Supervised Learning - Regression

Course:
INFO-6145 Data Science and Machine Learning

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Current Section

Supervised Learning

Supervised Learning involves training a model using input objects and their corresponding output values. The goal is to build a function f that can map new input data to predicted output values.

$$\hat{y} = f(x)$$

Where:

- $x^{(i)}$ are the input objects (feature vectors).
- $y^{(i)}$ are the output values (labels).
- $\hat{y}$ is the predicted output.

Example:

$$x = (x_1, x_2, \ldots, x_n)$$

Where $x_1, x_2, \ldots, x_n$ are the features, and the model predicts the output y based on these features.

Predictive Models: Regression

In predictive models, there are two main types:

Regression

The outcomes y are continuous values.

$$y \in \mathbb{R}$$

Example: Predicting house prices.

Classification

The outcomes y are discrete values.

$$y \in \{1, 2, \ldots, k\}$$

Example: Classifying emails as spam or not spam.

Predictive Models: Classification

Classification models deal with discrete outcomes. The outputs can belong to different categories or classes.

$$y \in \{1, 2, \ldots, k\}$$

Examples of classification:

- Binary classification: $y \in \{0, 1\}$ (e.g., true/false or spam/not spam).
- Multiclass classification: $y \in \{\text{green}, \text{red}, \text{black}\}$ (e.g., classifying different types of flowers).

Current Section

Regression Overview

Regression models are used for predicting continuous outcomes. The goal is to find a relationship between input features x and a continuous target variable y.

Some common types of regression models include:

- Linear regression
- Polynomial regression
- Support vector regression (SVR)
- Regression trees

Regression

Regression tasks involve predicting continuous output. Examples of regression tasks include:

- Predicting house prices
- Estimating stock prices

Some common types of regression models:

Linear Regression

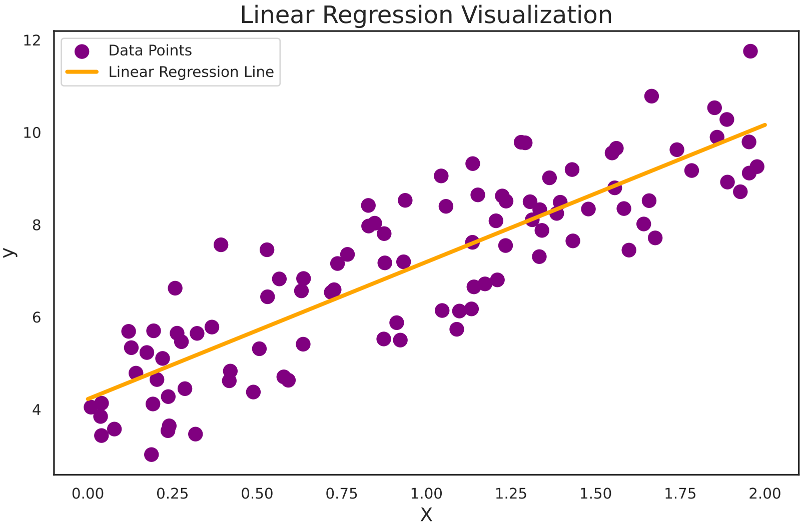
Simple yet highly effective for linear relationships.

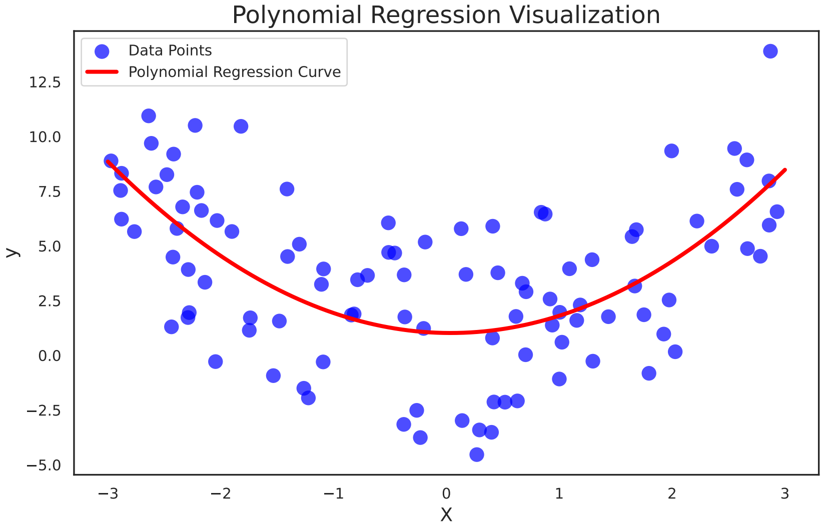
Regression Tree

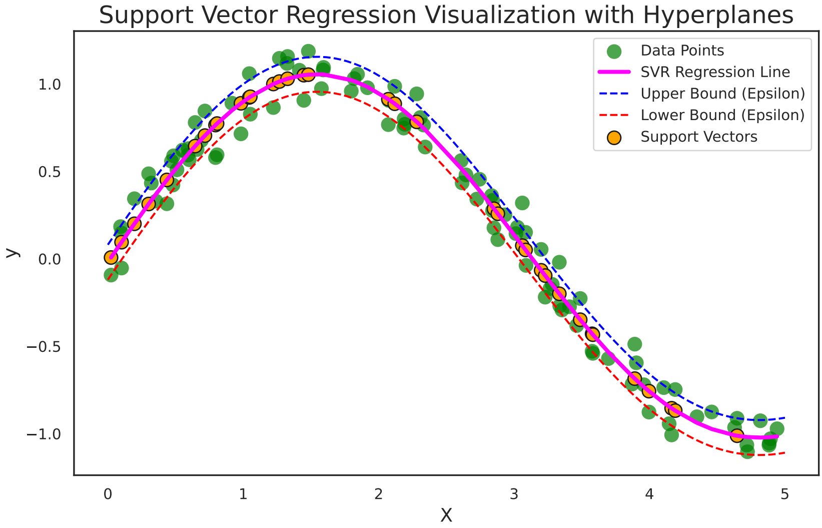
Uses tree-based structures to partition the data into subgroups.

Support Vector Regression (SVR)

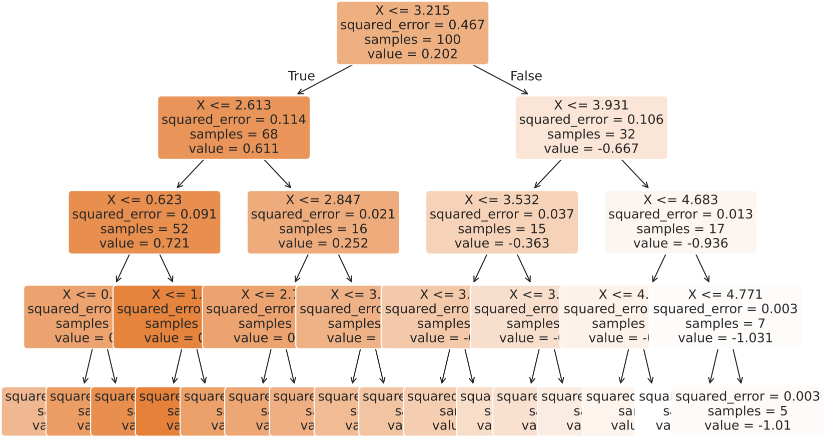
Attempts to fit the best hyperplane that minimizes error.

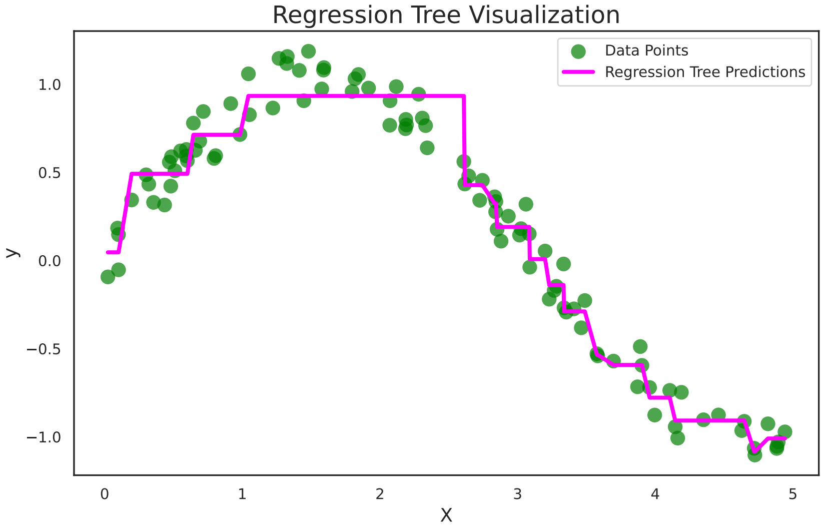

Linear Regression Visualization
Data Points
Linear Regression Line
12
10
8
6
4
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
X


Polynomial Regression Visualization
Data Points
Polynomial Regression Curve
12.5
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
y
−3
−2
−1
0
1
2
3
X


Support Vector Regression Visualization with Hyperplanes
Data Points
SVR Regression Line
Upper Bound (Epsilon)
Lower Bound (Epsilon)
Support Vectors
y
1.0
0.5
0.0
−0.5
−1.0
X
0
1
2
3
4
5

Decision Tree Structure




Regression Tree Visualization
Data Points
Regression Tree Predictions
y
X
1.0
0.5
0.0
−0.5
−1.0
0
1
2
3
4
5

Linear Regression

Linear regression is the simplest form of regression, where the relationship between the input and output is linear:

$$y = w_0 + w_1 x_1$$

Where:

- y is the predicted output.
- w_0 is the bias term (intercept).
- w_1 is the weight (slope) for the input feature x_1.

Example: If $w_0 = 10$ and $w_1 = 2$, for $x_1 = 5$:

$$y = 10 + 2 \times 5 = 20$$

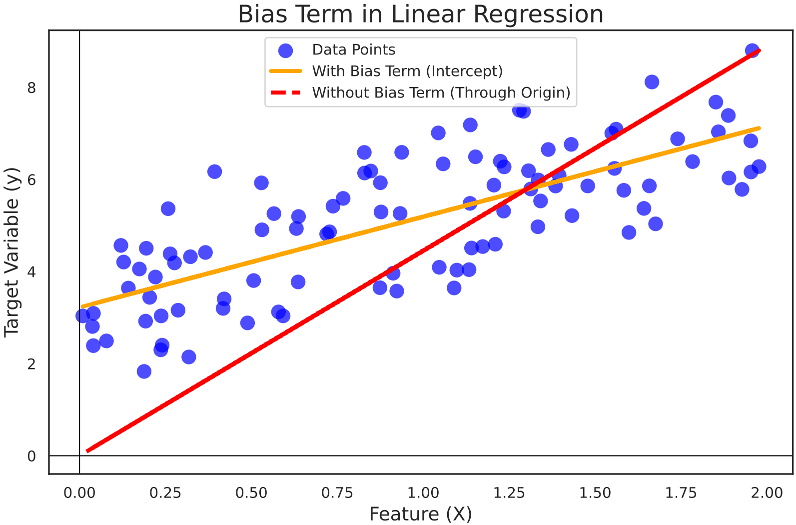
Bias Term in Linear Regression

The bias term w_0 in linear regression is an important constant that shifts the line up or down. It allows the model to fit the data better, even when the target variable y does not intersect the origin.

$$y = w_0 + w_1 x_1$$

If $w_0 = 0$, the regression line will always pass through the origin.

Bias Term in Linear Regression



Question: What is the effect of weight 0?

Q: What happens when an inputâs weight w_1 is 0?
If $w_1 = 0$, the model becomes:

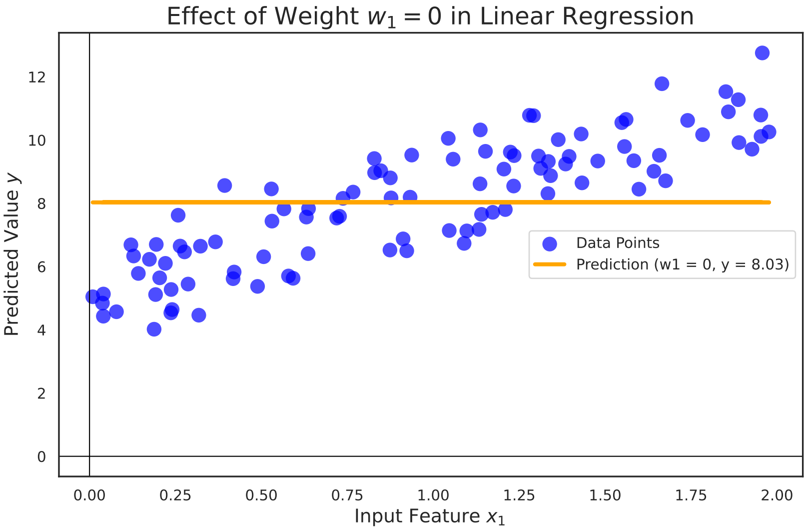
$$y = w_0$$

This means that the modelâs prediction is simply the bias term, and the input feature x_1 has no effect on the outcome.
Example:

$$y = 10 + 0 \times x_1 = 10$$

No matter the value of x_1, y will always be 10.

Question: What is the effect of weight 0?



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How Good is Our Model?

To evaluate how well a model fits the data, we use a **loss function.** The loss function measures the error between the actual value y and the predicted value $\hat{y}$.

$$\text{Loss} = L(y, \hat{y})$$

Common loss functions include:

- Mean Squared Error (MSE)
- Mean Absolute Error (MAE)

Mean Squared Error (MSE)

Mean Squared Error (MSE) is a commonly used loss function that measures the average squared difference between the actual values Y and the predicted values $\hat{Y}$.

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

Where:

- n is the number of observations.
- y_i is the actual value.
- $\hat{y}_i$ is the predicted value.

Example: If $y = [3, 5, 7]$ and $\hat{y} = [2.5, 5, 7.5]$:

$$MSE = \frac{1}{3}((3 - 2.5)^2 + (5 - 5)^2 + (7 - 7.5)^2) = 0.083$$

Mean Absolute Error (MAE)

Mean Absolute Error (MAE) is another common loss function that measures the average of the absolute differences between actual values Y and predicted values $\hat{Y}$.

$$MAE = \frac{1}{n}\sum_{i=1}^{n}|y_i - \hat{y}_i|$$

Where:

- n is the number of observations.
- y_i is the actual value.
- $\hat{y}_i$ is the predicted value.

Example: If $y = [3, 5, 7]$ and $\hat{y} = [2.5, 5, 7.5]$:

$$MAE = \frac{1}{3}(|3 - 2.5| + |5 - 5| + |7 - 7.5|) = 0.333$$

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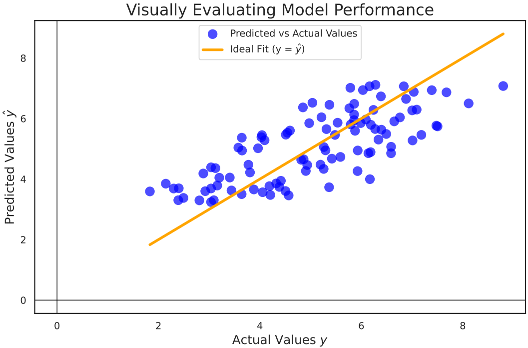
Visually Evaluating the Model

A visual way to evaluate the model's performance is by plotting the actual values y against the predicted values $\hat{y}$. The closer the points lie to the line $y = \hat{y}$, the better the model fits the data.
If the points scatter far from this line, it indicates poor predictions.



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Ridge Regression

Ridge Regression is a type of linear regression that includes a regularization penalty to avoid overfitting.

Key Concept

Ridge regression adds a penalty equal to the sum of the squared values of the coefficients to the loss function.

Formula

The objective function in ridge regression is:

$$\text{minimize} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 + \lambda \sum_{j=1}^{p} \beta_j^2$$

where λ is the regularization parameter.

Ridge Regression

Limitation

Ridge regression does not perform feature selection. All features contribute to the model.

Lasso Regression

Lasso regression (Least Absolute Shrinkage and Selection Operator) adds a regularization term to perform both variable selection and regularization.

Key Concept

Lasso regression includes a penalty equal to the absolute value of the coefficients.

Formula

The objective function in lasso regression is:

$$\text{minimize} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 + \lambda \sum_{j=1}^{p} |\beta_j|$$

where λ controls the strength of the regularization.

Lasso Regression

Warning

Lasso regression can shrink some coefficients to exactly zero, effectively performing feature selection.

Elastic Net Regression

Elastic Net combines both Lasso and Ridge regression penalties. It balances between Lasso's feature selection and Ridge's regularization.

Key Concept

Elastic Net applies both L1 and L2 regularization, combining the strengths of ridge and lasso.

Formula

The objective function for elastic net is:

$$\text{minimize} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 + \lambda_1 \sum_{j=1}^{p} |\beta_j| + \lambda_2 \sum_{j=1}^{p} \beta_j^2$$

where λ_1 and λ_2 control the regularization strength.

Elastic Net Regression

Usage

Elastic Net is useful when there are multiple correlated features.

Quantile Regression

Quantile regression estimates the conditional median or other quantiles of the response variable, providing a more robust alternative to linear regression.

Key Concept

Instead of minimizing the sum of squared residuals, quantile regression minimizes the sum of asymmetrically weighted absolute residuals.

Formula

The objective function for quantile regression is:

$$\text{minimize} \sum_{i=1}^{n} \rho_\tau(y_i - \hat{y}_i)$$

where ρ_τ is the quantile loss function.

Quantile Regression

Advantage

Quantile regression is robust to outliers and provides a more comprehensive view of the data distribution.

Bayesian Linear Regression

Bayesian Linear Regression provides a probabilistic approach to linear regression, incorporating prior distributions over the model parameters.

Key Concept

In Bayesian linear regression, we estimate the posterior distribution of the coefficients given the data, using Bayes' Theorem.

Formula

The posterior distribution is given by:

$$P(\beta|X,y) \propto P(y|X,\beta)P(\beta)$$

where $P(\beta)$ is the prior distribution, and $P(y|X,\beta)$ is the likelihood.

Bayesian Linear Regression

Benefit

Bayesian methods provide not just a point estimate but a full distribution for model parameters, giving a measure of uncertainty.