Assignment 5 of Computational Astrophysics in NTHU

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1 Programming Assignments

Q1 : Solve $x^3 + 1.5x^2 - 5.75x + 4.37$

I first try to search *Wolfram* [1] to find out the approximately solution of this equation, the root is in range -3 and -4.(Fig.1)

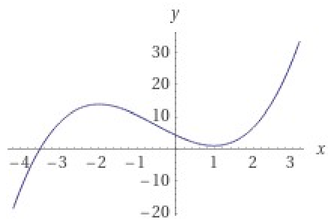


Figure 1: The prediction of *Wolfram*

In Fig.2 we can see Newton's method has the fastest performance of those methods. But if we given a wider range (as Fig.3 of 0 to -4), Newton's method will first go to the right side and has poorer performance.(because the concavity is down and tendency is to right) So we can say that: if we can first know the tendency of solution, the Newton's and secant method will be better, but if we don't, they may won't be the best choice.

(a) Convergence solution (b) Convergence error (c) Convergence rate

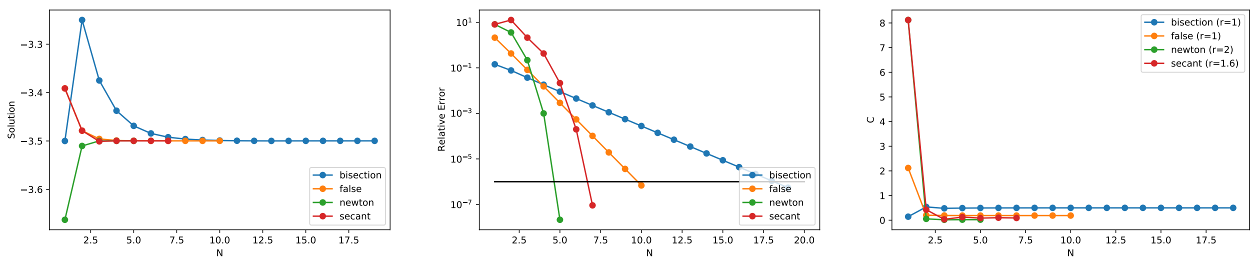


Figure 2: Compared the performance of given a closer($-3 \quad -4$) range of initial value

(a) Convergence solution (b) Convergence error (c) Convergence rate

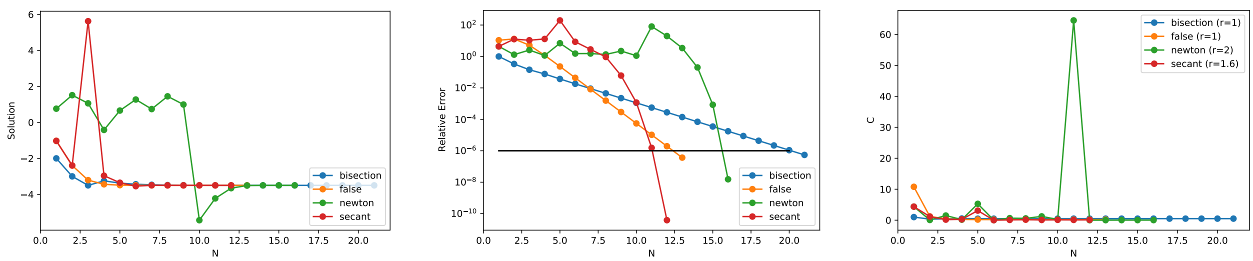


Figure 3: Compared the performance of given a wider($0 \quad -4$) range of initial value

[1] Wolfram:www.wolframalpha.com

Q2 : Doing the open integral

$$I = \int_0^\infty \frac{1}{x^4+x^2+1}dx \tag{1}$$

Now we can decompose the Eq.1 into a simple form to do integral:

$$\frac{1}{x^4+x^2+1} = \frac{1}{(x^2+x+1)(x^2-x+1)} = \frac{1}{2}\frac{(x+1)}{(x^2+x+1)} - \frac{1}{2}\frac{(x-1)}{(x^2-x+1)} \tag{2}$$

$$= \frac{1}{4}\frac{(2x+1)}{(x^2+x+1)} - \frac{1}{4}\frac{(2x-1)}{(x^2-x+1)} + \frac{1}{4}\frac{1}{(x^2+x+1)} - \frac{1}{4}\frac{-1}{(x^2-x+1)}$$

So we can integral the front two elements as **ln**; the last two elements as **atan**.

$$\int \frac{1}{4}\frac{(2x+1)}{(x^2+x+1)}dx - \int \frac{1}{4}\frac{(2x-1)}{(x^2-x+1)}dx + \int \frac{1}{4}\frac{1}{(x^2+x+1)}dx + \int \frac{1}{4}\frac{1}{(x^2-x+1)}dx$$

$$= \frac{1}{4}ln\left|x^2+x+1\right| - \frac{1}{4}ln\left|x^2-x+1\right| + \frac{1}{4}\int \frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}dx + \frac{1}{4}\int \frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}dx + c$$

$$= \frac{1}{4}ln\left|x^2+x+1\right| - \frac{1}{4}ln\left|x^2-x+1\right| + \frac{1}{2\sqrt{3}}tan^{-1}(\frac{2x+1}{\sqrt{3}}) + \frac{1}{2\sqrt{3}}tan^{-1}(\frac{2x-1}{\sqrt{3}}) + c$$

$$I_{finite} = \int \frac{1}{x^4+x^2+1}dx = \frac{1}{4}ln\left|\tfrac{x^2+x+1}{x^2-x+1}\right| + \frac{1}{2\sqrt{3}}tan^{-1}(\frac{\sqrt{3}x}{1-x^2}) + c \tag{3}$$

I divide the Eq.1 into two parts with 2, so integral become 0∼2 & 2∼ ∞:

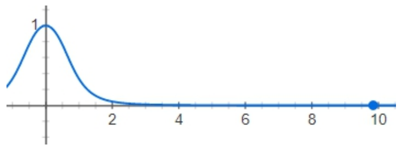


Figure 4: The shape of $\frac{1}{x^4+x^2+1}$

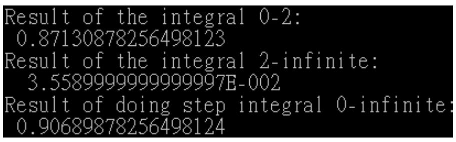
The first part is closed interval, so the value can be easily calculated (see Fig.4), I used the Simpson's rule to do integral, and get the result 0.8713...

$$\text{Simpson's rule} \int_a^b f(x)dx \sim \frac{(b-a)}{6}(f(a)+4f(\frac{a+b}{2})+f(b)) \tag{4}$$

But the second part need to use the Eq.3 to do calculation.

$$x\to\infty, \quad ln\frac{\infty}{\infty}\to 0; \quad atan\frac{1}{-\infty}\to 0 \quad \Rightarrow \quad I_{finite}\|_\infty \to 0$$

$$\int_2^\infty \frac{1}{x^4+x^2+1}dx = I_{finite}\|_\infty - I_{finite}\|_2 = -I_{finite}\|_2 = 0.03559$$



```
Result of the integral 0-2:
 0.87130878256498123
Result of the integral 2-infinite:
  3.5589999999999997E-002
Result of doing step integral 0-infinite:
 0.90689878256498124
```

Figure 5: The result of integral $\frac{1}{x^4+x^2+1}$ from 0 to ∞

Q3 : Stefan-Boltzmann constant

3a.

$$\frac{\sigma_B T^4}{\pi} = \int_0^\infty B_\nu(T)d\upsilon \quad \text{where} \quad B_\nu(T) = \frac{2h\upsilon^3}{c^2}\frac{1}{e^{\frac{h\upsilon}{KT}}-1} \tag{5}$$

$$\frac{\sigma_B T^4}{\pi} = \int_0^\infty \frac{2h\upsilon^3}{c^2}\frac{1}{e^{\frac{h\upsilon}{KT}}-1}d\upsilon \tag{6}$$

Then, we let $x \equiv \frac{h\upsilon}{kT}$, so $\upsilon = \frac{kT}{h}x$, the differential of ν by dx become $d\upsilon = \frac{kT}{h}dx$ and substitute it into Eq.6.

$$\frac{\sigma_B T^4}{\pi} = \int_0^\infty \frac{2hk^3T^3}{h^3c^2} \frac{x^3}{e^x - 1} \frac{kT}{h} dx$$

$$\Rightarrow \sigma_B = \int_0^\infty \frac{2\pi k^4}{h^3c^2} \frac{x^3}{e^x - 1} dx \tag{7}$$

So the main problem will be doing the integral of $\frac{x^3}{e^x-1}$. Google give us the shape of the function.(Fig.6)

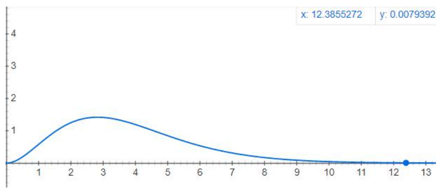
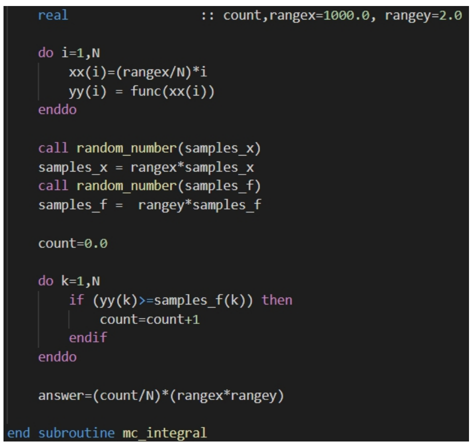


Figure 6: The shape of $\frac{x^3}{e^x-1}dx$

In my program(Fig.7), I make the boundary of MC integral in region : x[0-2] ; y[0-1000], and record the number of points which values are lower than the function $\frac{x^3}{e^x-1}$.

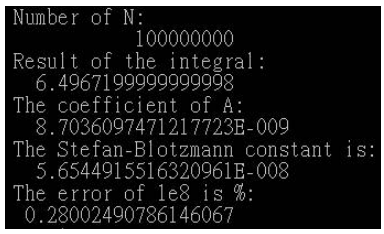
Set the number of $N = 10^8$, and we know the Stefan-Boltzmann constant=$5.670367 \times 10^{-8} kgs^{-3}K^{-4}$, the error of this program compare to the real constant value is 0.28%.

(a) program



```
    real                  :: count,rangex=1000.0, rangey=2.0

    do i=1,N
        xx(i)=(rangex/N)*i
        yy(i) = func(xx(i))
    enddo

    call random_number(samples_x)
    samples_x = rangex*samples_x
    call random_number(samples_f)
    samples_f =  rangey*samples_f

    count=0.0

    do k=1,N
        if (yy(k)>=samples_f(k)) then
            count=count+1
        endif
    enddo

    answer=(count/N)*(rangex*rangey)

end subroutine mc_integral
```

(b) result



```
Number of N:
          100000000
Result of the integral:
  6.4967199999999998
The coefficient of A:
  8.7036097471217723E-009
The Stefan-Blotzmann constant is:
  5.6544915516320961E-008
The error of 1e8 is %:
 0.28002490786146067
```

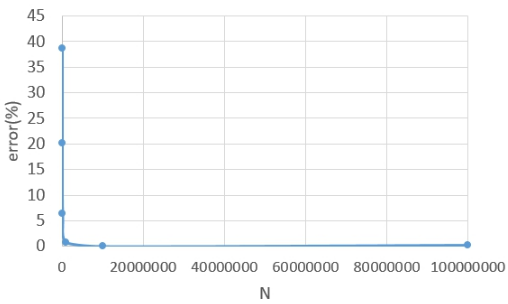
Figure 7: The program and result in fortran

3b. $\frac{1}{\sqrt{N}}$ law of MC integral

Then, I try different numbers of N(from 1e8 to 1e3)(Fig.11.), and make a form in EXCEL to plot the relation(Fig.8.) between error and number of N.

The result of $\frac{1}{\sqrt{N}}$ and error is a highly positive correlation, the Pearson's r=0.9554, proof the $\frac{1}{\sqrt{N}}$ law.

(a) N vs Error



(b) $1/\sqrt{N}$ vs Error

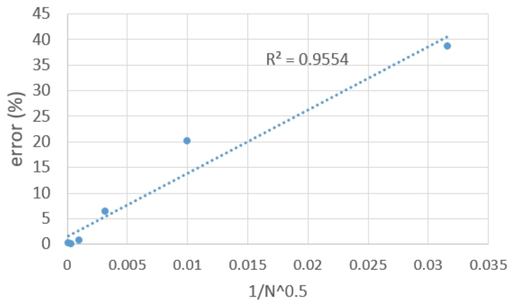


Figure 8: Compared the accuracy of given different number of N (Plot by EXCEL)

Q4 : N-dimension of Newton's method

$$f(x+s)\ f(x)+J_f(x)s \tag{8}$$

Where $J_f(x)$ is the Jacobian matrix of f and $\{J_f(x)\}_{ij} = \frac{\partial f_i(x))}{\partial x_j}$, and if s satisfies the linear system $J_f(x)s = -f(x)$, then x + s is taken as an approximate zero of f.

So we can say Jacobian matrix increase the dimensions of the original matrix but also make a nonlinear system become a linear system at the same time.

$$f(x) = \begin{bmatrix} x_1 + 2x_2 - 2 \\ x_1^2 + 4x_x^2 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

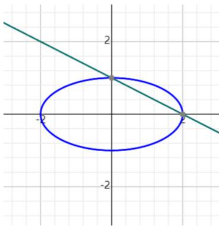
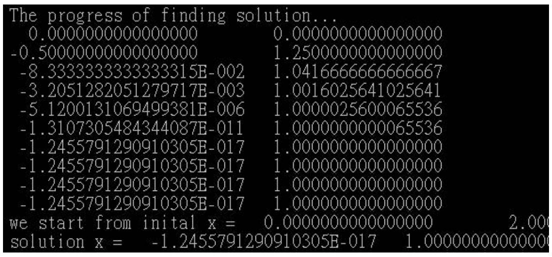


Figure 9: The shape of Eq.9

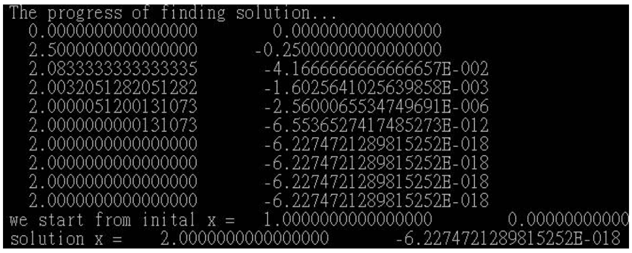
Fig.9 say there are two solutions in this nonlinear system, so I set the initial condition to (0,2)(Fig.10(a)) & (1,0)(Fig.10(b)) to find the solutions (0,1) & (2,0). Although the results still have some error, but the order of magnitude is small enough to ignore.(1e-17 to (0,1) & 1e-18 to (2,0))

(a) solution (0,1)



```
The progress of finding solution...
 0.0000000000000000          0.0000000000000000
-0.50000000000000000         1.2500000000000000
 -8.3333333333333315E-002    1.0416666666666667
 -3.2051282051279717E-003    1.0016025641025641
 -5.1200131069499381E-006    1.0000025600065536
 -1.3107305484344087E-011    1.0000000000065536
 -1.2455791290910305E-017    1.0000000000000000
 -1.2455791290910305E-017    1.0000000000000000
 -1.2455791290910305E-017    1.0000000000000000
 -1.2455791290910305E-017    1.0000000000000000
we start from inital x =    0.0000000000000000          2.00
solution x =   -1.2455791290910305E-017    1.00000000000000
```

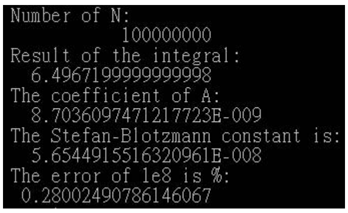
(b) solution (2,0)



```
The progress of finding solution...
 0.0000000000000000          0.0000000000000000
 2.5000000000000000        -0.25000000000000000
 2.0833333333333335         -4.1666666666666657E-002
 2.0032051282051282         -1.6025641025639858E-003
 2.0000051200131073         -2.5600065534749691E-006
 2.0000000000131073         -6.5536527417485273E-012
 2.0000000000000000         -6.2274721289815252E-018
 2.0000000000000000         -6.2274721289815252E-018
 2.0000000000000000         -6.2274721289815252E-018
 2.0000000000000000         -6.2274721289815252E-018
we start from inital x =    1.0000000000000000          0.00000000000
solution x =    2.0000000000000000          -6.2274721289815252E-018
```

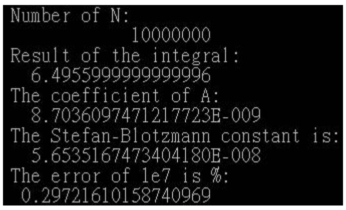
Figure 10: Solution of f(x) by using n-dimension Newton's method

(a) N=1e8



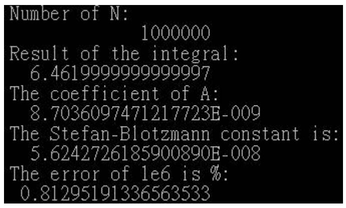
```
Number of N:
            100000000
Result of the integral:
  6.4967199999999998
The coefficient of A:
  8.7036097471217723E-009
The Stefan-Blotzmann constant is:
  5.6544915516320961E-008
The error of 1e8 is %:
 0.28002490786146067
```

(b) N=1e7



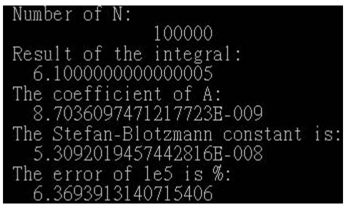
```
Number of N:
             10000000
Result of the integral:
  6.4955999999999996
The coefficient of A:
  8.7036097471217723E-009
The Stefan-Blotzmann constant is:
  5.6535167473404180E-008
The error of 1e7 is %:
 0.29721610158740969
```

(c) N=1e6



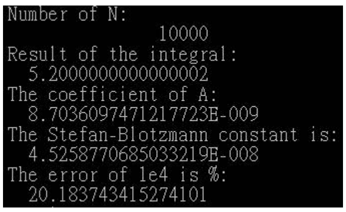
```
Number of N:
              1000000
Result of the integral:
  6.4619999999999997
The coefficient of A:
  8.7036097471217723E-009
The Stefan-Blotzmann constant is:
  5.6242726185900890E-008
The error of 1e6 is %:
 0.81295191336563533
```

(d) N=1e5



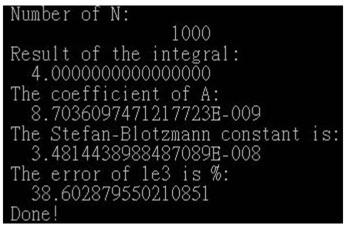
```
Number of N:
               100000
Result of the integral:
  6.1000000000000005
The coefficient of A:
  8.7036097471217723E-009
The Stefan-Blotzmann constant is:
  5.3092019457442816E-008
The error of 1e5 is %:
  6.3693913140715406
```

(e) N=1e4



```
Number of N:
                10000
Result of the integral:
  5.2000000000000002
The coefficient of A:
  8.7036097471217723E-009
The Stefan-Blotzmann constant is:
  4.5258770685033219E-008
The error of 1e4 is %:
  20.183743415274101
```

(f) N=1e3



```
Number of N:
                 1000
Result of the integral:
  4.0000000000000000
The coefficient of A:
  8.7036097471217723E-009
The Stefan-Blotzmann constant is:
  3.4814438988487089E-008
The error of 1e3 is %:
  38.602879550210851
Done!
```

Figure 11: Running the MC integral in different N size