Computational Astrophysics

ASTR 660, Spring 2021
計算天文物理

Lecture 10

PDE: Hydrodynamics

Instructor: Prof. Kuo-Chuan Pan
kuochuan.pan@gapp.nthu.edu.tw

Class website

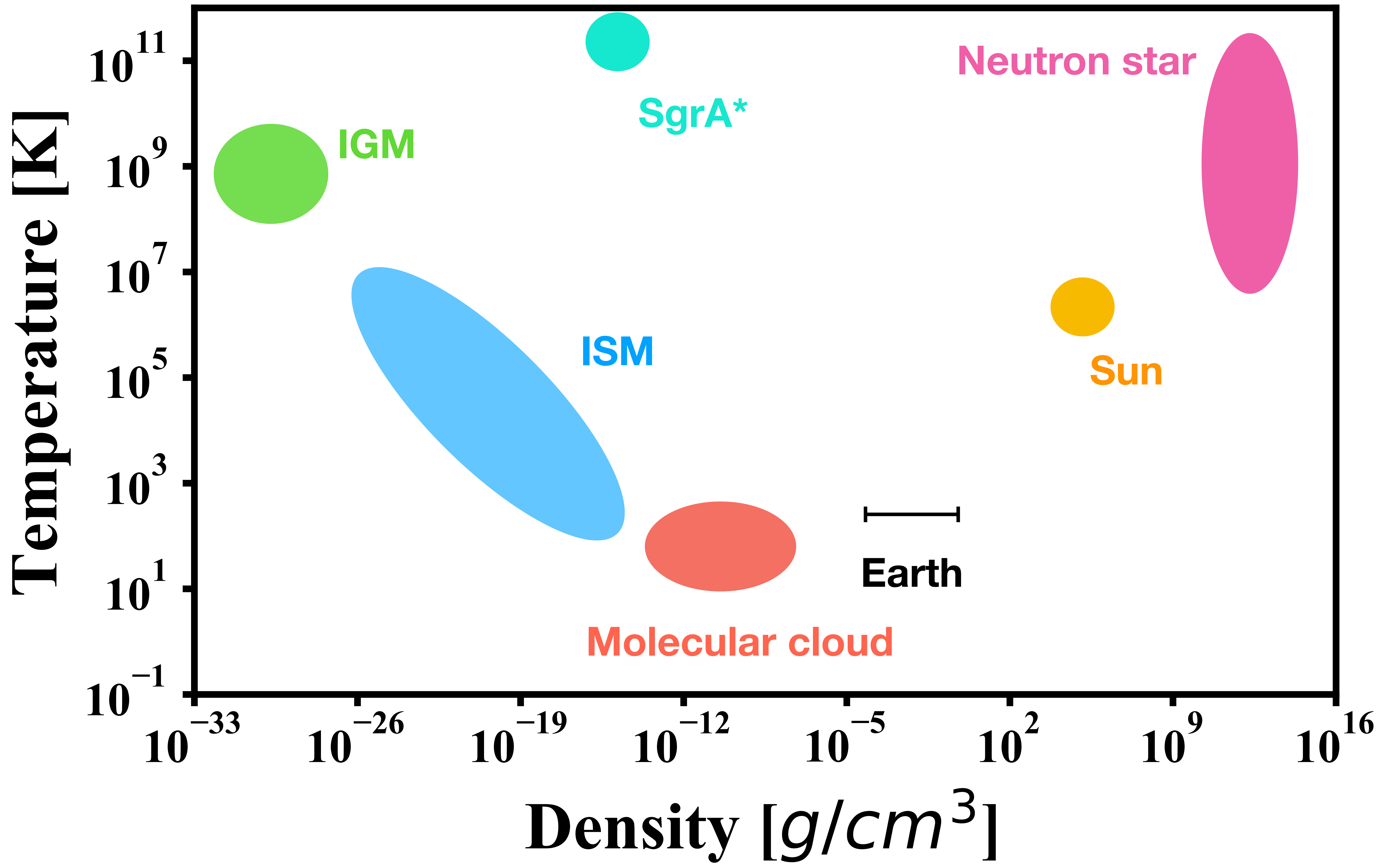
https://kuochuanpan.github.io/courses/109ASTR660_CA/

Plan for today

- Astrophysical fluids
- Hydrodynamics equations
- Finite difference method for a simple Lagrangian hydro code
- Lab: supernova fireball simulation
- Finite Volume method for solving HD
- Lab: Antares code

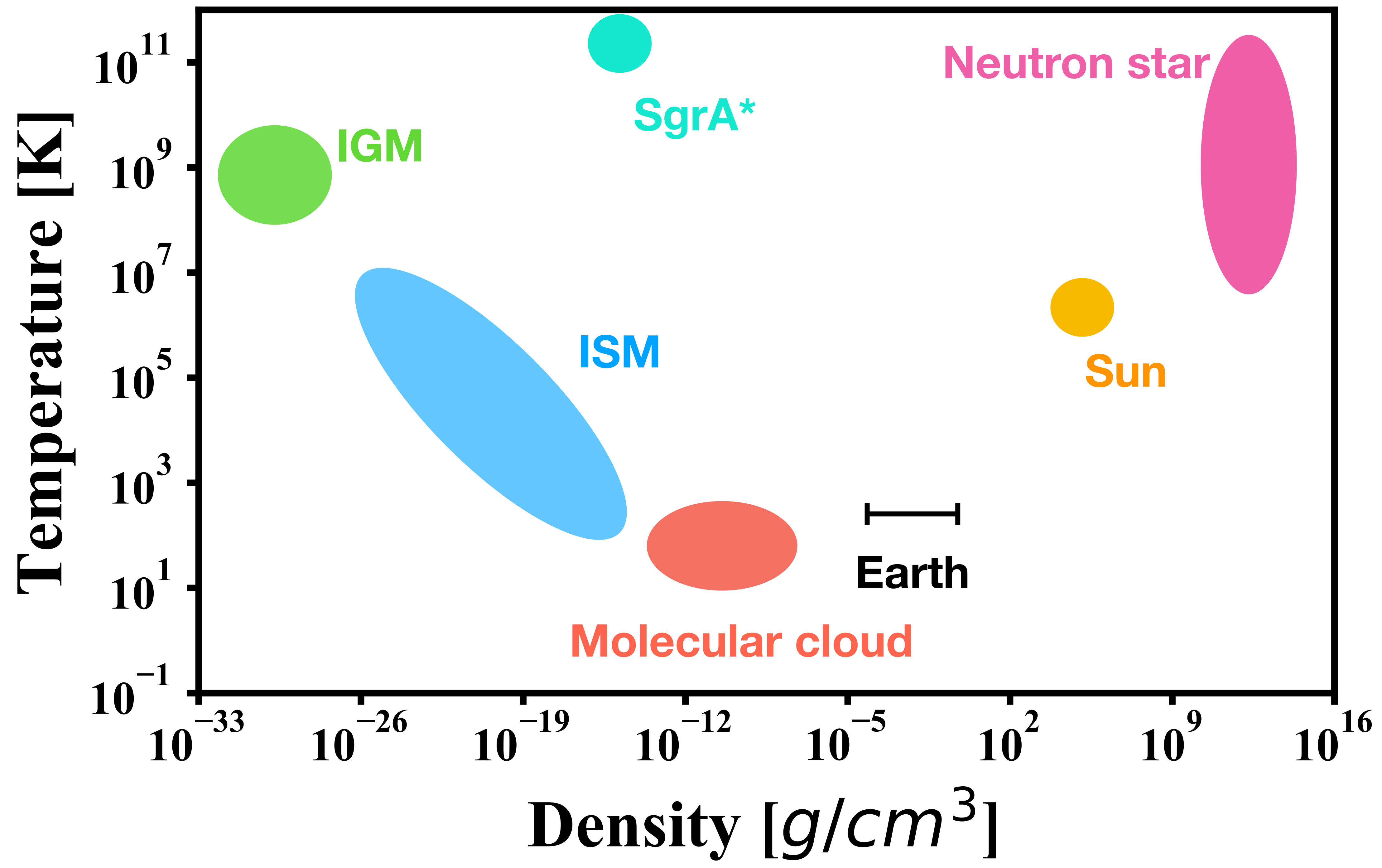
Astrophysical fluids

Astrophysical fluids



- Different with Earth-bound fluid
- Wide range of physical conditions

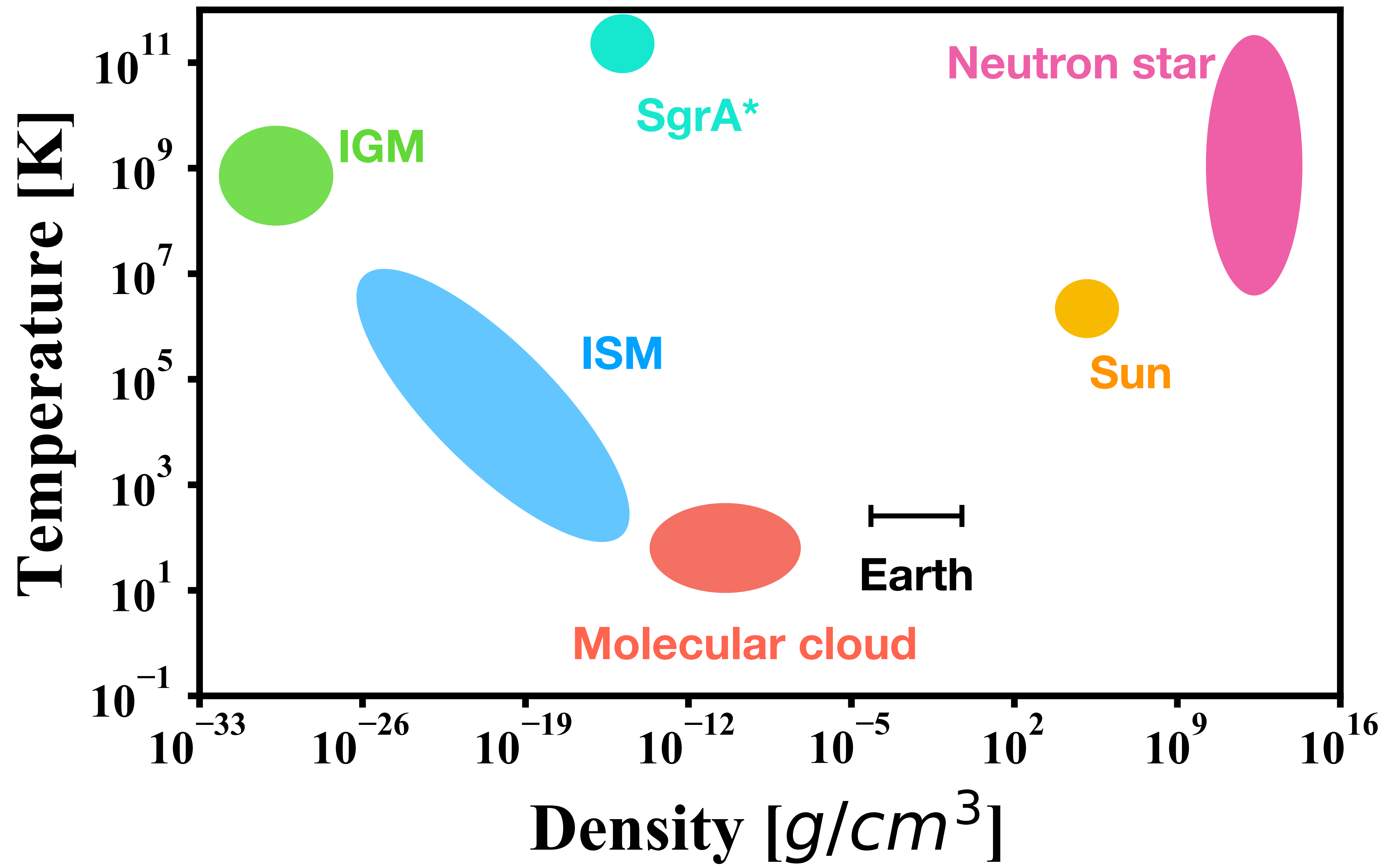
Astrophysical fluids



- Different with Earth-bound fluid
- Wide range of physical conditions

- Magnetic fields are often important
- Compressibility is often important
- Occasionally self-gravitating and relativistic
- Non-ideal processes are often unimportant
- Sometimes involves interactions with non-thermal particles and strong radiation fields

Astrophysical fluids



EX: Viscosity

Reynolds number: Re

$$\mathrm{Re} = \frac{Lv}{\nu}$$

- Simmer: $L \sim 10^2 \mathrm{cm}, v \sim 10^2 \mathrm{cm/s}, \nu = 0.01 \mathrm{cm}^2/\mathrm{s}$ $\quad \mathrm{Re} \sim 10^6$
- Sun: $L \sim 10^{11} \mathrm{cm}, v \sim 10^4 \mathrm{cm/s}, \nu = 10 \mathrm{cm}^2/\mathrm{s}$ $\quad \mathrm{Re} \sim 10^{13}$

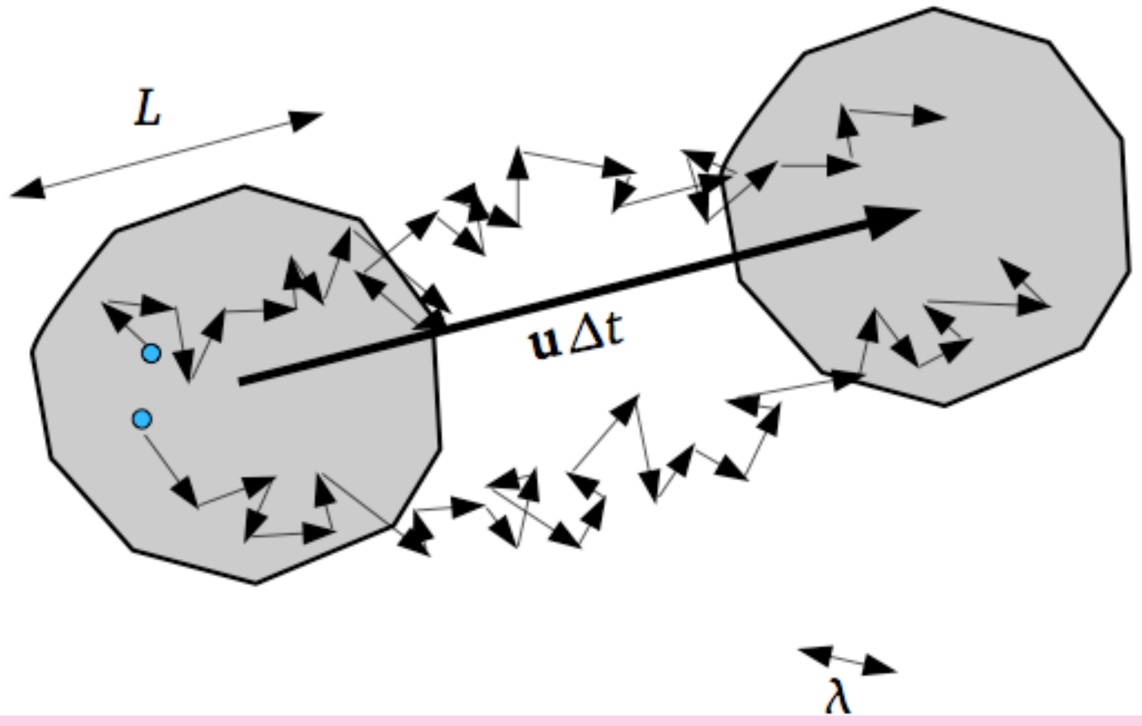
Fluid Approximation

- To solve the gas dynamics equations, one could use N-body method if we know the force between two particles.
- However, the number of particles is usually huge ($N_A = 10^{23}$). It is impossible to use direct N-body method to study the gas dynamics. -> Fluid

Ideal Fluid Approximation

$L >> \lambda_{\rm mfp}$, Mean free path

$T >> \tau_{\rm coll}$, Collision time scale



Exercise: Consider the air in this room

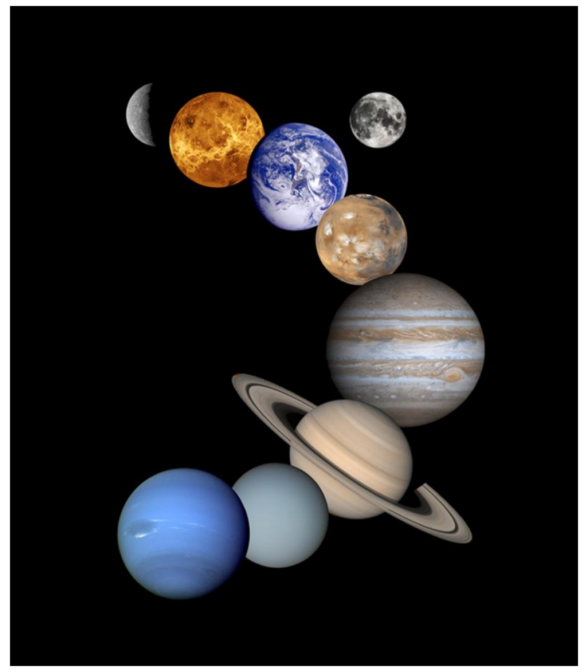
$$\rho \sim 10^{-3}\ {\rm g/cm^3},\ \mu \sim 28 m_p,\ T \sim 300\ {\rm K}$$

Cross section $\sigma \sim 3 \times 10^{-15}\ {\rm cm^2}$

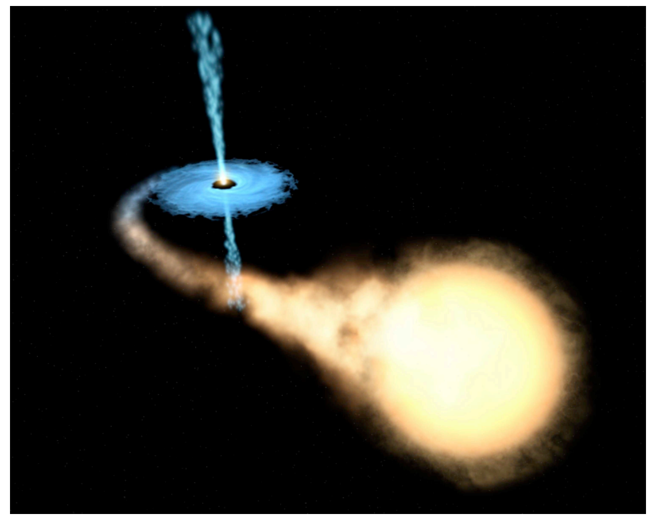
RMS speed of nitrogen $v \sim \sqrt{\frac{8kT}{\pi\mu}} \sim 5 \times 10^4\ {\rm cm/s}$

$$\tau_{\rm coll} \sim (n\sigma v)^{-1} \sim 3 \times 10^{-10}\ {\rm sec} \qquad \lambda_{\rm mfp} \sim (n\sigma)^{-1} \sim 3 \times 10^{-5}\ {\rm cm}$$

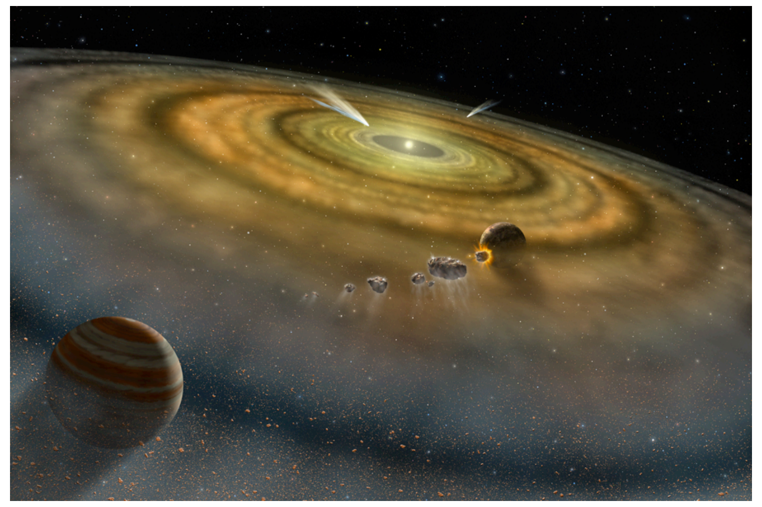
Astrophysical fluids



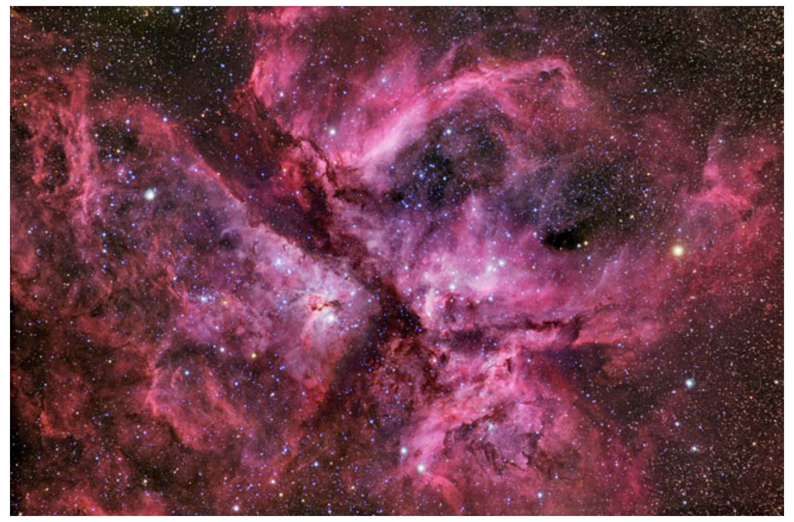
Planets
(~ Earth radius)



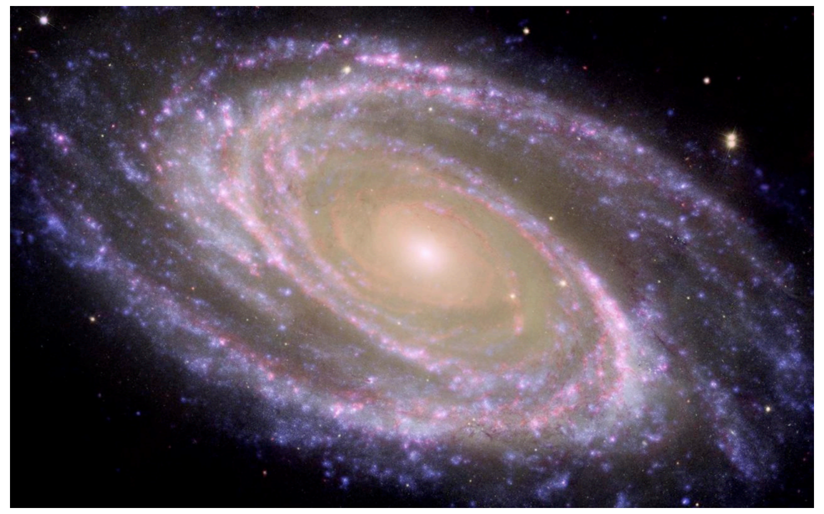
Stars/Binaries
(~0.1-1000 AU)



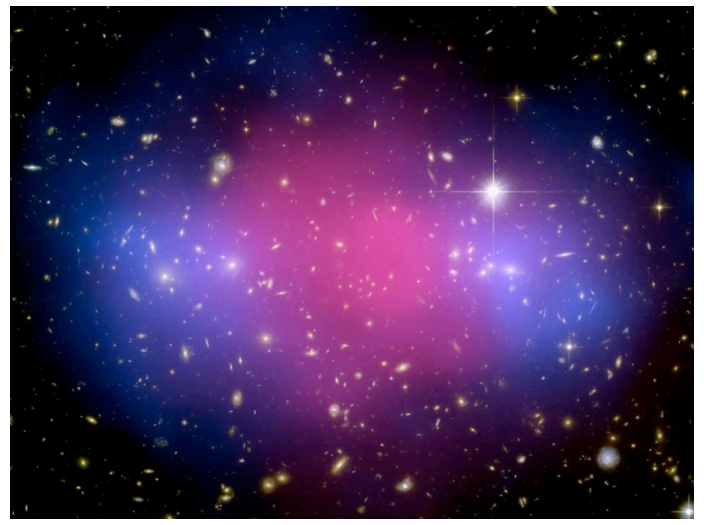
Planetary disk
(~100-1000 AU)



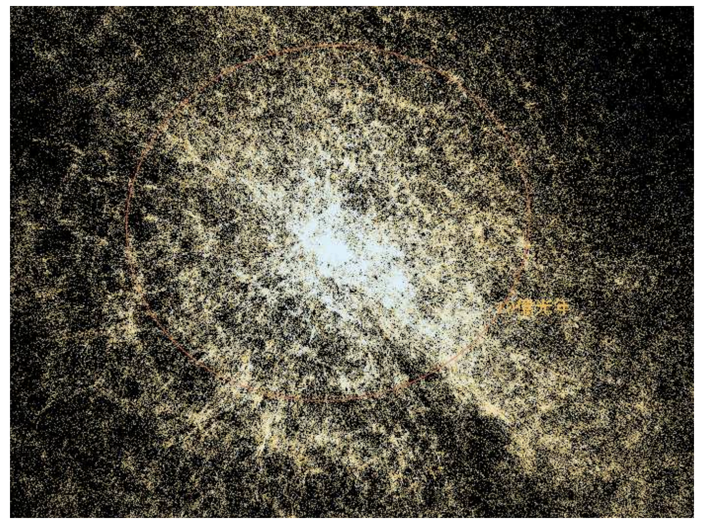
Interstellar medium
(~10-100 pc)



Galaxy
(~50 kpc)

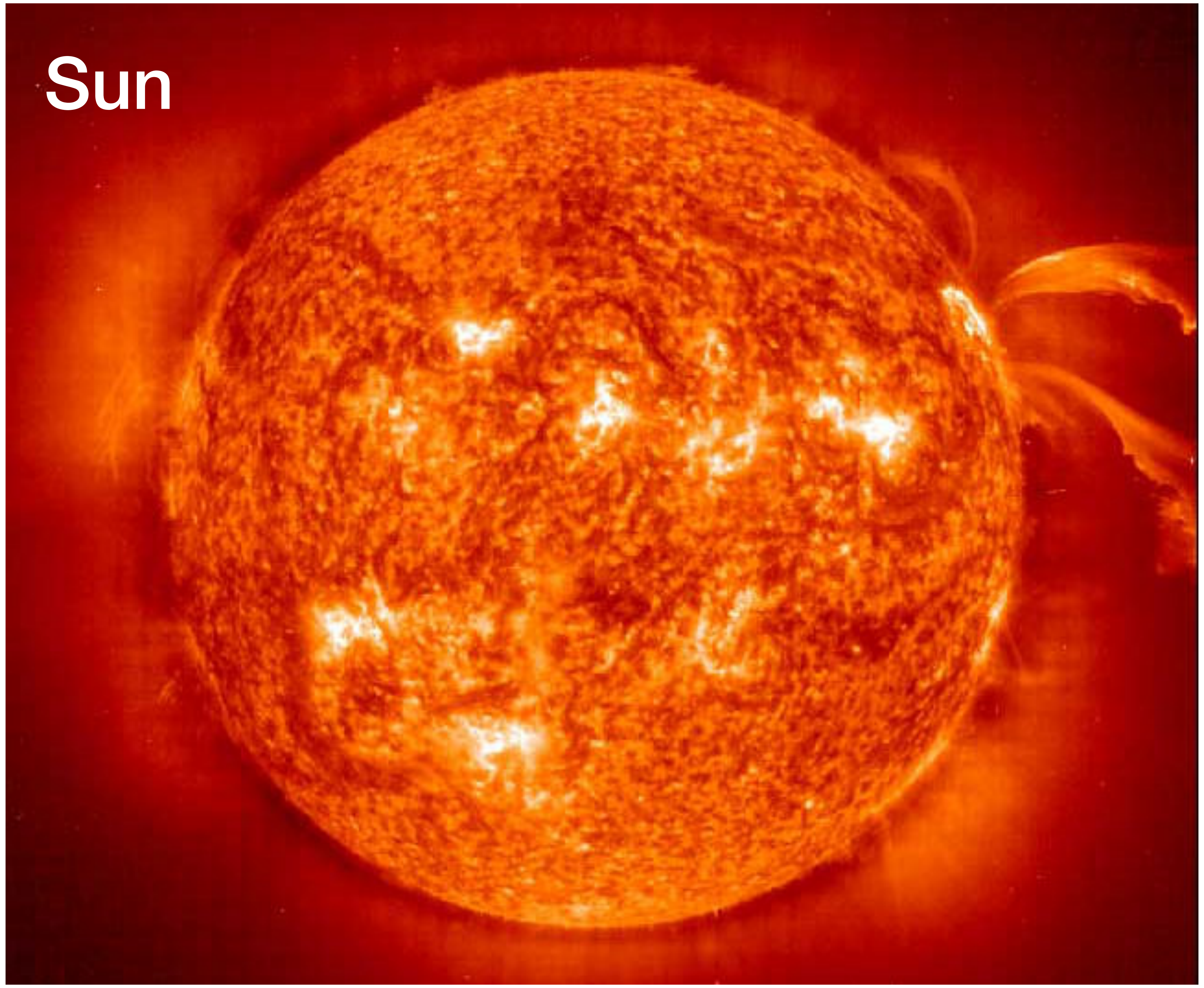


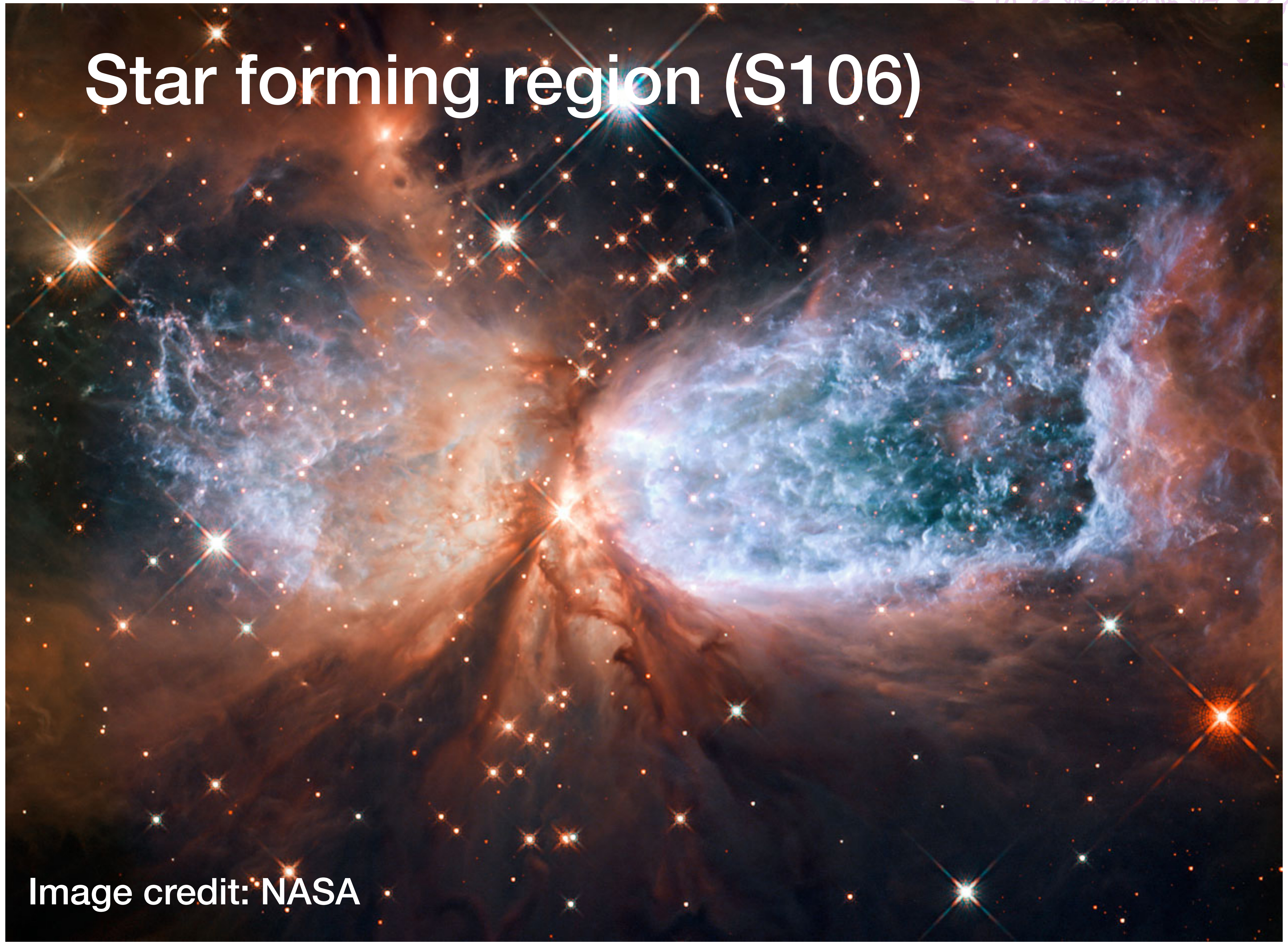
Galaxy clusters
(~1 Mpc)



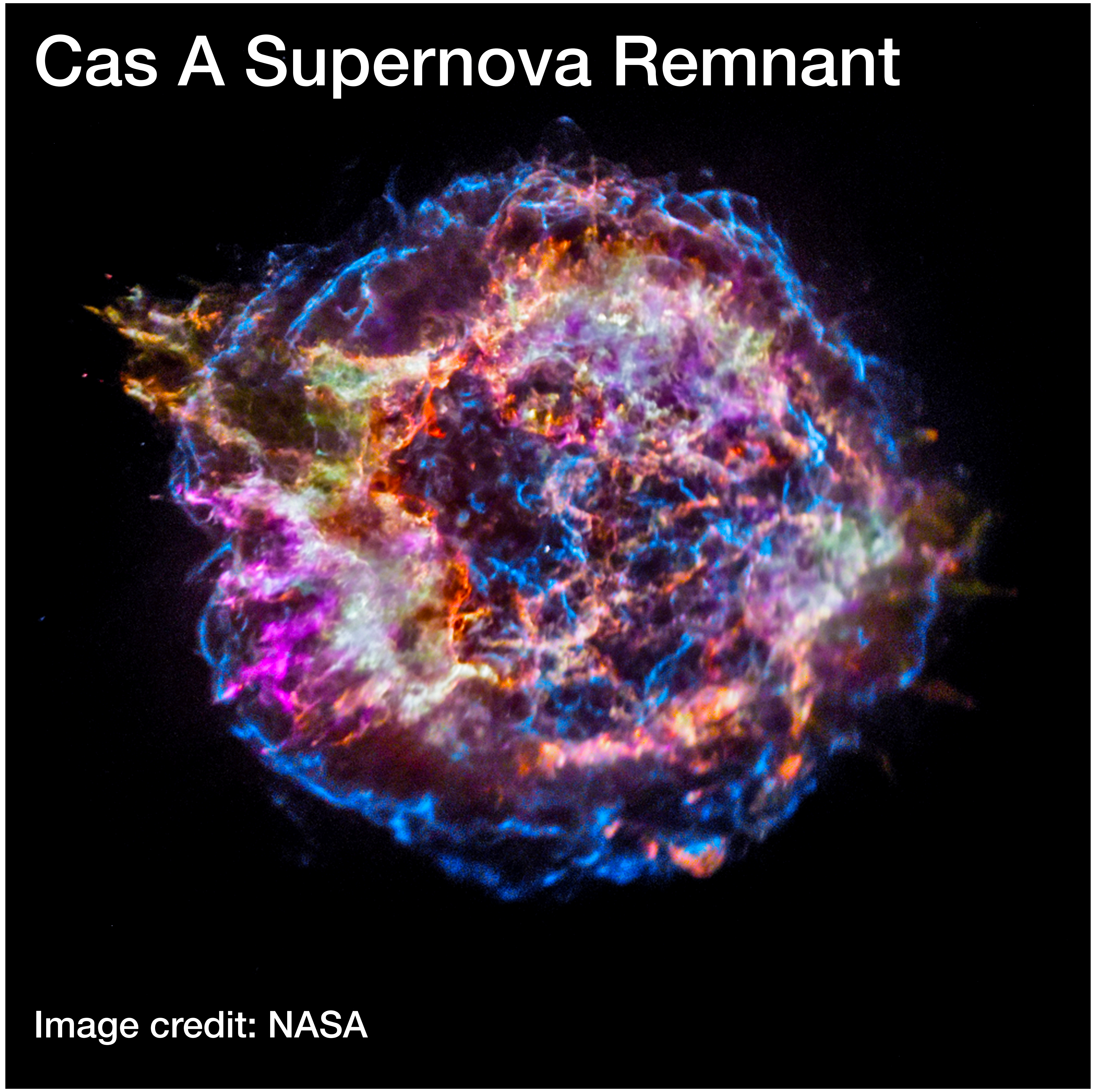
Large scale structure
(~1 Gpc)

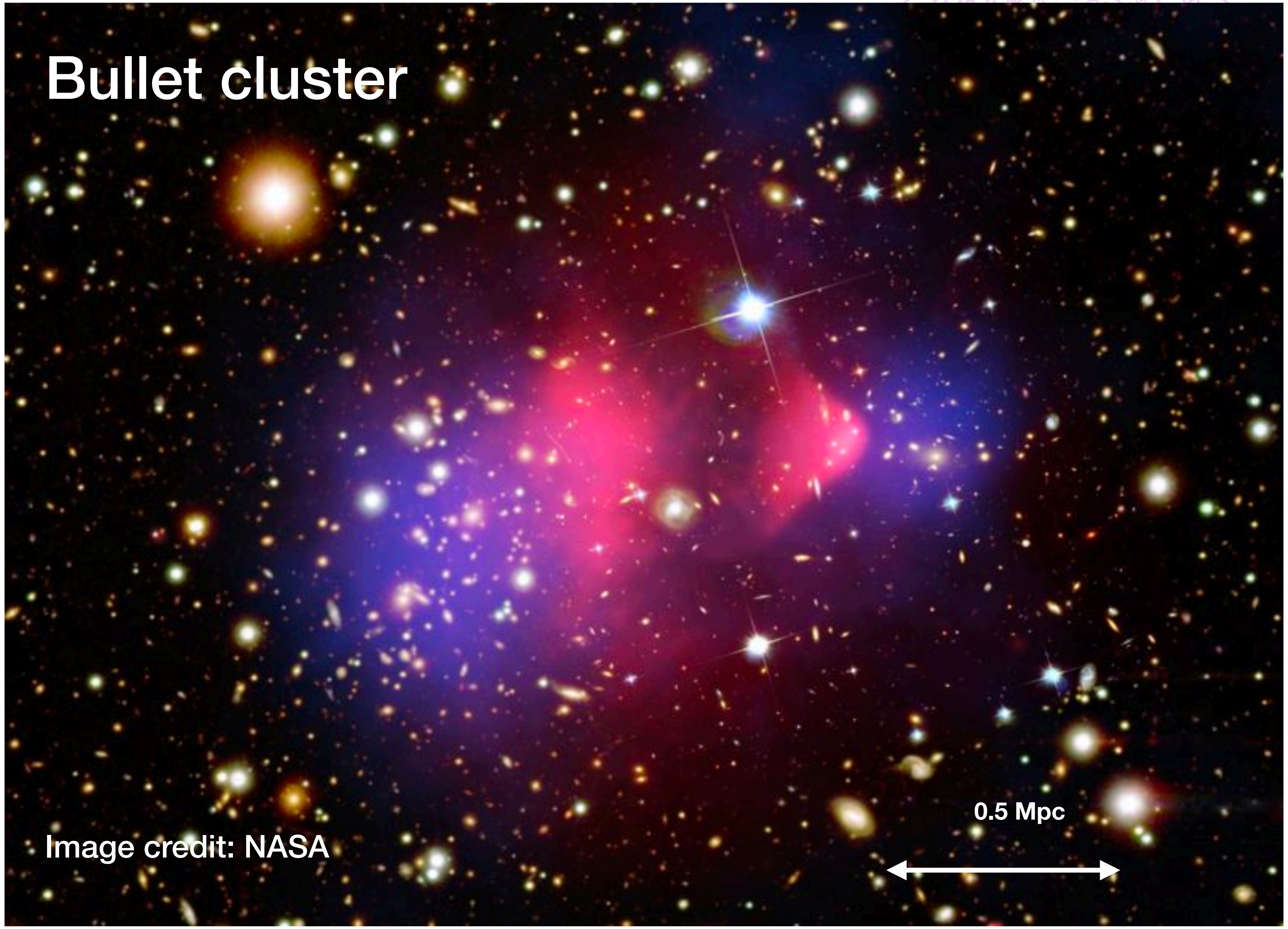
Astrophysical fluids





Astrophysical fluids





Fluid dynamics

https://youtu.be/ICxC5ekWnUc

Hydrodynamics Equations

Governing equations

The governing equations for ideal (inviscid) hydrodynamics. The main physical ideas are simple:

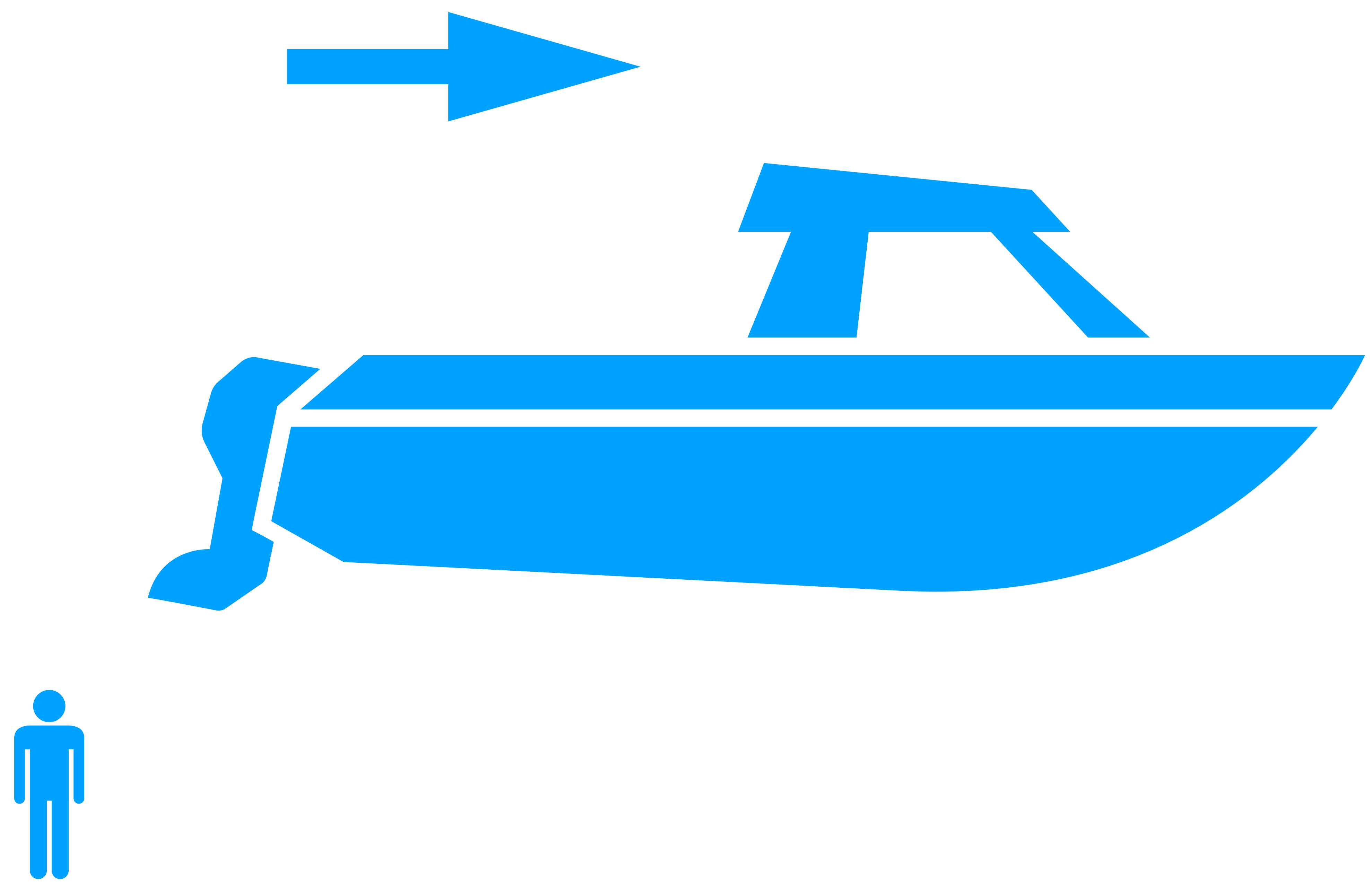
- Mass is conserved (1 constraint)
- Momentum is conserved (3 constraints)
- Energy is conserved (1 constraint)

$$\text{variables: } \rho, \boldsymbol{v}, \text{ and } u$$

ρ	Gas density
$\boldsymbol{v}$	Gas velocity
u	Gas internal energy

Coordinates

Eulerian



Lagrangian

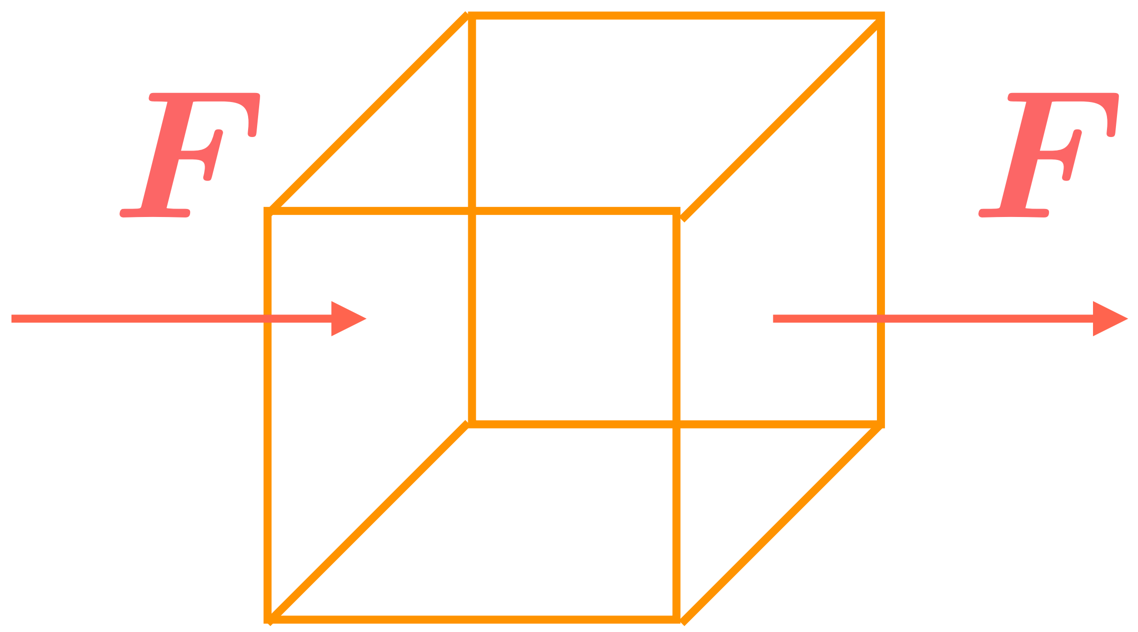
Conservational laws

The conservational laws can be written in what is called "conservation form":

$$\partial_t(U) = -\nabla \cdot \boldsymbol{F}$$

"Gauss's Theorem" or "divergence theorem"

$$\int_V (\nabla \cdot \boldsymbol{F})\, dV = \oint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$$



U Density of a quantity

$\boldsymbol{F}$ Flux density for that quantity

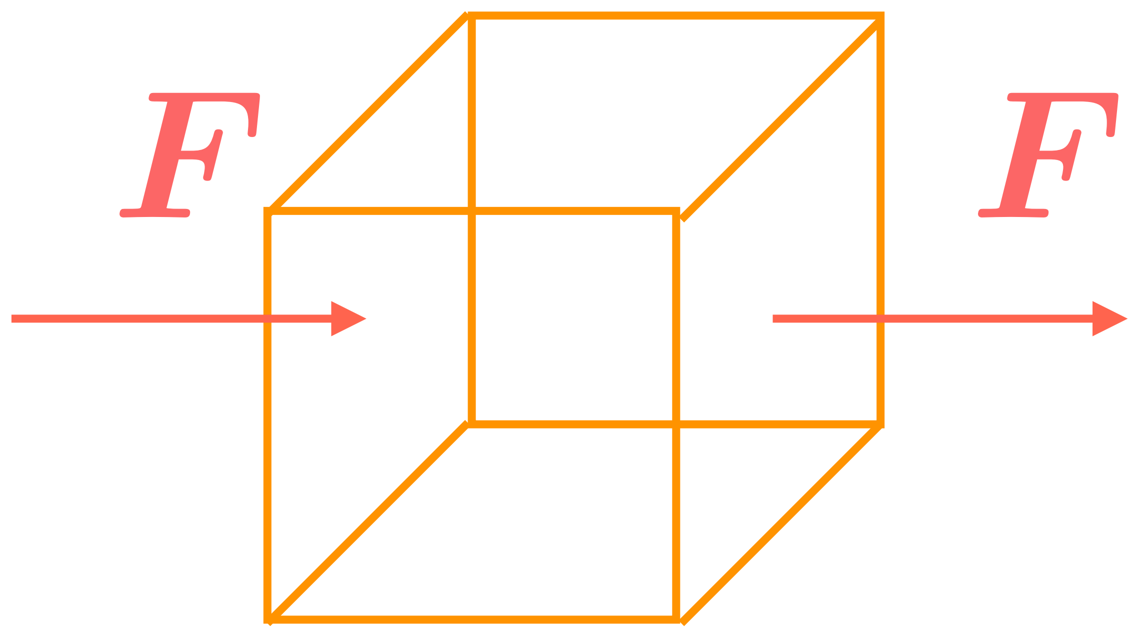
Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

ρ	Gas density
$\boldsymbol{v}$	Gas velocity

“Gauss’s Theorem" or "divergence theorem”

$$\int_V (\nabla \cdot \boldsymbol{F})\, dV = \oint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$$



U	Density of a quantity
$\boldsymbol{F}$	Flux density for that quantity

Convective derivative

- Consider a fluid element

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}, \qquad \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt},$$

$$\begin{aligned}\boldsymbol{a} &= \frac{d\boldsymbol{v}(\boldsymbol{r},\boldsymbol{t})}{dt}\\ &= \frac{\partial v}{\partial t} + \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial t}\\ &= \frac{\partial v}{\partial t} + (v\cdot\nabla)v\\ &\equiv \frac{Dv}{Dt}\end{aligned}$$

Convective derivate

- Rewrite continuity equation

$$\begin{aligned}\partial_t\rho &= -\nabla\cdot(\rho\boldsymbol{v})\\ &= -(v\cdot\nabla)\rho - \rho(\nabla\cdot v)\end{aligned}$$

$$\rightarrow \quad \partial_t\rho + (v\cdot\nabla)\rho = -\rho(\nabla\cdot v)$$

$$\rightarrow \quad \frac{D\rho}{Dt} = -\rho(\nabla\cdot v)$$

Continuity equation

Momentum equation

- What accelerations act on a fluid element?

$$\boldsymbol{a} = -\frac{\nabla p}{\rho}$$

- This gives us the Euler equations

$$\frac{D\boldsymbol{v}}{Dt} = -\frac{\nabla p}{\rho}$$

or

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P \cdot \boldsymbol{I}) = 0$$

Your homework!

Internal Energy Equation

- From first law of thermodynamics,

$$d\epsilon = T\cancel{ds}^{\,0} - pdV, \qquad dV = d\left(\frac{1}{\rho}\right) = \frac{-d\rho}{\rho^2}$$

Volume per unit mass

$$d\epsilon \equiv d\left(\frac{u}{\rho}\right) = \underline{\frac{du}{\rho}} - u\frac{d\rho}{\rho^2} = -pdV = p\frac{d\rho}{\rho^2}$$

$$\frac{Du}{Dt} = \frac{u+p}{\rho}\frac{D\rho}{Dt} = \frac{\partial u}{\partial t} + (v\cdot\nabla)u = -(u+p)\nabla\cdot v$$

- Combine internal energy equation with the momentum equation

$$\frac{\partial \rho E}{\partial t} + \nabla\cdot[(\rho E + P)\boldsymbol{v}] = 0 \qquad \boxed{E = \frac{1}{2}v^2 + \epsilon}$$

Total Energy equation

Summary: Hydrodynamics equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

Continuity equation

Stress tensor　　Unit tensor

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P \cdot \boldsymbol{I}) = 0$$

Momentum equation / Eular equations

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot [(\rho E + P)\boldsymbol{v}] = 0$$

Energy equation

ρ	Gas density
$\boldsymbol{v}$	Gas velocity
p	Gas pressure
ϵ	Gas specific internal energy
E	Gas specific total energy
u	Gas internal energy

$$u = \rho \epsilon$$

$$E = \frac{1}{2} v^2 + \epsilon$$

Summary: Hydrodynamics + Gravity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

Continuity equation

Stress tensor   Unit tensor

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \boldsymbol{v} + P \cdot \boldsymbol{I}) = -\rho \nabla \Phi$$

Momentum equation / Eular equations

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot [(\rho E + P)\boldsymbol{v}] = -\rho \boldsymbol{v} \cdot \nabla \Phi$$

Energy equation

$$\nabla^2 \Phi = 4\pi G \rho$$

Poisson equation

ρ	Gas density
$\boldsymbol{v}$	Gas velocity
p	Gas pressure
ϵ	Gas specific internal energy
E	Gas specific total energy
u	Gas internal energy
Φ	Gravitational potential

$$u = \rho \epsilon$$

$$E = \frac{1}{2} v^2 + \epsilon$$

Summary: Lagrangian form

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot v$$

Continuity equation

$$\frac{Dv}{Dt} = -\frac{\nabla p}{\rho} - \nabla \Phi$$

Momentum equation

$$\frac{Du}{Dt} = -(u + p) \nabla \cdot v$$

Energy equation

ρ Gas density

$\boldsymbol{v}$ Gas velocity

p Gas pressure

ϵ Gas specific internal energy

E Gas specific total energy

u Gas internal energy

Φ Gravitational Potential

$$u = \rho \epsilon$$

$$E = \frac{1}{2} v^2 + \epsilon$$

Equation of State (EoS)

- Need an Equation of State to close system of equations.

$n = \rho/m$	Number density
T	Temperature
k_B	Boltzmann constant
$\gamma = C_v/C_p$	Adiabatic index

For ideal gas,

$$P = (\gamma - 1)u = nk_\mathrm{B}T$$

- Isothermal EoS: $\gamma = 1$ $\quad P = \rho c_s^2$
- Polytropic EoS: $\quad P = K\rho^\gamma$

Finite Difference method: A simple Lagrangian code

In Spherical Coordinates

Consider a mass shell

$$dM_r = 4\pi r^2 \rho dr$$

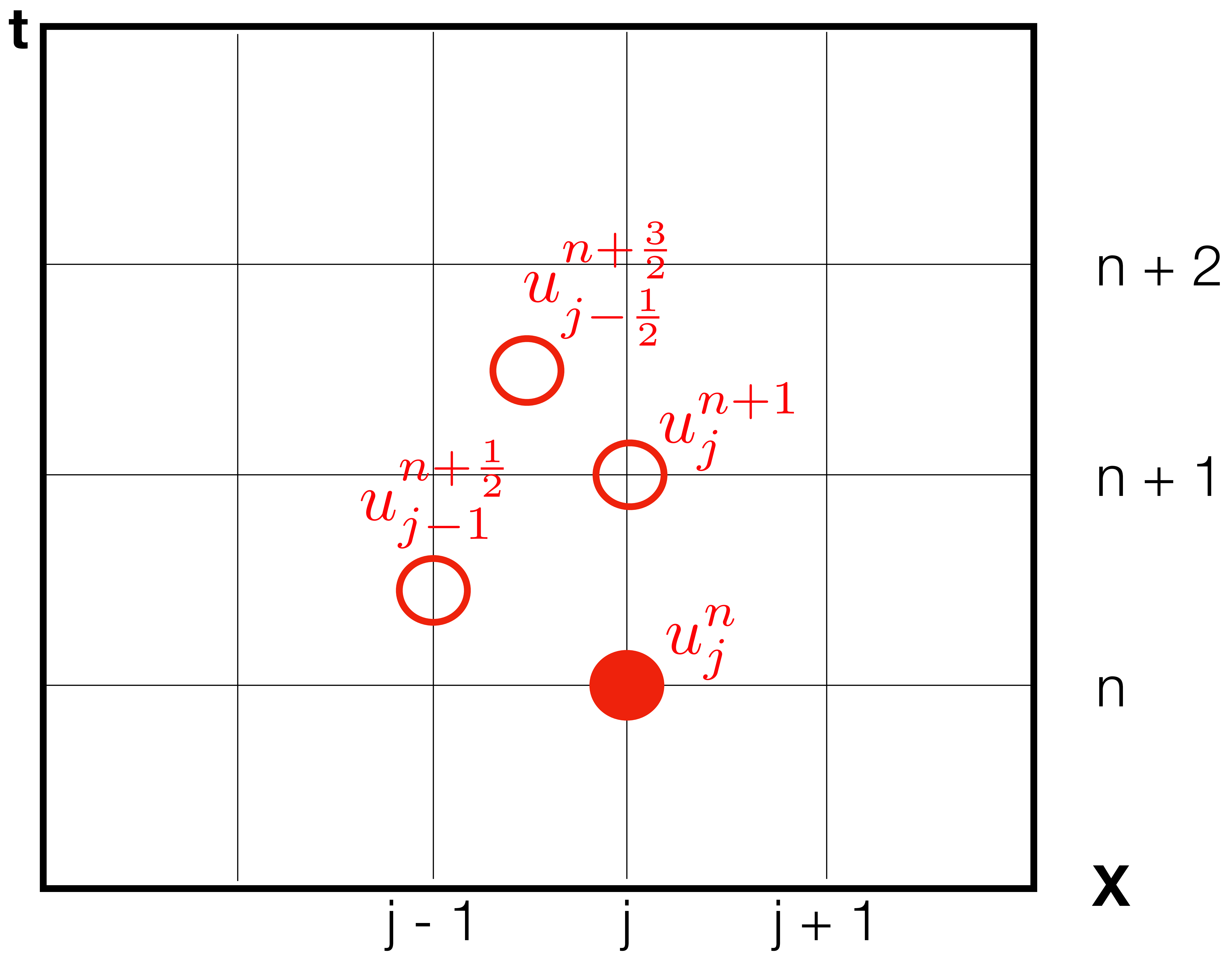
Hydrodynamics equations in Lagrangian coordinates become

$$\begin{cases} \frac{1}{\rho} = 4\pi r^2 \frac{dr}{dM_r} \\ \frac{dv}{dt} = -4\pi r^2 \frac{dP}{dM_r} \\ \frac{d\epsilon}{dt} = -4\pi P \frac{d(r^2 v)}{dM_r} \\ \frac{dr}{dt} = v. \end{cases}$$

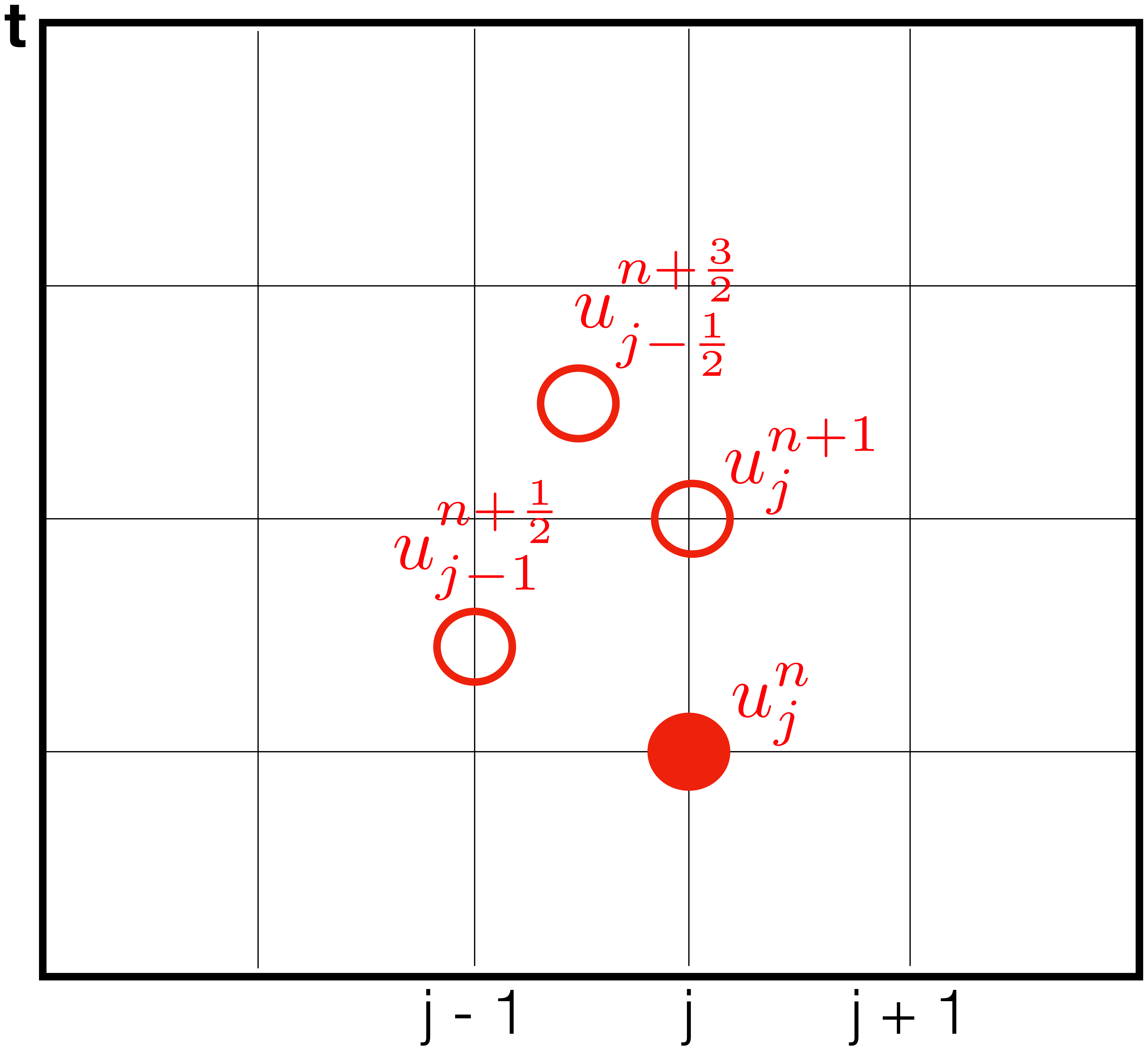
Four equations; five unknown $(\rho, r, v, P, \epsilon)$

Eos: $P = (\gamma - 1)u = (\gamma - 1)\rho\epsilon$

Staggered grid



Staggered grid



n + 2 r_j^n Cell corner

n + 1 $v_j^{n+\frac{1}{2}}$ Center of vertical edges

n $\rho_{j+\frac{1}{2}}^n$ $P_{j+\frac{1}{2}}^n$ $\epsilon_{j+\frac{1}{2}}^n$

x Center of horizontal edges

Staggered grid

Define

$$\begin{cases} \Delta m_{j+1/2} = \frac{4\pi}{3}\rho^0_{j+1/2}\left[(r^0_{j+1})^3 - (r^0_j)^3\right] \\ \Delta m_j = 0.5 \times (\Delta m_{j+1/2} + \Delta m_{j-1/2}) \\ A^n_j = 4\pi(r^n_j)^2 \\ \Delta t^n = 0.5 \times (\Delta t^{n-1/2} + \Delta t^{n+1/2}) \end{cases}$$

Staggered grid

$$\begin{cases} \frac{1}{\rho} = 4\pi r^2 \frac{dr}{dM_r} \\ \frac{dv}{dt} = -4\pi r^2 \frac{dP}{dM_r} \\ \frac{d\epsilon}{dt} = -4\pi P \frac{d(r^2 v)}{dM_r} \\ \frac{dr}{dt} = v. \end{cases}$$

Governing equations become

$$\begin{cases} v_j^{n+1/2} = v_j^{n-1/2} - A_j^n (P_{j+1/2}^n - P_{j-1/2}^n) \frac{\Delta t^n}{\Delta m_j} \\ r_j^{n+1} = r_j^n + v_j^{n+1/2} \Delta t^{n+1/2} \\ \rho_{j+1/2}^{n+1} = \frac{3}{4\pi} \frac{\Delta m_{j+1/2}}{(r_{j+1}^{n+1})^3 - (r_j^{n+1})^3} \\ \epsilon_{j+1/2}^{n+1} = \epsilon_{j+1/2}^n - \underbrace{P_{j+1/2}^{n+1/2}}_{\text{Unknown}} \left(A_{j+1}^{n+1/2} v_{j+1}^{n+1/2} - A_j^{n+1/2} v_j^{n+1/2} \right) \frac{\Delta t^{n+1/2}}{\Delta m_{j+1/2}} \end{cases}$$

Staggered grid

Equation of state gives

$$P_{j+1/2}^{n+1/2} = (\gamma - 1)\rho_{j+1/2}^{n+1/2}\epsilon_{j+1/2}^{n+1/2} = 0.5(\gamma - 1)\rho_{j+1/2}^{n+1/2}(\epsilon_{j+1/2}^{n} + \epsilon_{j+1/2}^{n+1})$$

Staggered grid

The simplest procedure consists of two steps

1. Approximate $P_{j+1/2}^{n+1/2}$ with $P_{j+1/2}^{n}$; solve the energy equation to get the first approximation for $\epsilon_{j+1/2}^{n+1}$; from EoS, calculate $P_{j+1/2}^{n+1}$

2. Approximate $P_{j+1/2}^{n+1/2}$ with $0.5(P_{j+1/2}^{n}+P_{j+1/2}^{n+1})$; solve the energy equation again to get the second approximation of $\epsilon_{j+1/2}^{n+1}$.

Given r and v, time step can be found from the CFL condition

Staggered grid

$$\begin{cases} v_j^{n+1/2} = v_j^{n-1/2} - A_j^n (P_{j+1/2}^n - P_{j-1/2}^n) \frac{\Delta t^n}{\Delta m_j} \\ r_j^{n+1} = r_j^n + v_j^{n+1/2} \Delta t^{n+1/2} \\ \rho_{j+1/2}^{n+1} = \frac{3}{4\pi} \frac{\Delta m_{j+1/2}}{(r_{j+1}^{n+1})^3 - (r_j^{n+1})^3} \\ \epsilon_{j+1/2}^{n+1} = \epsilon_{j+1/2}^n - P_{j+1/2}^{n+1/2} \left(A_{j+1}^{n+1/2} v_{j+1}^{n+1/2} - A_j^{n+1/2} v_j^{n+1/2} \right) \frac{\Delta t^{n+1/2}}{\Delta m_{j+1/2}} \end{cases}$$

This scheme works only as long as there are no shock waves in the flow.

With shocks, we have to add artificial viscosity.

Artificial viscosity

- Physically, a shock is not infinitely sharp, but spread out over a few particle mean free paths (MFP).
- MFP is far too small to be resolved on a typical grid.
- Numerically, a shock has to be smeared over a few grid cells. -> needs artificial viscosity

Artificial viscosity

- Von Neumann and Richtmyer (1950)

Artificial pressure term

$$Q = \begin{cases} q^2 \rho (\Delta x)^2 |\frac{\partial v}{\partial x}|^2 & \text{if} \frac{\partial v}{\partial x} < 0 \\ 0 & \text{if} \frac{\partial v}{\partial x} > 0 \end{cases},$$

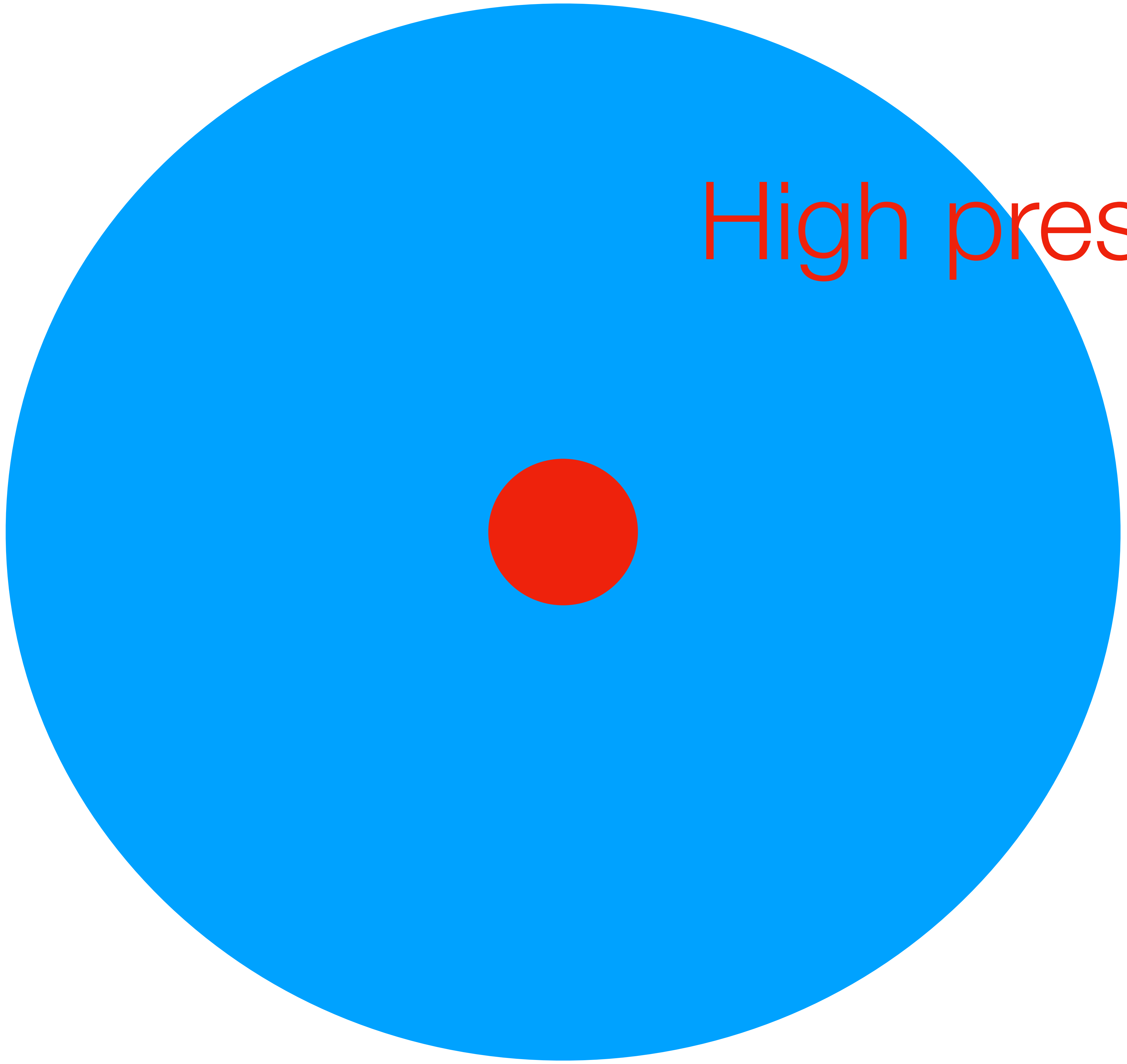
where q is the dimensionless artificial viscosity parameter

In most cases, $0.05 < q < 2$

The general strategy is replace dP/dx to d(P+Q)/dx in the momentum equation, and replace P to (P+Q) in the energy equation

Lab: a simple Lagrangian code

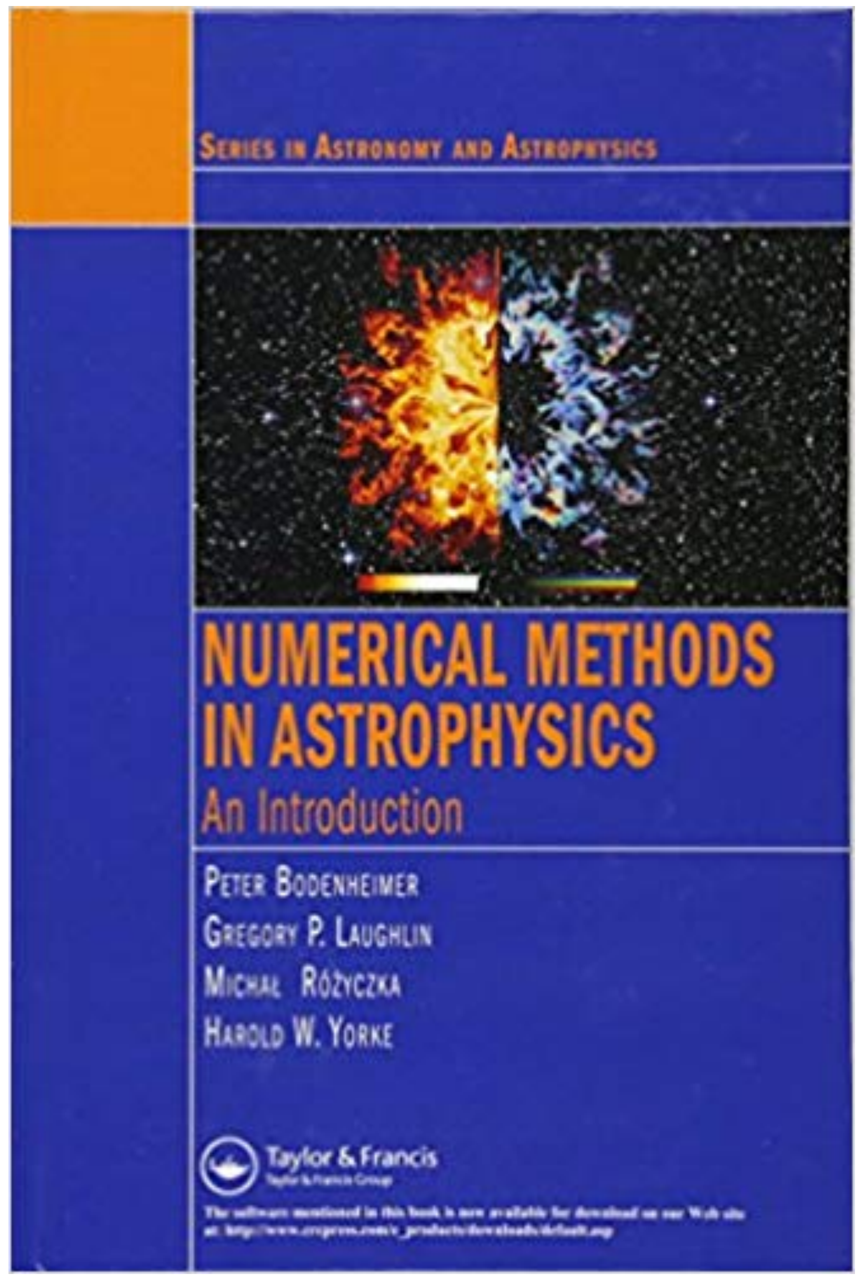
LAB: Supernova fireball simulation



High pressure ejecta

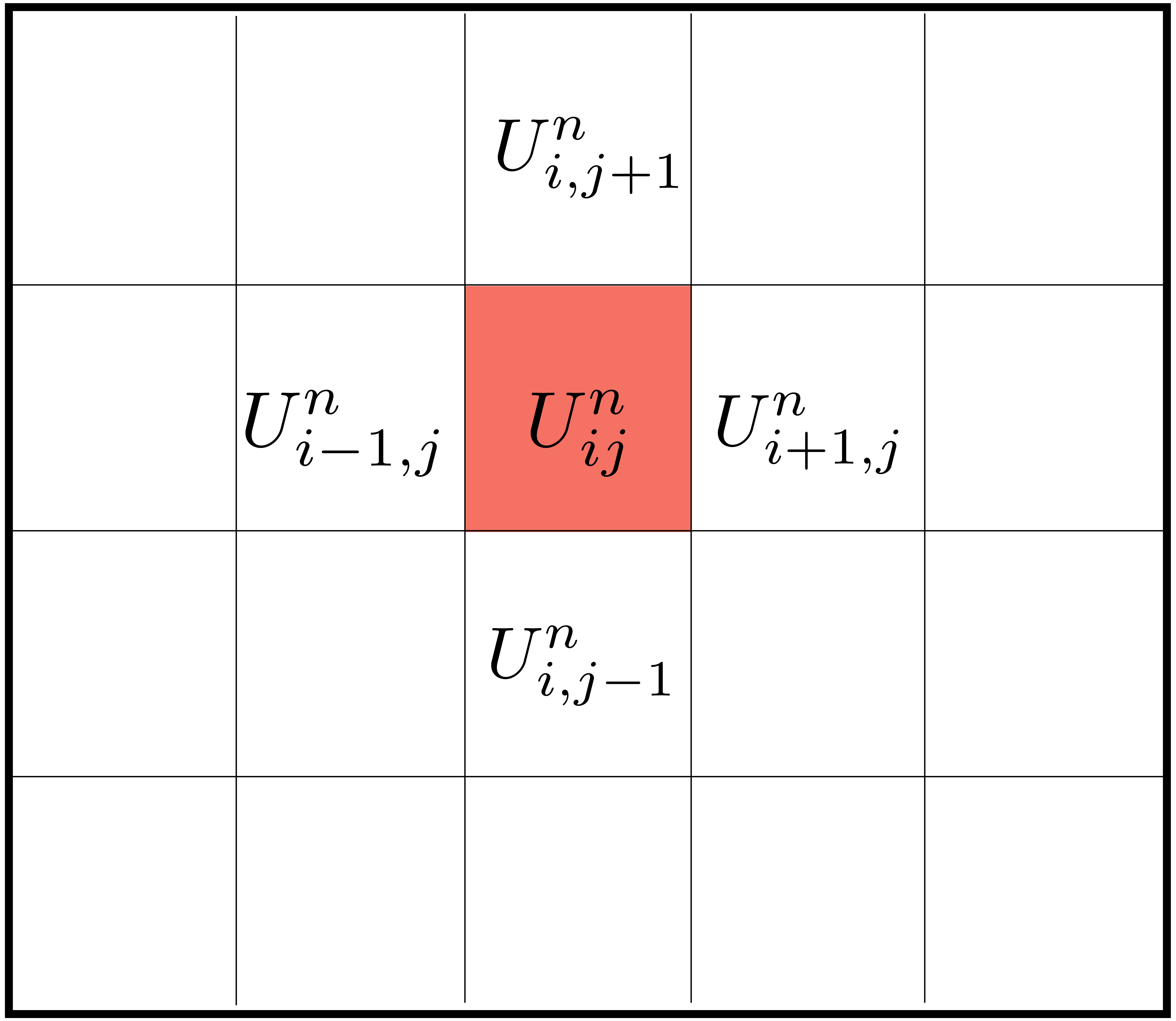
Low pressure ambient

See the code `1_lagrangian`



Finite Volume Method

Finite Volume Method

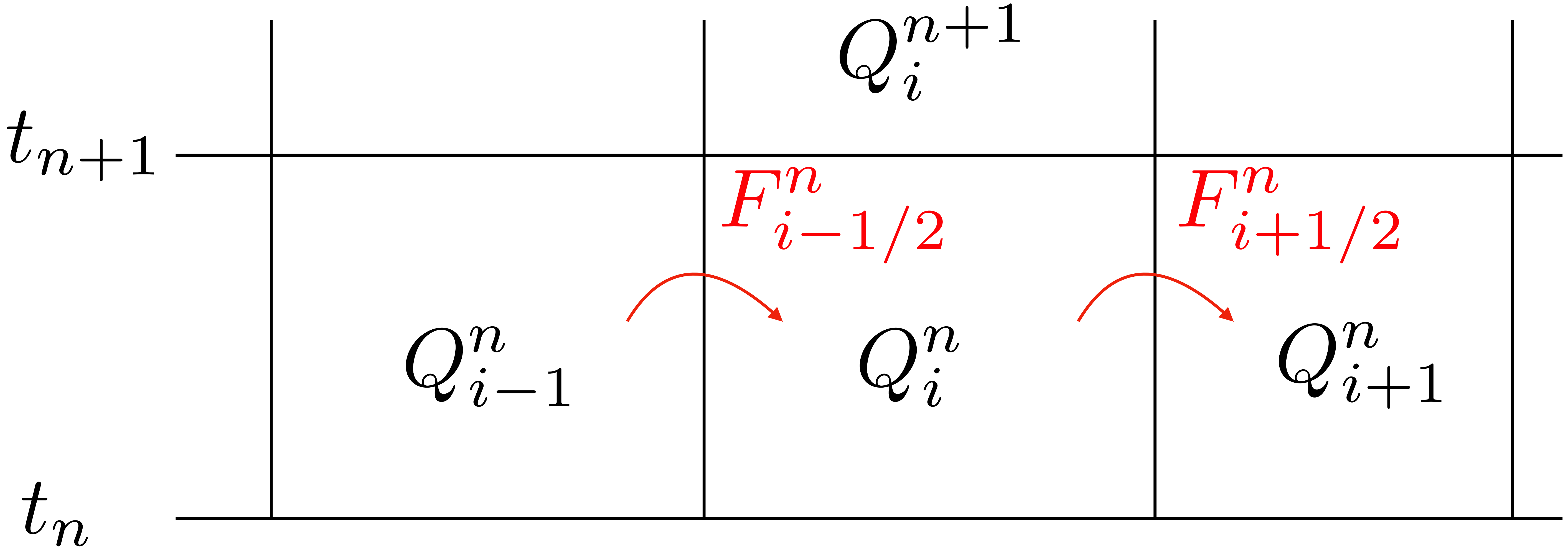


- Initial conditions of all hydrodynamics variables at time step n=0 (a given EoS)
- Boundary conditions at simulation boundaries
- Evolve hydrodynamics variables with a time step dt

$$\frac{\partial}{\partial t}\underbrace{\begin{pmatrix}\rho \\ \rho U \\ E\end{pmatrix}}_{W(x,t)}+\frac{\partial}{\partial x}\underbrace{\begin{pmatrix}\rho U \\ \rho U^2+p \\ (E+p)U\end{pmatrix}}_{F(W)}=0,$$

in 1D

Finite Volume Method



$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n+1/2}\right)$$

- Problem -> How to approximate the flux at cell edge

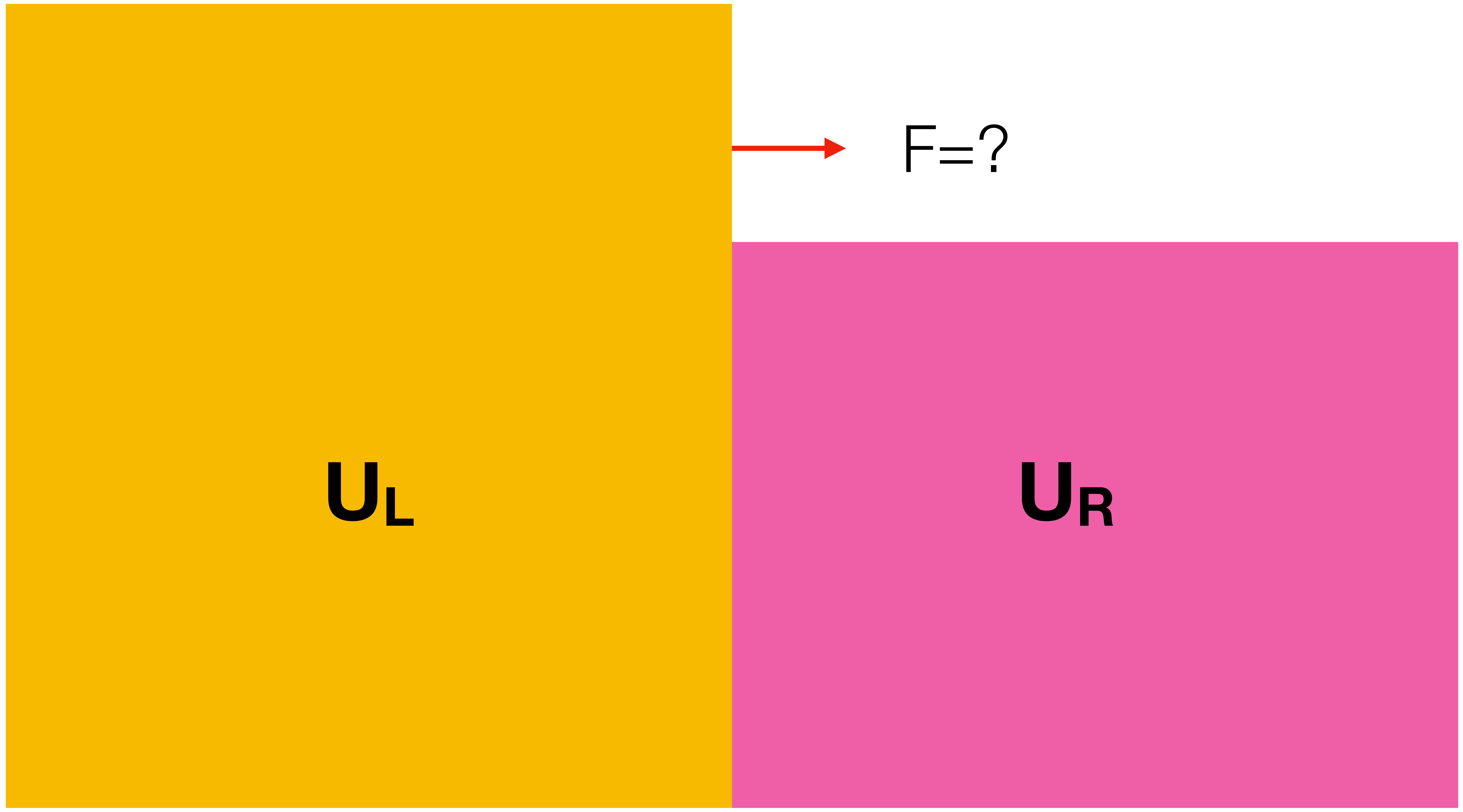
Finite Volume Method

- Integrate over the cell volume and time interval

$$U^n_{i,j,k} = \frac{1}{\Delta x \Delta y \Delta z} \int_{z_{k-1/2}}^{z_{k+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} U(x,y,z,t^n) dxdydz$$

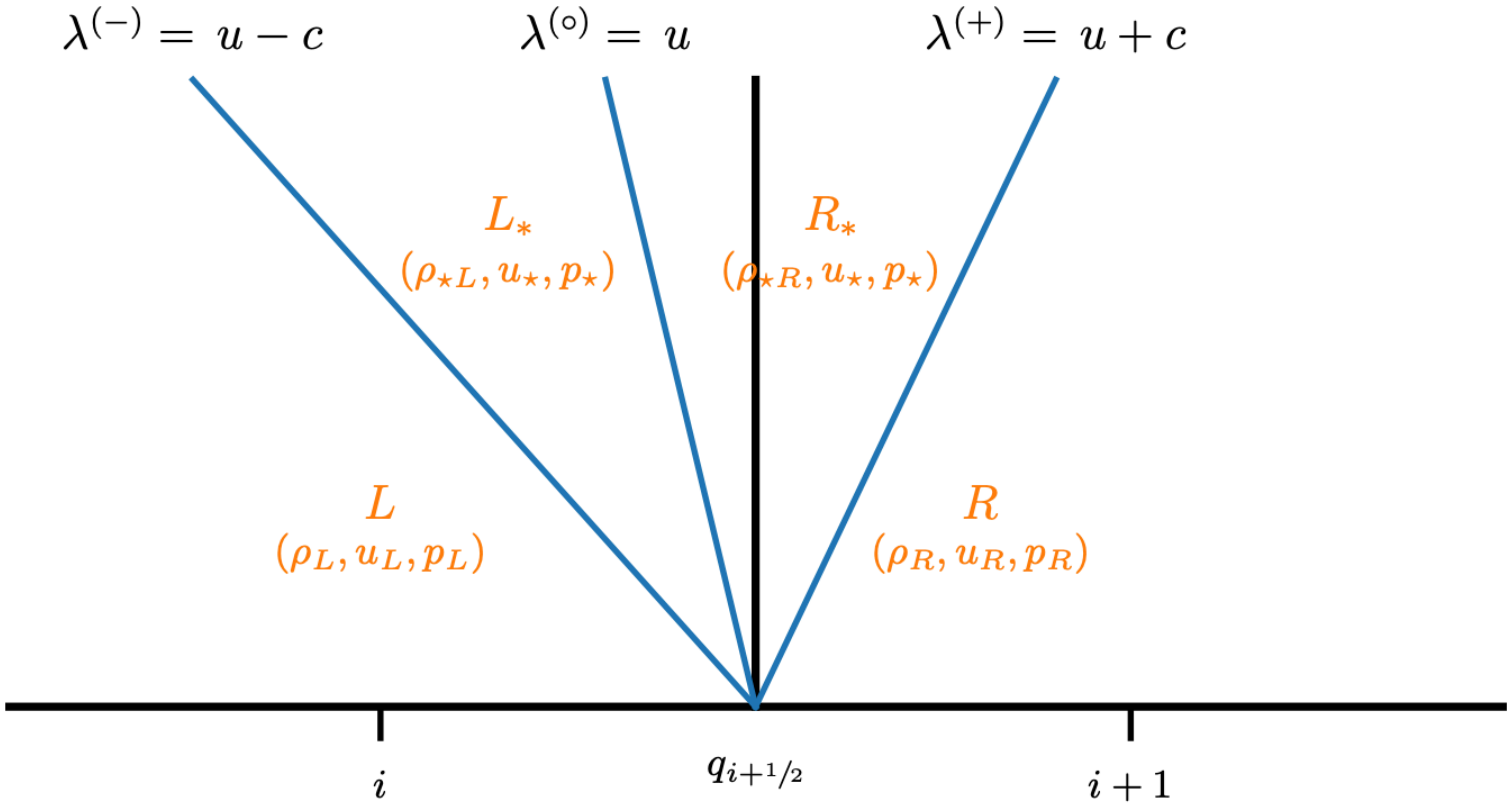
$$F^{n+1/2}_{x,i-1/2,j,k} = \frac{1}{\Delta t \Delta y \Delta z} \int_{t^n}^{t^{k+1}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} F(x_{i-1/2},y,z,t) dtdydz$$

Riemann Problem

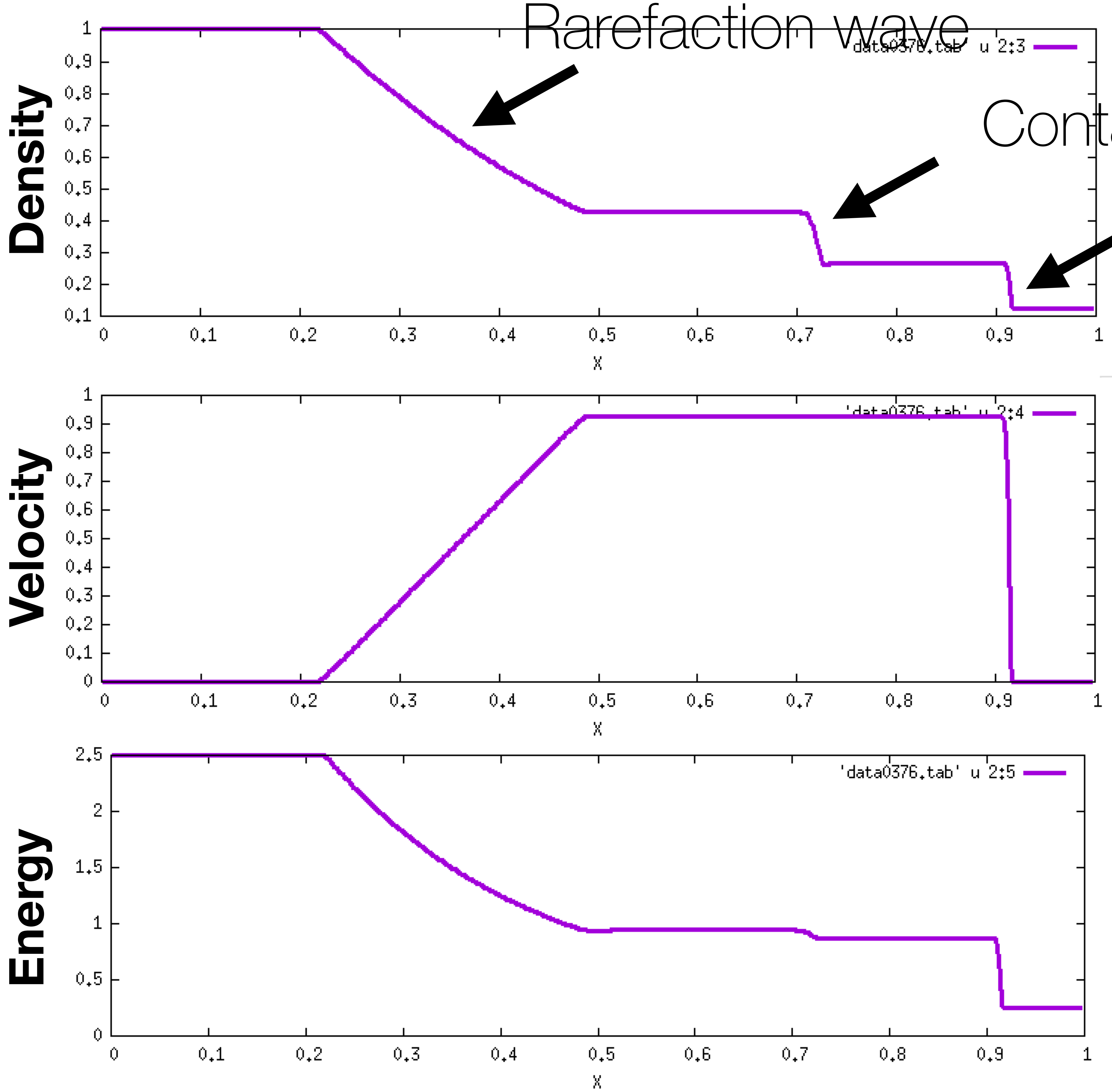


Characteristics

- For Euler equations, there are three eigenvalues



Sod Shock Tube Problem

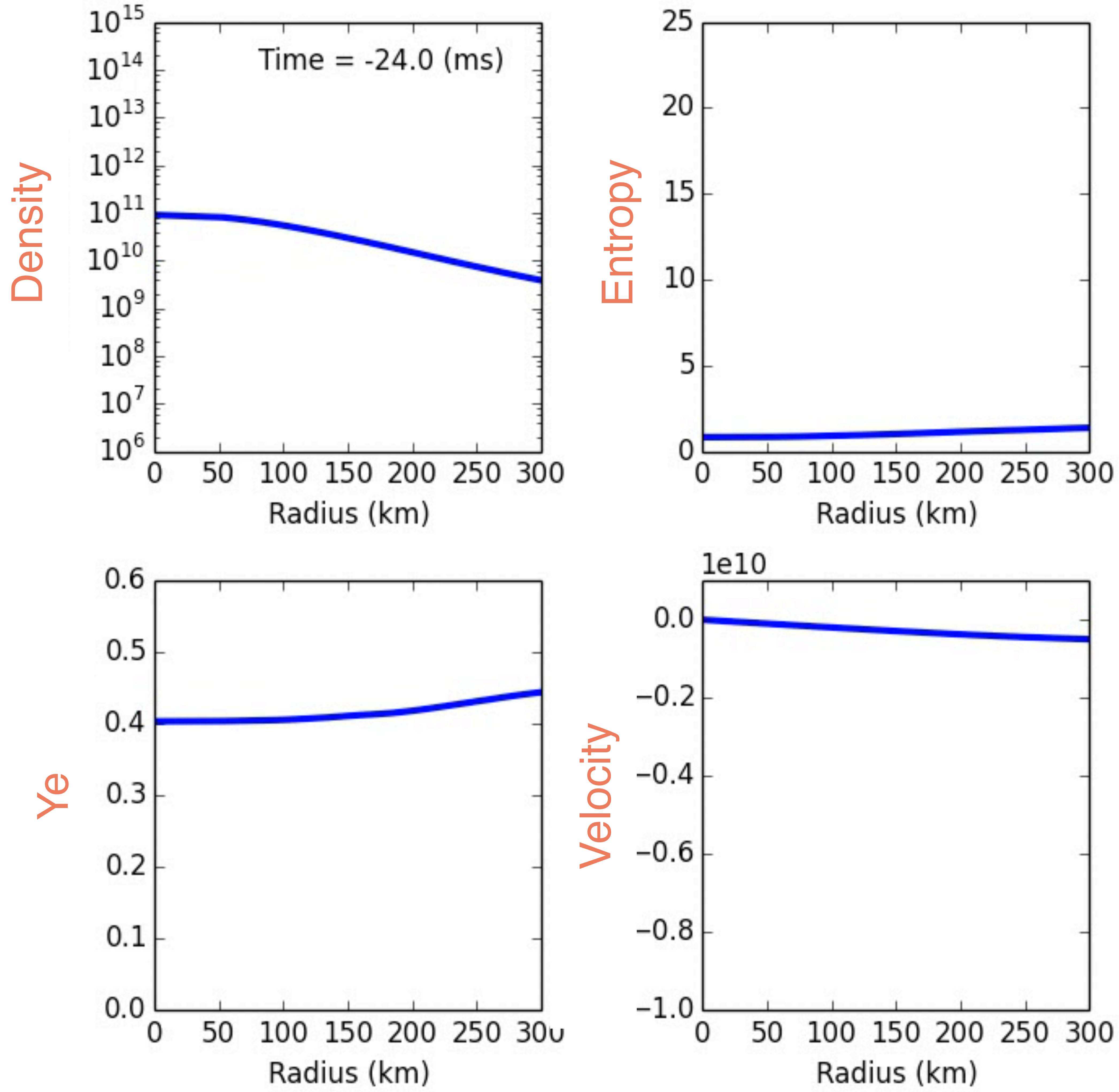


$$\begin{bmatrix} \rho_L \\ v_L \\ e_L \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0 \\ 2.5 \end{bmatrix}$$

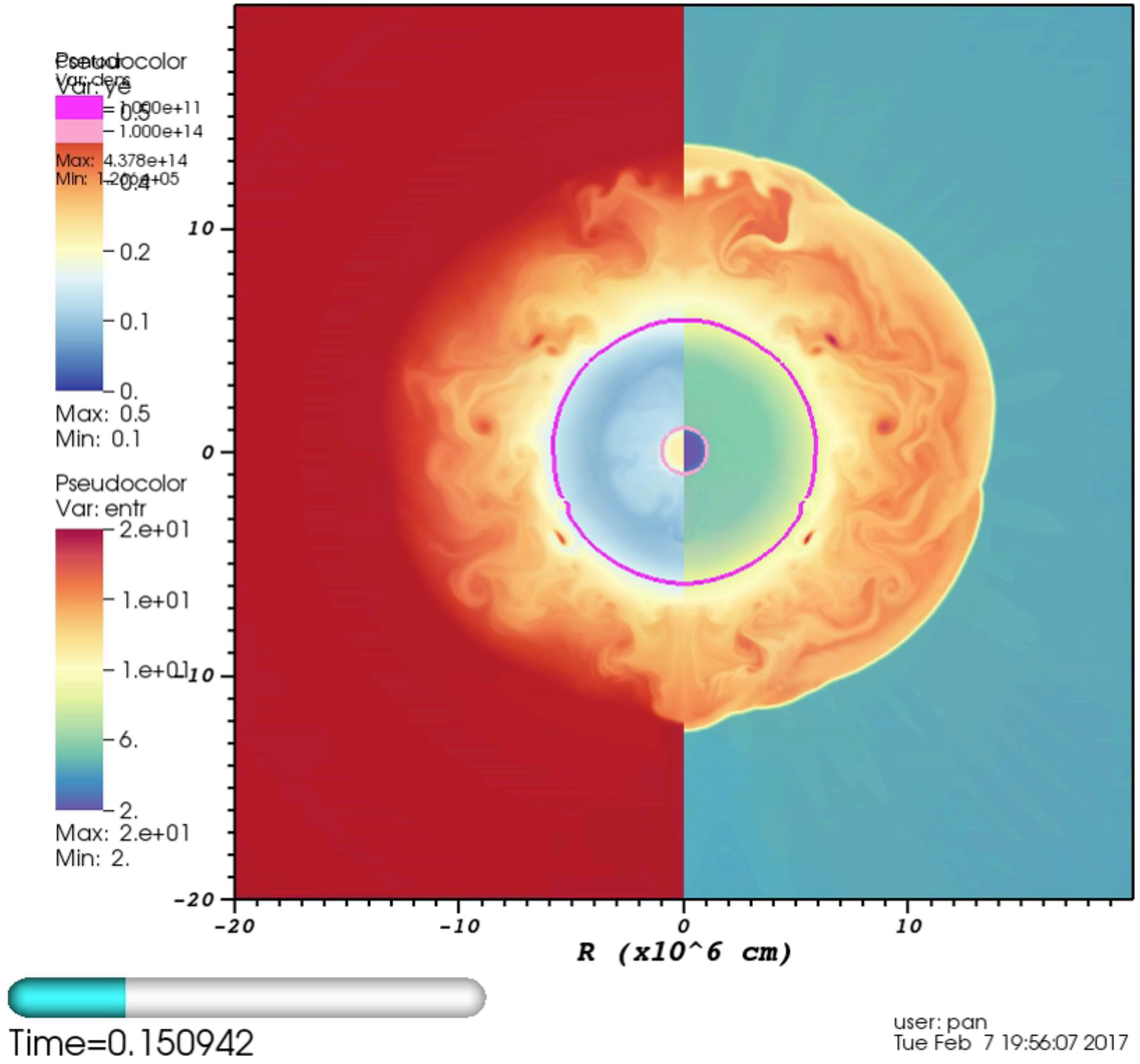
$$\begin{bmatrix} \rho_R \\ v_R \\ e_R \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0 \\ 0.25 \end{bmatrix}$$

Hydro codes

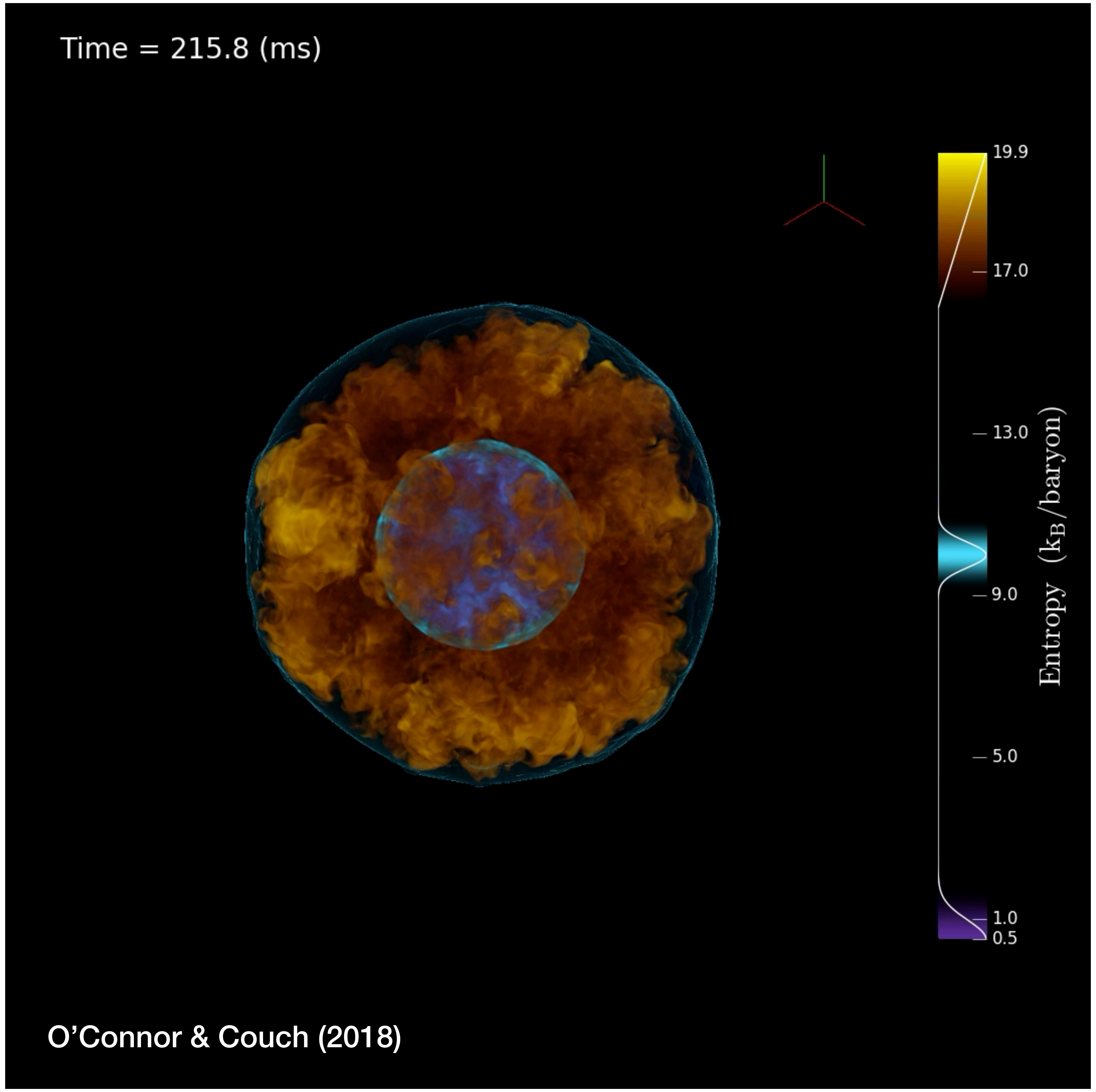
Supernova (Agile-Bolztran)



Supernova (FLASH)



Supernova (FLASH)



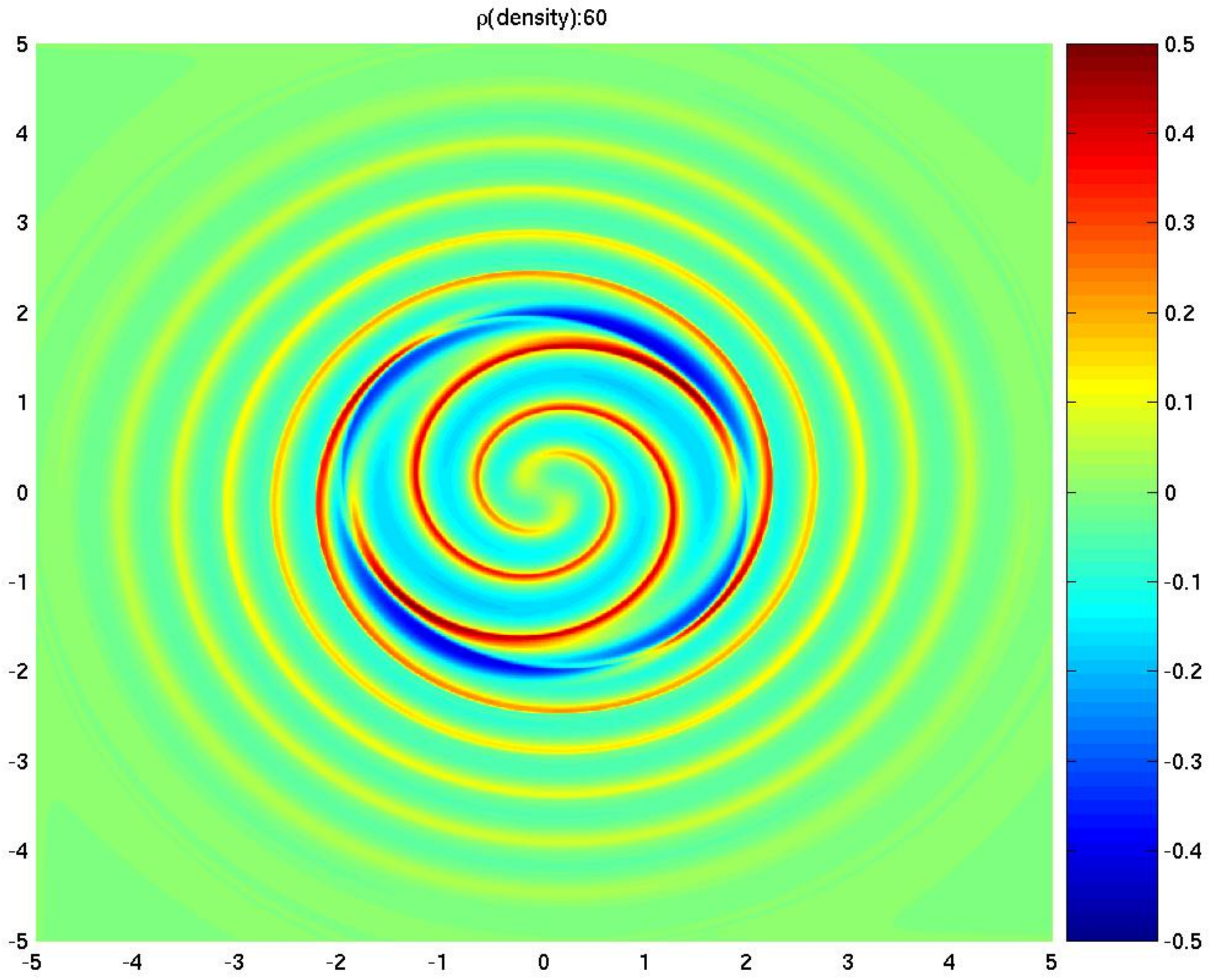
Incompressible fluids

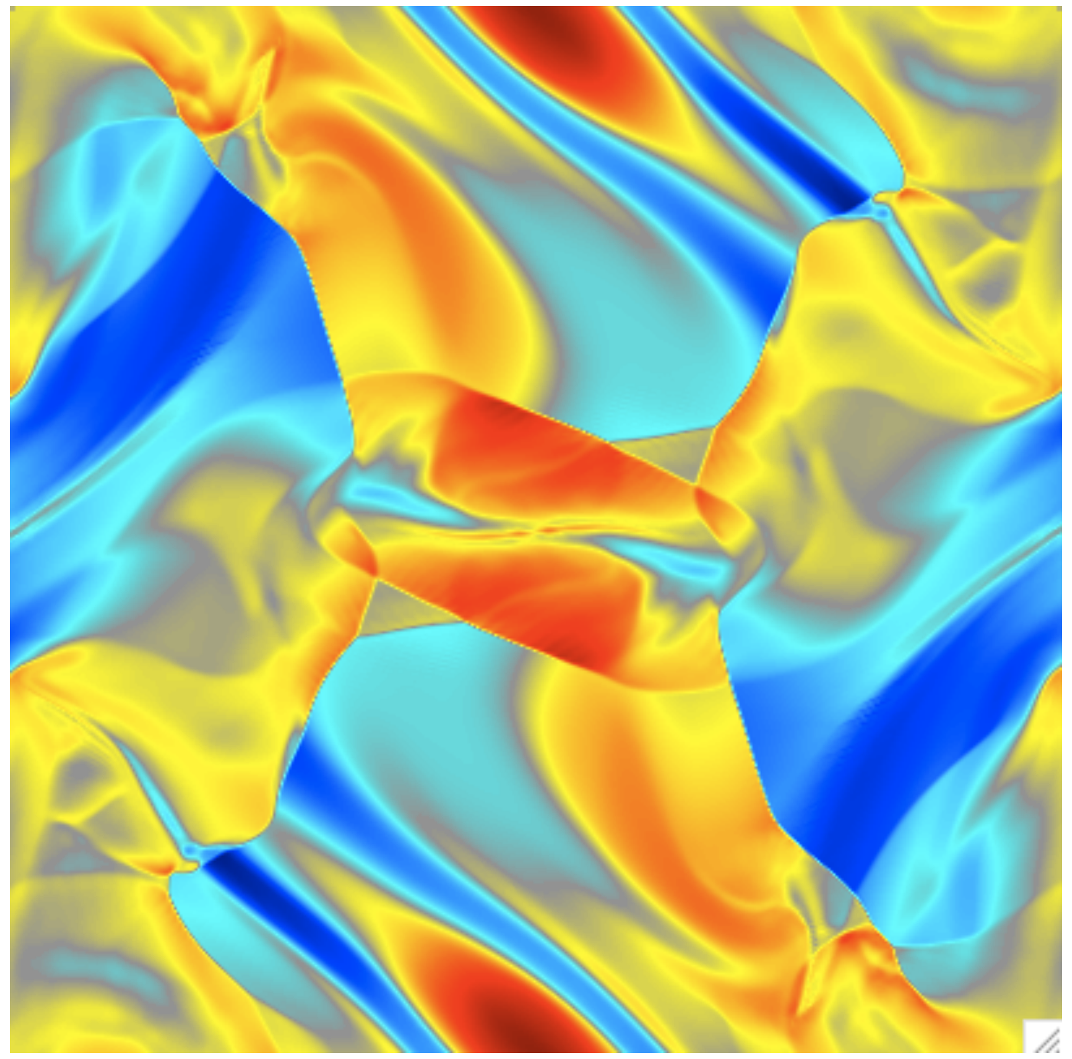
Incompressible fluids

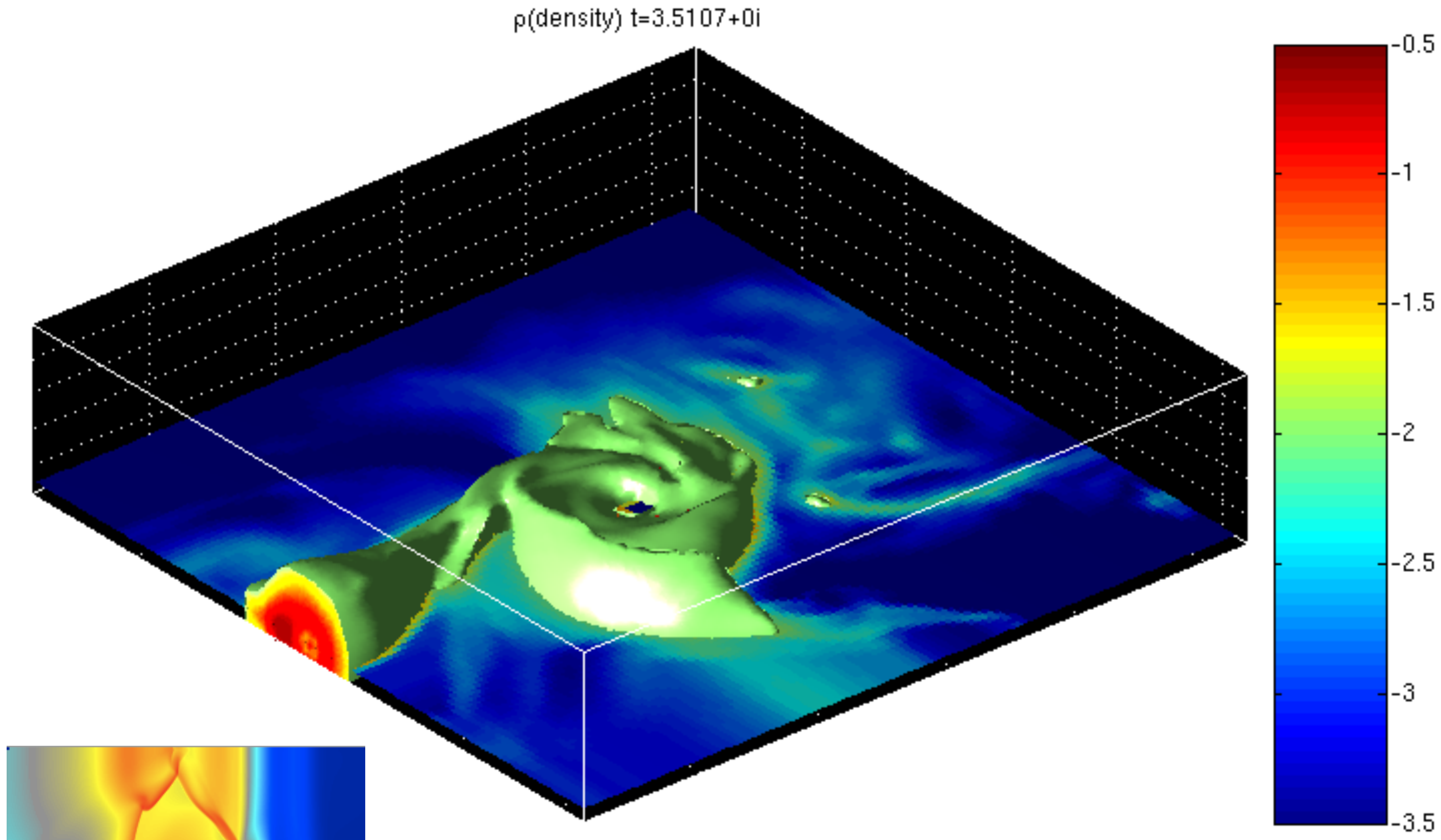
Lab: Antares Code

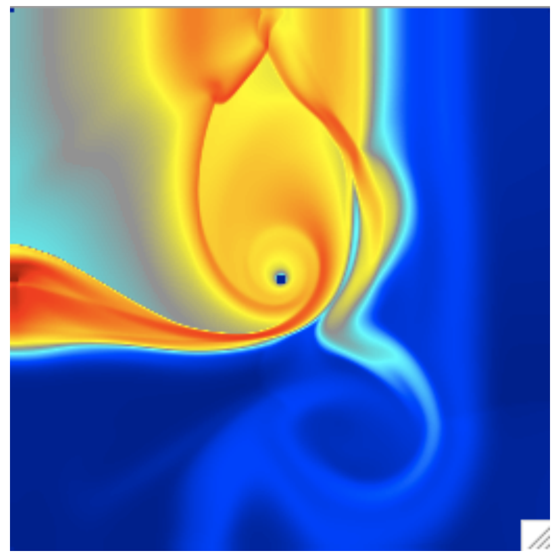
Antares Code

- Originally developed by David Yen & Chi Yuan
- Modified by H.H. Wang and K.C. Pan









Problem Set 9

https://kuochuanpan.github.io/courses/109ASTR660_CA/

ImageMagick

https://imagemagick.org/index.php

Features and Capabilities • News • Community

Use ImageMagick® to create, edit, compose, or convert bitmap images. It can read and write images in a variety of formats (over 200) including PNG, JPEG, GIF, HEIC, TIFF, DPX, EXR, WebP, Postscript, PDF, and SVG. Use ImageMagick to resize, flip, mirror, rotate, distort, shear and transform images, adjust image colors, apply various special effects, or draw text, lines, polygons, ellipses and Bézier curves.

ImageMagick is free software delivered as a ready-to-run binary distribution or as source code that you may use, copy, modify, and distribute in both open and proprietary applications. It is distributed under a derived Apache 2.0 license.

ImageMagick utilizes multiple computational threads to increase performance and can read, process, or write mega-, giga-, or tera-pixel image sizes.

The current release is ImageMagick 7.0.10-13. It runs on Linux, Windows, Mac Os X, iOS, Android OS, and others.

The authoritative ImageMagick web site is https://imagemagick.org. The authoritative source code repository is https://github.com/ImageMagick. We maintain a source code mirror at https://gitlab.com/ImageMagick. We continue to maintain the legacy release of ImageMagick, version 6, at https://legacy.imagemagick.org.

Next lecture

- Parallel Programing