1/19/24

<u>Problem 1:</u> Discrete, probably
In five card study, a poker player is dealt five cards from a standard deck of 52 cards.

1. How many different 5-card hands are there? (Remember, in poker the order in which the cards are received does not matter.)

<u>Solution:</u>

$$\binom{N}{i} = \frac{N!}{i!(N-i)!}$$

$$\binom{52}{5} = \frac{52!}{5!(52-5)!}$$

$$= \frac{8.0658175\,2 \times 10^{67}}{120(2.5862324 \times 10^{59})}$$

$$= \frac{8.06581752 \times 10^{67}}{3.1034789 \times 10^{61}}$$

$$= \underline{2598960}$$

2. What is the probability of being dealt four of a kind (a card of the same rank from each suit in a five card hand)?

Solution:

A 2 3 4 5 6 7 8 9 10 J Q K A

This means that the number of combinations are:

$^{13}C_1 \cdot {}^{4}C_4 \cdot {}^{12}C_1 \cdot {}^{4}C_1 = 624$

$$\frac{624}{\frac{52!}{5!\,47!}}$$

$$= \frac{624}{2598960}$$

$= 0.024\%$

3. What is the probability of being dealt a flush (five cards of the same suit)?

Solution:

A 2 3 4 5 6 7 8 9 10 J Q K A

Combinations:

$$\underbrace{{}^{4}C_1}_{\text{4 suits}} \cdot \underbrace{{}^{13}C_5}_{\text{13 to choose from}} - \underbrace{{}^{9}C_1 \cdot {}^{4}C_1}_{\text{straight flush}} - \underbrace{{}^{4}C_1}_{\text{regal flush}}$$

$$= 4 \times 1287 - 9 \cdot 4 - 4$$

$$= 5108$$

$$\frac{5108}{\frac{52!}{5!\,47!}}$$

$$= \frac{5108}{2598960}$$

$$= 0.20\%$$

Scanned with CamScanner

$P(x) = xe^{-2x}, \quad 0 \leq x < \infty$

1. Is $P(x)$ normalized? Normalized: $\int_0^\infty P(x) = 1$

$\int_0^\infty xe^{-2x}\,dx$ Integrate by parts: $\int u\,dv = uv - \int v\,du$

$u = x \quad v = \int e^{-2x}dx = \int -\frac{1}{2}e^{u}du = -\frac{1}{2}e^{-2x}$

$du = 1dx \quad dv = e^{-2x}dx$

$u = -2x$

$du = -2dx$

$dx = -\frac{1}{2}du$

$\int_0^\infty xe^{-2x}dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}dx$

$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^\infty$

$= \left(-\frac{1}{2}(\infty)\overset{0}{e^{-2(\infty)}} - \frac{1}{4}\overset{0}{e^{-2(\infty)}}\right) - \left(\cancel{-\frac{1}{2}(0)e^{-2(0)}} - \frac{1}{4}\overset{1}{e^{-2(0)}}\right)$

$= 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$

$\int P(x) \neq 1$, so needs to be normalized

$\tilde{P}(x) = \frac{P(x)}{\int_0^\infty P(x)\,dx} = \frac{xe^{-2x}}{\frac{1}{4}} = \boxed{4xe^{-2x}}$

See python code for plot

2. Most probable value of x?

Find maximum $P(x)$, corresponding value of x is most probable. See python code for one way to do this.

$\boxed{\text{Most probable} = 0.5}$

3. What is the expectation value of x?

$\langle x \rangle = \int_0^\infty x \cdot \tilde{P}(x)\, dx$

$u = x^2 \quad \cancel{du} = -\frac{1}{2}e^{-2x}$

$du = 2x \quad dv = e^{-2x}$

$= \int_0^\infty 4x^2 e^{-2x}\, dx$

$4\int_0^\infty x^2 e^{-2x}\, dx$

same as not normalized P(x)

$= 4\left(-\frac{1}{2}x^2 e^{-2x} + \int x e^{-2x}\, dx\right)$

$= 4\left(-\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right)\Big|_0^\infty$

$\left(-2x^2 e^{-2x} - 2x e^{-2x} - 1 \cdot e^{-2x}\right)\Big|_0^\infty$

$-e^{-2x}\left(2x^2 + 2x + 1\right)\Big|_0^\infty$

$\underbrace{-e^{-\infty}}_{0}(2\infty^2 + 2\infty + 1) - \left(-e^0(2 \cdot 0^2 + 2 \cdot 0 + 1)\right)$

$\langle x \rangle = 0 - (-1) = \boxed{1}$

4. What is the variance of x?

$\langle x\rangle^2 = 1^2 = 1$

Variance $\sigma^2 = \langle x^2\rangle - \langle x\rangle^2$

$\langle x^2\rangle = \int_0^\infty x^2 \cdot \hat{P}(x)$

$u = x^3 \quad v = -\frac{1}{2}e^{-2x}$

$du = 3x^2 dx \quad dv = e^{-2x}dx$

$= \int_0^\infty 4x^3 e^{-2x}\, dx$

$= 4\left(\frac{-1}{2}x^3 e^{-2x} - \int \frac{-3}{2}x^2 e^{-2x}\, dx\right)$

same as chunk in $\langle x\rangle$

$= 4\left(-\frac{1}{2}x^3 e^{-2x} + \frac{3}{2}\int x^2 e^{-2x}\, dx\right)$

$= 4\left(-\frac{1}{2}x^3 e^{-2x} + \frac{3}{2}\left(-\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right)\right)\Big|_0^\infty$

$= 4\left(-\frac{1}{2}x^3 e^{-2x} - \frac{3}{4}x^2 e^{-2x} - \frac{3}{4}xe^{-2x} - \frac{3}{8}e^{-2x}\right)\Big|_0^\infty$

$= -2x^3 e^{-2x} - 3x^2 e^{-2x} - 3xe^{-2x} - \frac{3}{2}e^{-2x}\Big|_0^\infty$

$= -e^{-2x}\left(2x^3 + 3x^2 + 3x + \frac{3}{2}\right)\Big|_0^\infty$

$= \left(\overset{0}{\cancel{-e^{\infty}}}\left(2\infty^3 + 3\infty^2 + 3\infty + \frac{3}{2}\right)\right) - \left(-e^0\left(2\cdot 0^3 + 3\cdot 0^2 + 3\cdot 0 + \frac{3}{2}\right)\right)$

$0 + \frac{3}{2} = \frac{3}{2}$

$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2$

$= \frac{3}{2} - 1$

$\boxed{\sigma^2 = 1/2}$

HW1_Solution

January 25, 2024

```
# This is formatted as code
```

1 Problem 2

1.1 Initialize python

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad
```

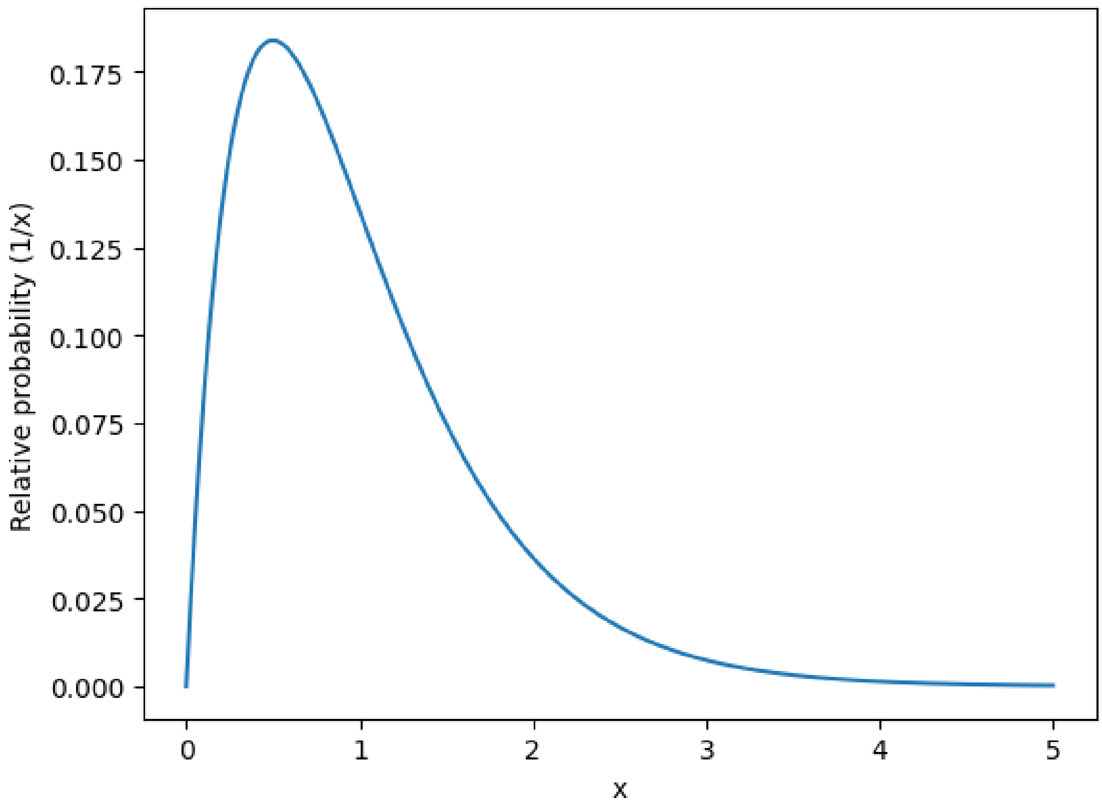
1.2 Define probability function

[ ]:
```
def phif(x):
    return x*np.exp(-2*x)
```

1.3 Plot out function

[ ]:
```
x = np.linspace(0,5,1000)
phi = phif(x)
plt.plot(x,phi)
plt.xlabel('x')
plt.ylabel('Relative probability (1/x)')
```

[ ]:
```
Text(0, 0.5, 'Relative probability (1/x)')
```



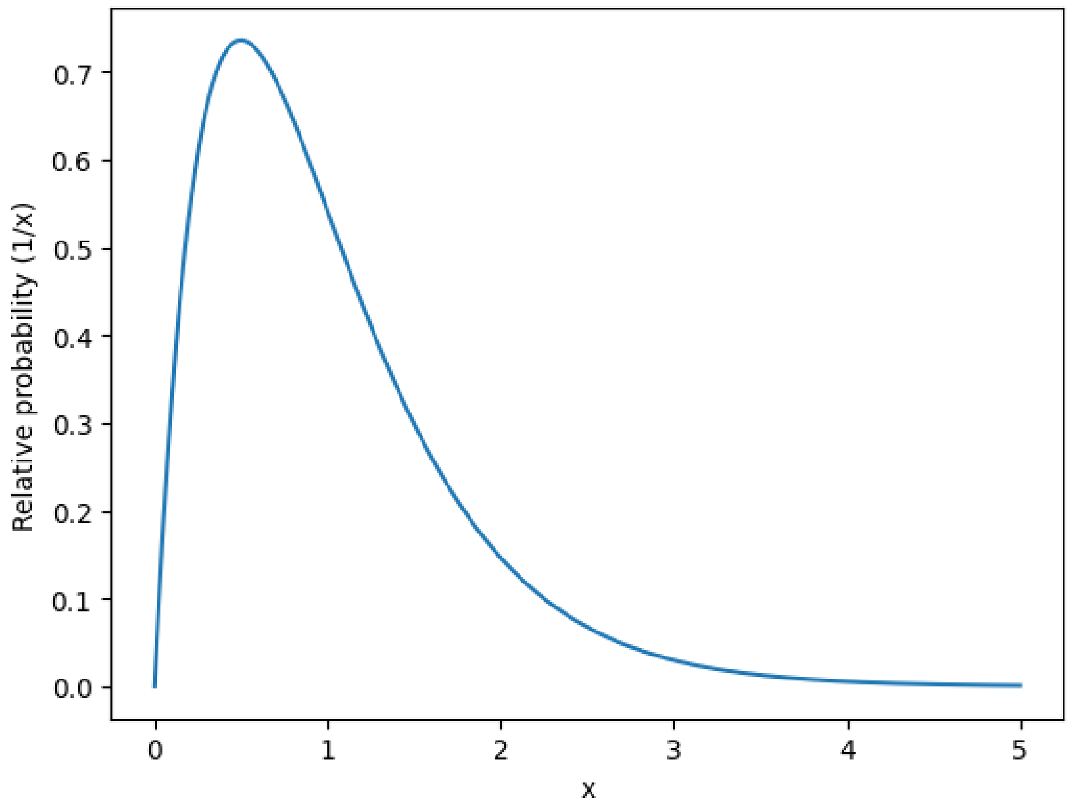
1.4 Normalize by numerical integration

[ ]:
```
N,err = quad(phif,0,5)

def phif_norm(x):
    return phif(x)/N

plt.plot(x,phif_norm(x))
plt.xlabel('x')
plt.ylabel('Relative probability (1/x)')
```

[ ]:
```
Text(0, 0.5, 'Relative probability (1/x)')
```



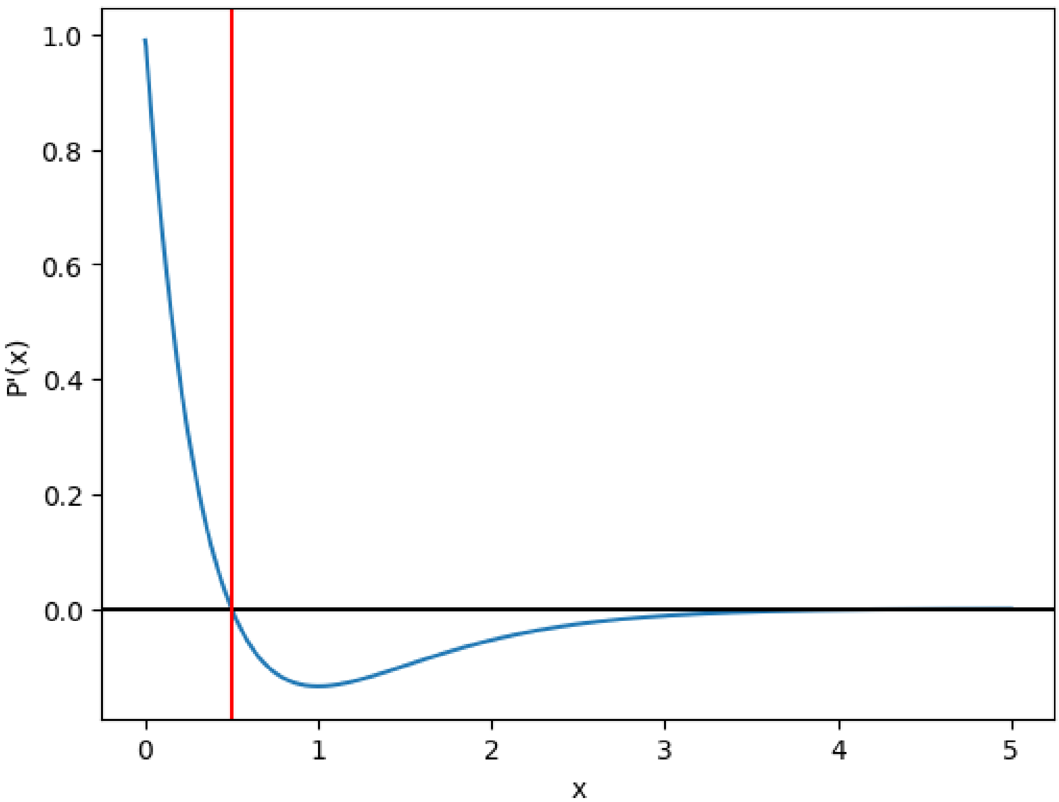
Most probable value of T????

Can find by determining the maximum of the curve, such as by taking the derivative and finding where it equals zero.

[ ]:
```
def Deriv(x):
    return np.gradient(phif(x),x)

plt.plot(x,Deriv(x))
plt.xlabel('x')
plt.ylabel("P'(x)")
plt.axhline(0,color ='k', ls='-')
plt.axvline(0.5,color = 'r', ls='-')
```

[ ]: <matplotlib.lines.Line2D at 0x7b1943216080>



Red line added to highlight that the graph is crossing the x-axis at P'(x) = 0.5

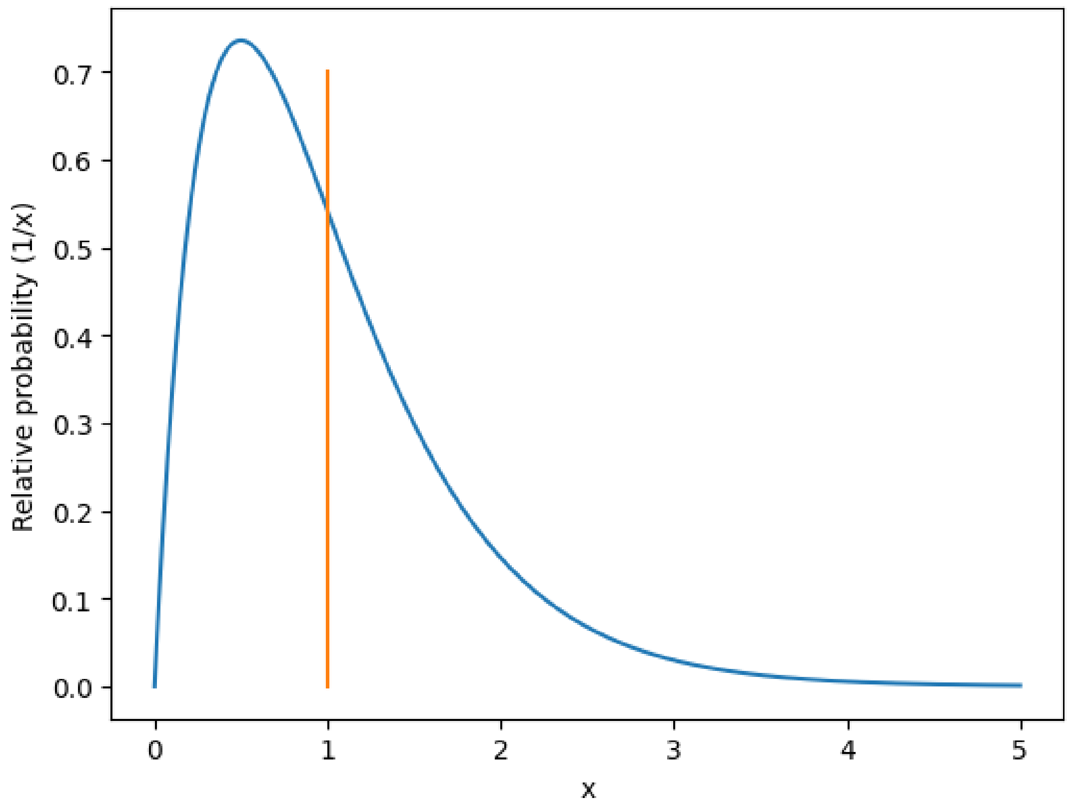
1.5 Expectation value of x:

[ ]:
```
def Xphif(x):
    return x*phif_norm(x)
Xbar,err = quad(Xphif,0,5)
print("Expectation value of x = {:4.1f}".format(Xbar))

plt.plot(x,phif_norm(x))
plt.xlabel('x')
plt.ylabel('Relative probability (1/x)')
plt.plot([Xbar,Xbar],[0,0.7],ls='-')
```

```
Expectation value of x =  1.0
```

[ ]:
```
[<matplotlib.lines.Line2D at 0x7b19430d9d20>]
```



1.6 Variance and standard deviation

[ ]:
```
def X2phif(x):
    return x*x*phif_norm(x)
X2bar,err = quad(X2phif,0,5)
print("Expectation value of X**2 = {:4.1f}".format(X2bar))

Xrms=np.sqrt(X2bar)
print("Room mean square T ={:4.1f}".format(Xrms))

variance = X2bar-Xbar*Xbar
print("Variance ={:4.1f}".format(variance))

stddev = np.sqrt(variance)
print("Standard deviation ={:4.1f}".format(stddev))
```

```
Expectation value of X**2 =  1.5
Room mean square T = 1.2
Variance = 0.5
Standard deviation = 0.7
```

[ ]:

```

```

##Problem 3

[ ]:

```
#It's late on a Friday night and people are stumbling up Notre Dame Ave. to␣
↪their dorms. You
#observe one particularly impaired individual who is taking steps of equal␣
↪length 1m to the north or
#south (i.e., in one dimension), with equal probability.


#1. What is the furthest distance could walk after 20 steps?

#set randomly either moving forward or backwards by 1 or -1 m (with equal␣
↪probability) with a max size of 20 m
steps = np.random.choice([-1, 1], size=20)


#the total distance is the sum of the steps
distance = np.sum(steps)

print("Furthest distance after 20 steps:", abs(distance))
```

```
Furthest distance after 20 steps: 6
```

[ ]:

```
#2. What is the probability that the person won't have traveled any net distance
#at all after 20 steps?

# Simulate multiple random walks to calculate the probability
num_simulations = 1000
no_travel_count = 0

for i in range(num_simulations):
    steps = np.random.choice([-1, 1], size=20)
    distance = np.sum(steps)

    #making an if statement, to see if that the total net is 0 in that␣
↪simulation, we add to the counter
    if distance == 0:
        no_travel_count += 1

#probability = no distance travelled / number of simulations
probability_no_travel = no_travel_count / num_simulations

print("Probability of not traveling any net distance:", probability_no_travel)
```

```
Probability of not traveling any net distance: 0.17
```

[ ]:
```python
#3. What is the probability that the person has traveled half the maximum
# distance after 20 steps?

# Simulate multiple random walks to calculate the probability
num_simulations = 1000
half_max_distance_count = 0

for _ in range(num_simulations):
    steps = np.random.choice([-1, 1], size=20)
    distance = np.sum(steps)
    if abs(distance) == 10:  # Half of the maximum distance after 20 steps
        half_max_distance_count += 1

probability_half_max_distance = half_max_distance_count / num_simulations
print("Probability of traveling half the maximum distance:",␣
 ↪probability_half_max_distance)
```

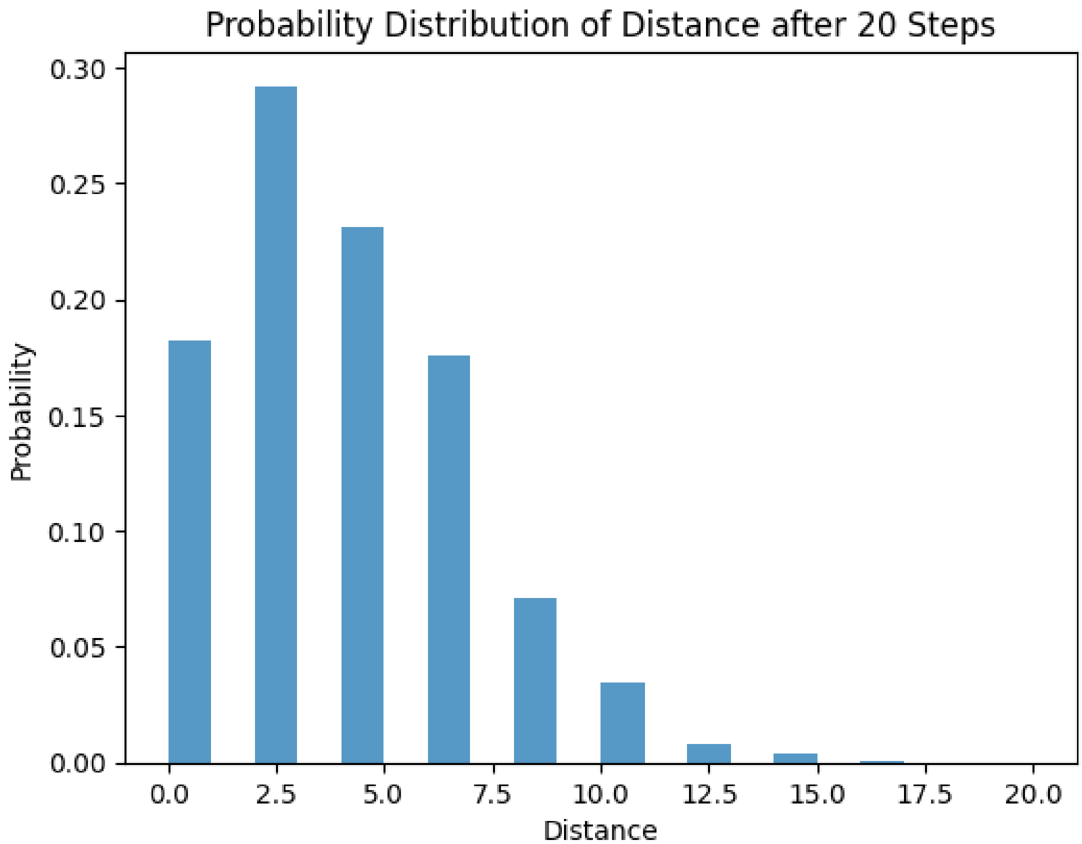
```
Probability of traveling half the maximum distance: 0.039
```

[ ]:
```python
#4. . Plot the probability of traveling a given distance vs distance. Does the
#probability distribution look familiar? You'll see it again when we talk about
#diffusion.

# Simulate multiple random walks to create the probability distribution
num_simulations = 1000
distances = []

for _ in range(num_simulations):
    steps = np.random.choice([-1, 1], size=20)
    distance = np.sum(steps)
    distances.append(abs(distance))

# Plotting the probability distribution
plt.hist(distances, bins=range(0, 21), density=True, alpha=0.75)
plt.xlabel("Distance")
plt.ylabel("Probability")
plt.title("Probability Distribution of Distance after 20 Steps")
plt.show()
```


Probability Distribution of Distance after 20 Steps
Probability
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Distance

[ ]:

HW # 1

Problem 4: Now this is what I call equilibrium

1. What is the expectation value of the velocity v of a particle?

Solution:

$$E \propto e^{-E/K_B T} \quad ; \quad \begin{array}{ll} m & \text{mass} \\ T & \text{temperature} \\ K_B & \text{Boltzmann constant} \end{array}$$

$$\bar{K} = \frac{mv^2}{2}$$

$$\langle v \rangle = \int_{-\infty}^{\infty} v \cdot P(v)\, dv$$

$$\langle v \rangle = \int_{-\infty}^{\infty} v \cdot \frac{1}{Z} e^{-mv^2/2K_B T}\, dv$$

where Z is a normalization constant

$$\langle v \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} v \cdot e^{-mv^2/2K_B T}\, dv$$

For the normalization constant:

$$Z = \int_{-\infty}^{\infty} e^{-mv^2/2K_BT}\, dv \quad ; \quad \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Gaussian integral

$$Z = \sqrt{\frac{2\pi K_BT}{m}}$$

Going back to the integral

$$\langle v \rangle = \frac{1}{\sqrt{\frac{2\pi K_BT}{m}}} \int_{-\infty}^{\infty} v\, e^{-mv^2/2K_BT}\, dv$$

$$\langle v \rangle = 0 \quad ; \quad \int_{-\infty}^{\infty} x e^{-ax^2} dx = 0$$

2. What is the expectation value of the Kinetic energy K of a particle? How does your answer depend on the particle mass? On temperature?

Solution:

$$K = \frac{mv^2}{2}$$

$$\langle K \rangle = \int_{0}^{\infty} \frac{1}{2} mv^2 \cdot P(v)\, dv$$

$$\langle K \rangle = \int_{0}^{\infty} \frac{1}{2} mv^2 \cdot e^{-mv^2/2K_BT}\, dv$$

$$\langle K \rangle = \frac{1}{2} m \int_0^{\infty} v^2 e^{-mv^2/2k_BT} \, dv$$

$$= \frac{1}{2} m \left[\frac{1}{4\left(\frac{m}{2k_BT}\right)} \right] \left(\frac{\pi}{m/2k_BT} \right)^{1/2}$$

$$= \frac{m}{\frac{8m}{2k_BT}} \sqrt{\frac{2\pi k_BT}{m}}$$

$$= \frac{2\cancel{m}k_BT}{8\cancel{m}} \sqrt{\frac{2\pi k_BT}{m}}$$

$$= \frac{k_BT}{4} \sqrt{\frac{2\pi k_BT}{m}}$$

Multiplying by the value of Z obtained before

$$\langle K \rangle = \frac{k_BT}{4} \sqrt{\frac{2\pi k_BT}{m}} \sqrt{\frac{m}{2\pi k_BT}}$$

$$\langle K \rangle = \frac{k_BT}{4}$$

proportional to the temperature