Multivariable Calculus: A Summary

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Table of Contents

Table of Contents 2

Background & Review 6

- Important Shapes in 2D & 3D 6
- Vectors 6
  - Dot Products 8
  - Cross Products 9

Vector-Valued Functions 11

- Lines & Planes as Vector-Valued Functions 11
  - Line 11
  - Plane 11
- Common Vector-Valued Functions 12
  - Circle 12
  - Helix 12
- Derivatives of Vector-Valued Functions 13
- Integrals of Vector-Valued Functions 15
- Reparameterization & Arc Length 15
- TNB Frame & Osculating Plane/Circle 16
  - T-Hat () 16
  - Curvature 16
  - N-Hat () 16
  - Osculating Plane/Circle & B-Hat () 18

Differential Multivariable Calculus 19

- Multivariable Functions 19
- Level Curves 19
- Quadric Surfaces 20
  - Paraboloid 20
  - Hyperboloid 20
    - Hyperboloid of One Sheet 20
    - Hyperboloid of Two Sheets 21
    - Cone 21
  - Hyperbolic Paraboloid 22
  - Ellipsoid 22

3D Cylinders 23
Circular Cylinder 23
Parabolic Cylinder 24
Parameterized Surfaces 24
Limits & Continuity in 3D 25
Open Delta Neighborhoods 25
Boundary Points, Open & Closed Sets 26
Limit & Continuity Definitions 26
Partial Derivatives 27
Partial Derivatives of x, y, and z 27
Tangent Planes 28
Linear Approximations 28
The Gradient 29
Gradient Properties 29
Linear Approximations with the Gradient 30
Gradient & C-Level Curves 30
Directional Derivatives 30
Optimization 31
Definitions 31
Critical Points 32
Second Derivative Test & Hessian Matrix 33
Gradient Descent 34
Lagrange Multipliers 35
Toy Example 35
Method of Lagrange Multipliers 36

Multiple Integrals 36

Double Integrals 36
Domain Regions & Fubini's Theorem 37
Average Values 39
Mean Value Theorem 39
Volume Between Surfaces 39
Plane Laminas 40
Triple Integrals 40
Fubini's Theorem for Z-Simple Regions 40
Plane Laminas 41

Curvilinear Coordinates 41

Review of Polar Coordinates 41
Circles 41
Lines 42
Integration 42
Gaussian Integral 42
Cylindrical Coordinates 43
Conversions 43
Integration 43
Spherical Coordinates 45
Conversions 45
Integration 45
Sphere Volume 46

Line & Surface Integrals 46

Vector Fields 46
Line Integrals 47
Line Integrals of Scalar Functions 47
Line Integrals of Vector Fields 47
Direction Matters 48
Circulations 48
Conservative Vector Fields 49
FTC for Line Integrals 50
Potential Functions 50
Test for a Conservative Vector Field 51
Surface Integrals 51
Surface Integrals of Scalar Functions 52
Surface Integrals of Vector Fields 53
Flux 53

Vector Analysis 54

Integral Curves 54
Divergence & Curl 55
Divergence 55
The Laplacian 56
Curl 56
Curl & Conservative Vector Fields 57
Divergence of Curl 58
Green’s Theorems 58

Green's Theorem for Circulation 58
Area of Closed Region 60
Green's Theorem for Flux 61
Divergence Theorem 62
Gauss's Laws 63
Electric Fields 63
Magnetic Fields 64
Stokes' Theorem 64
Faraday's Law of Induction & Ampere's Law 65

Additional Materials **66**
Line Integral Flowchart 66
Surface Integral Flowchart 66
Worked Test Questions 67
Test 1 67
Test 2 70
Test 3 74
Test 4 77
Online Resources 81

Background & Review

Important Shapes in 2D & 3D

In 2D space ($\mathbb{R}^2$) there are lines and circles.

- Lines have a form like $x = 1$ or $y = 2x + 1$.
- Circles have a form like $(x - 3)^2 + y^2 = 4$.

In 3D space ($\mathbb{R}^3$) there are planes, cylinders, and spheres.

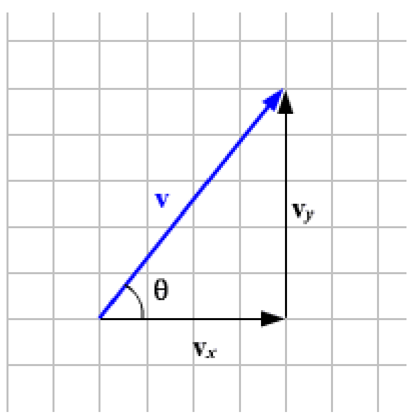
- Planes have forms like $x = 1$ or $y = 2x + 1$.
  Note the similarity to lines in $\mathbb{R}^2$. This is because the cross-section of a plane in a line.
- Cylinders have forms like $(x - 3)^2 + y^2 = 4$.
  Note the similarity to circles in $\mathbb{R}^2$. This is because the cross-section of a cylinder is a circle.

Vectors

A vector is a quantity with both direction and magnitude. One can think of it as a directed line segment. For multivariable calculus, we will deal almost exclusively with vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Scalars have vector analogues, many of which show up in physics. Speed becomes velocity, distance becomes displacement, and mass becomes the force of weight.

Say we have a vector, $\vec{v} = \langle v_x, v_y \rangle$.



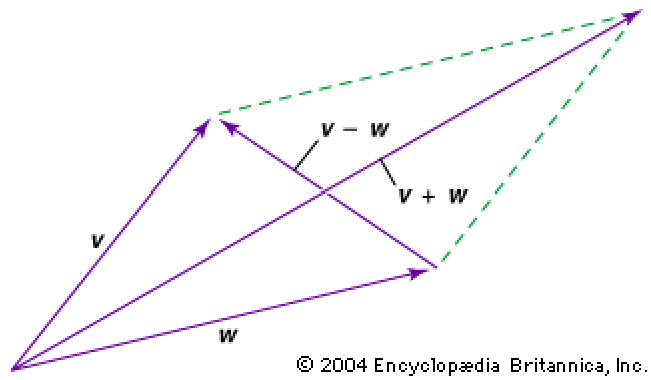
Its magnitude, length, notated $||\vec{v}||$ is $\sqrt{v_x^2 + v_y^2}$. This patterns of the square root of the sum of squares of the components continues into $\mathbb{R}^3$ and beyond.

The angle it forms with the horizontal, θ, is $\tan^{-1}\left(\frac{a_y}{a_x}\right)$. There is not as useful a version in $\mathbb{R}^3$ and beyond.
Using the θ and $||\vec{v}||$, we see that $a_x = ||\vec{a}|| \cos\theta$ and $a_y = ||\vec{a}|| \sin\theta$.

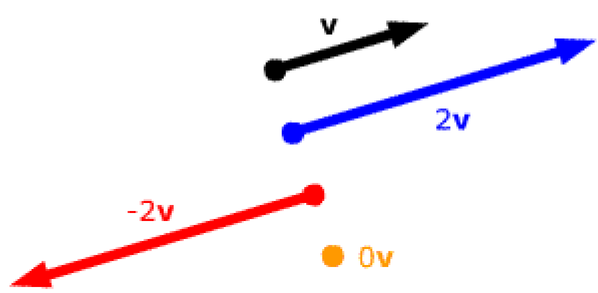
Vectors can be added together, getting back another vector. Each of the resultant vector's components is the sum of the components of the summed vectors.

For example, if $\vec{a} = \langle 1, 3\rangle$ and $\vec{b} = \langle 4, 7\rangle$, then $\vec{a} + \vec{b} = \langle 5, 10\rangle$.

Visually, you can think of aligning $\vec{w}$ at the end of $\vec{v}$. $\vec{v} + \vec{w}$ is the vector connecting the back of $\vec{v}$ with the end of $\vec{w}$. Further, if you put the backs of $\vec{v}$ and $\vec{w}$ together, then $\vec{v} - \vec{w}$ is the vector connecting the front of $\vec{w}$ with the front of $\vec{v}$.

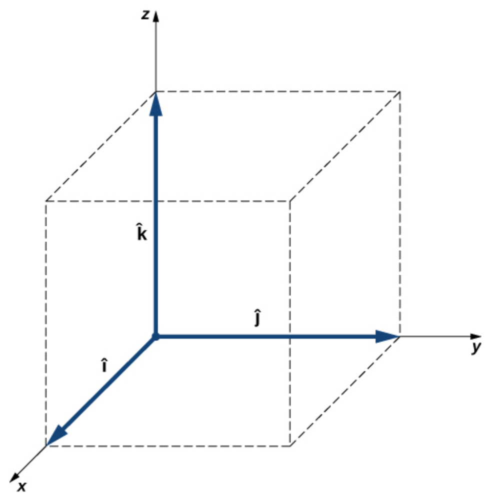


We can multiply vectors by scalars, getting back another vector. Each component is multiplied by the scalar. This multiplies the magnitude of the vector by the multiplied scalar, and keeps the direction the same, or opposite if the scalar is negative.



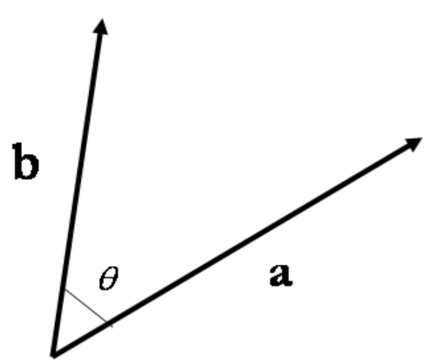
A unit vector is a vector of magnitude 1. Rather than using arrows like for other vectors, unit vectors have a carat (^) over top, like $\hat{\imath}$, which is read "i hat". We can normalize a vector by dividing each component by its magnitude. This normalized vector will point in the same direction as the original longer (or shorter) vector.

It is common in mathematics for $\hat{\imath} = \langle 1, 0, 0 \rangle$ to be the unit vector in x-direction, $\hat{\jmath} = \langle 0, 1, 0 \rangle$ to be the unit vector in the y-direction, and $\hat{k} = \langle 0, 0, 1 \rangle$ to be the unit vector in the z-direction. Together, these are called basis vectors because they can be added together to get all other vectors.



Dot Products

A dot product takes two vectors and gives a scalar. $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$. One way to think of the cross product is a measure of how much two vectors point in the same direction. It can also be shown through the law of cosines that $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + ... + a_nb_n$. Knowing the lengths of two vectors and their dot product, we can calculate the angle between them as $\theta = \cos^{-}1\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$.



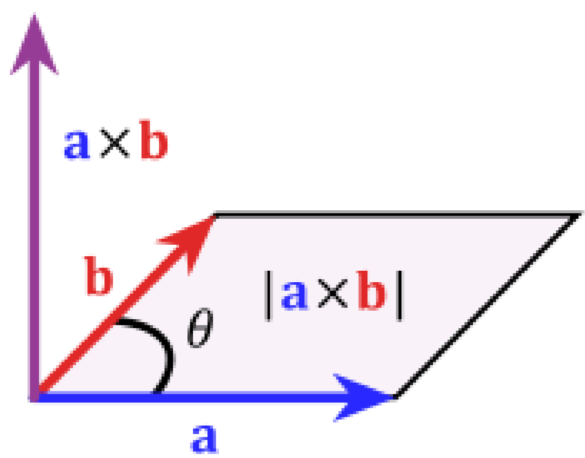
Dot products have some properties that differ them from scalars.

- The operation is commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
- It is distributive: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b}$.
- It is not associative, as $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is a nonsense expression. However, it is scalar associative: $(c\vec{a}) \cdot \vec{b} = \vec{a} \cdot (c\vec{b})$.

Cross Products

A cross product gives a vector when multiplying two vectors. Technically, the cross product only works for 3D vectors, but we will first look at a "fake" cross product in 2D that gives a scalar to build an intuition.

$\vec{a} \times \vec{b} = a_1 b_2 - b_1 a_2$. This is the area for the parallelogram spanned by $\vec{a}$ and $\vec{b}$ and is equal to $\|\vec{a}\|\|\vec{b}\| \sin\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$. Another way to think of the magnitude of the cross product, both in 2D and 3D, is how perpendicular two vectors are.



In 3D, $\vec{a} \times \vec{b}$ is a vector, and similar to the 2D case, this resultant vector's magnitude is equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.

$\vec{a} \times \vec{b} = \langle a_2 b_3 - b_2 a_3, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2 \rangle$ and $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

Each component give the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ in some plane: The x-component of $\vec{a} \times \vec{b}$ gives the area of the parallelogram in the yz-plane (x=0 plane).

$\vec{a} \times \vec{b}$ is perpendicular, also called "normal to", the plane containing $\vec{a}$ and $\vec{b}$. Its direction is determined by the right hand rule.

The multiplication table of the basis vectors is useful and give some insight into the properties of the cross product operation.

$\overrightarrow{row} \times \overrightarrow{col}$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$-\hat{k}$	0	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	0

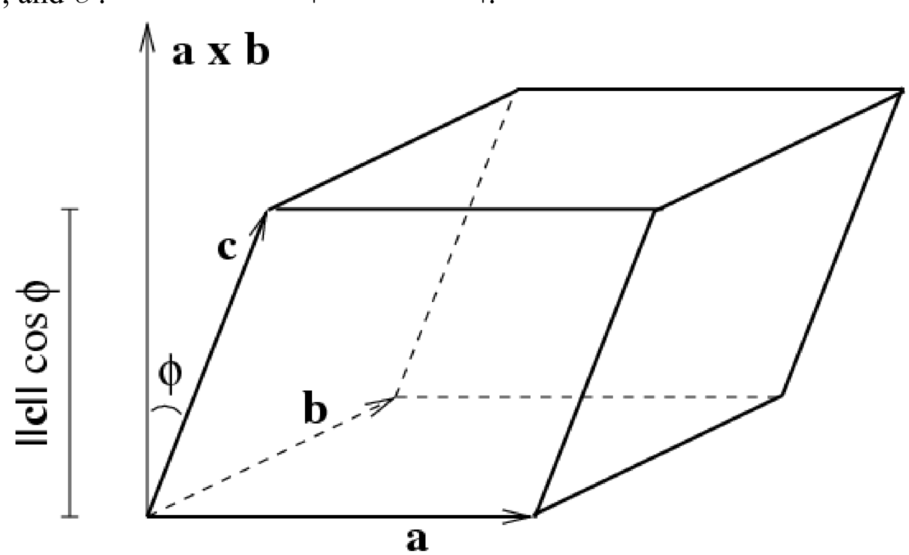
- The cross product is not commutative, but is antisymmetric: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
- Like the dot product, the cross product is also scalar associative: $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b})$.
- It is also distributive like the dot product: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

One can also think of the 2D cross product as the determinant of a matrix.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Now that we have a dot product and a cross product, we can put the two together using three vector into the scalar triple product, which finds the volume of a parallelepiped spanned by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$



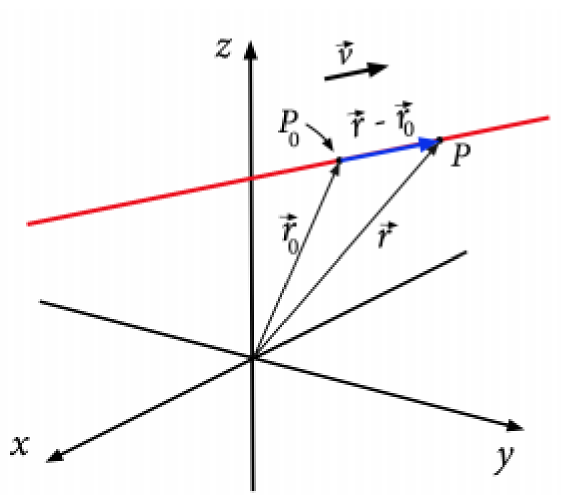
Vector-Valued Functions

Lines & Planes as Vector-Valued Functions

vector-valued functions are parametric equations that take one input value and return a set of one or more output values as a vector. We can draw a 1D curve by putting the tail of the putpute vector at the origin and tracing the location of the tip of the output vector as our input value varies.

Line

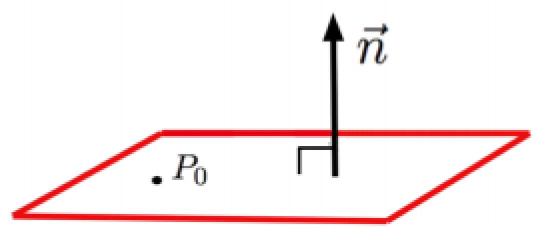
The simplest of these vector-valued functions in 3D is a strait line. We can define a straight line in 3D using a point and direction vector, similar to slope intercept form in 2D. Letting our point be P and our direction be $\vec{v}$, $\vec{r}(t) = \vec{P} + t\vec{v}$ where $\vec{P}$ is vector whose components are the same and the elements of P. This function starts at $\vec{P}$ at $t = 0$ and move in the direction of $\vec{v}$ as t increases.



Another common definition of a line is two points. To find the line in this case, let $\vec{v}$ be the vector connecting tow two points, and let $\vec{P}$ be one of the two points. This comes out to $\vec{r}(t) = \vec{P}_0 + t(\vec{P}_1 - \vec{P}_0)$. Where P_0 and P_1 are two points on the line.

Plane

A plane can also be defined by a point and a vector. The point, P_0, is one in the plane, and the vector, $\vec{n}$, is normal to the plane. All vectors $\langle x, y, z \rangle$ originating from P_0 will also be perpendicular to $\vec{n}$, so their dot product with $\vec{n}$ would be 0. This gives us an equation for the point-normal form of a plane as all points x, y, and z such that $\vec{n} \cdot (\langle x, y, z \rangle - P_0) = 0$.



There are other setups that also define a plane, like 3 points in the plane. One can still take advantage of point-normal form here by choosing 1 point to be P_0 and drawing vectors from this point to the two other points. The cross product of these two vectors is $\vec{n}$.

Another definition of a plane is a point and a line in the plane. One can get this setup into point-normal form by choosing two points along the line, and then proceeding like a 3-point setup using the given point and the two chosen points.

A third definition of a plane is two intersecting lines, $\vec{r_1}(t) = \vec{P_0} + t\vec{v_1}$ and $\vec{r_2}(t) = \vec{P_0} + t\vec{v_2}$ where $\vec{P_0}$ is the intersection of the two lines, and $\vec{v_1} \times \vec{v_2}$ is proportional to the normal vector.

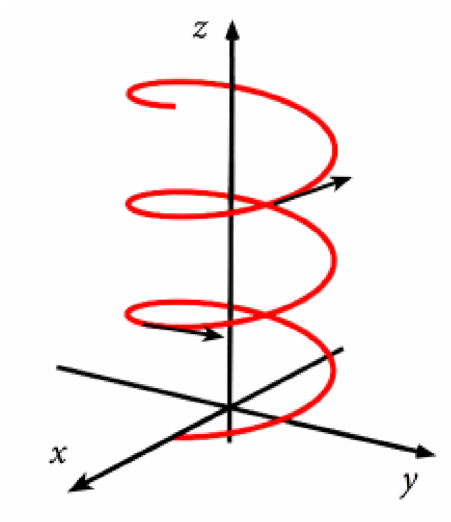
Common Vector-Valued Functions

Circle

You should already recognize $x^2 + y^2 = r^2$ as the equation for a circle of radius r centered at the origin. It can equivalently be written as a vector-valued function in $\mathbb{R}^2$ as $\vec{r}(t) = r\cos t, r\sin t\rangle$. In $\mathbb{R}^3$, a constant term, c, is added to the z-component of the vector-valued function to define the plane parallel to the xy-plane in which the circle lies.

Helix

A helix is a $\mathbb{R}^3$ form that looks like a spring and appears identical to a circle when viewed top-down. It's form is $\vec{r}(t) = \langle r\cos t, r\sin t, at \rangle$ where $a \in \mathbb{R}$ and defines the "tightness" between consecutive windings.



Since vector-valued functions are a way to package multiple functions together, their domain is that on which all components are defined.

For example, if $\vec{r}(t) = \langle \tan t, 6t, \ln{(16 - t^2)} \rangle$,

- The x-component is defined except when t is equivalent to $\pm\pi/2$ radians.
- The y-component is defined for all real numbers.
- The z-component is defined for $t \in (-4, 4)$

The intersection of these domains is $(-4, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 4)$, which is the domain of $\vec{r}(t)$.

Derivatives of Vector-Valued Functions

Just like normal functions with one input variable and one output variable, we can take the derivative of vector-valued functions. In fact, the limit definition of the derivative is nearly identical.

Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

$$\vec{r'}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$\vec{r'}(t) = \lim_{h \to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right\rangle$$

The limit distributes inside of the vector such that $\vec{r'}(t) = \langle x'(t), y'(t), z'(t) \rangle$.

We can see that the derivative of a vector-valued function, $\vec{r'}(t)$, is a vector-valued function where each component is the derivative of the corresponding component of $\vec{r}$ with respect to t.

Like regular derivatives of position, the derivative of vector-valued functions representing positions gives the velocity, and the magnitude gives the speed, which is often notated $v(t)$ (a scalar function). Similarly, the second derivative of a vector-valued position function gives acceleration.

There are 5 important properties for the derivative of vector-valued functions that are similar to regular derivatives. Let $\vec{r}(t)$ and $\vec{s}(t)$ be a vector-valued functions, $a(t)$ be a scalar function, and a be a scalar.

1. $\frac{\mathrm{d}}{\mathrm{d}t}a\vec{r}(t) = a\vec{r'}(t)$ Linearity of Derivative
2. $\frac{\mathrm{d}}{\mathrm{d}t}a(t)\vec{r}(t) = a(t)\vec{r'}(t) + \vec{r}(t)a'(t)$ Product Rule for Scalar Functions
3. $\frac{\mathrm{d}}{\mathrm{d}t}\vec{s}(t) \cdot \vec{r}(t) = \vec{s}(t) \cdot \vec{r'}(t) + \vec{r}(t) \cdot \vec{s'}(t)$ Dot Product Rule
4. $\frac{\mathrm{d}}{\mathrm{d}t}\vec{s}(t) \times \vec{r}(t) = \vec{s'}(t) \times \vec{r}(t) + \vec{s}(t) \times \vec{r'}(t)$ Cross Product Rule
5. $\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(a(t)) = \vec{r'}(a(t))a'(t)$ Chain Rule

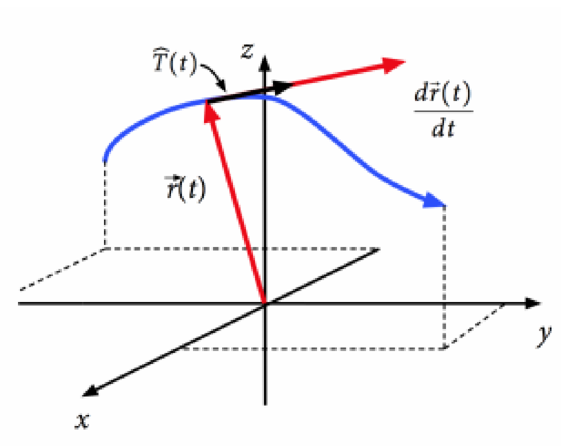
Note: A quotient rule does not make sense, as we do not have an operation for dividing two vector-valued functions.

Just like normal, single-variable functions, we can use the derivative of vector-valued functions to find the tangent line to the curve. Similar to how $f'(a)$ represents the slope at a, $\vec{r'}(a)$ represents the direction vector of the tangent line at a. Remembering our equation for a line, we get the tangent line to $\vec{r}$ at a as $l(t) = \vec{r}(t) + t\vec{r'}(t)$.

In fact, tangent lines appear so often, that we have a special vector for normalized $\vec{r'}(t)$.

$$\hat{T}(t) = \frac{\vec{r'}(t)}{\left|\left|\vec{r'}(t)\right|\right|}$$

You can remember $\hat{T}$ as the “tangent” vector.



Integrals of Vector-Valued Functions

vector-valued functions can also be integrated. The idea of the operation distributing inside the vector works for integrals, both indefinite and definite.

$$\int \vec{r}(t)\mathrm{d}t = \left\langle \int x(t)\mathrm{d}t, \int y(t)\mathrm{d}t, \int z(t)\mathrm{d}t \right\rangle$$

Note that for indefinite integrals, the answer will have a vector of constants added.

Reparameterization & Arc Length

Vector-valued functions can be reparameterized to represent the same shape while changing the speed at which the function traces out the path. One can replace t in $\vec{r}(t)$ with any non-decreasing function of t to get the same shape. This can often come in handy to change the limits of an integration problem to be more convenient.

One can integrate the derivative of vector-valued functions to find the displacement vector between two points because $\int_a^b \vec{r'}(t)\mathrm{d}t = \vec{r}(b) - \vec{r}(a)$. This is exactly like how $\text{displacement} = \text{velocity} \cdot \text{time}$.

If we integrate the magnitude of $\vec{r'}(t)$, we can use the relationship $\text{distance} = \text{speed} \cdot \text{time}$ to find the arclength of $\vec{r}(t)$ as $s = \int \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t = \int \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} dt$.

We can also write this as an arclength function $s(t) = \int_0^t \left|\left|\vec{r'}(\tau)\right|\right| \mathrm{d}\tau$.

If we have a function $f(t) = s = \int_0^t \left|\left|\vec{r'}(\tau)\right|\right| \mathrm{d}\tau$ where s is strictly increasing, then f has an inverse by the horizontal line test. That is, $t(s) = f^{-1}(s)$ exists and is also nondecreasing.
If we reparameterize $\vec{r}(t)$ to $\vec{r}(t(s))$ — called the arc length parameterization — this parameterization will have constant speed.

TNB Frame & Osculating Plane/Circle

T-Hat ($\hat{T}$)

This idea of arcelegth parameterizations also allows us to have another way to find $\hat{T}$.
$\hat{T}(t) = \frac{\vec{r'}(t)}{\left|\left|\vec{r'}(t)\right|\right|} = \frac{\mathrm{d}\vec{r}/dt}{\mathrm{d}s/dt} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}$

Curvature

We can also use $\hat{T}$ to find 1 over the radius of the circle that best approximates $\vec{r}(t)$ at a point, the curvature.
$\kappa(t) = \left|\left|\frac{\mathrm{d}\hat{T}}{\mathrm{d}s}\right|\right| = \left|\left|\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1}\right|\right| = \left|\left|\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}\right|\right|\frac{1}{v(t)}$

For example, let's find $\kappa(t)$ for the circle in the yz-plane: $\vec{r}(t) = \langle 7, R\sin t, R\cos t\rangle$.

$\vec{r'}(t) = \langle 0, R\cos t, -R\sin t\rangle$, $v(t) = \sqrt{0^2 + (R\cos t)^2 + (-R\sin t)^2} = |R|$

$\hat{T}(t) = \frac{1}{R}\vec{r'}(t) = \langle 0, \cos t, -\sin t\rangle$, $\frac{\mathrm{d}\hat{T}}{\mathrm{d}t} = \langle 0, -\sin t, -\cos t\rangle$, $\left|\left|\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}\right|\right| = 1$

$\kappa(t) = \frac{1}{R}$ This relationship is true of all circles.

N-Hat ($\hat{N}$)

We can write $\vec{r'}(t)$ as $\vec{r'}(t) = v(t)\hat{T}(t)$
Taking the derivative, $\vec{r''}(t) = v(t)\hat{T}'(t) + \hat{T}(t)v'(t)$
$\hat{T}(t) \perp \hat{T}'(t)$ because…

$\hat{T}(t) \cdot \hat{T}(t) = 1$

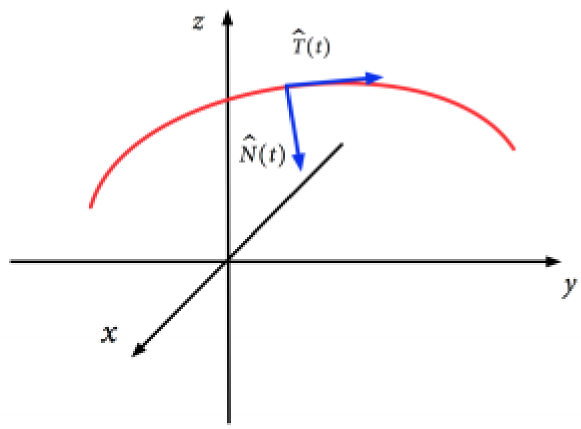
$\frac{d}{dt}\hat{T}(t)\cdot\hat{T}(t) = 2\hat{T}(t)\cdot\hat{T}'(t)$ and $\frac{d}{dt}\hat{T}(t)\cdot\hat{T}(t) = \frac{d}{dt}1 = 0$

So, $2\hat{T}(t)\cdot\hat{T}'(t) = 0$

And a 0 dot product implies that two vectors are $\perp$ if nonzero.

So, we can say $\hat{N}(t) = \frac{\hat{T}'(t)}{\left|\left|\hat{T}'(t)\right|\right|}$ is $\perp$ to $\hat{T}(t)$.

$\hat{N}$ is a unit normal vector perpendicular to $\hat{T}$ that points in the direction that the curve curls into. It is in the same plane is r', $\hat{T}$, and r''.

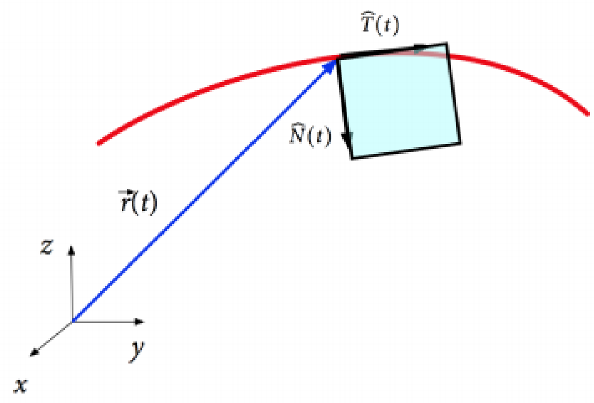


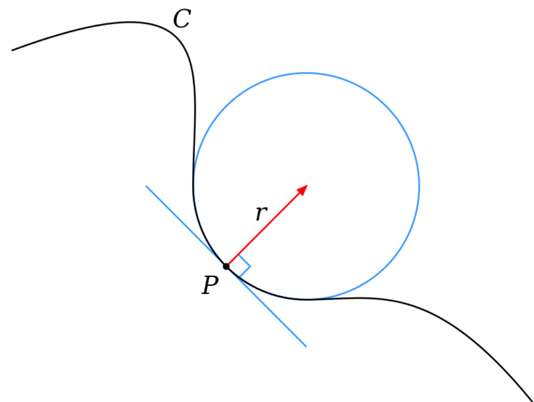
$\hat{N}$ allows us to rewrite r'': $\vec{r}''(t) = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{T}(t) + v^2(t)\kappa(t)\hat{N}(t)$

If $r(t)$ represents position, then $\frac{dv}{dt}$ is the component of linear acceleration. It is greater when the particle's speed is changing. $v^2(t)\kappa(t)$ is the component of centripetal acceleration, which is greater when the particle direction is changing. Physics students might recognize that if we let $R(t) = \frac{1}{\kappa(t)}$, the centripetal acceleration component is $\frac{v^2(t)}{R(t)}$, which is similar to the physics equation for centripetal acceleration: $a_c = \frac{v^2}{r}$.

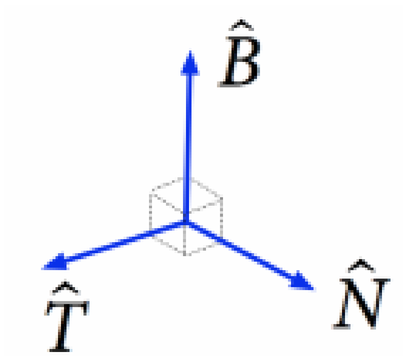
Osculating Plane/Circle & B-Hat ($\hat{B}$)

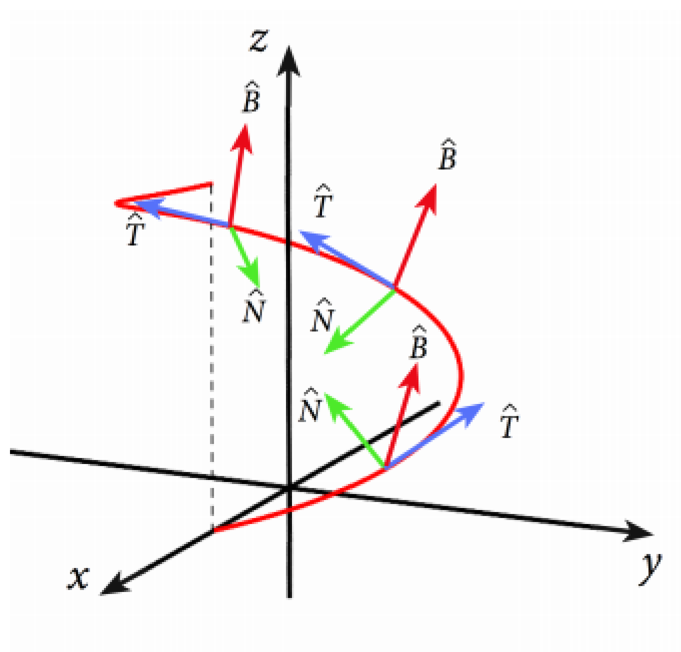
The osculating plane is the plane at $\vec{r}(t)$ containing the vectors $\hat{T}(t)$ and $\hat{N}(t)$. It is only defined when $\hat{N}(t) \neq \vec{0}$. One can also define the osculating circle, which lives in the osculating plane, is centered at $\vec{r}(t) + \dfrac{\hat{N}(t)}{\kappa(t)}$, and has radius $\dfrac{1}{\kappa(t)}$.





The unit normal vector that defines this plane is $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$, which is called the binormal vector. Together, $\hat{T}$, $\hat{N}$, and $\hat{B}$ create the Frenet Serret Frame, also called the TNB Frame.





Using $\hat{B}$, we can define is osculating plane as $\hat{B}(t) \cdot (\langle x, y, z \rangle - \vec{r}(t)) = 0$.

Differential Multivariable Calculus

Multivariable Functions

Multivariable take several variables an input and give a single value as output.

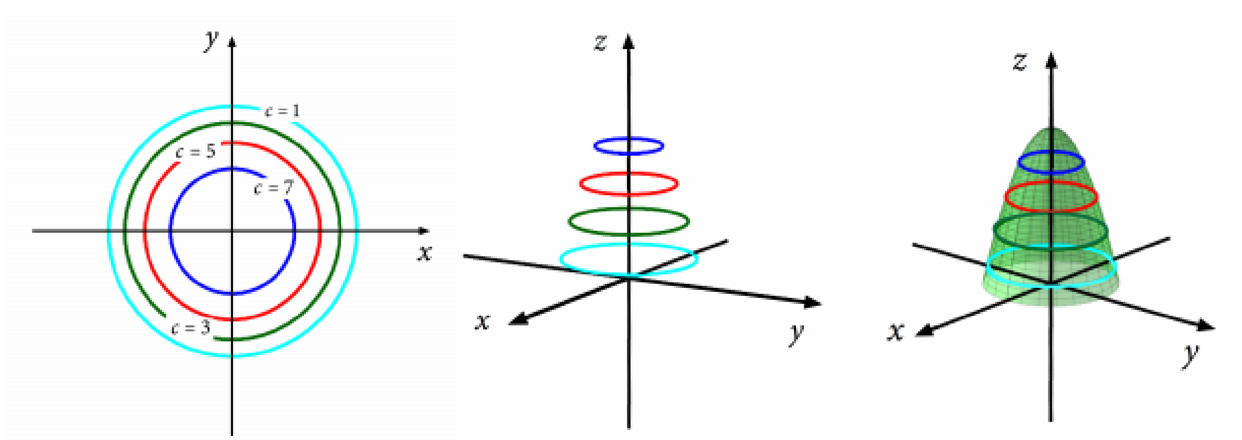
For example, $z = x^2 + y^2$ takes $\mathbb{R}^2 \to \mathbb{R}$.
Although we can only graph up to $\mathbb{R}^2 \to \mathbb{R}$ as a surface, we can imagine $\mathbb{R}^3 \to \mathbb{R}$ as a heatmap in 3D space, and the mathematics of theses functions works for any dimension.

The domain of a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$ is the largest set of points in $\mathbb{R}^n$ on which f is defined
.

For example, if $f(x, y) = \ln\left(9 - x^2 - y^2\right)$, the domain of $f(x, y)$ is
$\left\{(x, y)\,|x^2 + y^2 < 9\right\}$.

Level Curves

We can look at different cross sections of a surface $f(x, y)$ by looking at the equation $f(x, y) = c$ where $c \in \mathbb{R}$. This curve in the xy-plane is called the C-level curve. We often visualize these cross sections in the $z = c$ plane as part of the surface.
In higher dimensions, like for $f(x, y, z)$ a C-level curve becomes a C-level surface.



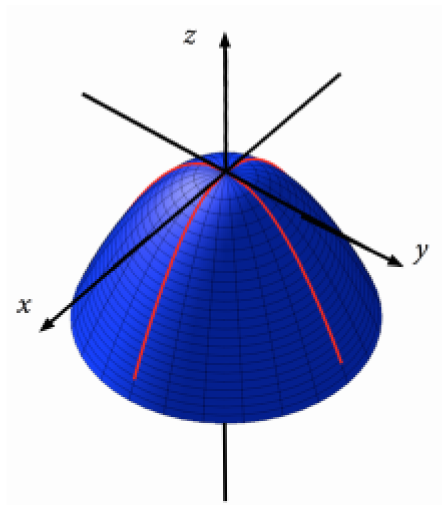
For example, the C-level surface of $e^{-\left(x^2+y^2+z^2\right)}$ is the sphere centered at the origin with radius $\sqrt{-\ln C}$: $-\ln C = x^2 + y^2 + z^2$.

Quadric Surfaces

Quadratic surfaces extend parabolas and other shapes that are composed of squared terms into 3-dimensional space in different ways.

Paraboloid

The paraboloid looks like a parabola that has been rotated about its center. It's radially symmetric, and its level curves are circles. It has the form $z = ax^2 + by^2$.

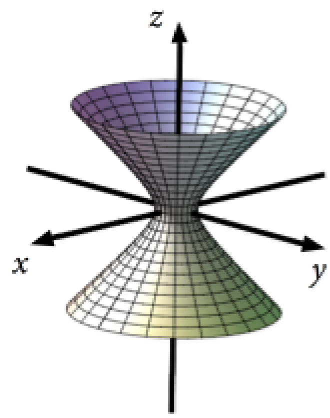


Hyperboloid

A Hyperboloid look like hyperbola that have been rotated about its center. So, it is also radially symmetric and has circular level curves. It has the form $C = \pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2}$ where one of the signs is different from the others. This gives rise to two different types of hyperboloids and a cone.

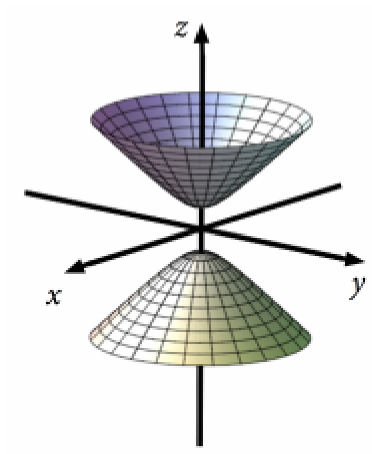
Hyperboloid of One Sheet

A hyperbolid of one sheet has 2 +'s and 1 -. This hyperboloid is made of one surface that is connected.



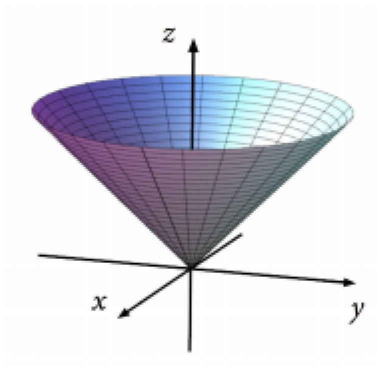
Hyperboloid of Two Sheets

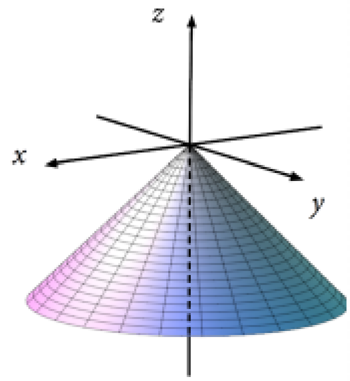
A hyperboloid of two sheets has 2 -'s and 1 +. This hyperboloid is made up of two disconnected surfaces.

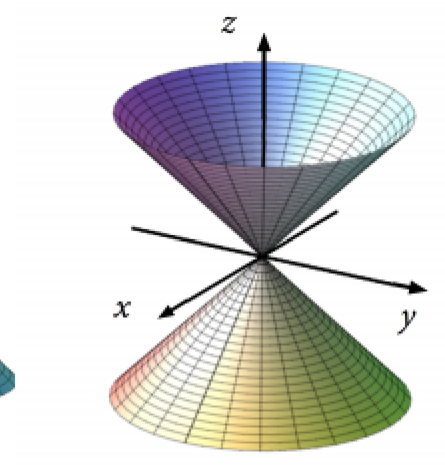


Cone

One of the cone's forms is $x^2 + y^2 - z^2 = 0$. This represents a transition between a hyperboloid of one and two sheets.

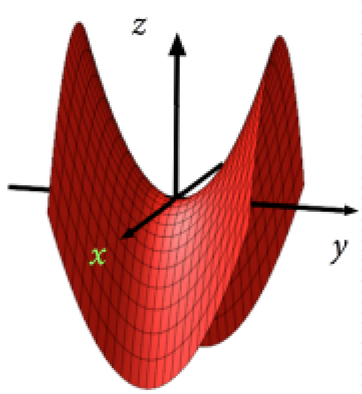






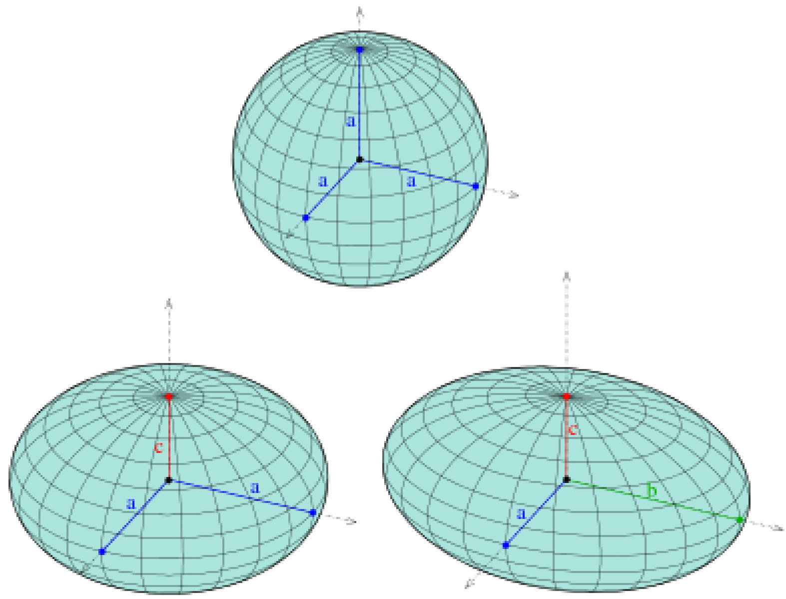
Hyperbolic Paraboloid

Hyperbolic paraboloids have the form $z = x^2 - y^2$. They are not radially symmetric. They look similar to a saddle or Pringles potato chip.



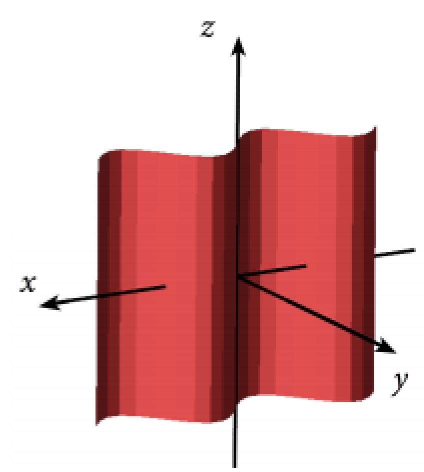
Ellipsoid

Ellipsoids look like ellipses that have been rotated along their axis. They have radial symmetry along this rotation axis. They have the general form $C = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$. Note that this is an identical form to a hyperboloid, but all of the signs are the same.



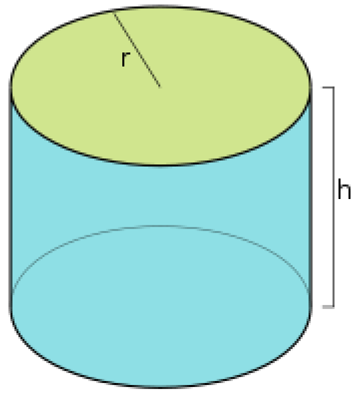
3D Cylinders

Although we normally only think of cylinders as being circular, one can create cylinders of any function of a single-variable by extruding the plane curve along an orthogonal axis. This means that cylinders can exist in 3 different forms depending on which axes the curve lives and is extruded along.



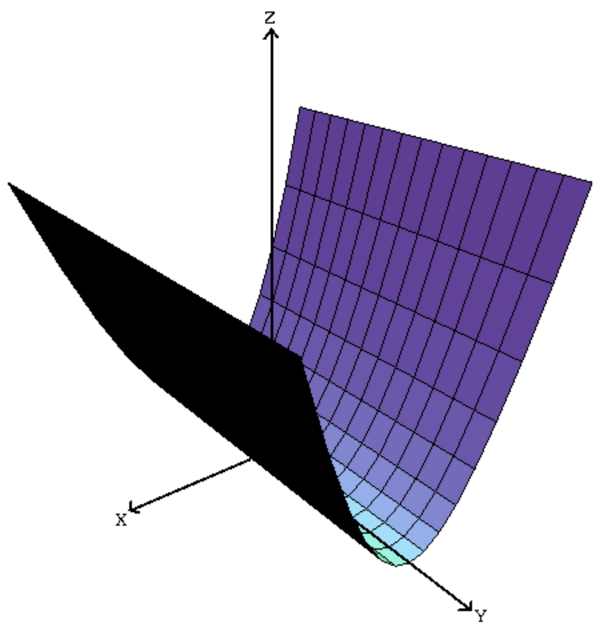
Circular Cylinder

A circular cylinder is what we usually think of as a cylinder. It is a circle extruded into 3D space. One of its forms is $x^2 + y^2 = R^2$ where R is the cylinder's radius.



Parabolic Cylinder

A parabolic cylinder looks like a parabola that has been extruded into 3D space. One of its forms is $f(x,y) = ay^2$.



Parameterized Surfaces

Parameterized surfaces are a natural extension of vector-valued functions that map $\mathbb{R}^n \to \mathbb{R}^m$ where usually $n < m$.

For example, a cylinder of radius 1 can be parameterized as $\vec{r}(u,v) = \langle \sin u, \cos u, v \rangle$ and maps $\mathbb{R}^2 \to \mathbb{R}^3$. The paraboloid $z = x^2 + y^2$ can be parameterized as $\vec{r}(u,v) = \langle u^2, v^2, u^2 + v^2 \rangle$.

A general trick when trying to parameterize a surface is to substitute u and v for x and y and find an expression for z. Although this doesn't always lead to the most useful parameterization, it can be a good starting point.

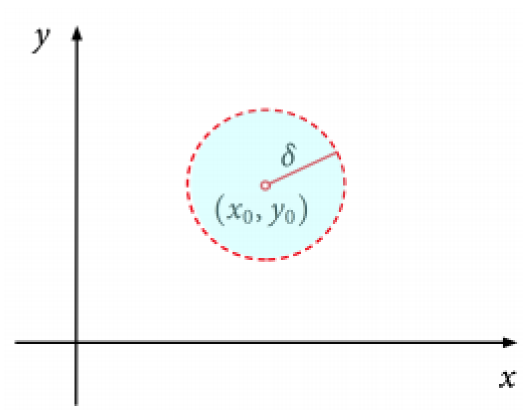
For example, if we wanted to parameterize the surface $y^2 = x^2 + z^2$ from $y = 1$ to $y = 9$, we could do $\vec{r}(u,v) = \langle u, \sqrt{u^2+v^2}, v\rangle$ where $1 \leq u^2 + v^2 \leq 9^2$. However, the bounds are a bit strange to work with because u and v are not independent. Recognizing that the surface is radially symmetric about the y-axis, we can let u be an angle and v be a radius to parameterize the surface as $\vec{r}(u,v) = \langle v\cos u, v, v\sin u\rangle$ where $0 \leq u \leq 2\pi$ and $1 \leq v \leq 9$. This parameterization has independent bounds, making operations like integration on the surface easier.

Limits & Continuity in 3D

Limits in single-variable calculus are relatively simple because there is a finite number of ways to approach any point on a curve. However, we have to be a bit more careful and formal in the definitions of limits and continuity in higher dimensions.

Open Delta Neighborhoods

An open δ-neighborhood of a point x_0 is defined as the set
$N(x_0, \delta) = \{x \in \mathbb{R}^n \mid ||x - x_0|| < \delta\}$



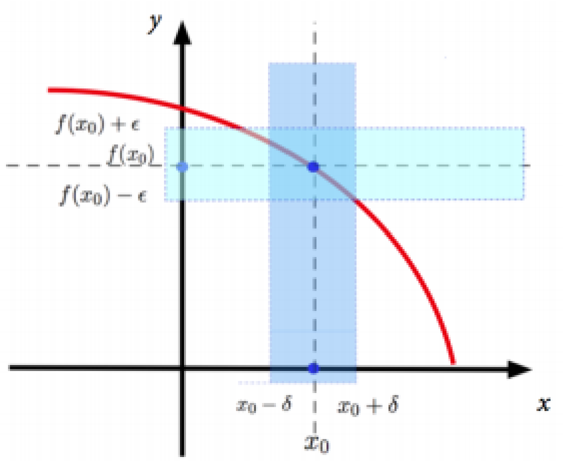
For example, $N(\langle 1,2\rangle, 7) = \{(x,y) \mid \sqrt{(x-1)^2 + (y-2)^2} < 7\}$ which is a ball (filled-in circle) of radius 7 centered at (1,2).

Boundary Points, Open & Closed Sets

Given some set, $\Omega \subset \mathbb{R}^n$, x is an interior point to Ω if $\exists \delta | N(x, \delta) \subset \Omega$. That is, x is an interior point to Ω if you can draw a circle with some non-zero radius around x such that the entire circle is in Ω.
All other points that are not interior points are boundary points. Formally, a point x is a boundary point of Ω if $\forall \delta \mid N(x, \delta) \not\subset \Omega$.
From here, we can define an open set to be one that doesn't contain any of its boundary points and a closed set to be one that contains all of its boundary points.

Limit & Continuity Definitions

Finally, we are ready to define a limit in multiple dimensions. We say that $\lim_{p \to p_0} f(p) = L$ if (L, ϵ) of L, $\exists N(p_0, \delta) \mid p \in N(p_0, \delta) \implies f(p) \in N(L, \epsilon)$.



Now with a limit definition, we can define continuity at a point. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at p_0 if $\lim_{p \to p_0} f(p) = f(p_0)$.

<u>Theorem</u>: Trigonometric functions, exponentials, logarithms, and sums, products, quotients, and compositions of such functions are continuous on their domain.

Although our definitions allow us to confirm that a value is the limit of a function, they do not give us any insight into how to find the value of a limit. For that, we have to approach our point of interest from every direction and see if the limit is the same. If any two outcomes are different, the limit doesn't exist.
We approach a function, f, by composing it with a single-variable path, $\vec{r}(t)$, that takes us over our point of interest, and then we find the limit.

If f is $f(x,y)$ and $\vec{r}(t)$ is $\langle x(t), y(t)\rangle$, then $f \circ \vec{r} = f(\vec{r}(t)) = f(x(t), y(t))$.

For example, let's try to find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^3}{x^2 + y^2}$.

We'll choose paths $\vec{r_1} = \langle t, 0\rangle$ and $\vec{r_2} = \langle t, t\rangle$ and find the limit as $t \to 0$ for both.

$$\lim_{t\to 0} f(\vec{r_1}(t)) = \lim_{t\to 0} \frac{t^2}{t^2} = 1$$

$$\lim_{t\to 0} f(\vec{r_2}(t)) = \lim_{t\to 0} \frac{t^2 - t^3}{t^2 + t^2} = \lim_{t\to 0} \frac{1}{2} - \frac{t}{2} = \frac{1}{2}$$

Since the limits on the two paths are not equal, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^3}{x^2 + y^2}\ \text{DNE}$.

Partial Derivatives

The single-variable calculus idea of tangent lines doesn't work in high dimensions because we can create many lines tangent to a point on a surface, depending on the plane that we use to slice the surface. That is, we can take to derivative at a point approaching from many directions.

Partial Derivatives of x, y, and z

It's common to look at the derivative from planes in the x, y, and z directions. These are called partial derivatives.

To compute $\dfrac{\partial}{\partial x} f(x,y)$, hold y constant and take the derivative with respect to x. Formally,
$\dfrac{\partial}{\partial x} f(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$ and $\dfrac{\partial}{\partial y} f(x,y) = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h}$.

We also use the shorthand $\dfrac{\partial}{\partial x} = f_x$ and $\dfrac{\partial}{\partial y} = f_y$. This shorthand can be extended so that $f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial x} f(x,y)\right)$.

Fubini's Theorem (also called Tonelli's or Clairaut's Theorem) says $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, and $f_{yz} = f_{zy}$. It extends into higher order mixed partial derivatives and says that two mixed partials are equal as long as they both differentiate the same number of variables the same number of times. This means that $f_{abcdab} = f_{aacdbb}$.

Tangent Planes

Although the tangent lines at a point on a surface can all be different depending on the direction one approaches a point from, all of these tangent lines lie in the same plane, defining the tangent plane. This means that the tangent plane to $z = f(x_0, y_0)$ has the following properties:

- The z-value of the tangent plane at (x_0, y_0) should be the same as (x_0, y_0).
- The value of the partial derivatives at (x_0, y_0) of the tangent plane should match those of $f(x_0, y_0)$.

The general form of a plane at (x_0, y_0, z_0) is $P(x, y) = A(x - x_0) + B(y - y_0) + z_0$.

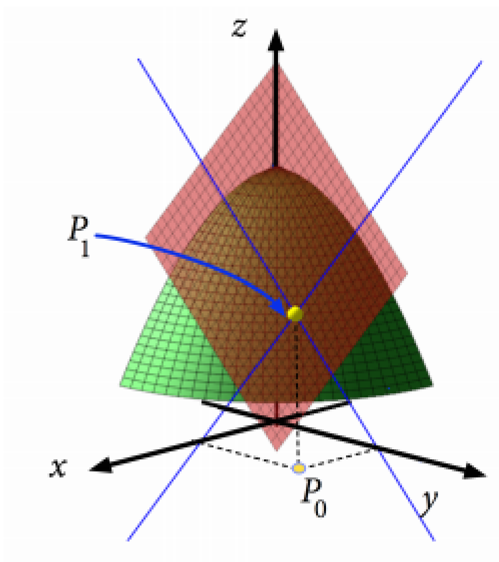
We want $\dfrac{\partial P}{\partial x} = \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial P}{\partial y} = \dfrac{\partial f}{\partial y}$.

This means that $P_x = f_x = A$ and $P_y = f_y = B$

Rewriting, $P(x, y) = f_x(x - x_0) + f_y(y - y_0) + z_0$

Normal Vector: $\langle \pm f_x, \pm f_y, \mp 1 \rangle$

So, the plane is $\langle -f_x, -f_y, 1 \rangle \cdot \langle x - x_0, y - y_0, z - f(x_0, y_0) \rangle = 0$.



Linear Approximations

Since $\partial z = f_x \partial x + f_y \partial y$, we can approximate Δz (the change in any function) as $\Delta z \approx f_x \Delta x + f_y \Delta y$. We use the tangent plane as the approximation for the function, similar to using the tangent line as an approximation in single-variable functions. We can also rewrite this as a dot product: $\Delta z \approx \langle f_x, f_y \rangle \cdot \langle \Delta x, \Delta y \rangle$.

For example, say a cylindrical can has radius $r = 1$ and height $h = 5$. If the radius is increased by .1 and the height by 1, what is the approximate ΔV?

$V = \pi r^2 h$, $V_r = 2\pi r h$, $V_h = \pi r^2$

$V_r(1,5) = 10\pi$, $V_h(1,5) = \pi$

$\Delta V \approx 10\pi(.1) + \pi(1) = 2\pi$

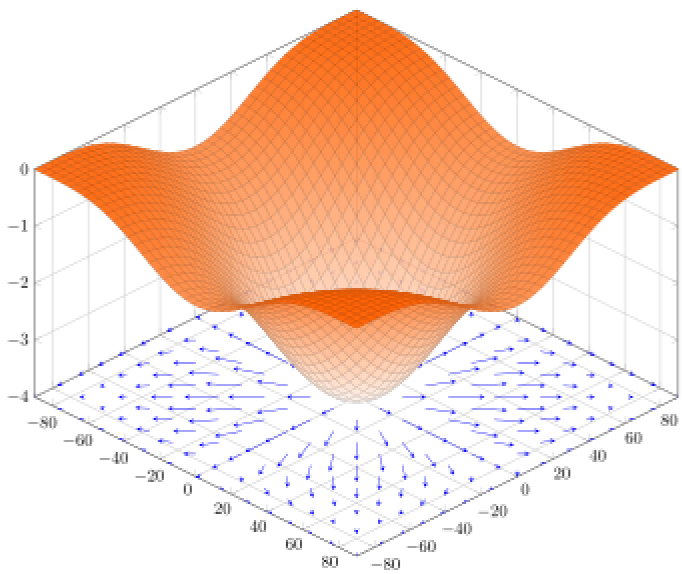
Comparing this to the real answer of 2.26π, we can see that our approximation is pretty good.

The Gradient

If I have a surface $f : \mathbb{R}^2 \to \mathbb{R}$, what direction, $\langle \Delta x, \Delta y \rangle$, should I go to maximize the change?

We saw earlier that $\Delta z \approx \langle f_x, f_y \rangle \cdot \langle \Delta x, \Delta y \rangle$. To maximize a dot product, $\langle \Delta x, \Delta y \rangle$ should be in the direction $\langle f_x, f_y \rangle$. We call this directional vector the gradient — the direction of steepest ascent.

Notated mathematically, $\nabla f(x,y) = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$.



Gradient Properties

The gradient has some important properties. Let f and g be functions of multiple variables and $\vec{r}$ be a vector-valued function.

1. $\nabla(f \pm g) = \nabla f \pm \nabla g$
2. $\nabla(af) = a\nabla f$
3. $\nabla(fg) = f\nabla g + g\nabla f$

4. $\nabla(f \circ \vec{r}(t)) = \nabla f \cdot \vec{r}(t)$

Linear Approximations with the Gradient

Property 4 can be generalized a bit further. Suppose we have $f(x, y, z)$, $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, and $g(u, v) = f \circ \vec{r}(u, v)$.

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

We can rewrite our linear approximation as $\Delta z \approx \nabla f \cdot \langle \Delta x, \Delta y\rangle$.

In fact, we have a way to find the derivatives of $f \circ \vec{r}(t)$.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \nabla f \cdot \langle x', y'\rangle.$$

Gradient & C-Level Curves

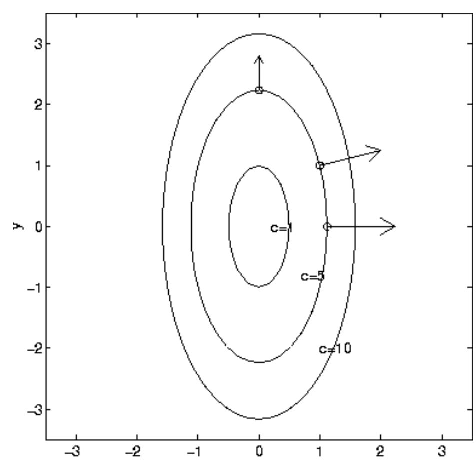
Let $\vec{r}(t)$ be the C-level curve of $f(x, y)$.

$f \circ \vec{r} = C$ and $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \vec{r}) = 0$

$\therefore \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \vec{r}) = \nabla f \cdot \vec{r'}(t) = 0$

$\therefore \nabla f \perp \vec{r'}(t)$

$\therefore \nabla f$ is perpendicular to the C-level curve of f.



Directional Derivatives

We already saw partial derivatives in the x, y, and z directions. However, we can go in any direction. This is called the directional derivative, $D_{\hat{u}}f$ where $\hat{u}$ is the direction.

The directional derivative at p_0 in the direction of $\hat{u}$: $D_{\hat{u}}f = \lim_{h\to 0}\frac{f(\vec{p_0}+h\hat{u})}{h}$.

If $\hat{u} = \langle a, b\rangle$, $D_{\hat{u}} = \lim_{h\to 0}\frac{f(x+ah, y+bh) - f(x,y)}{h}$.

Note that $D_{\hat{i}}f = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial f}{\partial x}$ and $D_{\hat{j}}f = \frac{\partial f}{\partial y}$.

Let's look at $D_{\hat{u}}f$.

$D_{\hat{u}}f = \lim_{h\to 0}\frac{f(x+ah, y+bh) - f(x+ah, y) + f(x+ah,y) - f(x,y)}{h}$

$= b\lim_{h\to 0}\frac{f(x+ah, y+bh) - f(x+ah,y)}{bh} + a\lim_{h\to 0}\frac{f(x+ah,y) - f(x,y)}{ah}$

$= b\frac{\partial f}{\partial y} + a\frac{\partial f}{\partial x}$

$= af_x + bf_y$

$\therefore D_{\hat{u}}f = \nabla f \cdot \hat{u}$

Optimization

What is the maximal rate of change of f. That is, when is $D_{\hat{u}}f$ maximized?

Since $D_{\hat{u}} = \nabla f \cdot \hat{u}$, $D_{\hat{u}}$ is maximised when $\hat{u} = \frac{\nabla f}{||\nabla f||}$ and $D_{\hat{u}} = ||\nabla f||$.

Similarly, the minimal rate of change is when $\hat{u}$ is $\frac{-\nabla f}{||\nabla f||}$ and $D_{\hat{u}} = -||\nabla f||$.

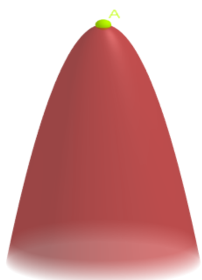
We can use these results and the gradient to find the minimums and maximums of functions. Often, we are concerned with minimizing a function that represents some sort of cost or inaccuracy. First, we need to establish some definitions.

Definitions

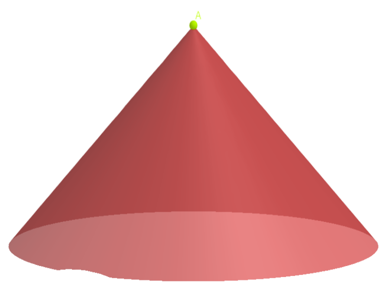
1. $f(x_0, y_0)$ is a local maximum of f if for some $\delta > 0$, $f(x_0,y_0) \geq f(x,y) \forall (x,y) \in N((x,y),\delta)$.
    a. That is, you can draw a circle in the domain of f such that every output point of f is less than $f(x_0, y_0)$.

2. $f(x_0, y_0)$ is a local minimum of f if for some $\delta > 0$, $f(x_0, y_0) \leq f(x, y) \forall (x, y) \in N((x, y), \delta)$.
    a. That is, you can draw a circle in the domain of f such that every output point of f is greater than $f(x_0, y_0)$.
3. $f(x_0, y_0)$ is a global max of f if $f(x_0, y_0) \geq f(x, y) \forall (x, y) \in D(f)$ where $D(f)$ is the domain of f.
4. $f(x_0, y_0)$ is a global min of f if $f(x_0, y_0) \leq f(x, y) \forall (x, y) \in D(f)$ where $D(f)$ is the domain of f.

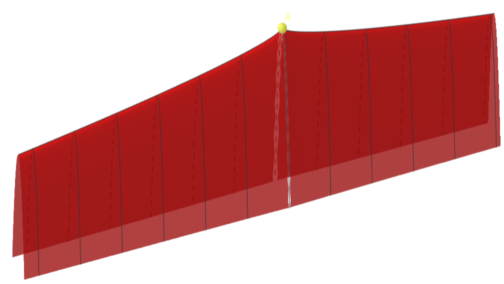
Theorem: If (x_0, y_0) is in the domain of f and a local extrema of $f(x, y)$, then $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ is either 0 or undefined.



$z = 9 - x^2 - y^2$
Both partials 0



$z = 9 - \sqrt{x^2 + y^2}$
Both Partials undefined



$z = 9 - x^{2/3} - y^2$
One partial defined, other undefined

Critical Points

Critical points are those the have the possibility of being a minimum or maximum. They are an extension of the critical points in single-variable calculus when the derivative is 0.

Definition: (x_0, y_0) is a critical point of $f(x, y)$ if f_x and f_y at (x_0, y_0) both DNE or are 0.

For example, consider the function $f(x, y) = x^2/2 - y^2/2 - xy - 2x - 2y$
$f_x = x - y - 2$ and is 0 when $y = x - 2$
$f_y = -y - x - 2$ and is 0 when $y = -x - 2$
$x - 2 = -x - 2$ when $x = 0$
When $x = 0$, $y = -2$
$\therefore (0, -2)$ is a critical point of f.

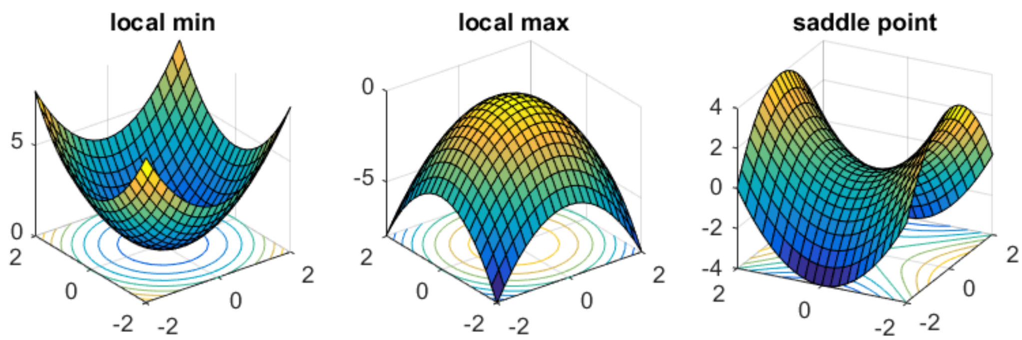
Second Derivative Test & Hessian Matrix

Recall from single-variable calculus that if we have a critical point (derivative is 0), we can find whether the point is a minimum or maximum by using the second derivative test. If the second derivative is < 0 at the critical point, then the critical point is a maximum, and if the second derivative is > 0 at the critical point, then the critical point is a minimum. A similar process works in higher dimensions.

In higher dimensions:

- If f_{xx} and $f_{yy} > 0$ at the critical point, then the critical point is a maximum.
- If f_{xx} and $f_{yy} < 0$ at the critical point, then the critical point is a minimum.
- If f_{xx} and f_{yy} don't agree on sign, then the point is a saddle point.

Although there are only 3 options for surfaces, we need a way that works for higher dimensional objects as well.



Definition: The Hessian Matrix (for $f(x, y)$): $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. $\det(H) = f_{xx}f_{yy} - f_{xy}^2$.

If (x_0, y_0) is a critical point:

- $\det(H(x_0, y_0)) > 0$ means (x_0, y_0) is an extrema.
  - If $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a minima.
  - If $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a maxima.

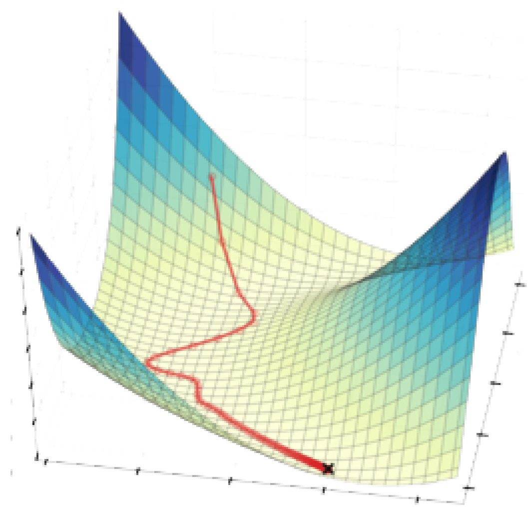
- $\det(H(x_0, y_0)) < 0$ means (x_0, y_0) is a saddle point.
- $\det(H(x_0, y_0)) > 0$ means the second derivative test is inconclusive.

Gradient Descent

Remember that if we have a multidimensional function, taking a step in the direction of the gradient results in the maximum possible increase of the function, and taking a step in the opposite direction of the gradient results in the maximum possible decrease of the function. Gradient descent is a method to find minima of functions.

Let's say we're trying to minimize $J(\vec{x})$ with gradient descent. Here are the steps we would take:

1. Pick (or guess) a starting point $\vec{x_0}$ and a learning rate (step size) δ.
2. $\vec{x_{n+1}} = \vec{x_n} - \delta \nabla J(\vec{x_n})$
3. Recursively repeat step 2 until some stopping criteria is met, like $||\delta \nabla J(\vec{x_n}) - \delta \nabla J(\vec{x_{n+1}})|| < .01$.



This method will lead you arbitrarily close to a local minimum, but does not guarantee finding the global minimum. More advanced versions of gradient descent exists that try to help with this, like giving the point “momentum” to be able to move out of local mins.
This method also has a tradeoff between speed and accuracy. Although increasing δ means fewer iterations of gradient descent are needed to narrow in on a local minimum, one is more likely to be stuck in a local min than had he or she used a smaller δ.

In the real world, your function is likely not going to be well defined enough to take its partial derivatives to find the gradient, so they too are approximated by doing something like

$$J_k \approx \frac{J(k+.0001, ...) - J(k, ...)}{.0001}.$$

See this presentation and this code for more on gradient descent.

Lagrange Multipliers

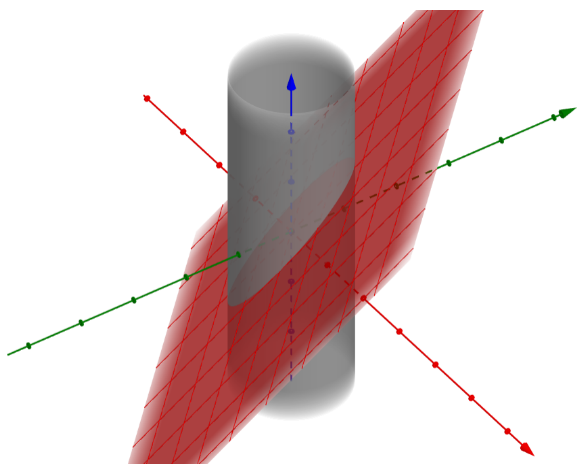
Lagrange multipliers are a method that allows us to find extrema subject to constraints in the domain.

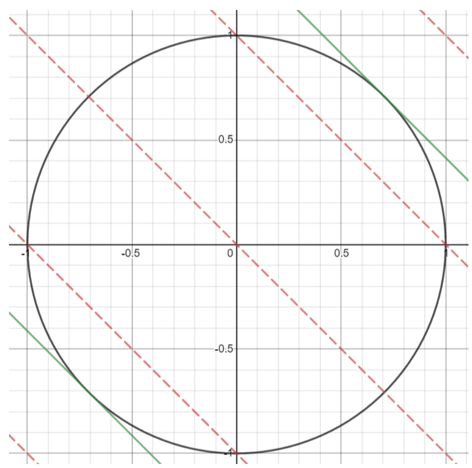
Toy Example

Say we wish to maximize $f(x, y) = x + y$ subject to the constraint $g(x, y) = x^2 + y^2 = 1$. We can do this by finding the C-level curve that is tangent to our constraint.

At this point, ∇f will be in the same direction as ∇g. That is, $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$.

We also add the constraint itself, $g(x, y) = k$, giving us a system of equations: $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = k \end{cases}$.





In our example above, $f_x = 1$, $g_x = 2x$, $f_y = 1$ and $g_y = 2y$ with constraint $x^2 + y^2 = 1$. Giving us a system

$$\begin{cases} 1 = \lambda 2x \\ 1 = \lambda 2y \\ x^2 + y^2 = 1 \end{cases} \implies x = y = \pm\frac{1}{\sqrt{2}} \text{ and } \lambda = \frac{1}{\sqrt{2}}.$$

This means that the max value of f constrained by g is $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$.

Method of Lagrange Multipliers

Given an objective function, $f(x, y)$ and a constraint, $g(x, y) = k$, define $F(x, y, \lambda) = f(x, y) + \lambda(k - g(x, y))$.

The solution, (x, y, λ), to $\nabla F = \vec{0}$ will be the solution to the constrained optimization problem.

For example, maximize $f(x, y) = xy$ subject to $(x-1)^2 + (y-1)^2 = 1$.

$$F(x, y, \lambda) = xy + \lambda\left(1 - (x-1)^2 - (y-1)^2\right)$$

$$\nabla F = \left\langle y + 2\lambda(x-1), x + 2\lambda(y-1), 1 - (x-1)^2 - (y-1)^2 \right\rangle = \vec{0}$$

$$\begin{cases} y - 2\lambda(x-1) = 0 \\ x - 2\lambda(y-1) = 0 \\ 1 - (x-1)^2 - (y-1)^2 = 0 \end{cases} \implies \begin{cases} y = 2\lambda(x-1) \\ x = 2\lambda(y-1) \\ (x-1)^2 + (y-1)^2 = 1 \end{cases}$$

$$y = 2\lambda(2\lambda(y-1) - 1) = 4\lambda^2 y - 4\lambda^2 - 2\lambda = \frac{2\lambda}{2\lambda + 1}$$

$$x = 2\lambda\left(\frac{2\lambda}{2\lambda+1} - 1\right) = \frac{2\lambda}{2\lambda+1} \implies x = y$$

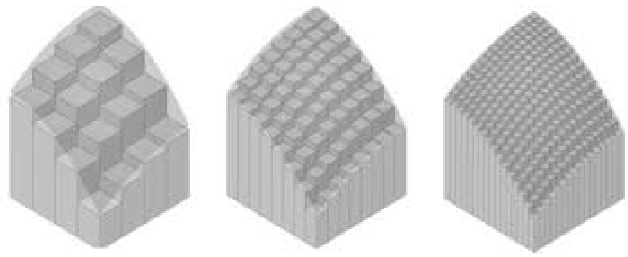
$$2(x-1)^2 = 1 \implies x = y = 1 \pm \frac{1}{\sqrt{2}}$$

Min/Max of $\frac{3 \mp 2\sqrt{2}}{2}$

Multiple Integrals

Double Integrals

Similar to how the limit of a Riemann Sum — the sum of the areas of small rectangles — is the area underneath a curve, we can find the volume underneath a surface by summing the volumes of small rectangular prisms.



In 2D (single-variable): $\Delta x = \dfrac{b-a}{n}$, $\displaystyle\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(a+i\Delta x)\Delta x$.

In 3D: $\Delta x = \dfrac{b-a}{n}$, $\Delta y = \dfrac{d-c}{n}$,

$$\int_c^d \int_a^b f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{m\to\infty}\sum_{j=0}^{m-1}\left(\lim_{n\to\infty}\sum_{i=0}^{n-1} f(a+i\Delta x, c+j\Delta y)\Delta x\right)\Delta y$$.

Domain Regions & Fubini's Theorem

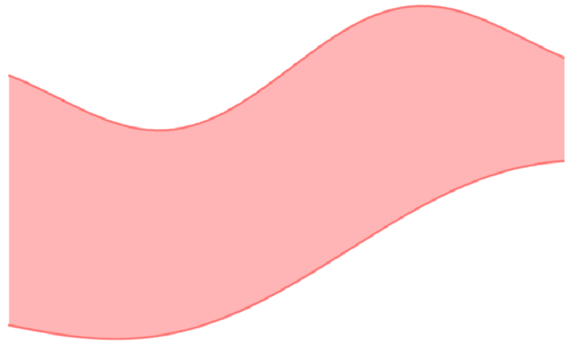
For example, let's find the volume under $f(x,y) = 9 - x^2 - y^2$, $(x,y) \in [0,1] \times [1,2]$. We'll do it twice: one with x first and then y and another with y first and then x and compare.

$$\begin{aligned}
V &= \int_0^1 \int_1^2 9 - x^2 - y^2 \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left[9y - x^2y - \frac{y^3}{3}\right]_1^2 \mathrm{d}x \\
&= \int_0^1 \frac{20}{3} - x^2 \mathrm{d}x \\
&= \left[\frac{20}{3}x - \frac{x^3}{3}\right]_0^1 \\
&= \frac{19}{3}
\end{aligned}$$

$$\begin{aligned}
V &= \int_1^2 \int_0^1 9 - x^2 - y^2 \mathrm{d}x\mathrm{d}y \\
&= \int_1^2 \left[9x - \frac{x^3}{3} - xy^2\right]_0^1 \mathrm{d}y \\
&= \int_1^2 \frac{26}{3} - y^2 \mathrm{d}y \\
&= \left[\frac{26}{3}y - \frac{y^3}{3}\right]_1^2 \\
&= \frac{19}{3}
\end{aligned}$$

Theorem: Fubini's Theorem states that the order of integration on a domain where the variables of integration (x, y, etc.) vary independently doesn't matter.

Let's look at a case where x and y are not independent. Specifically, where the upper and lower bounds on y are a function of x.



This is called a Type I Region. Formally, a Type I Region is a domain, D, where
$D = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq h(x)\}$.

Theorem: Let D be a Type I Region in $\mathbb{R}^2$. Fubini's Theorem for Type I Regions says that

$$\iint_D f(x,y)\mathrm{d}A = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\mathrm{d}y\mathrm{d}x.$$

There is also the case where y varies independently and x is bounded as a function of y. This is called a Type II Region. Formally a Type II Region is a domain, D, where
$D = \{(x, y) \mid g(y) \leq x \leq h(y), a \leq y \leq b\}$.

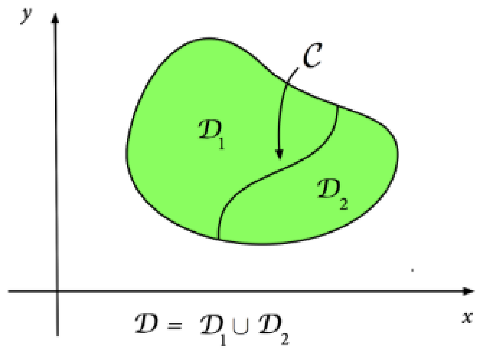
Theorem: Let D be a Type II Region. Fubini's Theorem for Type II Regions says that

$$\iint_D f(x,y)\mathrm{d}A = \int_a^b \int_{g(y)}^{h(y)} f(x,y)\mathrm{d}x\mathrm{d}y.$$

Sometimes, a region can be described as both Type I and Type II. One should pick whichever interpretation is most convenient.

Both of the previous two theorems can be summarized as dependent variables need to be integrated before the variables that they depend on. This core idea extends into higher dimensions where classifying regions becomes tedious and not very helpful.

One can also split a larger, harder to describe domain into smaller chunks. Let $D_1 \cup D_2 = D$.



$$\iint\limits_{D} f(x,y)\mathrm{d}A = \iint\limits_{D_1} f(x,y)\mathrm{d}A + \iint\limits_{D_2} f(x,y)\mathrm{d}A - \iint\limits_{D_1 \cap D_2} f(x,y)\mathrm{d}A.$$

Average Values

We can think of the average value of a function over some interval as the answer to the question: "If I flattened this function into a box over the interval, what would the height of the box be?"

For single-variable functions, the answer was $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{\int_a^b f(x)\mathrm{d}x}{\int_a^b \mathrm{d}x}$.
This idea of summing the function over a domain and dividing by the size of the domain can be generalized into multi-variable as $\bar{f} = \frac{\iint\limits_{D} f(x,y)\mathrm{d}A}{\iint\limits_{D} \mathrm{d}A}$.

Mean Value Theorem

<u>Theorem</u>: If a function, f, is continuous on a domain, D, then there exists some point, $p \in D$ such that $f(p) = \bar{f}$

Volume Between Surfaces

Similar to how we could find the area between two curves in single-variable calculus, we can find the volume between two surfaces.

$$V_{\text{btwn}} = \iint\limits_D f \mathrm{d}A - \iint\limits_D g \mathrm{d}A = \iint\limits_D (f-g)\mathrm{d}A$$
.

If g is normally below f in domain D_1 but crosses above g in a domain D_2, we can still find the volume between by splitting our integral into two domains.

$$V_{\text{btwn}} = \iint\limits_D |f-g|\,\mathrm{d}A = \iint\limits_{D_1} (f-g)\mathrm{d}A + \iint\limits_{D_2} (g-f)\mathrm{d}A$$

Plane Laminas

Definition: A plane lamina is an idealized 2D object with mass the occupies a region $D \subset \mathbb{R}^2$.

Some questions one may ask about a plane lamina are about the total mass and center of mass of the lamina.

We can think of the mass as $M = \iint\limits_D \sigma(x,y)\mathrm{d}A$.

The center of mass is: $\bar{x} = \frac{M_y}{M} = \frac{\iint\limits_D x\sigma(x,y)\mathrm{d}A}{M}$ and $\bar{y} = \frac{M_x}{M} = \frac{\iint\limits_D y\sigma(x,y)\mathrm{d}A}{M}$. Where M_x is the moment about the x-axis, and M_y is the moment about the y-axis.

Triple Integrals

Triple integrals work much the same way as single and double integrals. They are still defined by a Riemann sum, and Fubini's theorems about independent domains and the order of integration still applies.

Fubini's Theorem for Z-Simple Regions

There is however another case of Fubini's Theorem that arises in 3D.

Theorem: Let $D \subset \mathbb{R}^3$. Let $\Omega = \{(x,y,z) \mid (x,y) \in D, g(x,y) \leq z \leq h(x,y)\}$ be a z-simple region. Fubini's Theorem for Z-Simple Regions says that:

$$\iiint\limits_\Omega f(x,y,z)\mathrm{d}V = \iint\limits_D \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\mathrm{d}z \right) \mathrm{d}A$$

. In other words, dependent variables must be integrated before independent ones, but the order of integration for independent variables doesn't matter.

For example, we can express the unit sphere is a z-simple region where $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $-\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2}$. Note that D is a Type I region.

$$V_{\text{sphere}} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \mathrm{d}z\mathrm{d}y\mathrm{d}x = \frac{4\pi}{3}$$

Plane Laminas

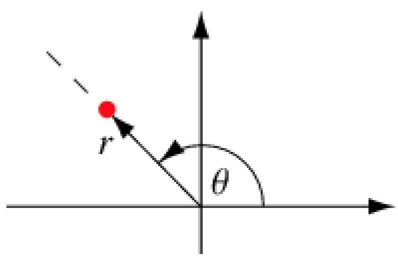
Similar to 2D laminas, $M = \iiint\limits_{\Omega} \sigma(x, y, z)\mathrm{d}V$, $\bar{x} = \dfrac{\iiint\limits_{\Omega} x\sigma(x, y, z)\mathrm{d}V}{M} = \dfrac{M_{yz}}{M}$, $\bar{y} = \dfrac{\iiint\limits_{\Omega} y\sigma(x, y, z)\mathrm{d}V}{M} = \dfrac{M_{xz}}{M}$, and $\bar{z} = \dfrac{\iiint\limits_{\Omega} z\sigma(x, y, z)\mathrm{d}V}{M} = \dfrac{M_{xy}}{M}$.

Curvilinear Coordinates

In 2D you should know about Cartesian (x, y) coordinates and polar (r, θ) coordinates. Cartesian extends into 3D as (x, y, z), but there are multiple ways to extend polar coordinates in 3D.

Review of Polar Coordinates

Polar coordinates represent every point in 2D space as a distance from the origin, r, and an angle from the horizontal, θ. This means that unlike rectangular (x, y) coordinates, different polar coordinates can represent the same point: $(2/-\pi/4) = (-2, 3\pi/4) = (2, 7\pi/4)$.



Polar coordinates can be transformed into rectangular coordinates by $x = r\cos\theta$ and $y = r\sin\theta$. This means that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

Circles

A circle of radius R center can be represented as $r = R$.

Circles off the origin require using our transformation equations.

$(x-a)^2+(y-b)^2=R^2 \implies (r\cos\theta-a)^2+(r\sin\theta-b)^2=R^2$

$r^2\cos^2\theta+r^2\sin^2\theta-2ra\cos\theta-2rb\sin\theta+a^2+b^2=R^2$

$r^2-2r(a\cos\theta+b\sin\theta)-R^2+a^2+b^2=0$

$r=(a\cos\theta+b\sin\theta)\pm\sqrt{R^2-a^2-b^2+(a\cos\theta+b\sin\theta)^2}$

Lines

- Lines through the origin can be represented as $\theta=\tan^{-1}m$ where m is the slope of the line.
- Lines of the form $x=a$ can be represented as $r=a\sec\theta$.
- Lines of the form $y=a$ can be represented as $r=a\csc\theta$.
- All other lines of rectangular form $y=ax+b$ can be represented as $r=\dfrac{b}{\sin\theta-a\cos\theta}$. This form covers the previous two ones as well.

Integration

The line element is $\mathrm{d}s^2=\mathrm{d}r^2+r^2\mathrm{d}\theta^2$, meaning that $s=\displaystyle\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)}\mathrm{d}\theta$.

The area element is $\mathrm{d}A=r\mathrm{d}r\mathrm{d}\theta$, meaning that $A=\displaystyle\int_{\theta_1}^{\theta_2}r^2\mathrm{d}\theta$.

Gaussian Integral

For example, let's compute $\displaystyle\iint_D e^{-x^2-y^2}\mathrm{d}A$ where D is the unit disk.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}e^{-x^2-y^2}\mathrm{d}y\mathrm{d}x=\int_0^1\int_0^{2\pi}e^{-r^2}r\mathrm{d}\theta\mathrm{d}r=\int_0^1 re^{-r^2}\mathrm{d}r\cdot\int_0^{2\pi}\mathrm{d}\theta$$

Let $u=-r^2$, $\mathrm{d}u=-2r\mathrm{d}r$.

$=\dfrac{1}{2}\displaystyle\int_0^{-1}e^u\mathrm{d}u\cdot 2\pi$

$=\pi\left(e^0-e^{-1}\right)$

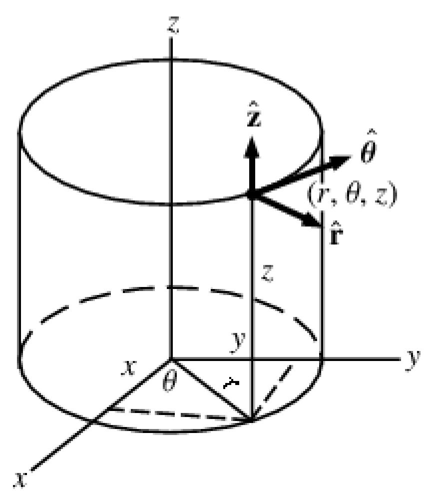
$=\pi\left(1-\dfrac{1}{e}\right)$

Now, let's have $D=\mathbb{R}^2$. This is called the Gaussian Integral.

$$= \int_0^\infty \int_0^{2\pi} re^{-r^2} \mathrm{d}\theta \mathrm{d}r$$
$$= \int_0^\infty re^{-r^2} \mathrm{d}r \cdot \int_0^{2\pi} \mathrm{d}\theta$$
Let $u = -r^2$, $\mathrm{d}u = -2r\mathrm{d}r$.
$$= \pi \int_0^{-\infty} e^u \mathrm{d}u$$
$$= \pi \left(\left(\lim_{a \to -\infty} e^a \right) - e^0 \right)$$
$$= \pi$$

Cylindrical Coordinates

Cylindrical coordinates are the expansion of polar coordinates by including a third term that represents the height from the xy-plane, z. All cylindrical coordinates have the form (r, θ, z). This system is called cylindrical because it's easy to describe shapes with cylindrical symmetry because integrations have constant, independent bounds.



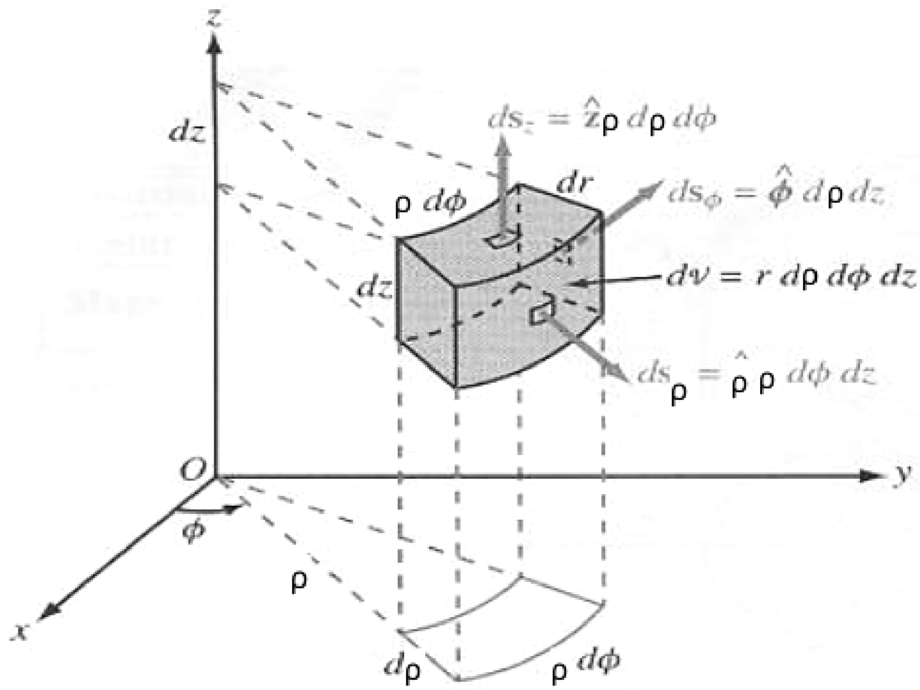
Conversions

From Cylindrical: $(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$

From Cartesian: $(x, y, z) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\dfrac{y}{x}, z\right)$

Integration

The volume element is $\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$.



For example, let's evaluate $\int_0^4 \int_0^{\sqrt{16-y^2}} \int_0^{16-x^2-y^2} \mathrm{d}z\mathrm{d}x\mathrm{d}y$, the volume under the paraboloid $z = 16 - x^2 - y^2$, by converting the integral to cylindrical coordinates.

$$= \int_0^4 \int_0^{\pi/2} \int_0^{16-r^2} r\mathrm{d}z\mathrm{d}\theta\mathrm{d}r$$

$$= \int_0^4 \int_0^{\pi/2} 16r - r^3\mathrm{d}\theta\mathrm{d}r$$

$$= \frac{\pi}{2}\int_0^4 16r - r^3\mathrm{d}r$$

$$= \frac{\pi}{2}\left[8r^2 - \frac{r^4}{4}\right]_0^4$$

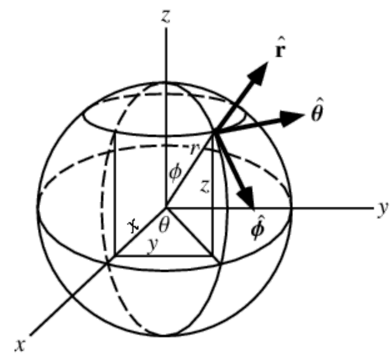
$$= 32\pi$$

For another example, let's find the average value of $f(x, y, z) = z$ on Ω which is bounded by $z = \sqrt{6 - x^2 - y^2}$ and $z = x^2 + y^2$.

$$\bar{f} = \frac{\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} zr\mathrm{d}z\mathrm{d}r\mathrm{d}\theta}{\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta} = \frac{11}{12\sqrt{6} - 17}$$ Work omitted for brevity

Spherical Coordinates

Unlike how cylindrical coordinates extend polar coordinates into 3D by adding a Cartesian term, spherical coordinates add an angular term, ϕ the azimuthal angle. All spherical coordinates have the form (ρ, θ, ϕ) where ρ is the distance from the origin, θ is the polar angle in the xy-plane, and ϕ is the azimuthal angle from the +z-axis. Shapes with spherical symmetry have constant bounds of integration in spherical coordinates.



Conversions

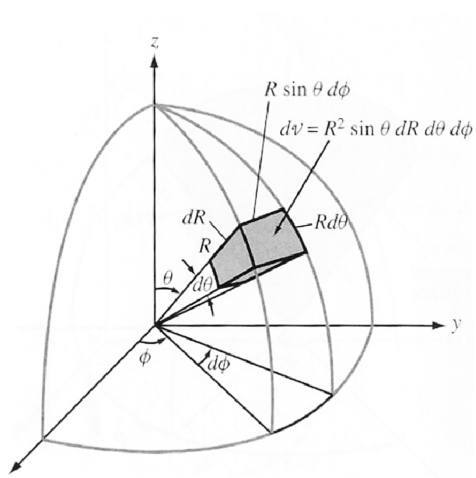
From spherical: $(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$

From Cartesian: $$(x, y, z) = \left(\sqrt{x^2+y^2+z^2}, \tan^{-1}\frac{y}{x}, \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

Integration

The area element is $\mathrm{d}A = \rho^2 \sin\phi \mathrm{d}\theta\mathrm{d}\phi$.

The volume element is $\mathrm{d}V = \rho^2 \sin\phi \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$.



Sphere Volume

Since a sphere of radius R is spherically symmetric, it should be easy to find its volume in spherical coordinates.

$$V = \int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho$$

$$= \int_0^R \rho^2 \mathrm{d}\rho \cdot \int_0^\pi \sin\phi \mathrm{d}\phi \cdot \int_0^{2\pi} \mathrm{d}\theta$$

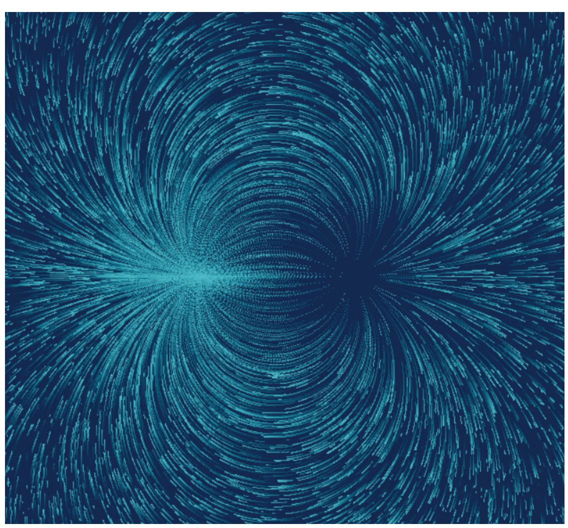
$$= \frac{R^3}{3} \cdot 2 \cdot 2\pi$$

$$= \frac{4\pi}{3} R^3$$

Line & Surface Integrals

Vector Fields

Vector fields are a function $f : R^n \to R^n$. This is generally conceptualized conceptualized as assigning a n-dimensional vector to every point in n-dimensional space.
Many physics concepts can be thought of as vector fields. The electric field due to some point change Q at some distance r from Q is given by $\vec{E}(x, y, z) = \frac{\epsilon_0 Q}{r^2}\hat{r}$ where $\hat{r}$ is a radial unit vector pointing away from Q.



Electric field between positive and negative charge.

A generic 2D vector field is $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. Vector fields work similarly to vector-valued functions in that they are added component-wise.

Line Integrals

Line Integrals of Scalar Functions

Definition: Let's say we have some simple (non self-intersecting) curve, C, in the xy-plane parameterized by $\vec{r}(t) = \langle x(t), y(t)\rangle$ and a surface $z = f(x,y)$ above C. We can extrude C up to f, forming a "curtain" with area A that can be found through integration.

$$\mathrm{d}A = \text{height of } f \cdot \text{small length of } C$$

$$\mathrm{d}A = f(x,y) \cdot \mathrm{d}s$$

$$s = \int \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t$$

$$\mathrm{d}s = \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t$$

$$\therefore A = \int_C f(x,y)\mathrm{d}s = \int \left(f \circ \vec{r}\right)(t) \cdot \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t$$, the line integral of $\vec{r}$ on f.

$(f \circ \vec{r})$ is called the "pullback".

For example, let's find the line integral of $y = x^2$ for $0 \le x \le \sqrt{2}$ on $f(x,y) = 2x$.

$$\vec{r}(t) = \langle t, t^2\rangle, 0 \le t \le \sqrt{2}$$

$$\int_C f(x,y)\mathrm{d}s = \int_0^{\sqrt{2}} \left(2x \circ \langle t, t^2\rangle\right) \cdot \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t = \int_0^{\sqrt{2}} 2t \cdot \sqrt{1+4t^2}\mathrm{d}t$$

$$= \frac{1}{4}\int_1^9 \sqrt{u}\mathrm{d}u = \frac{13}{3}$$

Line Integrals of Vector Fields

One can think of line integral of vector fields as the total work done by the vector field as it moves along some simple path.

$$W = \int_C \left(\vec{F} \circ \vec{r}\right) \cdot \hat{T}\mathrm{d}s = \int_C \left(\vec{F} \circ \vec{r}\right) \cdot \frac{\vec{r'}(t)}{\left|\left|\vec{r'}(t)\right|\right|} \cdot \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t$$

$$= \int_C \left(\vec{F} \circ \vec{r}\right) \cdot \vec{r'}(t)\mathrm{d}t = \int_C \vec{F} \cdot \mathrm{d}\vec{r}$$.

For example, let's find the line integral of $\vec{r}(t) = \langle t, t^2, t\rangle$ for $0 \le t \le 1$ in the vector field $\vec{F}(x, y, z) = \langle e^z, \sqrt{1-x^2}, \sin x\rangle$.

$\vec{F} \circ \vec{r} = \langle e^t, \sqrt{1-t^2}, \sin t\rangle$

$\vec{r}'(t) = \langle 1, 2t, 1\rangle$

$$\int_0^1 \langle e^t, \sqrt{1-t^2}, \sin t\rangle \cdot \langle 1, 2t, 1\rangle \mathrm{d}t$$

$$= \int_0^1 e^t + 2t\sqrt{1-t^2} + \sin t \mathrm{d}t$$

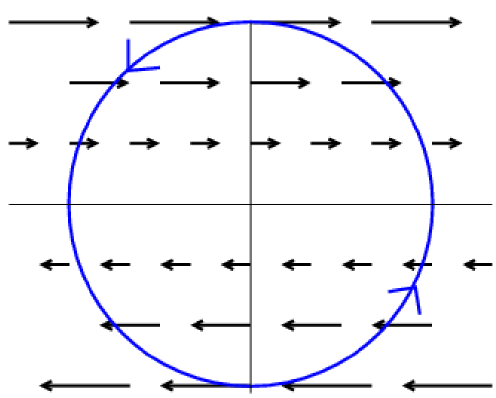
$$= e - \cos 1 + \frac{2}{3}$$

Direction Matters

<u>Lemma</u>: $\int_{-C} \vec{F} \cdot \mathrm{d}\vec{r} = -\int_C \vec{F} \cdot \mathrm{d}\vec{r}$. That is the direction in which one takes a line integral matters. You can think of this as the wind helping you sail in one direction (positive line integral) but fighting against you in the opposite direction (negative line integral).

Circulations

<u>Definition</u>: If C is a simple, closed curve, then $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ is the circulation of $\vec{F}$ on C. We notate that we are taking a circulation as $\oint_C \vec{F} \cdot \mathrm{d}\vec{r}$.



For example, let's find the circulation of $\vec{F}(x, y, z) = \langle yz, xz, xy\rangle$ on the circle of radius 1 centered at (0,0,1) in the z=1 plane in the counter-clockwise direction.

$\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$, $0 \le t \le 2\pi$

$\vec{F} \circ \vec{r} = \langle \sin(t), \cos(t), \sin(t)\cos(t) \rangle$

$\left(\vec{F} \circ \vec{r}\right) \cdot \vec{r}'(t) = \cos(2t)$

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{2\pi} \cos(2t)\mathrm{d}t$$

$$= \frac{1}{2}\sin(2t)\Big|_0^{2\pi} = 0$$

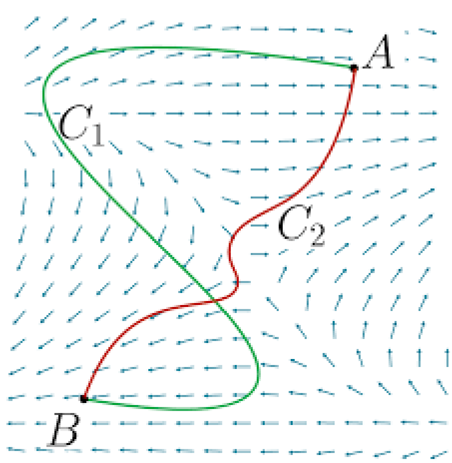
Conservative Vector Fields

Definition: A vector field, $\vec{F}$, is conservative if $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ is the same for all C connecting the same endpoints.

It's easy to see from this definition that vector fields of constant direction and magnitude, like $\vec{F}(x, y, z) = \langle a, a, a \rangle$ is conservative, as its line integral only depends on the curve.

Theorem: If $\vec{F}$ is conservative, then $\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$.

Proof: We can break the simple, closed curve, C, into two simple curves, C_1 and C_2, that have the same endpoints and direction such that $C = C_1 - C_2$.



So, $\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} - \int_{C_2} \vec{F} \cdot \mathrm{d}\vec{r}$

Since C_1 and C_2 have the same direction and endpoints, and $\vec{F}$ is conservative, the line integrals have the same value, L.

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = L - L = 0$$
.

FTC for Line Integrals

We saw earlier that $\mathrm{d}z = \nabla f \cdot \langle \mathrm{d}x, \mathrm{d}y \rangle$. This can be written as $\mathrm{d}z = \nabla f \cdot \mathrm{d}\vec{r}$ where $\vec{r}(t)$ parametrize a simple curve, C, and $a \leq t \leq b$.

So, $\Delta z = \int_C \nabla f \cdot \mathrm{d}\vec{r} = \int_a^b (\nabla f \circ \vec{r}) \cdot \vec{r}\mathrm{d}t = f \circ \vec{r}\Big|_a^b = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$.

$\therefore \int_C \nabla f \cdot \mathrm{d}\vec{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$, the fundamental theorem of calculus for line integrals.

Let's test this be computing a line integral directly and with the FTC. Let the path be the top-half semicircle connecting (1,0) to (-1,0), and let $f(x, y) = 12 - 3x - y$.

Directly:

$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$

$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$

$\nabla f = \langle -3, -1 \rangle$

$$\int_0^\pi 3\sin t - \cos t\mathrm{d}t = -3\cos t - \sin t\Big|_0^\pi = 6$$

By FTC:

$\vec{r}(0) = \langle 1, 0 \rangle$, $\vec{r}(\pi) = \langle -1, 0 \rangle$

$f\left(\vec{r}(0)\right) = 9$, $f\left(\vec{r}(\pi)\right) = 15$

$15 - 9 = 6$

Potential Functions

Note that the FTC for line integrals doesn't care about the specific path, only the starting and ending points. Any path that started and ended in the same place would have line integral of 0.

So, $\oint_C \nabla f \cdot \mathrm{d}\vec{r} = 0$. Therefore, by our theorem about conservative vector fields, all vector fields, $\vec{F}$, such that $\vec{F} = \nabla f$ are conservative. Such an f is called the potential function of $\vec{F}$.

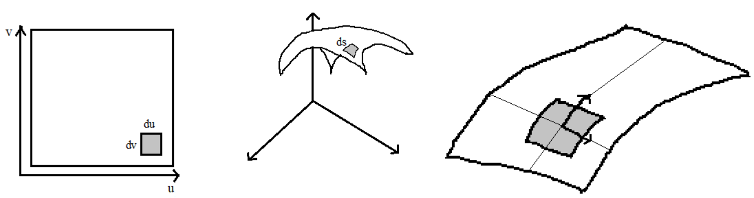
Test for a Conservative Vector Field

Since we know that conservative vector fields have potential functions, we can devise a test for conservative vector fields. $\vec{F}_{\text{Conservative}} = \langle f_x, f_y \rangle$, so $f_{xy} = f_{yx}$. That is, a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is conservative if $P_y = Q_z$.
For 3D vector fields, it a bit more complicated. A vector field $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is conservative if $P_y = Q_x$, $Q_z = R_y$, and $R_x = P_z$.

For example, let's see if $\langle yz, xz, xy \rangle$ is conservative.
$\frac{\partial}{\partial y}yz = z = \frac{\partial}{\partial x}xz$, $\frac{\partial}{\partial z}xz = x = \frac{\partial}{\partial y}xy$, and $\frac{\partial}{\partial x}xy = y = \frac{\partial}{\partial z}yz$.
So, the vector field is conservative.

Surface Integrals

We can parameterize any surface as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ (or some other coordinate system). This is because we are essentially taking one 2D surface, like a uv-plane, and transforming it into another surface in a way that areas near each other in the uv-plane are near each other in our new surface. The fact that areas stay close together allow us to make statements about complicated surfaces while working in the simple uv-plane.



The change of the surface in the u-direction is $\frac{\partial \hat{r}}{\partial u}\mathrm{d}u$
The change of the surface in the v-direction is $\frac{\partial \hat{r}}{\partial u}\mathrm{d}v$
So, the area on the surface in relation to u and v is the area of the parallelogram spanned by these two surfaces: a cross product. $\mathrm{d}s = \|\langle \frac{\partial \vec{r}}{\partial u}\mathrm{d}u \rangle \times \langle \frac{\partial \vec{r}}{\partial}\mathrm{d}v \rangle\| = \|\vec{r}_u \times \vec{r}_v\|\mathrm{d}u\mathrm{d}v$.

Definition: For a surface S parameterized by $\vec{r}(u,v)$ and $(u,v) \subset D$, then the surface area is

$$A(s) = \iint_S \mathrm{d}s = \iint_D \|\vec{r}_u \times \vec{r}_v\| \mathrm{d}u\mathrm{d}v$$

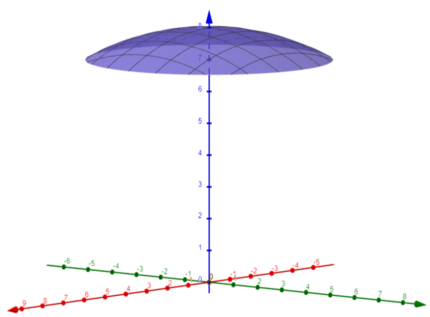
.

Surface Integrals of Scalar Functions

Definition: The surface integral of a scalar function is

$$\iint_S f(x,y,z)\mathrm{d}s = \iint_D (f \circ \vec{r})\, \|\vec{r}_u \times \vec{r}_v\| \mathrm{d}A$$

Let's apply a surface integral to a real problem. Let the cap of the sphere or radius $8m$ centered at the origin between z=7 and z=8 have charge density $\sigma(x,y,z) = z\ \mu C/m^2$. Find the total charge on the cap.



We will parametrize the sphere in spherical coordinates.

$$D = \left\{ (\rho, \theta, \phi) \mid \rho = 8, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \cos^{-1}\left(\frac{7}{8}\right) \right\}$$

$$\vec{r}(\theta, \phi) = \langle 8 \sin\phi \cos\theta, 8 \sin\phi \sin\theta, 8 \cos\phi \rangle$$

$$\vec{r}_\theta = \langle -8 \sin\phi \sin\theta, 8 \sin\phi \cos\theta, 0 \rangle$$

$$\vec{r}_\phi = \langle 8 \cos\phi \cos\theta, 8 \cos\phi \sin\theta, -8 \sin\phi \rangle$$

$$\sigma \circ \vec{r} = 8 \cos\phi$$

$$\|\vec{r}_\theta \times \vec{r}_\phi\| = 64 \sin\phi$$

$$Q = \int_0^{2\pi} \int_0^{\cos^{-1}\frac{7}{8}} 64 \sin\phi \cdot 8 \mathrm{d}\phi \mathrm{d}\theta$$

$$= 8^3 2\pi \int_0^{\cos^{-1}\frac{7}{8}} \sin\phi \mathrm{d}\phi$$

$$= 8^3 2\pi \left(-\cos\phi\right)\Big|_0^{\cos^{-1}\frac{7}{8}} = 8^3 2\pi \left(\frac{-7}{8} + 1\right) = 128\pi\ \mu C$$

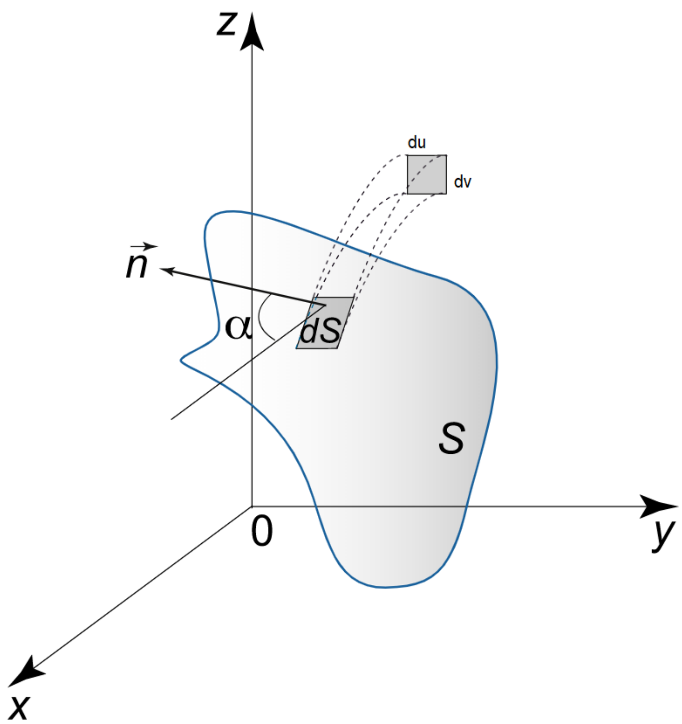
Surface Integrals of Vector Fields

Definition: The surface integral of a vector field, $\vec{F}$, through a surface, S, is $\iint\limits_S \vec{F} \cdot \hat{n} \mathrm{d}s$ where $\hat{n}$ is the normal vector to the surface.

This integral can also be written as $\iint\limits_S \vec{F} \cdot \mathrm{d}\vec{s}$.

Flux

We can rewrite this surface integral because $\hat{n} = \dfrac{\vec{r_u} \times \vec{r_v}}{\|\vec{r_u} \times \vec{r_v}\|}$ and $\mathrm{d}s = \|\vec{r_u} \times \vec{r_v}\|$.



So, $\iint\limits_S \vec{F} \cdot \mathrm{d}\vec{s} = \iint\limits_D \left(\vec{F} \circ \vec{r}\right) \cdot (\vec{r_u} \times \vec{r_v})\, \mathrm{d}u\mathrm{d}v$ where D is the domain of the surface S in uv-space (a uv-plane), and $\vec{r}(u, v)$ parameterizes the surface S. This quantity is called the "directed surface area integral" or "flux" through S.

Flux tells us how much a vector field penetrates a surface. If the field is parallel to the surface (perpendicular to the normal vector) then the flux is 0. As the field and normal vector to the surface become more aligned, the flux increases.

Flux has many practical applications. One of the fundamental equations governing electricity and magnetism talks about electric flux: the amount of an electric field that goes through a surface. We will investigate this equation later.

For example, let an electric field $\vec{E} = \langle x, y, 0 \rangle \text{ N/C}$. Compute the electric flux through the paraboloid $z = 25 - x^2 - y^2$ above the xy-plane. Assume a unit distance of 1 meter.
We will attack the problem thinking in cylindrical coordinates
$\vec{r}(u, v) = \langle u \cos v, u \sin v, 25 - u^2 \rangle$ where $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$.
$\vec{r_u} = \langle \cos v, \sin u, -2u \rangle$ and $\vec{r_v} = \langle -u \sin v, u \cos v, 0 \rangle$.
$\vec{r_u} \times \vec{r_v} = \langle 2u^2 \cos v, 2u^2 \sin v, 0 \rangle$
$\vec{E} \circ \vec{r} = \langle u \cos v, u \sin v, 0 \rangle$
$\left(\vec{E} \circ \vec{r} \right) \cdot (\vec{r_u} \times \vec{r_v}) = 2u^3$

$$\Phi_E = \int_0^{2\pi} \int_0^5 2u^3 \mathrm{d}u \mathrm{d}v = 2\pi \left(\frac{u^4}{2} \right) \Big|_0^5 = 625\pi \text{ Nm}^2/\text{C}$$

Vector Analysis

Integral Curves

An integral curve, $\vec{r}$ of a vector field, $\vec{F}$, is a vector valued function such that
$\vec{r'}(t) = \left(\vec{F} \circ \vec{r} \right)(t)$.

For example, let's find an integral curve to $\vec{F}(x.y) = \langle x, 2y \rangle$.

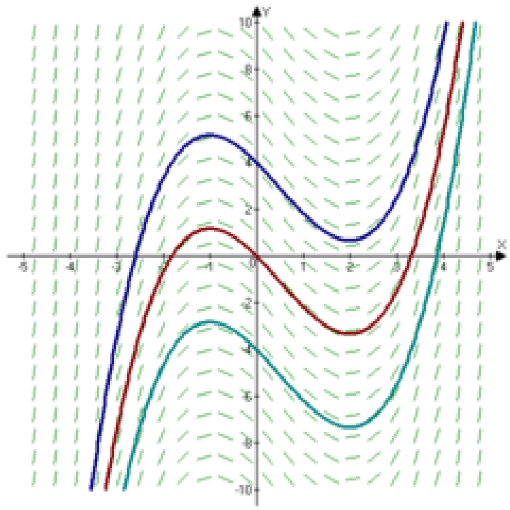
$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{r} = \vec{F} \circ \vec{r}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle x(t), y(t) \rangle = \langle x(t), 2y(t) \rangle$$

$$\begin{cases} x'(t) = x(t) \\ y'(t) = 2y(t) \end{cases}$$

$$\therefore \vec{r}(t) = \langle C_1 e^t, C_2 e^{2t} \rangle$$

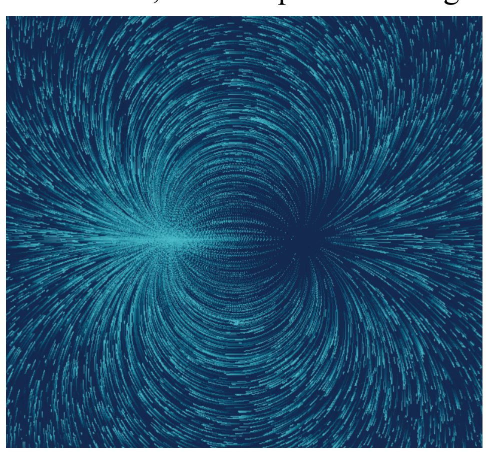
These integral curves have applications in physics and show up naturally in nature alongside vector fields. In physics, the integral curves for an electric field are called "field lines" and integral curves of the velocity field of a fluid like water or air are called "streamlines." You may recognize these curves as tracing out a slope field.



Divergence & Curl

Divergence

In 2D, we define the divergence of a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ as $\operatorname{div}(\vec{F}) = P_x + Q_y$. This tells how the separation between particles in the vector field change over time. Positive divergence at some point means that particles tend to move away from each other, and that point is acting like a "source." Negative divergence at some point means that particles tend to move towards each other, and that point is acting like a "sink."



Source on left; sink on right

This operation extends to higher dimensions. We define the "del operator" as $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots \rangle$ so that $\mathrm{div}(\vec{F}) = \nabla \cdot \vec{F}$.

The del operator is not coordinate system independent. The version there works for Cartesian coordinates. For 3D spherical coordinates,

$\nabla \cdot \vec{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2 F_\rho)}{\partial \rho} + \frac{1}{\rho \sin\theta}\frac{\partial}{\partial \theta}(F_\theta \sin\theta) + \frac{1}{\rho \sin\theta}\frac{\partial F_\phi}{\partial \phi}$ where $\vec{F} = \langle F_\rho, F_\theta, F_\phi \rangle$.

Thankfully, vector fields in spherical coordinates are rare for vector fields, and it's usually easier to convert to Cartesian coordinates before doing any calculations.

Definition: If $\nabla \cdot \vec{F} = 0$, then $\vec{F}$ is incompressible.

This aligns with the idea of incompressible fluids in physics and can simplify or remove the need for some calculations.

The Laplacian

The Laplacian is the higher dimension version of concavity. Let f be a scalar function, We say that $\nabla^2 f = \nabla \cdot (\nabla f) = \mathrm{div}(\mathrm{grad}(f))$.

Written out more fully, $\nabla^2 f = f_{xx} + f_{yy} + \ldots$.

Curl

In 2D, we define the curl of a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$, as

$\nabla \times \vec{F} = \frac{\partial}{\partial x}Q(x, y) - \frac{\partial}{\partial y}P(x, y)$. Note that this result is a scalar for the 2D case.

If $\frac{\partial}{\partial x}Q > 0$, then the field lines accelerate upwards and to the right together: a counter-clockwise rotation.

If $\frac{\partial}{\partial y}P > 0$, then the field lines accelerate rightwards as the particles moves up: a clockwise rotation.

Therefore, $\nabla \times \vec{F}$ tells us the net counter-clockwise rotation at a point.

We use the same del operator from divergence for curl. So $\nabla \times \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \ldots \rangle \times \vec{F}$. This means that for 3 dimensions and greater, curl is a vector. This vector is perpendicular to the plane of rotation of the vector field.

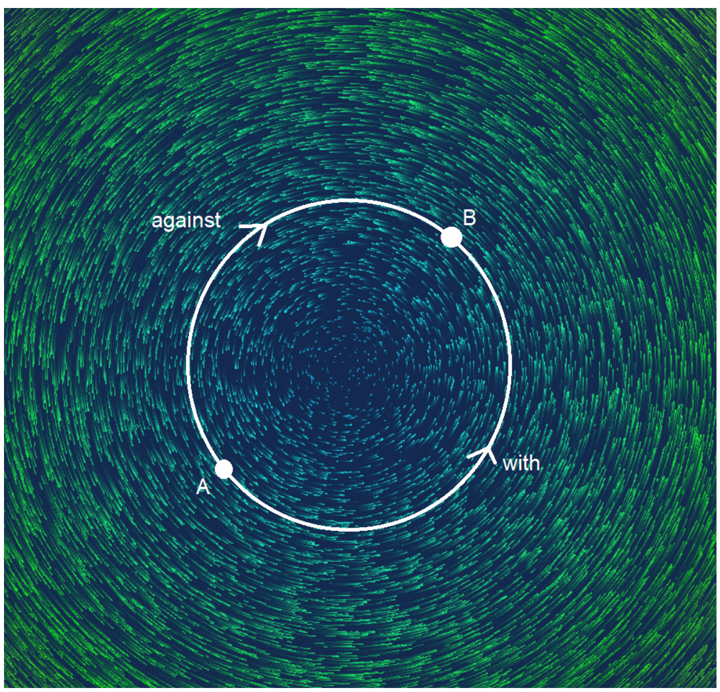
For example, let's compute the curl of $\vec{F}(x,y) = \langle -y, x \rangle$

$$\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 1 - (-1) = 2.$$

Curl & Conservative Vector Fields

Note that the above vector field is not conservative because $\frac{\partial}{\partial y}(-y) \neq \frac{\partial}{\partial x}(x)$. We can use the curl to come up with another test for a vector field being conservative that is easier to apply in higher dimensions.

Theorem: $\vec{F}$ is conservative $\Leftrightarrow \nabla \times \vec{F} = 0$.
Partial Proof: Let's prove the 2D case and provide intuition for higher dimensions.
Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ be a conservative vector field.
Since $\vec{F}$ is conservative, $P_y = Q_x$.
$\therefore \nabla \times \vec{F} = Q_x - P_y = 0$.
Thinking back to what it means for a vector field to be conservative, a vector field must have path independence between all points to have a potential function and be conservative. If in some plane there is a net rotation, there cannot be path independence because one can choose one path that goes "with" the field and another that goes "against" the field.



Divergence of Curl

Theorem: Let $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a twice differentiable vector field. Then $\nabla \cdot (\nabla \times \vec{F}) = 0$.

Proof:

$\nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

$\nabla \cdot (\nabla \times \vec{F}) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$

We know by Fubini's Theorem that $P_{yz} = P_{zy}$, $Q_{xz} = Q_{zx}$, and $R_{xy} = R_{yx}$.

So, $\nabla \cdot (\nabla \times \vec{F}) = 0$.

Green's Theorems

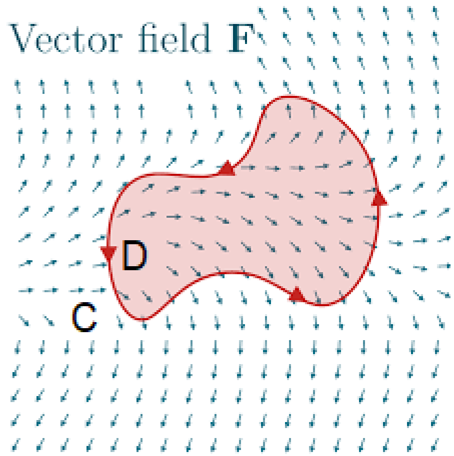
Similar to how we had the fundamental theorem of calculus (FTC) for single-variable calculus, there are several higher-dimensional versions of the FTC.

Recall that the fundamental theorem of calculus is $\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$. One way to think of this statement is "summing up the derivative on a closed interval is the same as the 'sum' of f on the boundary." We will see that many of these higher versions of the FTC deal with intervals and boundaries.

Green's Theorem for Circulation

Theorem: Let C be a closed, counter-clockwise oriented curve in $\mathbb{R}^2$. For any differentiable vector field $\vec{F}(x, y) = \langle Q(x, y), R(x, y)\rangle$, $\oint_C R\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y}\right) \mathrm{d}x\mathrm{d}y$ where D is the interior of C.

In more modern notation, $\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_D \nabla \times \vec{F}\mathrm{d}A$.



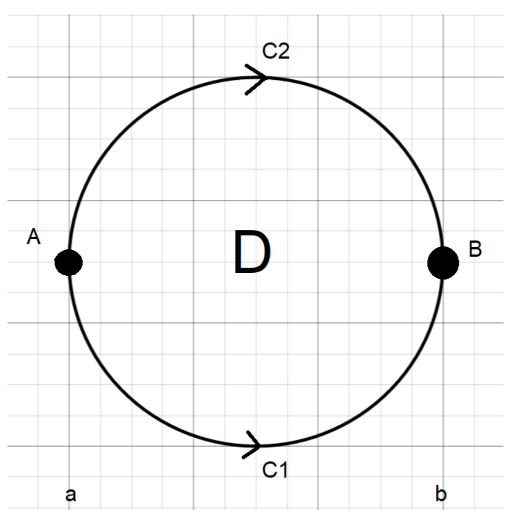
This is saying that summing up the interior of the derivative is the to summing up the boundary of the function, very similar to the FTC.
One can think of this in a physical sense as saying that the work done by the vector field in moving a particle counter-clockwise on C is equal to the rotation (curl) inside of C (D).
This theorem also relates the idea of path independence and curl of a conservative vector field that we proved the 2D case for. The left side shows path independence and will be 0 for conservative vector fields, and the right side shows curl, which will also be 0 for conservative vector fields.

Proof

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \oint_C \langle P(x,y), 0 \rangle \mathrm{d}\vec{r} + \oint_C \langle 0, Q(x,y) \rangle \mathrm{d}\vec{r}$$

We show a 2D case here, but the argument can easily be generalized.

If D is convex, we can break C into two curves on the same interval. C_1 and C_2 such that $C_1 : \vec{r}\langle x, h_1(x) \rangle$ and $C_2 : \vec{r}\langle x, h_2(x) \rangle$ where $x \in [a, b]$.



$$= \int_a^b P(x, h_1(x))\mathrm{d}x - \int_a^b P(x, h_2(x))\mathrm{d}x$$

$$= -\int_a^b P(x, h_2(x)) - P(x, h_1(x))\mathrm{d}x$$

$$= -\int_a^b P(x, y)\Big|_{y=h_1(x)}^{y=h_2(x)}\mathrm{d}x$$

$$= -\int_a^b \int_{h_1(x)}^{h_2(x)} \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x$$

$$= -\iint\limits_D \frac{\partial P}{\partial y}\mathrm{d}A + \iint\limits_D \frac{\partial Q}{\partial x}\mathrm{d}A$$

$$= \iint\limits_D \nabla \times \vec{F}\mathrm{d}A \blacksquare$$

For example, let's use Green's Theorem for Circulation to compute $\oint_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F} = \langle x^2, xy + x^2 \rangle$ and C is the unit circle

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint\limits_D \nabla \times \vec{F}\mathrm{d}A$$

$$\nabla \times \vec{F} = y + 2x$$

Since C is the unit circle, we'll evaluate in polar coordinates. So, $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$.

$$\iint\limits_D \nabla \times \vec{F}\mathrm{d}A = \int_0^{2\pi} \int_0^1 (r\sin\theta + 2r\cos\theta) r\mathrm{d}r\mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{\sin\theta}{3} + \frac{2\cos\theta}{3}\mathrm{d}\theta$$

$= 0$ Although the line integral is 0, the field isn't conservative everywhere because the curl is not 0.

Area of Closed Region

The area inside of D is $A = \iint\limits_D \mathrm{d}x\mathrm{d}y$.

$$= \iint\limits_D \left(\nabla \times \vec{F}\right)\mathrm{d}x\mathrm{d}y$$

if $\nabla \times \vec{F} = 1$. One such field is $\vec{F} = \langle -y/2, x/2 \rangle$

$= \oint_C (-y/2)\mathrm{d}x + (x/2)\mathrm{d}y$ by Green's Theorem for Circulation

$= \frac{1}{2}\oint_C xy' - yx'$

So, if we have some counter-clockwise parametric function $(x(t), y(t))$ that defines C, from t_0 to t_1, then $A = \frac{1}{2}\int_{t_0}^{t_1} (x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t})\mathrm{d}t$.

We can also choose different vector fields so that $A = \oint_C xy' = -\oint_C yx'$.

For a more visual explanation of this topic that includes a discrete version of this idea, see Mathologer's video on YouTube: [Gauss's magic shoelace area formula and its calculus companion](#).

For example, let's compute the area of a circle with radius R.

We'll parameterize the circle as $(R\cos t, R\sin t), t \in [0, 2\pi]$.

$x' = -R\sin t$ and $y' = R\cos t$.

$A = \frac{1}{2}\int_0^{2\pi} ((R\cos t)(R\cos t) - (R\sin t)(-R\sin t))\mathrm{d}t$

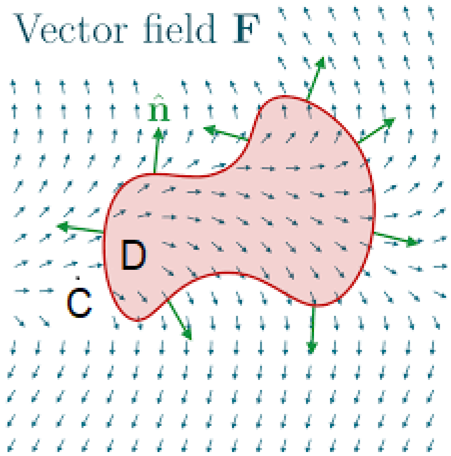
$= \frac{1}{2}\int_0^{2\pi} R^2\mathrm{d}t$

$= \frac{1}{2}R^2 2\pi$

$\therefore A = \pi R^2$

Green's Theorem for Flux

Theorem: Let C be a closed, counter-clockwise oriented curve in $\mathbb{R}^2$ and let D be the region contained in C. For any differentiable vector field $\vec{F}(x, y)$, $\iint_D \nabla \cdot \vec{F}\mathrm{d}A = \oint_C \vec{F} \cdot \hat{n}\mathrm{d}s$



This is saying that the sum of the divergence within D is equal to the flux through C.
Our intuition for this is that divergence is the tendency for integral curves of $\vec{F}$ to spread out, and flux would then be these integral curves crossing the boundary curve C.

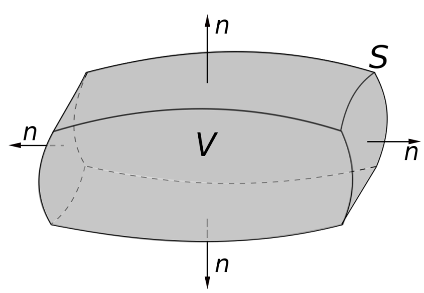
This ability to convert between a line integral and surface integral often makes flux problems easier to solve. For example, one would have to calcalcutate four integrals to find the flux outside of a rectangular box, but only a single double integral over the simple interior region.

Divergence Theorem

<u>Theorem</u>: Let V be a compact solid, and let S be its boundary surface. For any differentiable vector field $\vec{F}(x, y, z)$,

$$\oint_S \oint \vec{F} \cdot \mathrm{d}\vec{s} = \iiint_V \nabla \cdot \vec{F} \mathrm{dV}$$

where $\mathrm{d}\vec{s} = \hat{n}\mathrm{d}s = (\vec{r_u} \times \vec{r_v})\mathrm{d}u\mathrm{d}v$ and $\mathrm{d}V$ is a volume differential.



This says that the flux through a closed surface is equal to the sum of the divergence inside that surface.
Intuitively, divergence describes how much a vector field is going in or out at a point, so summing it up inside some solid would tell us the amount the vector field is going in or out on the solid's boundary, which is flux.

For example, let's find the flux through the unit sphere centered at the origin from the vector field $\vec{F}(x, y, z) = \langle x, y, z^2 \rangle$.

$S = \{(\rho, \theta, \phi) \mid \rho = 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$

$V = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$

$$\text{Flux} = \oint_S \oint \vec{F} \cdot \mathrm{d}\vec{s}$$

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} \nabla \cdot \langle x, y, z^2 \rangle \rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho$$

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} (2 + 2z)\rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho$$

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} (2 + 2\rho\sin\phi)\rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho$$

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} 2\rho^2 \sin\phi + 2\rho^3 \sin\phi \cos\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho$$

$$= 2\pi \int_0^1 \int_0^\pi 2\rho^2 \sin\phi + 2\rho^3 \sin\phi \cos\phi \mathrm{d}\phi \mathrm{d}\rho$$

$$= 4\pi \int_0^\pi \frac{1}{3} \sin\phi + \frac{1}{4} \sin\phi \cos\phi \mathrm{d}\phi$$

$$= \frac{8\pi}{3}$$

Gauss's Laws

Electric Fields

Gauss's Law is an important application of Divergence Theorem in physics. Gauss's Law says that $\Phi_E = \dfrac{Q_{\text{in}}}{\epsilon}$. That is, the electric flux through a closed surface (whether real or hypothetical) is equal to the charge contained within the surface divided by the permittivity of the space.

Rewritten in calculus, $\displaystyle\oint_S \oint \vec{E} \cdot \mathrm{d}\vec{s} = \iiint_V \frac{Q_{\text{in}}}{\epsilon} \mathrm{d}V$

So, $\displaystyle\iiint_V \nabla \cdot \vec{E} \mathrm{d}V = \iiint_V \frac{\nu}{\epsilon} \mathrm{d}V$ where ν is the charge density by Divergence Theorem.

If we let $V \to 0$, $\nabla \cdot \vec{E} = \frac{\nu}{\epsilon}$. This is the differential form of Gauss's Law and the first of Maxwell's Equations.

Magnetic Fields

Gauss's Law for magnetism says that the magnetic flux through a closed surface is 0.

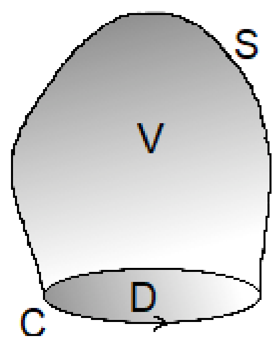
Rewritten in calculus, $\oint_S \oint \vec{B} \cdot \mathrm{d}\vec{s} = 0$.

So, $\iiint_V \nabla \cdot \vec{B}\mathrm{d}V = \iiint_V 0\mathrm{d}V$.

If we let $V \to 0$, $\nabla \cdot \vec{B} = 0$. This means that there are no sources or sinks in magnetic fields: magnetic field flow is incompressible (solenoidal). This is the second of Maxwell's Equations.

Stokes' Theorem

Let C be a closed, counter-clockwise oriented curve in R^2, and let D be the region contained in C. Let S be an open surface with opening boundary C, and let V be the region contained inside $\tilde{S} = D \cup S$.



Finding the flux of $\nabla \times \vec{F}$,

$$\oint_{\tilde{S}} \oint \nabla \times \vec{F} = \iiint_V \nabla \cdot (\nabla \times \vec{F})\mathrm{d}V$$ by Divergence Theorem.

Since $\nabla \cdot (\nabla \times \vec{F}) = 0$, $\iint_D \nabla \times \vec{F}\mathrm{d}A = \iint_S \nabla \times \vec{F}\mathrm{d}\vec{s}$.

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \nabla \times \vec{F}\mathrm{d}\vec{s}$$ by Green's Theorem for Circulation. This is Stokes' Theorem.

For example, let's use Stokes' Theorem to evaluate $\iint\limits_S \nabla \times \vec{F} \mathrm{d}\vec{s}$ where S is the hemisphere $x^2 + y^2 + z^2 = 4, x \geq 0$ and $\vec{F}(x, y, z) = \langle yz, x \sin z, xyz^2 \rangle$.

$S = \left\{(x, y, z) \mid x^2 + y^2 + z^2 = 2^2, x \geq 0\right\}$

$C = \ (y, z) \mid y^2 + z^2 = 2^2\big\} = \{(r, \theta) \mid r = 2, 0 \leq \theta \leq 2\pi\}$ where θ is in the yz-plane.

$C = \vec{r}(t) = \langle 0, 2\cos t, 2\sin t \rangle$, $0 \leq t \leq 2\pi$.

$\vec{F} \circ \vec{r} = \langle 4\cos t \sin t, 0, 0 \rangle$

$\vec{r'}(t) = \langle 0, -2\sin t, 2\cos t \rangle$

$(\vec{F} \circ \vec{r}) \cdot \vec{r'} = 0$

$\therefore \iint\limits_S \nabla \times \vec{F} \mathrm{d}A = \oint_0^{2\pi} 0 \mathrm{d}t = 0$

Faraday's Law of Induction & Ampere's Law

Faraday's Law of Induction quantifies the idea that changing magnetic flux in a coil induces a current in the coil. More precisely, it induces a voltage, the potential function of electric field ($\vec{E} = \nabla V$), and this field will oppose the magnetic field that induced it.

In the language of calculus, $-\frac{\partial}{\partial t} = \iint\limits_S \vec{B} \mathrm{d}\vec{s} = \oint\limits_C \vec{E} \mathrm{d}\vec{r}$

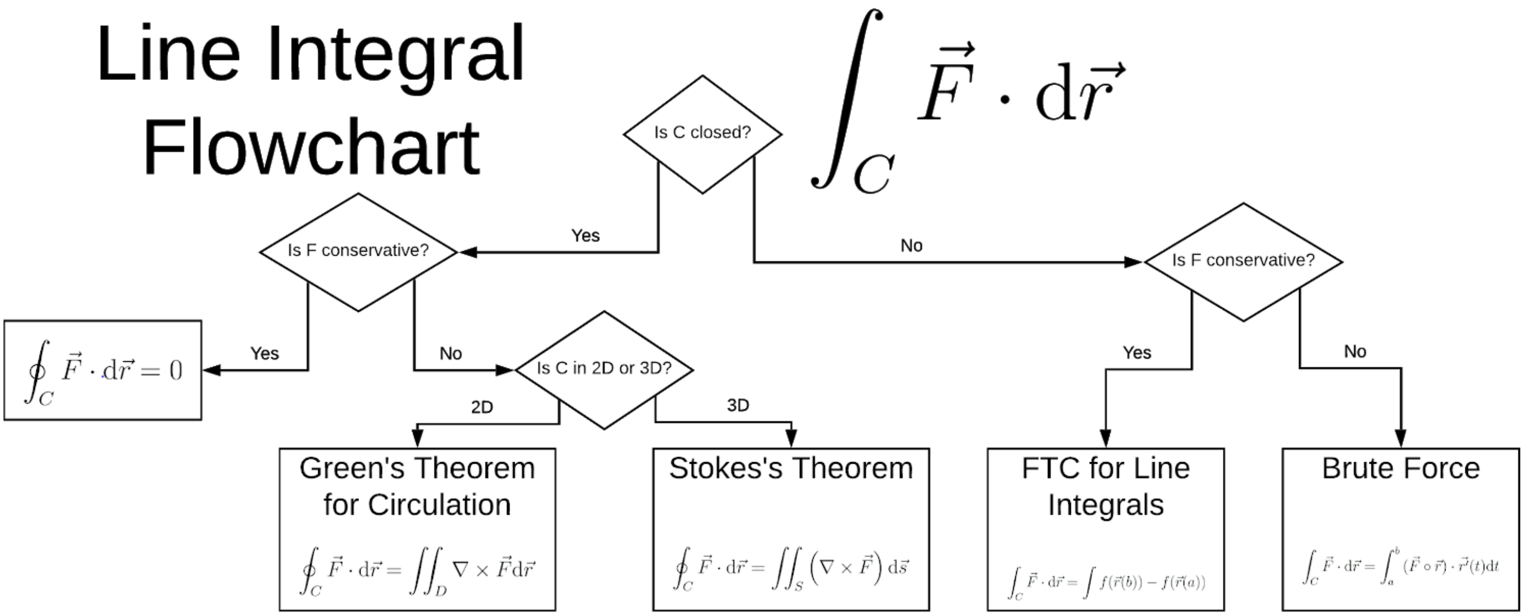
$-\iint\limits_S \frac{\partial}{\partial t} \vec{B} \mathrm{d}\vec{s} = \iint\limits_S \nabla \times \vec{E} \mathrm{d}\vec{s}$ by Stokes' Theorem

As S collapses to a point, $\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$. This is the Maxwell-Faraday Equation. This is the third of Maxwell's Equations.

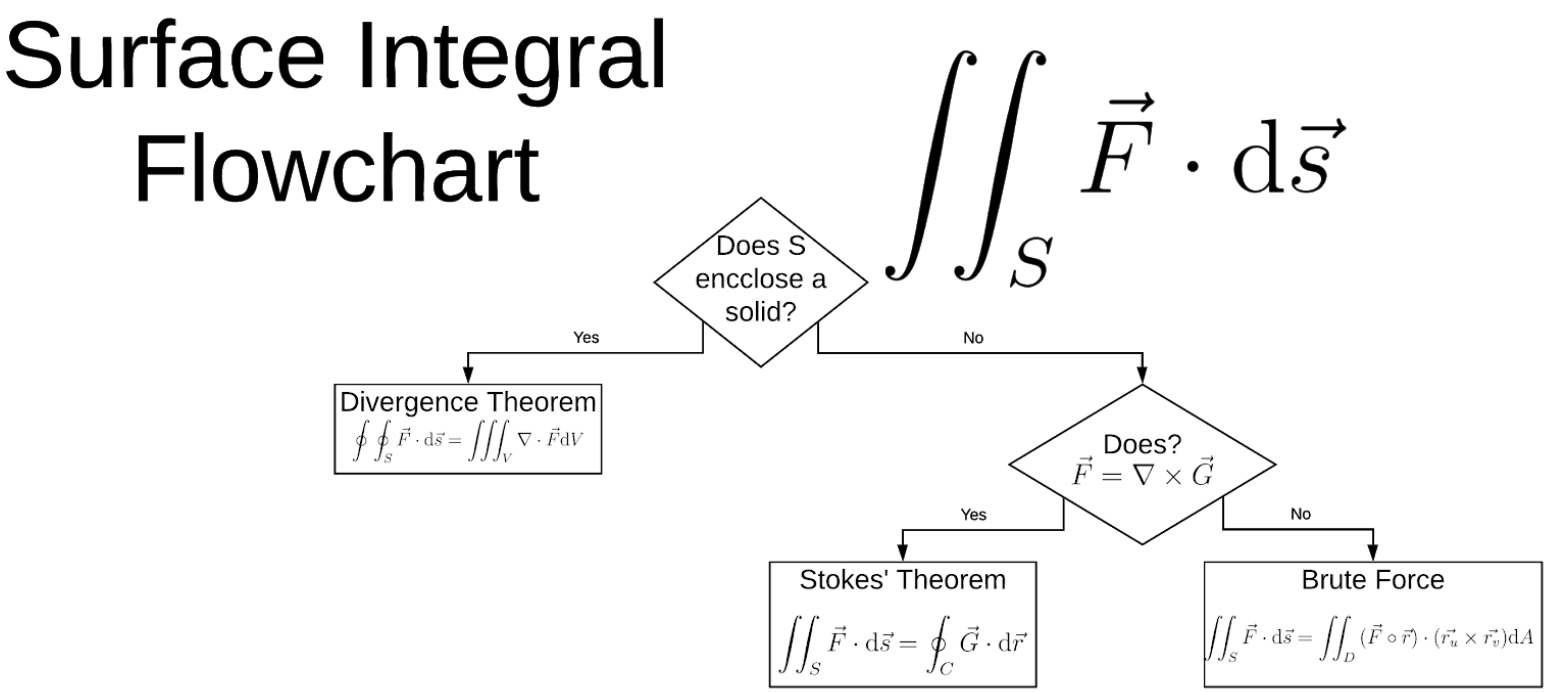
The final of Maxwell's Equations is Ampere's Law (with Maxwell's correction). It says that $\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 J$ where J is current density and μ_0 is the permeability of free space. This also gives the speed of light — $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ — the speed at which all electromagnetic waves propagate.

Additional Materials

Line Integral Flowchart



Surface Integral Flowchart



Worked Test Questions

Test 1

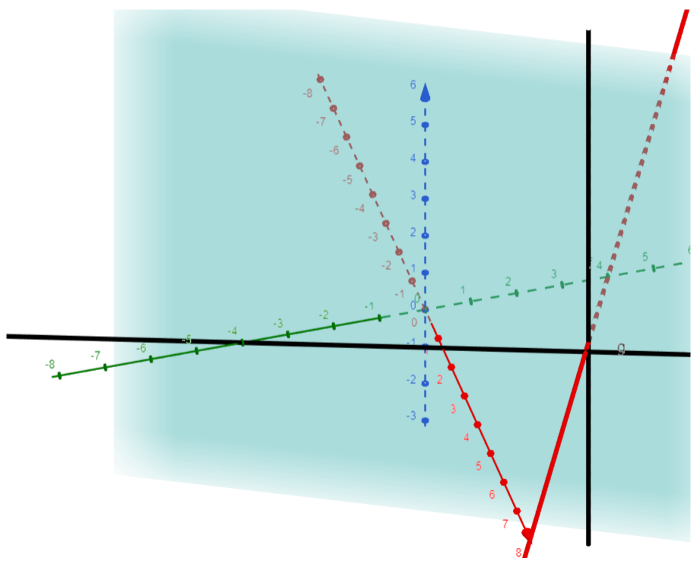
1. Consider the two intersecting lines $\vec{r_1}(t) = \langle 2, 3, 4t \rangle$ and $\vec{r_2}(t) = \langle 2+t, 3+2t, 0 \rangle$. Give the direction vector of each line. Find the equation of the pane which contains both lines. Draw a diagram of the lines, the plane, and the relevant vectors.

   The direction vector of a line is the derivative of the position vector.
   - Direction 1: $\langle 0, 0, 4 \rangle$
   - Direction 2: $\langle 1, 2, 0 \rangle$

   A normal vector is the cross product of the direction vectors.
   - $\vec{n} = \langle 0, 0, 4 \rangle \times \langle 1, 2, 0 \rangle = \langle -8, 4, 0 \rangle$

   The lines intersect at: $(2, 3, 0)$

   So, the plane equation is: $\langle -8, 4, 0 \rangle \cdot \langle x-2, y-3, z \rangle = 0$.



2. Given the vector-valued function $\vec{r}(t) = \langle 10t, 7\cos t, 7\sin t \rangle$,
   a. Compute the unit tangent vector $\hat{T}(t)$ and the unit normal vector $\hat{N}(t)$.

   $$\hat{T} = \frac{\vec{r'}(t)}{\|\vec{r'}(t)\|}$$

   $$\vec{r'}(t) = \langle 10, -7\sin t, 7\cos t \rangle$$

   $$\|\vec{r'}(t)\| = \sqrt{10^2 + (-7\sin t)^2 + (7\cos t)^2} = \sqrt{149}$$

$\hat{T}(t) = \dfrac{1}{\sqrt{149}}\langle 10, -7\sin t, 7\cos t\rangle$

$\hat{N}(t) = \dfrac{\mathrm{d}\hat{T}/\mathrm{d}t}{\|\mathrm{d}\hat{T}/\mathrm{d}t\|}$

$\dfrac{\mathrm{d}\hat{T}}{\mathrm{d}t} = \dfrac{1}{\sqrt{149}}\langle 0, -7\cos t, -7\sin t\rangle$

$\|\dfrac{\mathrm{d}\hat{T}}{\mathrm{d}t}\| = \dfrac{1}{\sqrt{149}}\sqrt{(-7\cos t)^2 + (-7\sin t)^2} = \dfrac{7}{\sqrt{149}}$

$\hat{N}(t) = \langle 0, -\cos t, -\sin t\rangle$

b. Show that $\hat{T} \perp \hat{N}$ for all t.
If $\hat{T} \perp \hat{N}$, then $\hat{T} \cdot \hat{N} = 0$ for all t.

$\dfrac{1}{\sqrt{149}}\langle 10, -7\sin t, 7\cos t\rangle \cdot \langle 0, -\cos t, -\sin t\rangle$

$\hat{T} \cdot \hat{N} = \dfrac{1}{\sqrt{149}}(0 + 7\sin t\cos t - 7\sin t\cos t) = 0$

$\therefore \hat{T} \perp \hat{N}$

3. A cannon fires cannonballs with a speed of $20\mathrm{m/s}$. Take acceleration due to gravity to be $g = 10\mathrm{m/s}^2$.

   a. Starting with a constant acceleration function $\vec{a} = \vec{r''}(t) = \langle 0, -g\rangle$, find the velocity and position functions ($\vec{r'}(t)$ and $\vec{r}(t)$, respectively) of the cannonball if the cannon is fired at an angle θ with respect to the horizontal. For simplicity, assume the cannon is initially positioned at the origin.

   We know that velocity is the integral of acceleration.
   $\vec{v}(t) = \vec{r'}(t) = \langle c_1, c_2 - gt\rangle$
   We know that the initial velocity is $20\mathrm{m/s}$ and the angle is θ.
   $v_0 = 20\langle\cos\theta, \sin\theta$
   $\vec{v}(t) = \langle 20\cos theta, 20\sin\theta - gt\rangle$

   We know that position is the integral of velocity; the particle starts at the origin.

   $\vec{r}(t) = \langle 20\cos\theta t + c_1, 20\sin\theta t - \dfrac{1}{2}gt^2 + c_2\rangle$

   $\vec{r}(t) = \langle 20\cos(\theta)t, 20\sin(\theta)t - \dfrac{1}{2}gt^2\rangle$

$\vec{r}(t) = \langle 20\cos(\theta)t, 20\sin(\theta) - 5t^2\rangle$

b. What angle θ should the cannon be fired at to hit a target on the ground at a distance 40m away?

We want to find a point where $y = 0$ and $x = 40$.
Solving when $y = 0$ for t, $20\sin(\theta)t - 5t^2 = 0$,
$t = 0 \text{ or } 4\sin(\theta)$

Solving plugging in t when $x = 40$,
$20\cos(\theta)\,(4\sin(\theta)) = 40$
$2\sin(\theta)\cos(\theta) = 1$
$\sin(2\theta) = 1$, $2\theta = \pi/2$
$\theta = \pi/4$

4. Consider the following particle trajectory: $\vec{r}(t) = \langle R\cos e^t, R\sin e^t, \frac{h}{2\pi}e^t\rangle$ for $t \geq 0$. The shape of the trajectory is a helix with radius R and a vertical spacing of h. Find the arc length function $s(t)$ of the trajectory starting at $t = 0$. Give the arc length reparameterization of the helix.

$$s(t) = \int_0^t \|\vec{r'}(\tau)\| \mathrm{d}\tau$$

$$\vec{r'}(t) = \langle -Re^t\sin e^t, Re^t\cos e^t, \frac{h}{2\pi}e^t\rangle$$

$$\|\vec{r'}(t)\| = \sqrt{(-Re^t\sin e^t)^2 + (Re^t\cos e^t)^2 + (\frac{h}{2\pi}e^t)^2}$$

$$\|\vec{r'}(t)\| = e^t\sqrt{R^2 + \frac{h^2}{4\pi^2}}$$

$$s(t) = \int_0^t e^\tau\sqrt{R^2 + \frac{h^2}{4\pi^2}}\mathrm{d}\tau = \sqrt{R^2 + \frac{h^2}{4\pi^2}}\left(e^t - 1\right)$$

Solving for t,

$$t = \ln\left(\frac{s}{\sqrt{R^2 + \frac{h^2}{4\pi^2}}} + 1\right)$$

$$\vec{r}(s) = \left\langle R\cos\left(\frac{s}{\sqrt{R^2 + \frac{h^2}{4\pi^2}}} + 1\right), R\sin\left(\frac{s}{\sqrt{R^2 + \frac{h^2}{4\pi^2}}} + 1\right), \frac{h}{2\pi}\left(\frac{s}{\sqrt{R^2 + \frac{h^2}{4\pi^2}}} + 1\right)\right\rangle$$

5. (Bonus) Let $\vec{r}(t)$ be the position function of a particle trapped on the surface of a sphere centered at the origin. Show that $\vec{r}(t) \perp \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t)$ for all t.
   Since $\vec{r}(t)$ is on a sphere, $\|\vec{r}(t)\| = R$ and $\vec{r}(t) \cdot \vec{r}(t) = R^2$.
   $\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}(t) \cdot \vec{r}(t)) = 2\vec{r}(t) \cdot \vec{r'}(t)$
   $\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}R^2 = 0$
   So, $2\vec{r}(t) \cdot \vec{r'}(t) = 0$ and $\vec{r}(t) \cdot \vec{r'}(t) = 0$
   $\therefore \vec{r}(t) \perp \vec{r'}(t)$

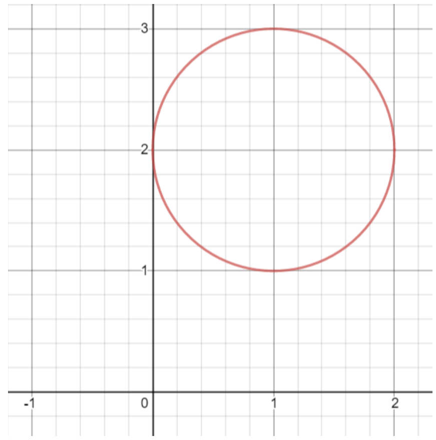
Test 2

1. Consider the function $f(x, y) = x^2 - 2x + y^2 - 4y + 7$.
   a. Find equations for and plot (if possible) the C-level curves of f for $C = 3$ and $C = 1$.

      We will find the curve for any C and the plug in 1 and 3.
      $x^2 - 2x + y^2 - 4y + 7 = C$
      $x^2 - 2x + 1 + y^2 - 4y + 4 = C - 2$
      $(x-1)^2 + (y-2)^2 = C - 2$: A circle of radius $\sqrt{C-2}$ centered at $(1, 2)$.

      For $C = 1$, the level curve does not exist because the circle would have radius $\sqrt{1-2} = \sqrt{-1}$.

      For $C = 3$, $(x-1)^2 + (y-2)^2 = 1$.



b. Compute ∇f.

$\nabla f = \langle f_x, f_y \rangle = \langle 2x - 2, 2y - 4 \rangle$

c. Find the equation of the plane tangent to the surface $z = f(x, y)$ at the point $(x_0, y_0, z_0) = (2, 4, 7)$.

$\vec{n} = \langle f_x, f_y, -1 \rangle = \langle 2x - 2, 2y - 4, -1 \rangle$
At $(2, 4, 7)$, $\vec{n} = \langle 2, 7, -1 \rangle$
So, the plane equation is $\langle 2, 7, -1 \rangle \cdot \langle x - 2, y - 4, z - 7 \rangle = 0$.

d. Perform one iteration of gradient descent on $f(x, y)$ with a learning rate $\delta = 1/4$ starting from the point $(x_0, y_0) = (2, 4)$.

$(x_n, y_n) = (x_{n-1}, y_{n-1}) - \delta \nabla f$
$(x_0, y_0) = (2, 4)$, $\delta = 1/4$, and $\nabla f = \langle 2x - 2, 2y - 4 \rangle$
$(x_1, y_1) = (2, 4) - 1/4 \langle 2(2) - 2, 2(4) - 4 \rangle$
$(x_1, y_1) = (3/2, 3)$

2. Recall that for a differentiable function $f(x, y)$ and the unit vector $\hat{u} = \langle a, b \rangle$. We proved that $D_{\hat{u}} f = \nabla f \cdot \hat{u}$
   a. Prove the statement "the gradient is the direction of steepest ascent" by showing that the directional derivative $D_{\hat{u}} f$ is maximized when $\hat{u} \parallel \nabla f$.

$D_{\hat{u}} f = \nabla f \cdot \hat{u} = \|\nabla f\| \|\hat{u}\| \cos\theta = \|\nabla f\| \cos\theta$

This value is maximized then θ is a multiple of 2π, meaning the angle between ∇f and $\hat{u}$ is 0. This means the maximum value of the directional derivative is when $\nabla f \parallel \hat{u}$ ■.

b. State the limit definition of the directional derivative $D_{\hat{u}}f$. Starting from the limit definition, prove that $D_{\hat{u}}f = \nabla \cdot f \cdot \hat{u}$.

$$D_{\hat{u}}f = \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$ and $\hat{u} = \langle a, b\rangle$.

$$= \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x+ah, y)}{h} + \frac{f(x+ah, y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} b\frac{f(x, y+bh) - f(x,y)}{bh} + a\frac{f(x+ah) - f(x,y)}{ah}$$

$$= bf_y + af_x = \langle f_x, f_y\rangle \cdot \langle a, b\rangle = \nabla f \cdot \hat{u}.$$

3. Use the method of Lagrange Multipliers to find the maximum of the product of two numbers x and y given that (x, y) is a coordinate pair in the first quadrant located in the unit circle centered at the origin. Begin by stating the constraint equation $g(x,y) = k$ and the objective function $f(x, y)$.

Objective function: $f(x,y) = xy$

Constraint function: $g(x,y) = x^2 + y^2 = 1$, $x \geq 0$ and $y \geq 0$.

$F(x, y, \lambda) = xy + \lambda(1 - x^2 - y^2)$

$\frac{\partial F}{\partial x} = y - 2\lambda x$, $\frac{\partial F}{\partial y} = x - 2\lambda y$, and $\frac{\partial F}{\partial x} = 1 - x^2 - y^2$.

$\langle y - 2\lambda x, x - 2y, 1 - x^2 - y^2\rangle = \vec{0}$

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2 + y^2 = 1 \end{cases} \implies \begin{cases} \lambda = 1/2 \\ x = 1/\sqrt{2} \\ y = 1/\sqrt{2} \end{cases}$$

So, the maximum product is $\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

4. The function $p(x, y)$, shown below, is the probability density function of a bivariate normal distribution with mean (a, b) and standard deviation $1/\sqrt{2}$. Show that the global maximum of $p(x, y)$ occurs at (a, b).

$$p(x, y) = \frac{1}{\pi} \exp\left(-(x-a)^2 - (y-b)^2\right)$$

$p_x = \frac{1}{\pi}(-2(x-a)) \exp\left(-((x-a)^2 + (y-b)^2)\right)$
$p_y = \frac{1}{\pi}(-2(y-b)) \exp\left(-((x-a)^2 + (y-b)^2)\right)$
$p_x = 0$ when $x = a$
$p_y = 0$ when $y = b$
$\implies$ (a, b) is a critical point.
$p_{xx} = \frac{1}{\pi}\left(4(x-a)^2 \exp -((x-a)^2 + (y-b)^2) - 2 \exp -((x-a)^2 + (y-b)^2)\right)$
$p_{xx} = \frac{1}{\pi}\left(4(x-a)^2 - 2\right) \exp -((x-a)^2 + (y-b)^2)$
$p_{yy} = \frac{1}{\pi}\left(4(y-b)^2 - 2\right) \exp -((x-a)^2 + (y-b)^2)$
$p_{xy} = p_{yx} = \frac{1}{\pi} 4(x-a)(y-b) \exp -((x-a)^2 + (y-b)^2)$
$p_{xx}(a, b) = \frac{-2}{\pi}$, $p_{yy}(a, b) = \frac{-2}{\pi}$, and $p_{xy} = p_{yx} = 0$
$H_{(a,b)} = \begin{bmatrix} \frac{-2}{\pi} & 0 \\ 0 & \frac{-2}{\pi} \end{bmatrix}$
$\det\left(H_{(a,b)}\right) = \frac{4}{\pi^2}$
(a, b) is an extrema because the determinant is positive.
Since $f_{xx}(a, b) < 0$ and $f_{yy}(a, b) < 0$, (a, b) is a maximum. (a, b) is a global maximum because $p(x, y)$ always decreases as (x, y) move away from (a, b).

5. (Bonus) Let the C-level curve of the function $f(x, y)$ be parameterized by the vector-valued function $\vec{r}(t) = \langle x(t), y(t) \rangle$. Use the chain rule of show that $\nabla f(\vec{r}(t)) \perp \vec{r'}(t)$ for all t.

   Since $\vec{r}(t)$ parameterizes a C-level curve of f, $f \circ \vec{r}(t) = C$.

$$\frac{\mathrm{d}}{\mathrm{d}t}f \circ \vec{r}(t) = \frac{\mathrm{d}}{\mathrm{d}t}C$$
$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$
$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t} \right\rangle = 0$$
$$\nabla f(\vec{r}(t)) \cdot \vec{r'}(t) = 0$$

So, $\nabla f(\vec{r}(t)) \perp \vec{r'}(t) \blacksquare$.

Test 3

1. Evaluate each of the following integrals as they appear or by changing coordinate systems. Sketch and/or describe the region geometrically to help in choosing an appropriate coordinate system.

   a. $\displaystyle\int_0^3 \int_0^2 \int_0^1 z e^{x+y+z^2} \mathrm{d}z \mathrm{d}y \mathrm{d}x$

   We can use a simple u-substitution.

   $u = z^2 + x + y$, $\mathrm{d}u = 2z\mathrm{d}z$

   $$= \int_0^3 \int_0^2 \int_{x+y}^{1+x+y} \frac{1}{2} e^u \mathrm{d}u \mathrm{d}y \mathrm{d}x$$
   $$= \frac{1}{2} \int_0^3 \int_0^2 e^{1+x+y} - e^{x+y} \mathrm{d}y \mathrm{d}x$$
   $$= \frac{1}{2} \int_0^3 \left((e^{3+x} - e^{2+x}) - (e^{1+x} - e^x) \right) \mathrm{d}x$$
   $$= \frac{1}{2} \left(e^6 - e^5 - e^4 + e^2 + e - 1 \right)$$

   b. $\displaystyle\int_0^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_0^5 z \mathrm{d}z \mathrm{d}y \mathrm{d}x$

   The region is a half-cylinder, so we will use cylindrical coordinates.

   $$= \int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{2}} \int_0^5 rz \mathrm{d}z \mathrm{d}r \mathrm{d}\theta$$

$$= \pi \int_0^{\sqrt{2}} r \mathrm{d}r \cdot \int_0^5 z \mathrm{d}z$$

$$= \pi \left(\frac{\sqrt{2}^2}{2} - \frac{0^2}{2} \right) \cdot \left(\frac{5^2}{2} - \frac{0^2}{2} \right)$$

$$= \frac{25\pi}{2}$$

c. $\int_{-3}^{3} \int_0^{9-x^2} \int_0^{\sqrt{9-x^2-y^2}} z \mathrm{d}z \mathrm{d}y \mathrm{d}x$

The region is a quarter sphere, so we will use spherical coordinates.

$$= \int_0^3 \int_0^{\pi} \int_0^{\pi/2} \rho^3 \sin\phi \cos\phi \mathrm{d}\phi \mathrm{d}\theta \mathrm{d}\rho$$

$$= \int_0^3 \rho^3 \mathrm{d}\rho \cdot \int_0^{\pi} \mathrm{d}\theta \cdot \int_0^{\pi/2} \sin\phi \cos\phi \mathrm{d}\phi$$

$$= \frac{3^4}{4} \cdot \pi \cdot \frac{1}{2} = \frac{81\pi}{8}$$

2. Let $\Omega \subset \mathbb{R}^3$ be the spherical ball of radius R centered at the origin. Set up and evaluate the integral $\iiint_{\Omega} \mathrm{d}V$.

Since we are finding the volume of a ball, we will use spherical coordinates.

$\mathrm{d}V = \rho^2 \sin\phi \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}\phi$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi \mathrm{d}\phi \mathrm{d}\theta \mathrm{d}\rho$$

$$= \int_0^R \rho^2 \mathrm{d}\rho \cdot \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^{\pi} \sin\phi \mathrm{d}\phi$$

$$= \frac{R^3}{3} \cdot 2\pi \cdot 2 = \frac{4\pi R^3}{3}$$

3. A plane lamina with density $\sigma(x, y) = \sqrt{x^2 + y^2}$ occupies the region D, where D is the region bounded by the Archimedian spiral $r = \theta$ and the half-line $\theta = \alpha$, $r \geq 0$

where α is an unknown angle in radians. Find α such that the average density σ_{avg} of the lamina is $\pi/2$.

$$\bar{f}=\frac{\iint\limits_D \sigma \mathrm{d}V}{\iint\limits_D \mathrm{d}V}=\frac{\pi}{2}$$

$D=\{(r,\theta)\mid 0\le\theta\le\alpha, 0\le r\le\theta\}$, $\sigma(r,\theta)=r$

$$\bar{f}=\frac{\int_0^\alpha\int_0^\theta r^2\mathrm{d}r\mathrm{d}\theta}{\int_0^\alpha\int_0^\theta r\mathrm{d}r\mathrm{d}\theta}$$

$$\bar{f}=\frac{\int_0^\alpha \frac{\theta^3}{3}\mathrm{d}\theta}{\int_0^\alpha \frac{\theta^2}{2}\mathrm{d}\theta}$$

$$\bar{f}=\frac{\alpha^4/12}{\alpha^3/6}=\frac{\alpha}{2}$$

$$\frac{\alpha}{2}=\frac{\pi}{2}\implies \alpha=\pi$$

4. Consider the 2D gaussian function $f(x,y)=e^{-(x^2+y^2)}$. Evaluate $\iint\limits_D f(x,y)\mathrm{d}A$ where D is the disk of radius a centered at the origin. Use the results the evaluate $\iint\limits_{\mathbb{R}^2} f(x,y)\mathrm{d}A$.

$$D=\{(r,\theta)\mid 0\le r\le a, 0\le\theta\le 2\pi\}$$

$$f(x,y)=\exp\left(-x^2-y^2\right)=\exp\left(r^2\right)$$

$$\iint\limits_D f(x,y)\mathrm{d}A=\int_0^{2\pi}\int_0^a e^{-r^2}r\mathrm{d}r\mathrm{d}\theta$$

$$=\frac{-1}{2}\int_0^{2\pi}\int_0^{-a^2}e^u\mathrm{d}u\mathrm{d}\theta$$

$$=\frac{-1}{2}\int_0^{2\pi}\left(e^{-a^2}-1\right)\mathrm{d}\theta$$

$$=\pi\left(1-e^{-a^2}\right)$$

$$\iint\limits_{\mathbb{R}^2} f(x,y)\mathrm{d}A = \lim_{a\to\infty} \pi\left(1 - e^{-a^2}\right) = \pi$$

Test 4

1. For each of the following, determine if $\vec{F}$ is conservative. Then evaluate $\int_C \vec{F}\cdot\mathrm{d}\vec{r}$.

   a. $\vec{F} = \langle xz, x^2z, xy^2z\rangle$ and C given by $\vec{r}(t) = \langle t, e^{-t}, e^t\rangle$ for $0 \le t \le 1$.

   $$\nabla\times\vec{F} = \langle 2xyz - x^2, x - y^2z, 2xz - 0\rangle$$

   Since $\nabla\times\vec{F} \neq \vec{0}$, $\vec{F}$ isn't conservative.

   $$\vec{F}\circ\vec{r} = \langle te^t, t^2e^t, te^{-2t}e^t\rangle$$
   $$\vec{r'}(t) = \langle 1, -e^{-t}, e^t\rangle\,\vec{r'}(t) = \langle 1, -e^{-t}, e^t\rangle$$
   $$\left(\vec{F}\circ\vec{r}\right)\cdot\vec{r'} = te^t - t^2 + t\left(\vec{F}\circ\vec{r}\right)\cdot\vec{r'} = te^t - t^2 + t$$
   $$\int_C \vec{F}\cdot\mathrm{d}\vec{r} = \int_0^1 te^t - t^2 + t\mathrm{d}t\int_C \vec{F}\cdot\mathrm{d}\vec{r} = \int_0^1 te^t - t^2 + t\mathrm{d}t$$
   $$= \frac{1}{2} - \frac{1}{3} + te^t - \int_0^1 e^t\mathrm{d}t = \frac{1}{2} - \frac{1}{3} + te^t - \int_0^1 e^t\mathrm{d}t$$
   $$= 1 + \frac{1}{2} - \frac{1}{3} = 1 + \frac{1}{2} - \frac{1}{3}$$

   b. $\vec{F} = \left\langle\sqrt{\frac{yz}{x}}, \sqrt{\frac{xz}{y}}, \sqrt{\frac{xy}{z}}\right\rangle$ and C given by $\vec{r}(t) = \langle\cos(t), \sin(t), \sin(4t)\rangle$ for $0 \le t \le 2\pi$.

   $\vec{F} = \nabla(2\sqrt{xyz}) \implies \vec{F}$ is conservative.

   $\vec{r}(0) = \vec{r}(2\pi) \implies C$ is a circulation.

   For a conservative vector field, $\oint_C \vec{F}\cdot\mathrm{d}\vec{r} = 0$

c. $\vec{F} = \langle yz, xz, xy \rangle$ and C given by $\vec{r}(t) = \langle 2t^2, e^{1-t^2}, \arctan(t^2/2) \rangle$ for $0 \leq t \leq \sqrt{2}$.

$\vec{F} = \nabla(xyz) \implies \vec{F}$ is conservative.

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \nabla f \cdot \mathrm{d}\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= \frac{4\pi}{4e} - 0 = \frac{\pi}{e}$$

2. Let the surface S be the portion of the paraboloid $z = 8 - \frac{x^2}{2} - \frac{y^2}{2}$ that lines in the xy-plane. Let $\vec{F}$ be given by $\vec{F}(x, y, z) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0 \right\rangle$

a. Parameterize S with appropriate bounds on the parameters u and v.

$x^2 + y^2 \leq 16$ is $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$.

$\vec{r}(u, v) = \langle u\cos v, u\sin v, 8 - \frac{1}{2}u^2 \rangle$

Where $0 \leq u \leq 4$ and $0 \leq v \leq 2\pi$

b. Compute the surface area of S.

$$SA = \iint_D \|\vec{r_u} \times \vec{r_v}\| \mathrm{d}A$$

$$\vec{r_u} = \langle \cos v, \sin u, -u \rangle$$

$$\vec{r_v} = \langle -u\sin v, u\cos v, 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle u^2\cos v, u^2\sin v, u \rangle$$

$$\|\vec{r_u} \times \vec{r_v}\| = u\sqrt{1+u^2}$$

$$SA = \int_0^{2\pi} \int_0^4 u\sqrt{1+u^2} \mathrm{d}u\mathrm{d}v$$

$$= 2\pi \int_0^4 u\sqrt{1+u^2} \mathrm{d}u$$

$$= \pi \frac{34\sqrt{17} - 2}{3}$$

c. Compute the flux of $\vec{F}$ through S.

$$\text{Flux} = \iint_D (\vec{F} \circ \vec{r}) \cdot (\vec{r_u} \times \vec{r_v}) \mathrm{d}A$$

$$\vec{F} \circ \vec{r} = \langle \cos v, \sin v, 0 \rangle$$

$$(\vec{F} \circ \vec{r}) \cdot (\vec{r_u} \times \vec{r_v}) = u^2$$

$$\text{Flux} = \int_0^{2\pi} \int_0^4 u^2 \mathrm{d}u \mathrm{d}v$$

$= \dfrac{128\pi}{3}$ for an outward-oriented surface

3. Consider a 3D vector field $\vec{F}(x, y, z) = \langle P, Q, R \rangle$ and a scalar function of two variables $f(x, y)$. Determine which of the following expressions is defined. If it is defined, evaluate it. If it is not defined, explain why. If you can deduce the value of the expression from a theorem, do so and state the theorem.

a. $\nabla f \cdot \vec{F}$

This operation is not defined because ∇f is a 2D vector, and $\vec{F}$ is a 3D vector.

b. $\nabla \times \nabla f$

If we allow the cross product to be generalized into 2D, then
$\nabla \times \nabla f = f_{yx} - f_{xy} = 0$.

c. $\nabla \times (\nabla \cdot \vec{F})$

This is not defined because $\nabla \cdot \vec{F}$ results in a scalar function, and the curl of a scalar function is not defined.

d. $\nabla \cdot (\nabla \times \vec{F})$

This is defined and has a value of 0 if $\vec{F}$ is twice differentiable, the proof of which is below.

Let $\vec{F} = \langle P, Q, R \rangle$.
$\nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$
$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$

4. Consider the vector field $\vec{F}(x, y, z) = \langle -z, 2y, x \rangle$. Find the integral curve of $\vec{F}$ with initial conditions $\vec{r}(0) = \langle 5, 1, 0 \rangle$.

$$\begin{cases} x' = -z \\ y' = 2y \\ z' = x \end{cases}$$

$y(t) = Ce^{2t} = e^{2t}$

$$\begin{cases} x' = -z \\ z' = x \end{cases}$$

$z(t) = A\cos t + B\sin t$ and $x(t) = B\cos t - A\sin t$
$\implies A = 0, B = 5$
$z(t) = 5\sin t$ and $x(t) = 5\cos t$
$\implies \vec{r}(t) = \langle 5\cos t, e^{2t}, 5\sin t \rangle$

5. State and prove the Fundamental Theorem of Calculus for Line Integrals.

State: $\displaystyle\int_C \nabla f \cdot \mathrm{d}\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$
Where $\vec{r}(t)$ parameterizes C for $a \leq t \leq b$.

Proof: $\displaystyle\int_C \nabla f \cdot \mathrm{d}\vec{r} = \int_a^b (\nabla f \circ \vec{r}) \cdot \vec{r}' \mathrm{d}t$
$\displaystyle = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \vec{r}) \mathrm{d}t$
$= f(\vec{r}(b)) - f(\vec{r}(a)) \blacksquare$.

Online Resources

Khan Academy — Multivariable Calculus
Paul's Online Notes — Calc III
Franke, Griggs, and Norris — Calculus for Engineers and Scientists Volume III
PatrickJMT — Calculus / Third Semester / Multivariable Calculus
MIT OpenCourseWare — Multivariable Calculus
Stewart — Single and Multivariable Calculus