Assignment 7 of Computational Astrophysics in NTHU

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1 Written Assignments

Q1 : Stellar structure - polytrope.

Q1.a. : polytropic constant K

$$K = \left[\frac{4\pi}{\xi^{n+1}(-\theta_n')^{n-1}}\right]^{\frac{1}{n}}_{\xi_1} \frac{G}{n+1} M^{1-\frac{1}{n}} R^{-1+\frac{3}{n}} \tag{1}$$

Q1.b. : central pressure P_c

$$P_c = \frac{8.952 \times 10^{14}}{(n+1)(\theta_n')^2_{\xi_1}} (\frac{M}{M_\odot})^2 (\frac{R}{R_\odot})^{-4} [dyne\ cm^{-2}] \tag{2}$$

Q1.c. : central temperature T_c

$$T_c = \frac{2.293 \times 10^7}{(n+1)(-\xi\theta_n')_{\xi_1}} \mu (\frac{M}{M_\odot})(\frac{R}{R_\odot})^{-1} [K] \tag{3}$$

Derivation

From the lecture in the class, we have derived the **Lane-Emden equation** :

$$\frac{(n+1)P_c}{4\pi G\rho_c^2} \frac{1}{r} \frac{d}{dr}(r^2 \frac{d\theta}{dr}) + \theta^n = \frac{1}{\xi^2} \frac{d}{d\epsilon}(\epsilon^2 \frac{d\theta_n}{d\xi}) + \theta_n^n = 0 \tag{4}$$

We also define a new **dimensionless radial coordinate** ξ and have Eq.7 and applied it to Eq.4 later part.

$$r = r_n \xi \text{ where we define } r_n^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2} \tag{5}$$

And we also have **the power law for pressure**:

$$P(r) = K\rho_c^{1+\frac{1}{n}} \theta^{1+\frac{1}{n}}(r) = P_c\ \theta^{1+\frac{1}{n}}(r) \tag{6}$$

Based on those information, we and find K when apply Pc in Eq.6 to Eq.5

$$r_n^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2} = \frac{(n+1)K\rho_c^{1+\frac{1}{n}}}{4\pi G\rho_c^2} = \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G} \tag{7}$$

And then we can consider another aspect - mass (implied by Eq.1 have M component), use the mass conservation mention in the lecture.

$$dM_r = 4\pi r^2 \rho dr$$

Then do integral and use the dimensionless radial coordinate Eq.5 ($r = r_n\xi$, $R = r_n\xi_1$, $\rho = \rho_c\theta_n^n$)

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi r_n^3 \rho_c \int_0^{\xi_1} \xi^2 \theta_n^n d\xi$$

Recall **Lane-Emden equation**(Eq.4) : $\frac{d}{d\epsilon}(\xi^2 \frac{d\theta_n}{d\xi}) = -\xi^2 \theta_n^n$ and can change variable for the integral.

$$M = -4\pi r_n^3 \rho_c \int_0^{\xi_1} d(\xi^2 \frac{d\theta_n}{d\xi}) = -4\pi r_n^3 \rho_c \xi_1^2 \frac{d\theta_n}{d\xi}|_{\xi_1} = -4\pi r_n^3 \rho_c \xi_1^2 \theta_n'|_{\xi_1} \tag{8}$$

By Eq.5, we can get $R = r_n \xi_1$, and substitute it into Eq.7 we can get K to the left side of the equation:

$$\frac{R^2}{\xi_1^2} = \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G} \Rightarrow K = \frac{4\pi G R^2}{(n+1)\xi_1^2}\rho_c^{1-\frac{1}{n}} \tag{9}$$

And then use Eq.8 to get $\rho^{1-\frac{1}{n}}$ and substitute to the equation:

$$\rho_c^{1-\frac{1}{n}} = \frac{M}{4\pi r_n^3 \xi_1^2 \theta_n'}|_{\xi_1}^{1-\frac{1}{n}}$$

After arrange the substitution and we can get the polytropic constant K (Eq.1)

$$K = \left[\frac{4\pi}{\xi^{n+1}(-\theta_n')^{n-1}}\right]_{\xi_1}^{\frac{1}{n}} \frac{G}{n+1} M^{1-\frac{1}{n}} R^{-1+\frac{3}{n}}$$

And we recall the derivation above,

$$P_c = K\rho_c^{1+\frac{1}{n}} \text{ in Eq.6}$$
$$K = \frac{4\pi G R^2}{(n+1)\xi_1^2}\rho_c^{1-\frac{1}{n}} \text{ in Eq.9}$$

So we need to get ρ_c and substitute it to let the form of P_c become Eq.2. From Eq.8, we have the relation between mass and density : $M = -4\pi r_n^3 \rho_c \xi_1^2 \theta_n'$ and can get ρ_c:

$$\rho_c = -\frac{M}{4\pi r_n^3 \xi_1^2 (\theta_n')_{\xi_1}} \tag{10}$$

After substituting, we can get the form of Eq.2:

$$P_c = \frac{4\pi G R^2}{(n+1)\xi_1^2}\rho_c^{1-\frac{1}{n}}\rho_c^{1+\frac{1}{n}} = \frac{4\pi G R^2}{(n+1)\xi_1^2}(\frac{M}{4\pi r_n^3 \xi_1^2 (\theta_n')_{\xi_1}})^2 = \frac{G/4\pi}{(n+1)(\theta_n')^2_{\xi_1}} M^2 R^{-4}$$

Ideal gas EoS:

$$P = \frac{\rho N_A k T}{\mu} \tag{11}$$

we can write:

$$T_c = K\rho_c^{\frac{1}{n}}(\frac{\mu}{N_A k}) \tag{12}$$

and substitute ρ_c(Eq.10) and K(Eq.9) we have get before, we can get the form of Eq.3:

$$T_c = \frac{4\pi G R^2}{(n+1)\xi_1^2}\rho_c^{1-\frac{1}{n}}\rho_c^{\frac{1}{n}}(\frac{\mu}{N_A k}) = \frac{4\pi G R^2}{(n+1)\xi_1^2}(-\frac{M}{4\pi r_n^3 \xi_1^2 (\theta_n')_{\xi_1}})(\frac{\mu}{N_A k})$$
$$\Rightarrow T_c = -\frac{G/N_A k}{(n+1)(\xi\theta_n')_{\xi_1}} \mu M R^{-1}$$

Q2 : Stellar structure - main-sequence stars.

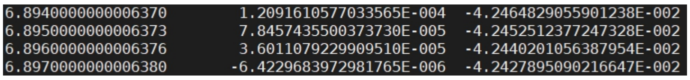
Now consider a main-sequence star: $1M_\odot = 1.989E30(kg)$, $1R_\odot = 6.9634E8(m)$ with polytrope n=3.

Calculate the central density ρ_c, central pressure P_c, and central temperature T_c. They can be gotten by the equation we have derived before, which is Eq.10, Eq.2 and Eq.3. Noted that Eq.10 can be substitute by the dimensionless radial coordinate we have defined: $R = r_n \xi_1$.

central density ρ_c

$$\rho_c = -\frac{M\xi_1}{4\pi R^3 (\theta_n')_{\xi_1}}$$

I use the code we build on class to calculate the value of $\xi_1 = 6.896$, $\theta_n' = -0.04243$ (from Fig.1)



6.8940000000006370	1.2091610577033565E-004	-4.2464829055901238E-002
6.8950000000006373	7.8457435500373730E-005	-4.2452512377247328E-002
6.8960000000006376	3.6011079229909510E-005	-4.2440201056387954E-002
6.8970000000006380	-6.4229683972981765E-006	-4.2427895090216647E-002

Figure 1: calculate the value of ξ_1, θ_n'.

$$\rho_c = -\frac{1.989 \times 10^{30} \times 6.896}{4\pi \times (6.9634 \times 10^8)^3 \times -0.04243} = 7.619 \times 10^4$$

$$P_c = \frac{8.952 \times 10^{14}}{(3+1)(-0.04243)^2} = 1.243 \times 10^{17}[dyne\ cm^{-2}] = 1.243 \times 10^{11}[bar]$$

And from the Sun Fact Sheet provided by the NASA [1], $\rho_c = 1.622 \times 10^5$, $P_c = 2.477 \times 10^{11} bar$, which say that we calculate almost half of the truth value of density and pressure.

For Eq.3, we should first know how to calculate the mean molecular weight μ [2]

First we consider the number density of particles,

$$n = \frac{\rho}{\mu m} \Rightarrow \frac{1}{\mu} = \frac{mn}{\rho}$$

Then this star has a uniform composition X=0.75(H), Y=0.25(He).

$$X = \frac{m_H n_H}{\rho}, Y = \frac{m_{He} n_{He}}{\rho}$$

Supposed that the star is fully ionized, so there will be three types of particles in this stars: electron, hydrogen, helium ($n = n_{e^-} + n_H + n_{He}$)

$$\frac{1}{\mu} = \frac{m_H n_e}{\rho} + \frac{m_H n_H}{\rho} + \frac{m_H n_{He}}{\rho}$$

We know that 1 hydrogen provide 1 electron, 1 helium provide 2 electron; 1 hydrogen has 1 amu, 1 helium has 4 amu.

$$n_{e^-} = n_H + 2n_{He} \ \mid \ m_{He} = 4m_H$$

$$\frac{1}{\mu} = (X + \frac{1}{2}Y) + (X) + (\frac{1}{4}Y) = 2X + \frac{3}{4}Y$$

Finally, we substitute X=0.75 Y=0.25, and get $\mu = 0.59259$

$$T_c = \frac{2.293 \times 10^7}{(3+1)(-6.896(-0.04243))} \times 0.59259 = 1.161 \times 10^7[K]$$

This value is closed to the value from the Sun Fact Sheet: $T_c = 1.571 \times 10^7[K]$

2 Programming Assignments

Q1 : Tolman-Oppenheimer-Volkoff (TOV).

$$\frac{dP}{dr} = -\frac{GM}{r^2}(\rho + \frac{P}{c^2})(1 + \frac{4\pi r^3 P}{Mc^2})(1 - \frac{2GM}{rc^2})^{-1} \tag{13}$$

Q1.a. : mass density with radius

Given $P = K\rho^2$, with $K = 1.455 \times 10^5$ and $\gamma = 2$. So Eq.13 can be written in the form which relates to ρ:

$$\frac{d\rho}{dr} = -\frac{GM}{2Kr^2}(1 + \frac{K}{c^2}\rho)(1 + \frac{4\pi K r^3 \rho^2}{Mc^2})(1 - \frac{2GM}{rc^2})^{-1} \tag{14}$$

and use PK4 algorithm we also need to derive the second order equation:

$$\frac{d^2\rho}{dr^2} = \frac{d}{dr}(-\frac{G}{2K}(Mr^{-2} + \frac{K}{c^2}M\rho r^{-2} + \frac{4\pi K}{c^2}r\rho^2 + \frac{4\pi K^2}{c^4}r\rho^3)(1 - \frac{2GM}{rc^2})^{-1}) \tag{15}$$

We need to notice that mass M(r) is a variable which will change by different radius r:

$$dM_r = 4\pi r^2 \rho d$$

so there are 3 variables will do differential: r, $\rho(r)$, $M(r)$

$$\begin{aligned}
\rho'' = -\frac{G}{2K}(((M'r^{-2} - 2Mr^{-3}) + \frac{K}{c^2}M\rho r^{-2} + \frac{4\pi K}{c^2}r\rho^2 + \frac{4\pi K^2}{c^4}r\rho^3) \\
+(Mr^{-2} + \frac{K}{c^2}(M'\rho r^{-2} + M\rho' r^{-2} - 2M\rho r^{-3}) + \frac{4\pi K}{c^2}r\rho^2 + \frac{4\pi K^2}{c^4}r\rho^3) \\
+(Mr^{-2} + \frac{K}{c^2}M\rho r^{-2} + \frac{4\pi K}{c^2}(\rho^2 + 2r\rho\rho') + \frac{4\pi K^2}{c^4}r\rho^3) \\
+(Mr^{-2} + \frac{K}{c^2}M\rho r^{-2} + \frac{4\pi K}{c^2}r\rho^2 + \frac{4\pi K^2}{c^4}(\rho^3 + 3r\rho^2\rho')))(1 - \frac{2GM}{rc^2})^{-1} \\
\times -\frac{G}{2K}(Mr^{-2} + \frac{K}{c^2}M\rho r^{-2} + \frac{4\pi K}{c^2}r\rho^2 + \frac{4\pi K^2}{c^4}r\rho^3)(1 - \frac{2GM}{rc^2})^{-2})(\frac{2G}{c^2})(M'r^{-1} - Mr^{-2})
\end{aligned}$$

I have tried my best but the result of code is strange and doesn't have the property of the neutral star...

(a) one case of trying (b) another case of trying

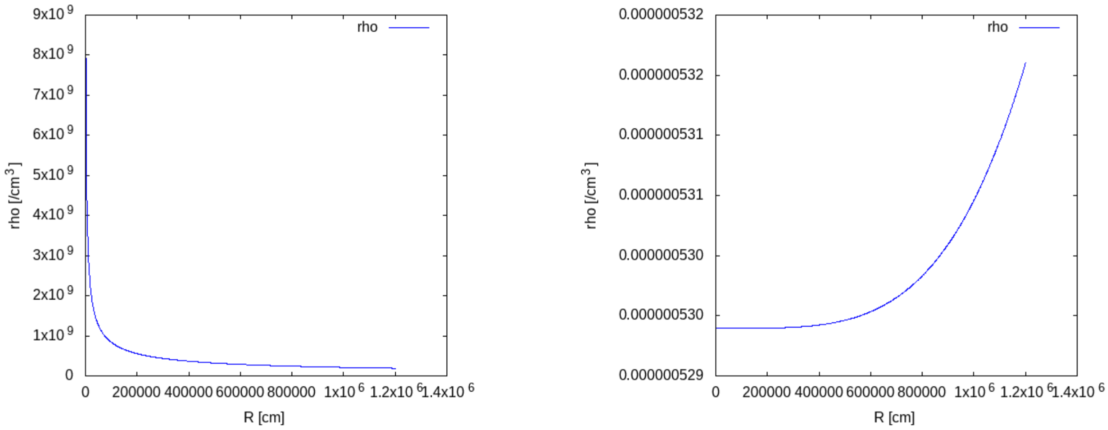


Figure 2: case of trying

Q1.b. : mass with radius

Because I stuck in question 1, so didn't accomplish this part.

References

[1] Sun Fact Sheet from NASA (https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html).

[2] Calculate the mean molecular weight μ https://web.njit.edu/ gary/321/Lecture7.html