1/19/24

Problem 1: Discrete, probably

In five card study, a poker player is dealt five cards from a standard deck of 52 cards.

1. How many different 5-card hands are there? (Remember, in poker the order in which the cards are received does not matter.)

Solution:

$$\binom{N}{i} = \frac{N!}{i!(N-i)!}$$

$$\binom{52}{5} = \frac{52!}{5!(52-5)!}$$

$$= \frac{8.06581752 \times 10^{67}}{120(2.5862324 \times 10^{59})}$$

$$= \frac{8.06581752 \times 10^{67}}{3.1034789 \times 10^{61}}$$

$$= 2598960$$

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2. What is the probability of being dealt four of a kind (a card of the same rank from each suit in a five card hand)?

<u>Solution:</u>

A 2 3 4 5 6 7 8 9 10 J Q K A

This means that the number of combinations are:

$$\underbrace{{}^{13}C_1}_{\text{we have 13 different cards}} \cdot \underbrace{{}^{4}C_4}_{\text{we have 4 suits to choose from (we are choosing all 4 cards)}} \cdot \underbrace{{}^{12}C_1}_{\text{12 ranks to choose from}} \cdot \underbrace{{}^{4}C_1}_{} = 624$$

For instance, if we have a Jack, we have 4 Jacks to choose from

$$\frac{624}{\frac{52!}{5!\,47!}}$$

$$= \frac{624}{2598960}$$

$$= 0.024\,\%$$

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3. What is the probability of being dealt a flush (five cards of the same suit)?

Solution:

A 2 3 4 5 6 7 8 9 10 J Q K A

Combinations:

$$\underbrace{{}^{4}C_{1}}_{\text{4 suits}} \cdot \underbrace{{}^{13}C_{5}}_{\text{13 to choose from}}$$

$= 4 \times 1287$

$= 5148$

$$\frac{5148}{\frac{52!}{5!\,47!}}$$

$$= \frac{5148}{2598960}$$

$= 0.20\%$

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Whatever method you want to use for integration, as long as you get to the right answer. I have the math written out as well as a solution using Python.

$P(x) = xe^{-2x}, \quad 0 \le x < \infty$

1. Is $P(x)$ normalized? Normalized: $\int_0^\infty P(x) = 1$

$\int_0^\infty xe^{-2x}\,dx$ Integrate by parts: $\int u\,dv = uv - \int v\,du$

$u = x \qquad v = \int e^{-2x}dx = \int -\frac{1}{2}e^{u}du = -\frac{1}{2}e^{-2x}$

$du = 1dx \quad dv = e^{-2x}dx$

$u = -2x$

$du = -2dx$

$dx = -\frac{1}{2}du$

$\int_0^\infty xe^{-2x}dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}dx$

$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^\infty$

$= \left(\cancelto{0}{-\frac{1}{2}(\infty)e^{-2(\infty)}} - \cancelto{0}{\frac{1}{4}e^{-2(\infty)}}\right) - \left(\cancelto{0}{-\frac{1}{2}(0)e^{-2(0)}} - \frac{1}{4}\cancelto{1}{e^{-2(0)}}\right)$

$= 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$

$\int P(x) \ne 1$, so needs to be normalized

$\tilde{P}(x) = \frac{P(x)}{\int_0^\infty P(x)\,dx} = \frac{xe^{-2x}}{\frac{1}{4}} = \boxed{4xe^{-2x}}$

See python code for plot

2. Most probable value of x?

Find maximum $P(x)$, corresponding value of x is most probable. See python code for one way to do this.

$\boxed{\text{Most probable} = 0.5}$

Can also compute dP(x)/dx = 0 and solve to get the same answer.

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3. What is the expectation value of x?

$$\langle x \rangle = \int_0^\infty x \cdot \tilde{P}(x)\, dx$$

$u = x^2 \quad dv = -\frac{1}{2}e^{-2x}$

$du = 2x \quad v = e^{-2x}$

$$= \int_0^\infty 4x^2 e^{-2x}\, dx$$

$$4\int_0^\infty x^2 e^{-2x}\, dx$$

same as not normalized P(x)

$$= 4\left(-\frac{1}{2}x^2 e^{-2x} + \int x e^{-2x}\, dx\right)$$

$$= 4\left(-\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right)\Big|_0^\infty$$

$$\left(-2x^2 e^{-2x} - 2x e^{-2x} - 1 \cdot e^{-2x}\right)\Big|_0^\infty$$

$$-e^{-2x}\left(2x^2 + 2x + 1\right)\Big|_0^\infty$$

$$\underset{0}{-e^{-\infty}}(2\infty^2 + 2\infty + 1) - \left(-e^0(2 \cdot 0^2 + 2 \cdot 0 + 1)\right)$$

$$\langle x \rangle = 0 - (-1) = \boxed{1}$$

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4. What is the variance of x?

$\langle x \rangle^2 = 1^2 = 1$

Variance $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

$\langle x^2 \rangle = \int_0^\infty x^2 \cdot \hat{P}(x)$

$u = x^3 \quad v = -\frac{1}{2}e^{-2x}$

$du = 3x^2 dx \quad dv = e^{-2x} dx$

$= \int_0^\infty 4x^3 e^{-2x} dx$

$= 4\left(-\frac{1}{2}x^3 e^{-2x} - \int -\frac{3}{2}x^2 e^{-2x} dx\right)$

same as chunk in $\langle x \rangle$

$= 4\left(-\frac{1}{2}x^3 e^{-2x} + \frac{3}{2}\int x^2 e^{-2x} dx\right)$

$= 4\left(-\frac{1}{2}x^3 e^{-2x} + \frac{3}{2}\left(-\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right)\right)\Big|_0^\infty$

$= 4\left(-\frac{1}{2}x^3 e^{-2x} - \frac{3}{4}x^2 e^{-2x} - \frac{3}{4}x e^{-2x} - \frac{3}{8}e^{-2x}\right)\Big|_0^\infty$

$= -2x^3 e^{-2x} - 3x^2 e^{-2x} - 3x e^{-2x} - \frac{3}{2}e^{-2x} \Big|_0^\infty$

$= -e^{-2x}\left(2x^3 + 3x^2 + 3x + \frac{3}{2}\right)\Big|_0^\infty$

$= \left(-\cancel{e^{-\infty}}^{\,0}\left(2\infty^3 + 3\infty^2 + 3\infty + \frac{3}{2}\right)\right) - \left(-e^0\left(2\cdot 0^3 + 3\cdot 0^2 + 3\cdot 0 + \frac{3}{2}\right)\right)$

$0 + \frac{3}{2} = \frac{3}{2}$

$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

$= \frac{3}{2} - 1$

$\boxed{\sigma^2 = 1/2}$

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HW1_Solution

January 25, 2024

```
# This is formatted as code
```

1 Problem 2

1.1 Initialize python

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad
```

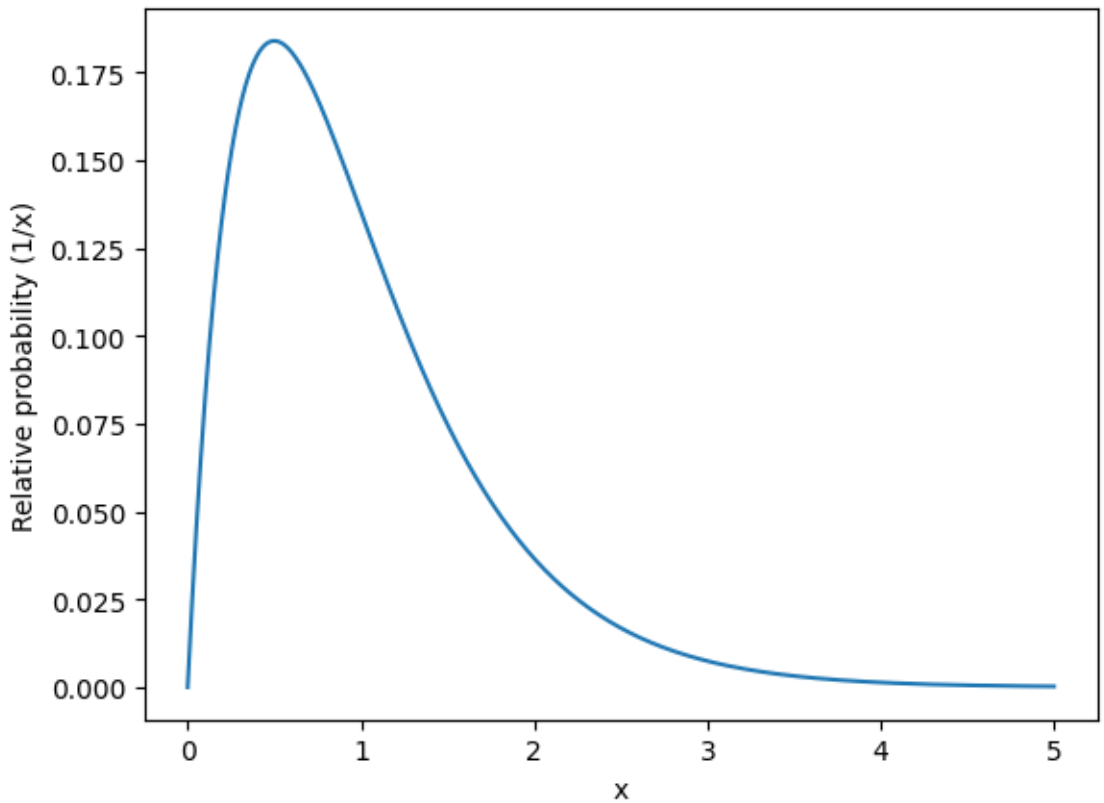
1.2 Define probability function

[ ]:
```
def phif(x):
    return x*np.exp(-2*x)
```

1.3 Plot out function

[ ]:
```
x = np.linspace(0,5,1000)
phi = phif(x)
plt.plot(x,phi)
plt.xlabel('x')
plt.ylabel('Relative probability (1/x)')
```

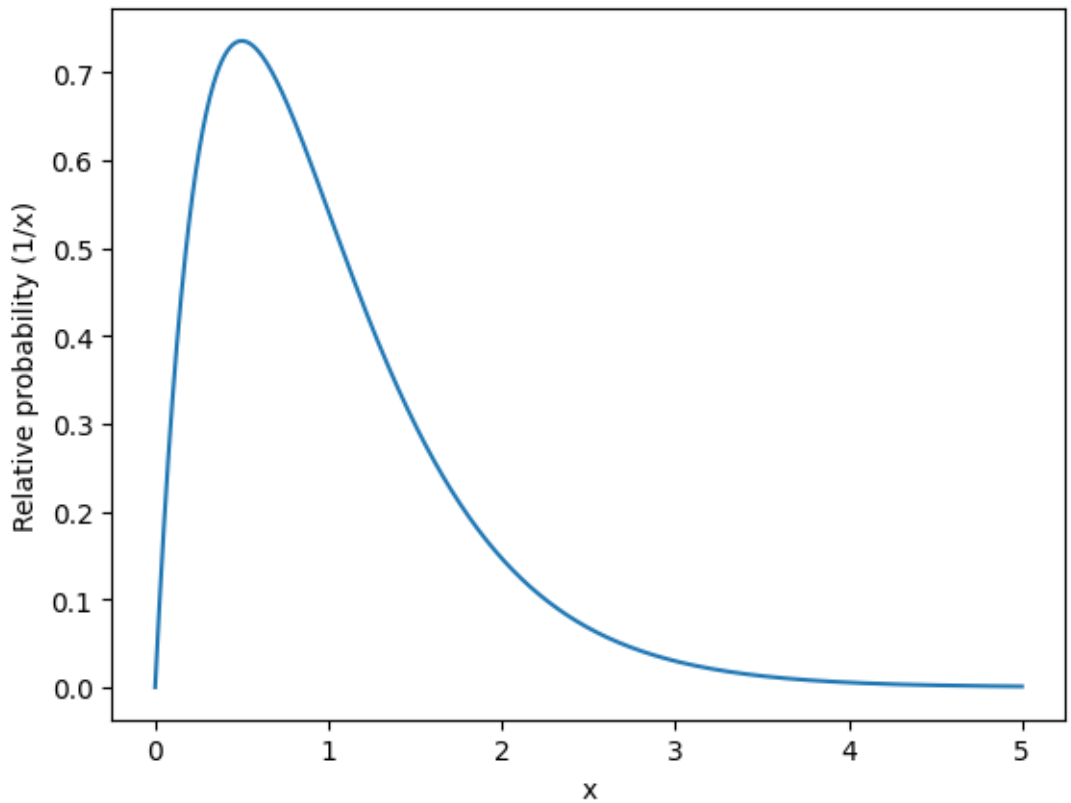
[ ]:
```
Text(0, 0.5, 'Relative probability (1/x)')
```



1.4 Normalize by numerical integration

[ ]:
```
N,err = quad(phif,0,5)

def phif_norm(x):
    return phif(x)/N

plt.plot(x,phif_norm(x))
plt.xlabel('x')
plt.ylabel('Relative probability (1/x)')
```

[ ]:
```
Text(0, 0.5, 'Relative probability (1/x)')
```

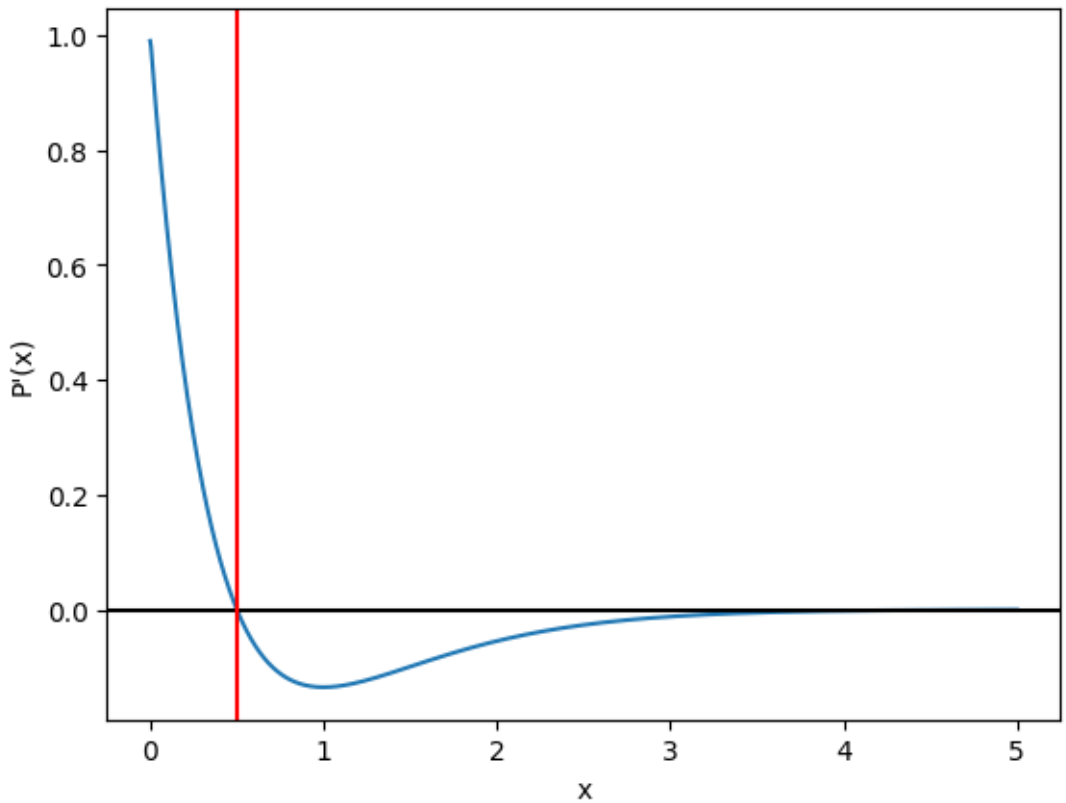


Most probable value of T????

Can find by determining the maximum of the curve, such as by taking the derivative and finding where it equals zero.

[ ]:
```
def Deriv(x):
    return np.gradient(phif(x),x)

plt.plot(x,Deriv(x))
plt.xlabel('x')
plt.ylabel("P'(x)")
plt.axhline(0,color ='k', ls='-')
plt.axvline(0.5,color = 'r', ls='-')
```

[ ]:
```
<matplotlib.lines.Line2D at 0x7b1943216080>
```



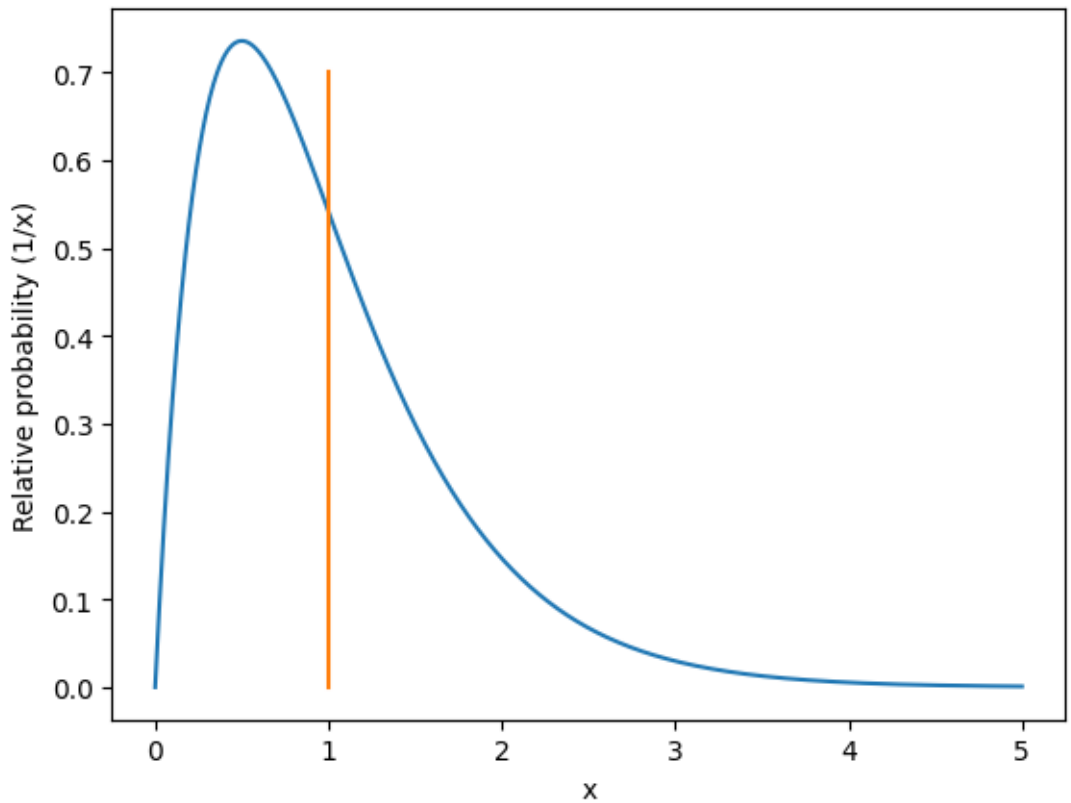
Red line added to highlight that the graph is crossing the x-axis at P'(x) = 0.5

1.5 Expectation value of x:

[ ]:
```
def Xphif(x):
    return x*phif_norm(x)
Xbar,err = quad(Xphif,0,5)
print("Expectation value of x = {:4.1f}".format(Xbar))

plt.plot(x,phif_norm(x))
plt.xlabel('x')
plt.ylabel('Relative probability (1/x)')
plt.plot([Xbar,Xbar],[0,0.7],ls='-')
```

```
Expectation value of x =  1.0
```

[ ]:
```
[<matplotlib.lines.Line2D at 0x7b19430d9d20>]
```



1.6 Variance and standard deviation

[ ]:
```
def X2phif(x):
    return x*x*phif_norm(x)
X2bar,err = quad(X2phif,0,5)
print("Expectation value of X**2 = {:4.1f}".format(X2bar))

Xrms=np.sqrt(X2bar)
print("Room mean square T ={:4.1f}".format(Xrms))

variance = X2bar-Xbar*Xbar
print("Variance ={:4.1f}".format(variance))

stddev = np.sqrt(variance)
print("Standard deviation ={:4.1f}".format(stddev))
```

```
Expectation value of X**2 =  1.5
Room mean square T = 1.2
Variance = 0.5
Standard deviation = 0.7
```

Problem #3

1. What is the furthest distance that the students could walk after 20 steps?

$$The\ students\ can\ take\ a\ random\ walk\ either + 1\ m\ or - 1\ m\ in\ either\ direction$$

Case 1: Maximum distance is when the students goes all the way north, without any change in direction

$$D_{max} = 20\ *\ 1 = 20\ m$$

Case 2: Maximum distance is when the students goes all the way south, without any change in direction

$$D_{max} = 20 *\ -1 =\ -20\ m$$

2. What is the probability that the person won't have traveled any net distance at all after 20 steps?

$$N_{possible} = \#\ of\ possible\ tracks\ the\ trajectories = 2^{20}$$

$$N_{equal} = Equal\ number\ of\ north(10) and\ south\ (10) = 20C_{10}$$

$$P(0) = \frac{N_{equal}}{N_{possible}} = \frac{20C_{10}}{2^{20}} = 0.175$$

3. What is the possibility that the person has traveled half the maximum distance after 20 steps?

$$N_{possible} = \#\ of\ possible\ tracks\ the\ trajectories = 2^{20}$$

$$1. Half\ maximum\ distance\ can\ be\ in\ either\ direction\ to\ north\ or\ south$$

$$D_{half} = \frac{20}{2} m = + or -\ 10\ m$$

$$2. split\ the\ probability\ into\ 2, north\ and\ south$$

$$P(10)_{north} =\ Possibility\ to\ the\ north\ for\ 10\ m = \frac{20C_{10}}{2^{20}}$$

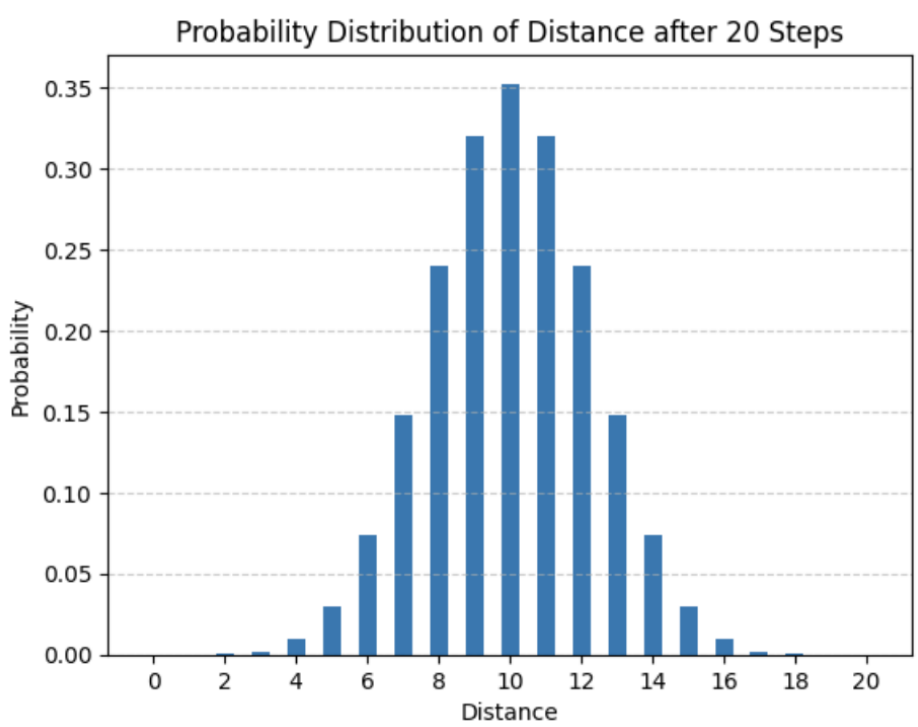
$$P(10)_{south} =\ Possibility\ to\ the\ south\ for\ 10\ m = \frac{20C_{10}}{2^{20}}$$

$$3. Add\ the\ probability\ to\ get\ the\ total$$

$$P(10)_{north} + P(10)_{south} = 2 * \frac{20C_{10}}{2^{20}} = 0.36$$

4. Plot the probability distribution (P in the y-axis and distance in the x-axis)

```
import math
import matplotlib.pyplot as plt

# Total number of steps
total_steps = 20

# Calculate probabilities for each distance
distances = range(total_steps + 1)  # Possible distances range from 0 to 20

probabilities = []

for d in distances:
      probabilities.append(2 * math.comb(total_steps, d) / (2 ** total_steps))

# Plotting the probability distribution
plt.bar(distances, probabilities, width=0.5)
plt.xlabel("Distance")
plt.ylabel("Probability")
plt.title("Probability Distribution of Distance after 20 Steps")
plt.xticks(range(0, total_steps + 1, 2))  # Adjust x-axis ticks for better visibility
plt.grid(axis='y', linestyle='--', alpha=0.7)
plt.show()
```



HW #1

Problem 4: Now this is what I call equilibrium

1. What is the expectation value of the velocity v of a particle?

<u>Solution:</u>

$$P(v) \propto e^{-1/2 mv^2/K_B T} \; ; \quad \begin{array}{ll} m & \text{mass} \\ T & \text{temperature} \\ K_B & \text{Boltzmann constant} \end{array}$$

Normalizing first:

$$Z = \underbrace{\int_{-\infty}^{\infty} e^{-mv^2/2K_B T}\, dv}_{\text{Gaussian integral}} \; ; \quad \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$Z = \sqrt{\frac{2\pi K_B T}{m}}$$ normalization constant

Then calculating $\langle v \rangle$:

$$\langle v \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} v \cdot P(v)\, dv$$

$$\langle v \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} v \cdot e^{-mv^2/2K_B T}\, dv$$

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$$\langle v \rangle = \frac{1}{\sqrt{\frac{2\pi K_B T}{m}}} \int_{-\infty}^{\infty} v e^{-mv^2/2K_BT} \, dv$$

$$\underline{\langle v \rangle = 0} \quad ; \quad \int_{-\infty}^{\infty} x e^{-ax^2} \, dx = 0$$

2. What is the expectation value of the Kinetic energy K of a particle? How does your answer answer depend on the particle mass? On temperature?

<u>Solution:</u>

$$\langle K \rangle = \frac{m}{2} \langle v^2 \rangle$$

Calculating $\langle v^2 \rangle$:

$$\langle v^2 \rangle = \frac{1}{Z} \cdot \int_0^{\infty} v^2 e^{-mv^2/2K_BT} \, dv$$

$$\langle v^2 \rangle = \frac{1}{\sqrt{\frac{2\pi K_B T}{m}}} \int_0^{\infty} v^2 e^{-mv^2/2K_BT} \, dv$$

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According to integral table,

$$\int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4 \cdot a^{3/2}} \quad ; \quad a = m/2k_BT$$

$$\langle v^2 \rangle = \frac{1}{\sqrt{\frac{2\pi k_B T}{m}}} \frac{\sqrt{\pi}}{4 \cdot \left(m/2k_BT\right)^{3/2}}$$

$$\langle v^2 \rangle = \sqrt{\frac{m}{2\pi k_B T}} \frac{\sqrt{\pi}}{4 \cdot \left(m/2k_BT\right)^{3/2}}$$

$$\langle v^2 \rangle = \frac{k_B T}{m}$$

$$\therefore \langle K \rangle = \frac{m}{2} \cdot \frac{k_B T}{m}$$

$$= \underline{1/2\, k_B T}$$

proportional to the temperature; does not depend on mass

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