + 1. \end{aligned}$$

If $\dim(\ker \Phi) > 2$ there will be more even parity covers that would create more restrictive constraints. However, if $\dim(\ker \Phi) = 2$, we have considered all even parity covers and have solved the MMP. ■

The 5×5 , 17×17 , 41×41 , 53×53 , 77×77 , grids are those of size $(6k - 1) \times (6k - 1)$ less than 100 with nullity 2. Thus, we find that the answers to the MMP for these grids are 15, 199, 1191, 1999, and 4239 respectively.

6 Conjectures

Conjecture 1. *There are infinitely many n such that $d(n) = 2$.*

We already know that the version of this conjecture with $d(n) = 0$ is true. Sutner mentions $d(2^k - 1) = 0$ for $k \in \mathbb{N}$ [7]. Theorem 2 tells us that there are infinitely many n with $d(n) \geq 2$. By computer search we found that for $n \leq 10000$, there are 497 such n such that $d(n) = 2$.

Conjecture 2. *If $d(n) = 2$, then $n \equiv -1 \pmod{6}$.*

That is, the even parity covers of all nullity 2 grids are tilings of the 5×5 even parity covers. If true, this would mean that we have solved the MMP for all nullity 2 grids. We performed a computer search, applying Theorem 2 in a sieve approach to eliminate grids we know have nullity greater than 2, and found no counterexamples for $n \leq 10000$.

7 Future Work

We'd like future work to focus on resolving these conjectures and solving the MMP on grids with nullities greater than 2. For any sized grid (in fact, any graph), we can apply the same technique as in theorems 1 and 3 of finding the answer to the MMP as the solution to an ILP. However, it seems we cannot apply the same technique of trying to solve the ILP with all constraints tight for larger nullities.

We'd also like future work to focus more on understanding the relationships between the nullities of different sized grids. Theorem 2 is a good start to this, but there seem to be many other relationships to discover. For example, Knuth shows that if an even parity cover of an $n \times n$ grid is perfect – every row and column contains elements from the cover – then we can construct a perfect even parity cover for a $(2n + 1) \times (2n + 1)$ grid [4]. All the even parity covers constructed in Theorem 2 are not perfect. Also, Sutner conjectures that $d(2n + 1) = 2d(n) + \delta_n$, where $\delta_n \in \{0, 2\}$, and $\delta_{2n+1} = \delta_n$ [7].

References

- [1] Marlow Anderson and Todd Feil, *Turning lights out with linear algebra*, Mathematics Magazine **71** (1998), no. 4, 300–303.
- [2] Henrik Eriksson, Kimmo Eriksson, and Jonas Sjöstrand, *Note on the lamp lighting problem*, Advances in Applied Mathematics **27** (2001), no. 2, 357–366.
- [3] John Goldwasser and William Klostermeyer, *Maximization versions of "Lights Out" games in grids and graphs*, Congressus Numerantium **126** (1997), 99–111.
- [4] Donald Knuth, *The art of computer programming, volume 4A: Combinatorial algorithms, part 1*, Addison-Wesley, 2011, Section 7.1.3, Exercises 190–194.
- [5] Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge, *The diameter of the Rubik's Cube graph is twenty*, SIAM Journal on Discrete Mathematics **27** (2013), 1082–1105.
- [6] Jaap Scherphuis, *The mathematics of Lights Out*, 2015, <https://www.jaapsch.net/puzzles/lomath.htm>.
- [7] Klaus Sutner, *Linear cellular automata and the Garden-of-Eden*, The Mathematical Intelligencer **11** (1989), no. 2, 49–53.