Computational Astrophysics

ASTR 660, Spring 2020
計算天文物理

Lecture 8

Boundary Value Problems

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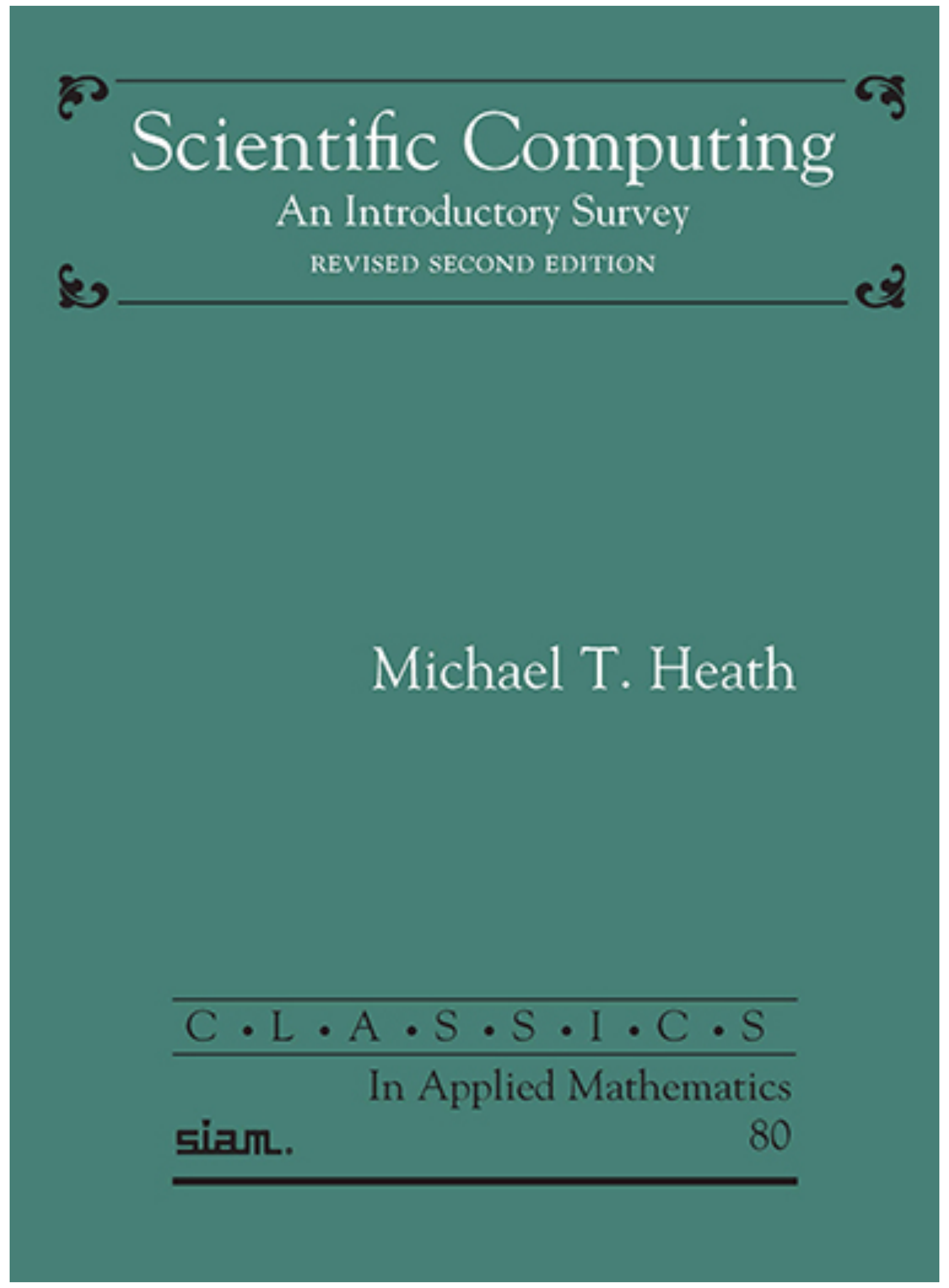
Class website

https://kuochuanpan.github.io/courses/109ASTR660_CA/

Plan for today

- Ordinary Differential Equations (ODEs, continues)
- ODE: Boundary Value Problems (BVPs)
- Stellar structure (Polytropes)
- Lab: Polytrope



Reference:
"Scientific Computing: An introductory survey", Michael Heath

https://books.google.com.tw/books?id=f6Z8DwAAQBAJ&hl=zh-TW

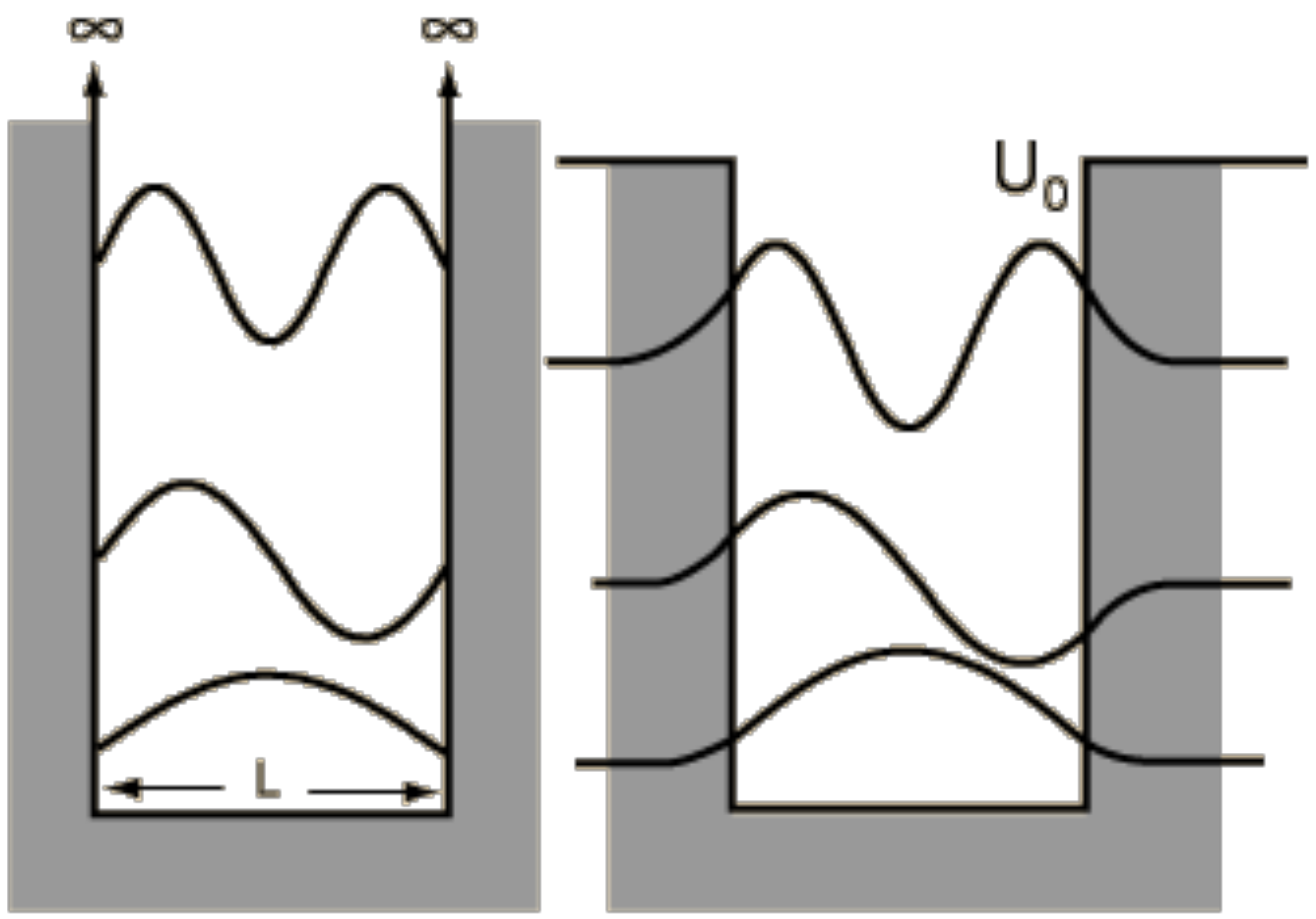
Boundary Value Problems (BVPs)

Boundary Value Problem

- Side conditions prescribing solution or derivative values at specified points are required to make solution of ODE unique
- For initial value problems, all side conditions are specified at a single point, say at t_0.
- For boundary value problem (BVP), side conditions are specified at more than one point
- K-th order ODE, or equivalent first-order system, requires k side conditions.
- Side conditions are typically specified at endpoints of interval [a,b], so we have two-point boundary value problem with boundary condition (BC) at a and b

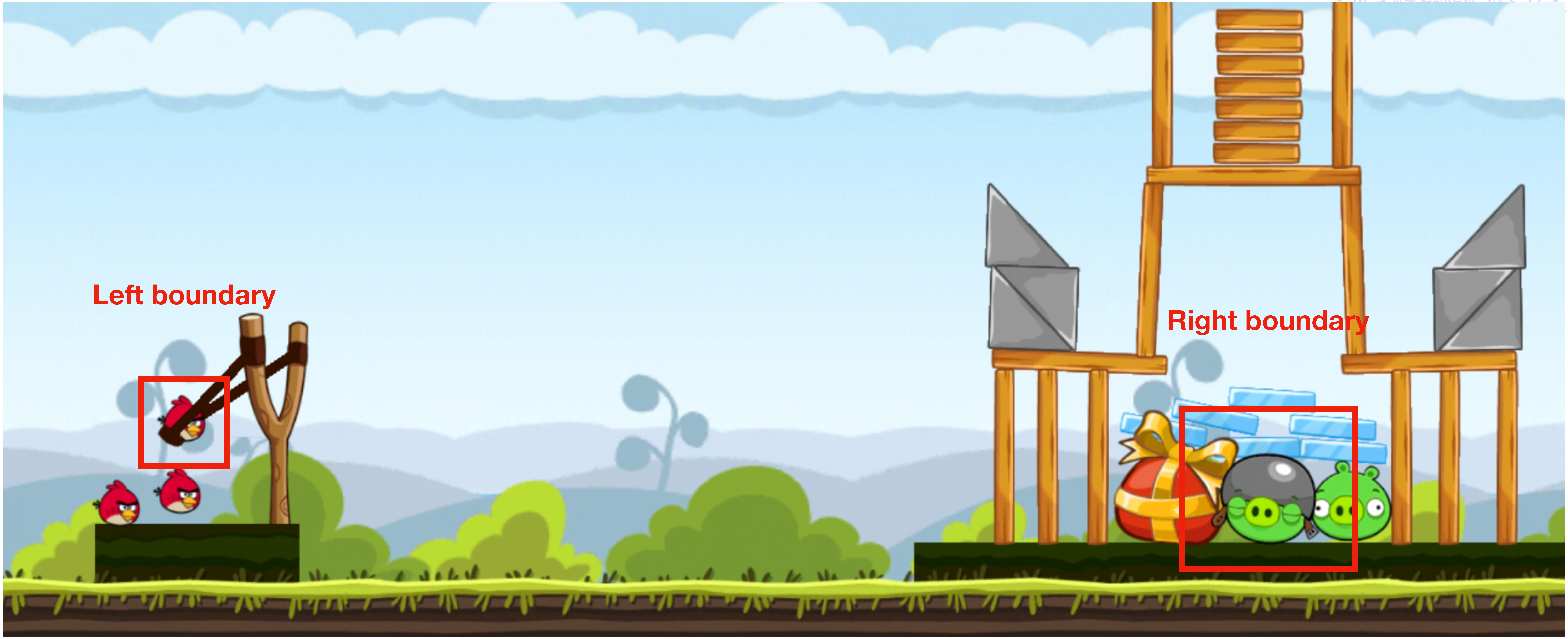
Example: Boundary Value Problem

- Schrödinger equation in an infinity potential well



$$-\frac{h^2}{2m}\frac{d^2\varphi}{dx^2}+V(x)\varphi=E\varphi$$

Example: Boundary Value Problem

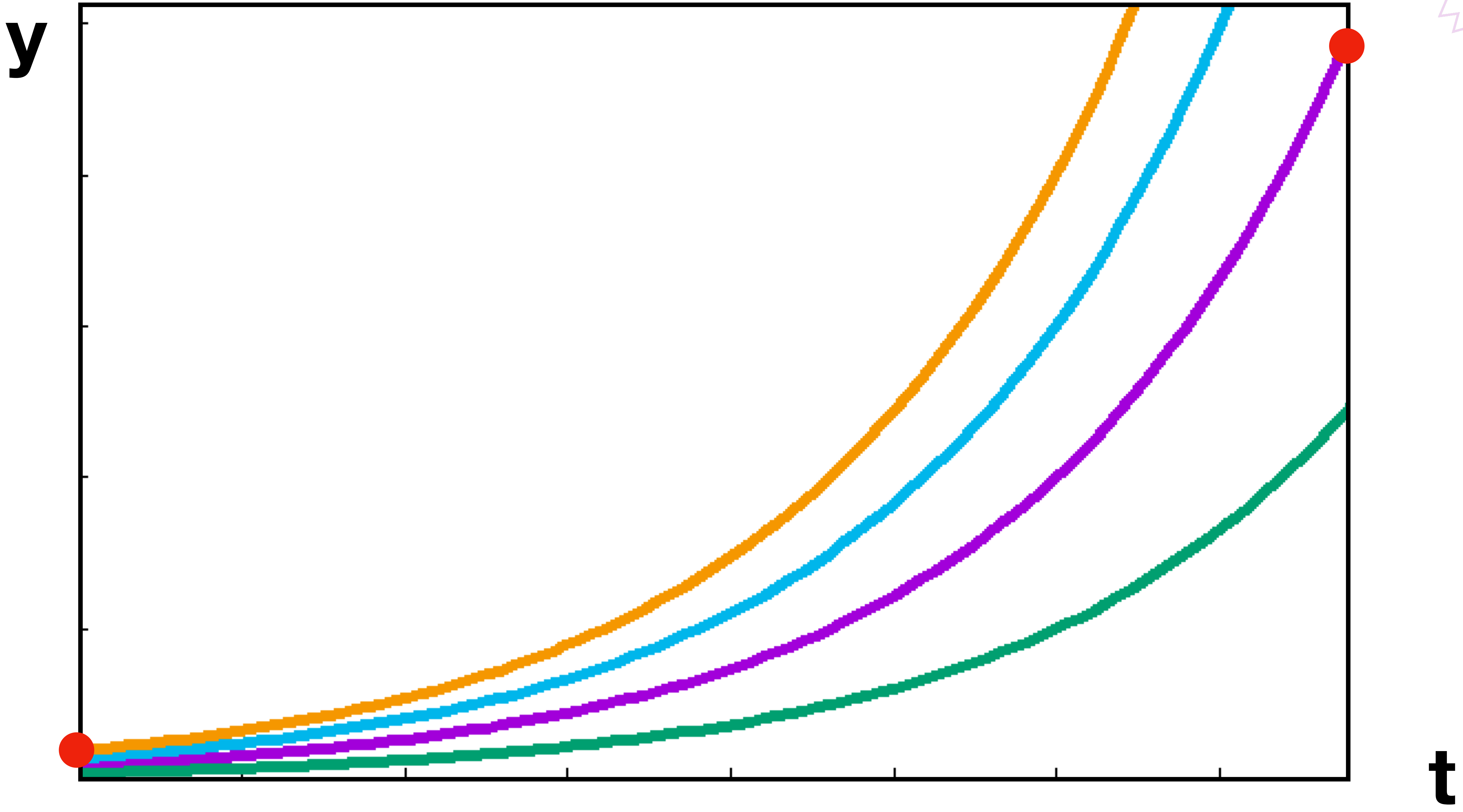


Shooting method

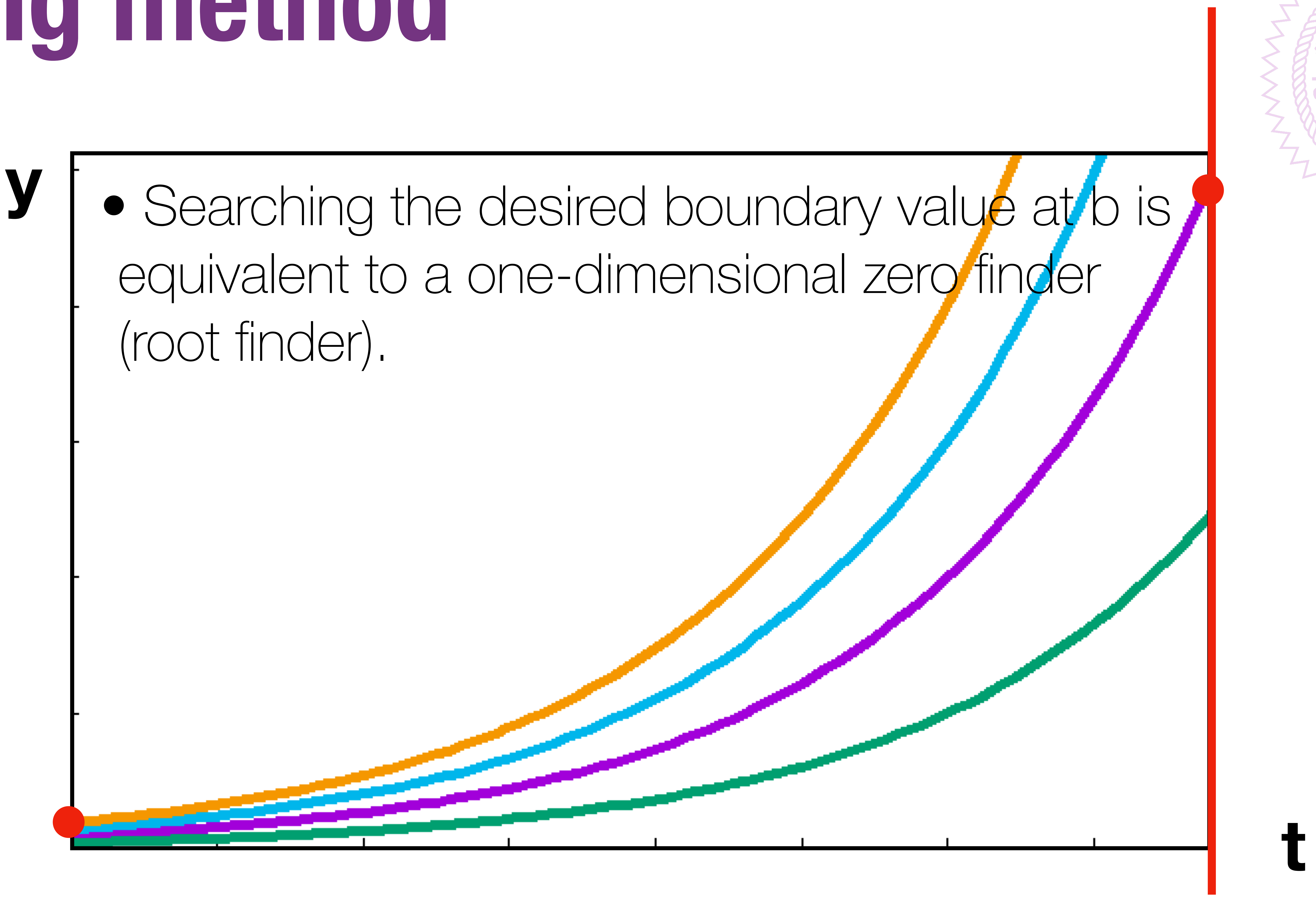
Shooting method

- If we know the value of u(a) but lacking u'(a)
- If we also know u'(a), then we would have IVP that we could solve be methods discussed in the previous lecture.
- Lacking that information, we could try sequence of increasingly accurate guesses until we find value of u'(a) such that when we solve resulting IVP, approximate solution value at t=b matches desired boundary values, u(b)

Shooting method



Shooting method



- Searching the desired boundary value at b is equivalent to a one-dimensional zero finder (root finder).

Exercise: shooting method

- Consider two-point BVP for second-order ODE

$$u'' = 6t \qquad 0 < t < 1$$

with BC

$$u(0) = 1 \qquad u(1) = 1$$

- For each guess of u'(0), we could integrate the resulting IVP using the classical Runge-Kutta method to determine how close we come to hitting desired solution value at t=1

Exercise: shooting method

- Consider two-point BVP for second-order ODE

$$u'' = 6t \qquad 0 < t < 1$$

with BC

$$u(0) = 1 \qquad u(1) = 1$$

$$\boldsymbol{y}'(t) = \begin{bmatrix} y_2 \\ 6t \end{bmatrix}$$

With initial guess $y_2(0) = 1$

Exercise: shooting method, part 1

- Solve ODE: $u'' = 6t$ or $\boldsymbol{y}'(t) = \begin{bmatrix} y_2 \\ 6t \end{bmatrix}$

  with BC $u(0) = 1$ $u(1) = 1$

- Step 1: Modify the example1.f90 to compute the corresponding IVP with an initial guess of $y_2(0) = 1$.
- Step 2: Change the initial guess of $y_2(0) = -1.5$
- Step 3: Continue vary the initial guess of $y_2(0)$ until $u(1)=1$ is satisfied.

Exercise: shooting method, part 2

- Solve ODE: $u'' = 6t$ or $\boldsymbol{y}'(t) = \begin{bmatrix} y_2 \\ 6t \end{bmatrix}$

  with BC $u(0) = 1$ $u(1) = 1$

- Step 1: Copy your example1.f90 to example2.f90
- Step 2: Implement a bisection search for the best $y_2(0)$
- Step 3: Compare that result with your solution in example1.f90

Finite Difference Method

Finite Difference Method

- Finite difference method converts BVP into system of algebraic equations by replacing all derivatives with finite difference approximations
- For example, to solve two-point BVP

$$u'' = f(t, u, u') \qquad a < t < b$$

with BC

$$u(a) = \alpha \qquad u(b) = \beta$$

We introduce mesh points $t_i = a + i\,h$

Finite Difference Method

- We replace derivatives by finite difference approximations such as

$$u'(t_i) \sim \frac{y_{i+1} - y_{i-1}}{2h}$$

$$u''(t_i) \sim \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

This yields system of equations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h})$$

to be solved for unknowns y_i

Finite Difference Method

- For these particular finite difference formulas, system to be solved is tridiagonal, which saves on both work and storage compared to general system of equations.
- This is generally true of finite difference methods: they yield sparse systems because each equation involves few variables

Example: Finite Difference Method

Solve

$$u'' = 6t \qquad\qquad 0 < t < 1$$

with BC

$$u(0) = 1 \qquad\qquad u(1) = 1$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 6t_i$$

If we take h=0.5, yielding three points: y_0, y_1 and y_2

$$\frac{1 - 2y_1 + 1}{(0.5)^2} = 6t_1 = 3$$

Example: Finite Difference Method

- In practice, much smaller step size and many more mesh points would be required to achieve acceptable accuracy
- We would therefore obtain system of equations to solve for approximate solution values at mesh points, rather than single equation as in this example

Exercise: Finite Difference Method

Solve

$$u'' = 6t \qquad 0 < t < 1$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 6t_i$$

with BC

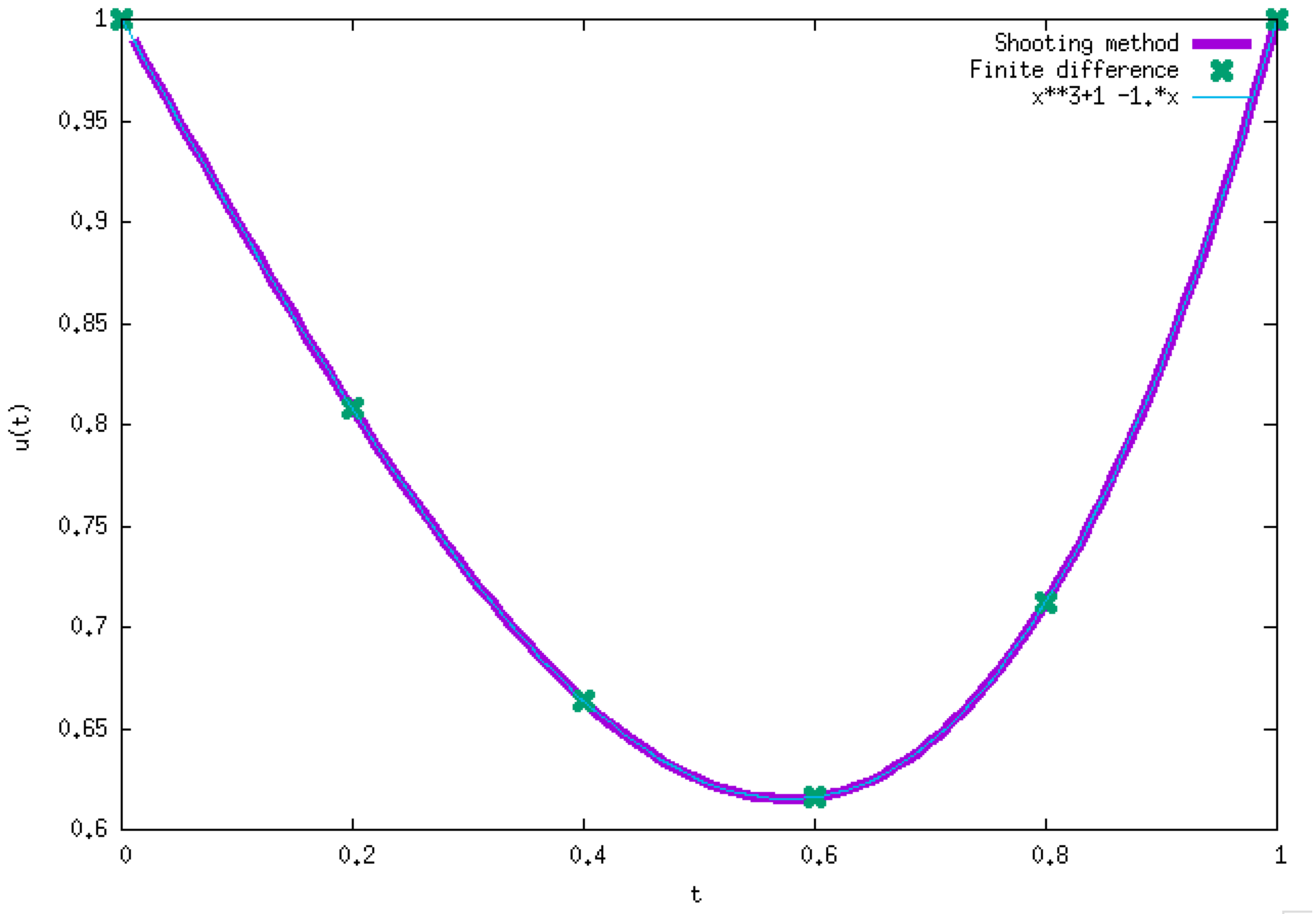
$$u(0) = 1 \qquad u(1) = 1$$

Take h=0.2, yielding

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} 4.8h^2 - 1 \\ 3.6h^2 \\ 2.4h^2 \\ 1.2h^2 - 1 \end{bmatrix}$$

Is a linear system. We know how to solve y_i

Exercise: Finite Difference Method



Stellar Structure

Stellar Structure

$$P = P(\rho, T, X)$$

$$L = L(\rho, T, X)$$

$$\kappa = \kappa(\rho, T, X)$$

$$\epsilon = \epsilon(\rho, T, X)$$

Stellar Structure

Hydrostatic equation

$$\frac{dP}{dr} = -\rho g = -\rho \frac{GM_r}{r^2}$$

$$dM_r = 4\pi r^2 \rho dr,$$

Mass conservation

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}$$

$$\frac{dr}{dM_r} = -\frac{1}{4\pi r^2 \rho}$$

$$\frac{dL_r}{dM_r} = E_{nuc} - \frac{d\epsilon}{dt} - P\frac{dV}{dt}$$

Energy conservation

Polytropic EoS and polytopes

- Pressure can be described by

$$P(r) = K\rho^{1+1/n}(r)$$

where n is the polytropic index (a constant),
K is the polytropic constant

Polytropic EoS and polytopes

- From the hydrostatic equation

$$\frac{dP}{dr} = -\rho g = -\rho \frac{GM_r}{r^2}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G \rho$$

Which is the Poisson's equation

Polytropic EoS and polytopes

- Define the dimensionless variable theta by

$$\rho(r) = \rho_c \theta^n(r)$$

The power law for pressure is then

$$P(r) = K\rho_c^{1+1/n}\theta^{n+1}(r) = P_c\theta^{1+n}(r)$$

Substitute these into Poisson's equation

$$\frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

Polytropic EoS and polytopes

- Define the dimensionless radial coordinate xi

$$r = r_n \xi$$

where we define $r_n^2 = \frac{(n+1)P_c}{4\pi G \rho_c^2}$

Poisson's equation becomes

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta_n}{d\xi}\right) = -\theta_n^n}$$

The Lane-Emden equation

Polytropic EoS and polytopes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_n}{d\xi}\right) = -\theta_n^n$$ The Lane-Emden equation

- If ideal gas EoS $P = \rho N_A kT/\mu$, solve for temperature
- Specify K, n and either rho$_c$ or P$_c$, yields all solutions

$$R = r_n\xi_1 = \left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]\xi_1 \qquad T = T_c\theta_n(r)$$

$$P_c = K\rho_c^{1+1/n} \qquad T_c = K\rho_c^{1/n}\left(\frac{N_A k}{\mu}\right)$$

Lane-Emden Equation

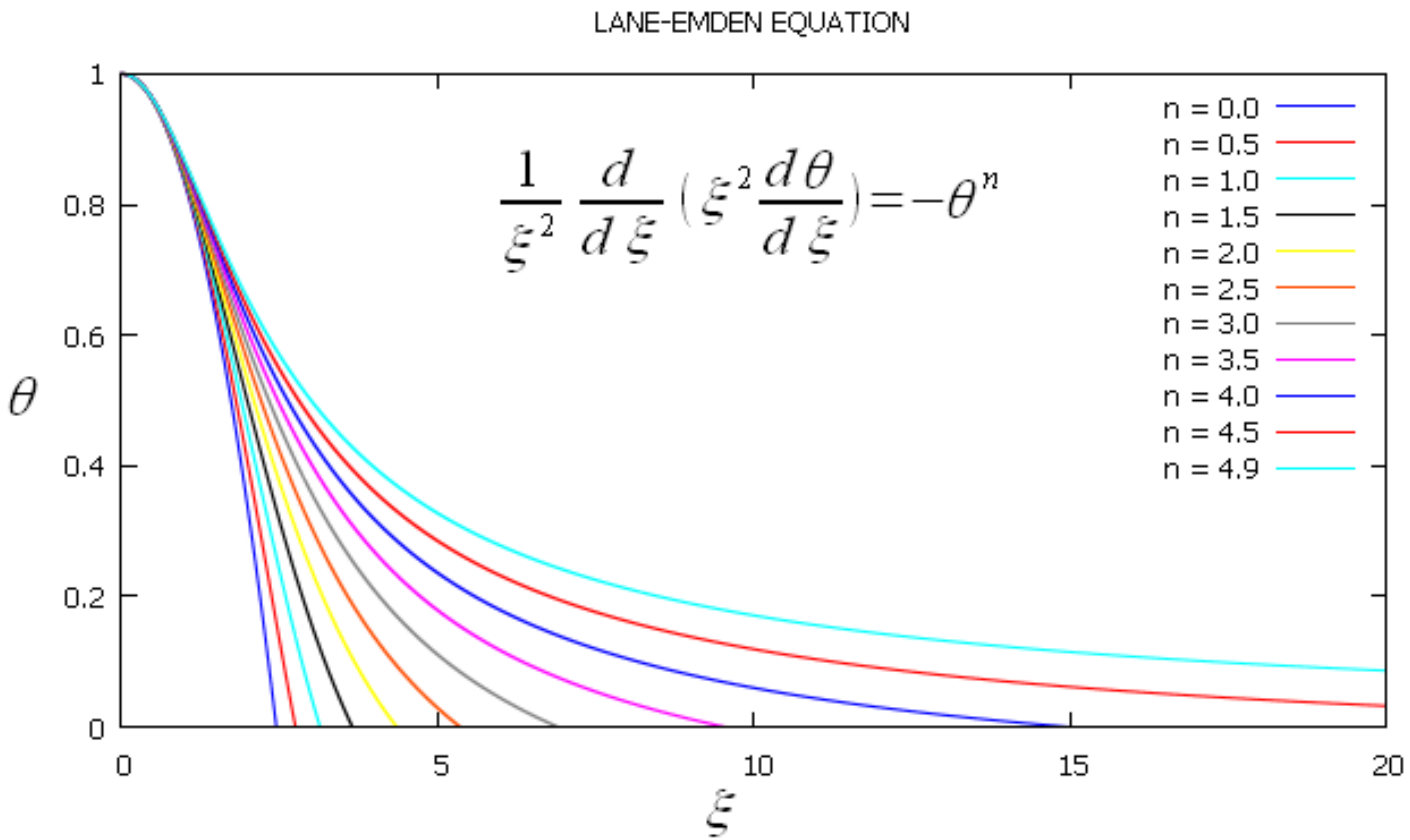
$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_n}{d\xi}\right) = -\theta_n^n$$ The Lane-Emden equation

- Boundary conditions: density and pressure are 0 at surface. At center, density is central density and pressure is the central pressure Pc

$\theta_n(0) = 1$ $\theta_n'(0) = 0$ at center

IVP !! $\theta_n(\xi = \xi_1) = 0$ at surface

Lane-Emden Equation



Exercise: Polytropes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_n}{d\xi}\right) = -\theta_n^n$$

The Lane-Emden equation

$$\theta_n(0) = 1 \qquad \theta_n'(0) = 0$$

at center

$$\theta_n(\xi = \xi_1) = 0$$

at surface

Exercise: Polytropes

$$\boldsymbol{y} = \begin{bmatrix} \theta_n = y1 \\ \frac{d\theta_n}{d\xi} = y2 \end{bmatrix} \qquad \boldsymbol{y}' = \begin{bmatrix} y2 \\ -y_1^n - \frac{2}{\xi} y_2 \end{bmatrix}$$

The Lane-Emden equation

$$\theta_n(0) = 1 \qquad \theta_n'(0) = 0$$

at center

$$\theta_n(\xi = \xi_1) = 0$$

at surface

- Step1: modify polytrop.f90 and compute the solution with different polytope index (n=1,1.5, 2,3,4)
- Step2: Visualize the solutions
- Step3: make y-axis in log scale

Polytropes

- However, the surface of the star could be very sharp
- IVP requires step size adjustment to resolve the envelope of the star
- In realistic cases, BCs of a stellar model might came from the surface region.

Exercise: Polytropes, Part 2

$$\boldsymbol{y} = \begin{bmatrix} \theta_n = y1 \\ \frac{d\theta_n}{d\xi} = y2 \end{bmatrix} \qquad \boldsymbol{y}' = \begin{bmatrix} y2 \\ -y_1^n - \frac{2}{\xi} y_2 \end{bmatrix}$$

The Lane-Emden equation

$$\theta_n(0) = 1 \qquad \theta_n'(0) = 0$$

at center

$$\theta_n(\xi = \xi_1) = 0$$

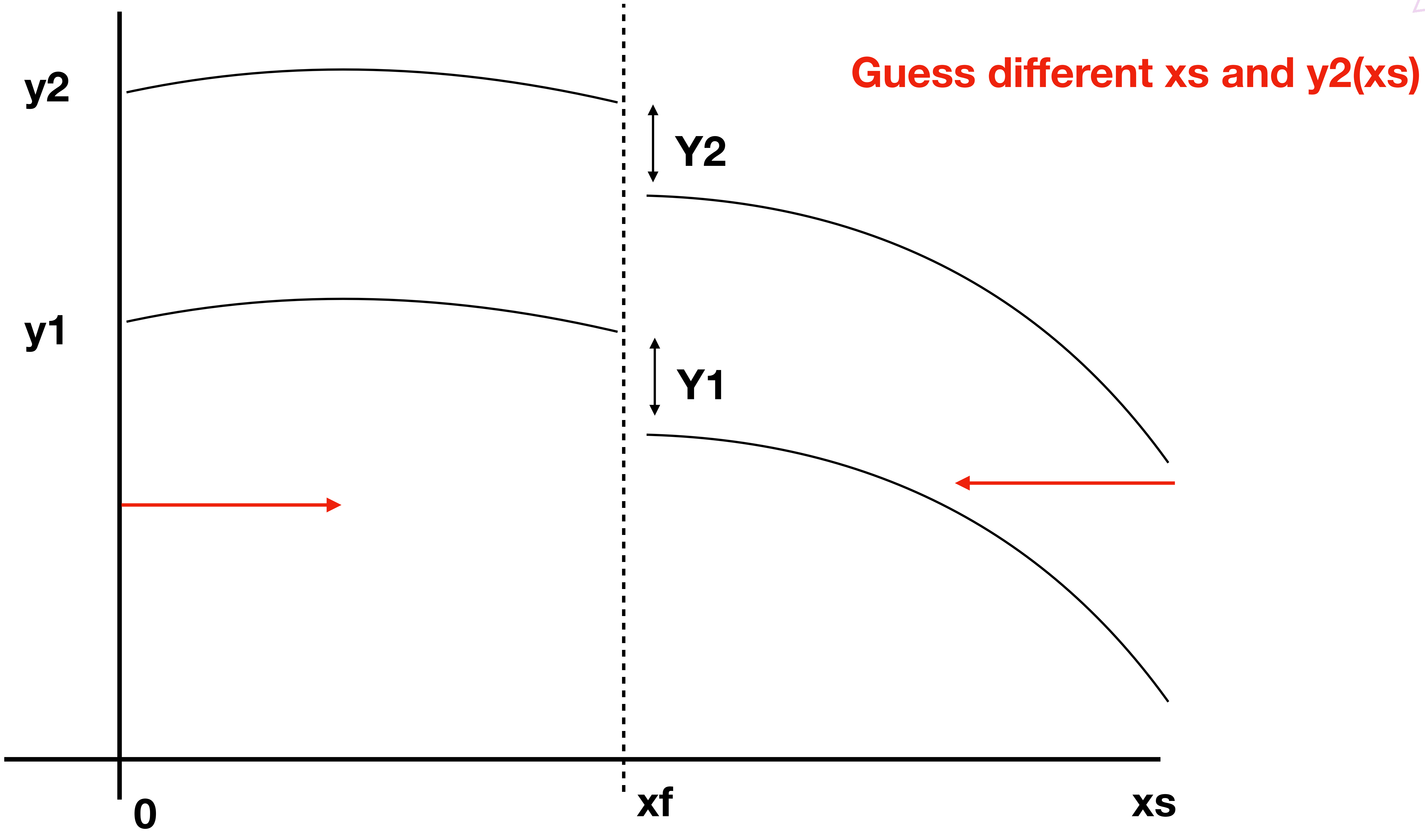
at surface

- Step1: copy your poltrope.f90 to polytrop2.f90
- Step2: set the polytrope index n = 1
- Step3: Use shooting method to integrate from the surface (assuming we know xi at surface is pi)

Lane-Emden Equation

- However, the surface of the star could be very sharp
- A better way to solve Lane-Emden equation is starting from both the center and the surface and try to match at some point -> Fitting method (multiple shooting).

Lane-Emden Equation: Fitting method



Lane-Emden Equation: Fitting method

$$Y1(x_s, y_{2s}) = y_{1i}(x_f) - y_{1,0}(x_f)$$

$$Y2(x_s, y_{2s}) = y_{2i}(x_f) - y_{2,0}(x_f)$$

$$x_s \to x_s + \delta x_s$$

$$y_{2s} \to y_{2s} + \delta y_(2s)$$

- Becomes an eigenvalue problem

Lane-Emden Equation: Fitting method

- Standard eigenvlaue problem for second-order ODE has form

$$u'' = \lambda f(y, u, u') \qquad a < t < b$$

with BC

$$u(a) = \alpha \qquad u(b) = \beta$$

- Scalar lambda is eigenvalue and solution u is corresponding eigenfunction for this two-point BVP

Summary

- Two points BVP for ODE specifies BC at both endpoints of interval
- Shooting method replaces BVP by sequence of IVPs, with missing initial conditions
- Finite difference method replaces derivatives in ODE by finite difference.

Problem Set 7

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Next lecture

- PDE: Hyperbolic systems