Assignment 2 of Computational Astrophysics in NTHU

Wei-Hsiang Yu 游惟翔

March 18, 2021

1 Written Assignments

Q1 : Derive the Kepler's Law

Kepler's Second Law

$$A = \frac{1}{2}\frac{L}{\mu}P \tag{1}$$

Where $A = \pi ab$ is the area of ellipse, L is angular momentum, μ is reduced mass, P is orbital period.
$< pf >$:
In polar coordinates,

$$\vec{r} = r\hat{r} = r(\hat{i}\cos\theta + \hat{j}\sin\theta) \tag{2}$$

Derivatives Eq.2. with respect to time t,

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}(\hat{i}\cos\theta + \hat{j}\sin\theta) + r(-\hat{i}\sin\theta\frac{\theta}{dt} + \hat{j}\cos\theta\frac{\theta}{dt}) = \dot{r}\hat{r} + r\omega\hat{\theta} \tag{3}$$

So the area swept by $\vec{r}$ during time dt is (Fig..) :

$$dA = \frac{1}{2}\left|\vec{r}\times\vec{v}\right| = \frac{1}{2}\left|r\hat{r}\times(\dot{r}\hat{r} + r\omega\hat{\theta})\right| = \frac{1}{2}r^2\omega dt \tag{4}$$

and we eliminate Eq.4. to get :

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega$$

Recall that $L = \vec{r}\times\vec{p} = \mu r^2\omega$, so we substitute and get Kepler's Second Law
(Here, angular momentum is produced by m_1 & m_2 relative to center of mass, so we use reduced mass μ)

$$A = \frac{1}{2}\frac{L}{\mu}P$$

Kepler's Third Law

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)}a^3 \tag{5}$$

Where a is the binary separation, and m_1 and m_2 are masses.
$< pf >$:
By center of mass, we can say that :

$$m_1a_1 = m_2a_2$$

and we can make $a_1 = \frac{m_2a_2}{m_1}$,so we will have Eq.6.

$$a = (\frac{m_2+m_1}{m_1})a_2 \tag{6}$$

We can see Fig.., m_2 has two force act on it : gravity & centripetal acceleration. They should be equal!

$$F_{gravity} = \frac{Gm_1m_2}{a^2} = F_{centripetal} = m_2a_2\omega^2 \tag{7}$$

Then, we substitute the binary separation a(Eq.6.) to Eq.7. and replace $\omega = \frac{2\pi}{P}$

$$\frac{Gm_1m_2}{a^2} = \frac{m_1m_2}{m_1+m_2}a(\frac{2\pi}{P})^2$$

After eliminating, we can get Kepler's Third Law

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)}a^3$$

2 Programming Assignments

Q1 : Sun-Earth system.

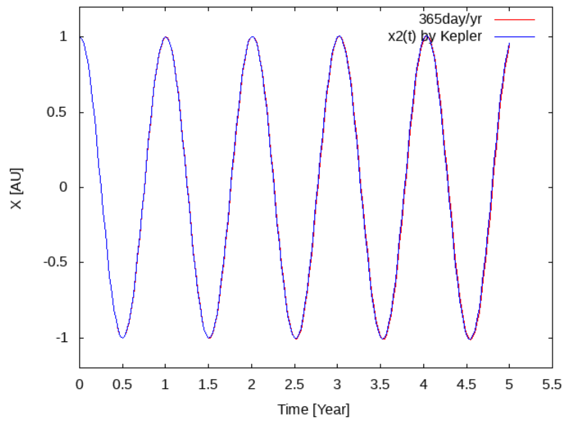


Figure 1: The trajectory of m_1 m_2 (when m_2 has a 1.25 factor of velocity).

The blue line is my computed result used by Kepler's Third Law. The red line uses 365 days as 1 year to run program. In Fig.1., we can say that it's not very obvious different by using two different period. But we can see the line's trough to the x axis, the fifth trough doesn't match in time=4.5yr as the first trough does in time=0.5yr.

Q2 : Consider more & some different perturbation

2a.

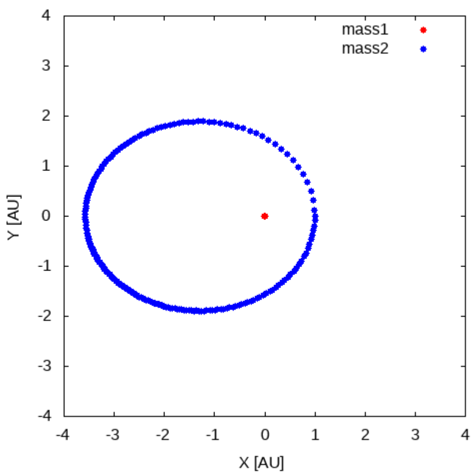


Figure 2: The trajectory of m_1 m_2 (when m_2 has a 1.25 factor of velocity).

2b.Perihelion and Aphelion

As the question say, we know that:

$$r_p = a(1-e) = 1au \tag{8}$$

r_p , the position of the perihelion, will be $1au$, which is the initial condition we give to the program.

Later, we need to find the position of mass2 at the aphelion in order to acquire "a" in Eq.8. We find the file[binary002.dat] to get the the closet two position to aphelion:

$$x_2 = -0.534935585707E + 14 \qquad y_2 = -0.211156452869E + 12$$

$$x_1 = -0.534872149361E + 14 \qquad y_1 = 0.609624905270E + 12$$

So after using interpolation, we can get the position of aphelion (at y=0) is $-0.5349192658994514E+14$, and substitute it to Eq.8. and get the eccentricity $e = 0.562909......$

$$Velocity\ at\ Perihelion \qquad v^2 = \frac{GM_\odot}{a}\frac{1+e}{1-e} \tag{9}$$

$$Velocity\ at\ Aphelion \qquad v^2 = \frac{GM_\odot}{a}\frac{1-e}{1+e} \tag{10}$$

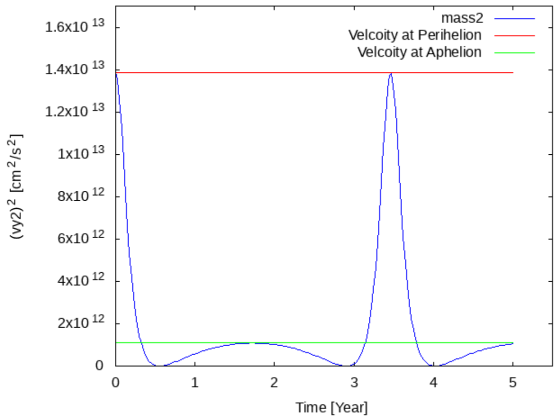


Figure 3: ${v_{y,2}}^2(t)$ and the velocity at Perihelion and Aphelion.

We can see that in Fig.3. star will move rapidly when it is closed to Perihelion (blue line is sharp) and will move slowly when it is closed to Aphelion (blue line become smooth).

2c.Conservation of $L(t)$ and $E(t)$

We know that the angular momentum is

$$L = \vec{r} \times \vec{p} \tag{11}$$

and in our program we use orthogonal coordinates to analyze binary system.

So we already have $\vec{r_x}$,$\vec{r_y}$,$\vec{v_x}$,$\vec{v_x}$ and substitute into Eq.11.

$$L = (\vec{r_x} + \vec{r_y}) \times m(\vec{v_x} + \vec{v_x}) = m(r_x v_y - r_y v_x)$$

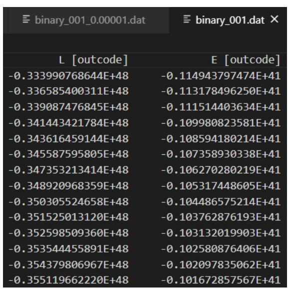
About energy, there is a equation [1] can calculate the total energy in the binary system,

$$E = K + U = \frac{1}{2}m\frac{GMm}{L}^2(e^2 - 1) \tag{12}$$

So I put this two equation into [output.f90] and modify some format of the output file. Remember that Eq.11. only get mass1 or mass2 independently, the total angular momentum will be $L_{m1} + L_{m2}$. However, Eq.12. represent the whole system.

The result is there are both "closed" to conservation! There are closed because the order of the value are all the same, but still have error. Furthermore, the error will also pass to the next step, and a larger time step will has larger error in the same step.(see Fig.4.)

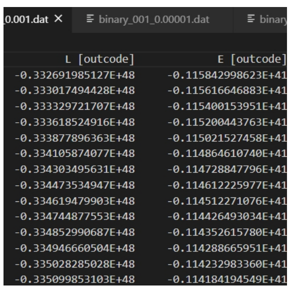
(a) dt=0.01yr



binary_001_0.00001.dat | binary_001.dat

L [outcode]	E [outcode]
-0.333990768644E+48	-0.114943797474E+41
-0.336585400311E+48	-0.113178496250E+41
-0.339087476845E+48	-0.111514403634E+41
-0.341443421784E+48	-0.109980823581E+41
-0.343616459144E+48	-0.108594180214E+41
-0.345587595805E+48	-0.107358930338E+41
-0.347353213414E+48	-0.106270280219E+41
-0.348920968359E+48	-0.105317448605E+41
-0.350305524658E+48	-0.104486575214E+41
-0.351525013120E+48	-0.103762876193E+41
-0.352598509360E+48	-0.103132019903E+41
-0.353544455891E+48	-0.102580876406E+41
-0.354379806967E+48	-0.102097835062E+41
-0.355119662220E+48	-0.101672857567E+41

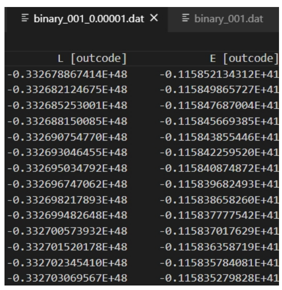
(b) dt=0.001yr



0.001.dat | binary_001_0.00001.dat | binary

L [outcode]	E [outcode]
-0.332691985127E+48	-0.115842998623E+41
-0.333017494428E+48	-0.115616646883E+41
-0.333329721707E+48	-0.115400153951E+41
-0.333618524916E+48	-0.115200443763E+41
-0.333877896363E+48	-0.115021527458E+41
-0.334105874077E+48	-0.114864610740E+41
-0.334303495631E+48	-0.114728847796E+41
-0.334473534947E+48	-0.114612225977E+41
-0.334619479903E+48	-0.114512271076E+41
-0.334744877553E+48	-0.114426493034E+41
-0.334852990687E+48	-0.114352615780E+41
-0.334946660504E+48	-0.114288665951E+41
-0.335028285028E+48	-0.114232983360E+41
-0.335099853103E+48	-0.114184194549E+41

(c) dt=0.00001yr



binary_001_0.00001.dat | binary_001.dat

L [outcode]	E [outcode]
-0.332678867414E+48	-0.115852134312E+41
-0.332682124675E+48	-0.115849865727E+41
-0.332685253001E+48	-0.115847687004E+41
-0.332688150085E+48	-0.115845669385E+41
-0.332690754770E+48	-0.115843855446E+41
-0.332693046455E+48	-0.115842259520E+41
-0.332695034792E+48	-0.115840874872E+41
-0.332696747062E+48	-0.115839682493E+41
-0.332698217893E+48	-0.115838658260E+41
-0.332699482648E+48	-0.115837777542E+41
-0.332700573932E+48	-0.115837017629E+41
-0.332701520178E+48	-0.115836358719E+41
-0.332702345410E+48	-0.115835784081E+41
-0.332703069567E+48	-0.115835279828E+41

Figure 4: L & E in 0.01,0.001,0.00001 time step

[1]the total energy in the binary system, page24-28 : https://web.njit.edu/ cao/hw4.pdf

Q3 : Three Body problem

3a./3b.Using ordinary/RK2 method to handle 3-body problem

In this part, I modify the code in the update() subroutine in the [physics.f90]. The most important thing is that there are 3 body in the system, so the relations we need to consider will be 3! = 6 probabilities.

There are position $x\&y$, square of distance rsq, angle, force, force in x & y direction. **All these variables need to consider 3 relations(1-2,1-3,2-3)**, not 6 because there will be 3 pairs of each variable, they will be the same value but one is positive and one is negative.

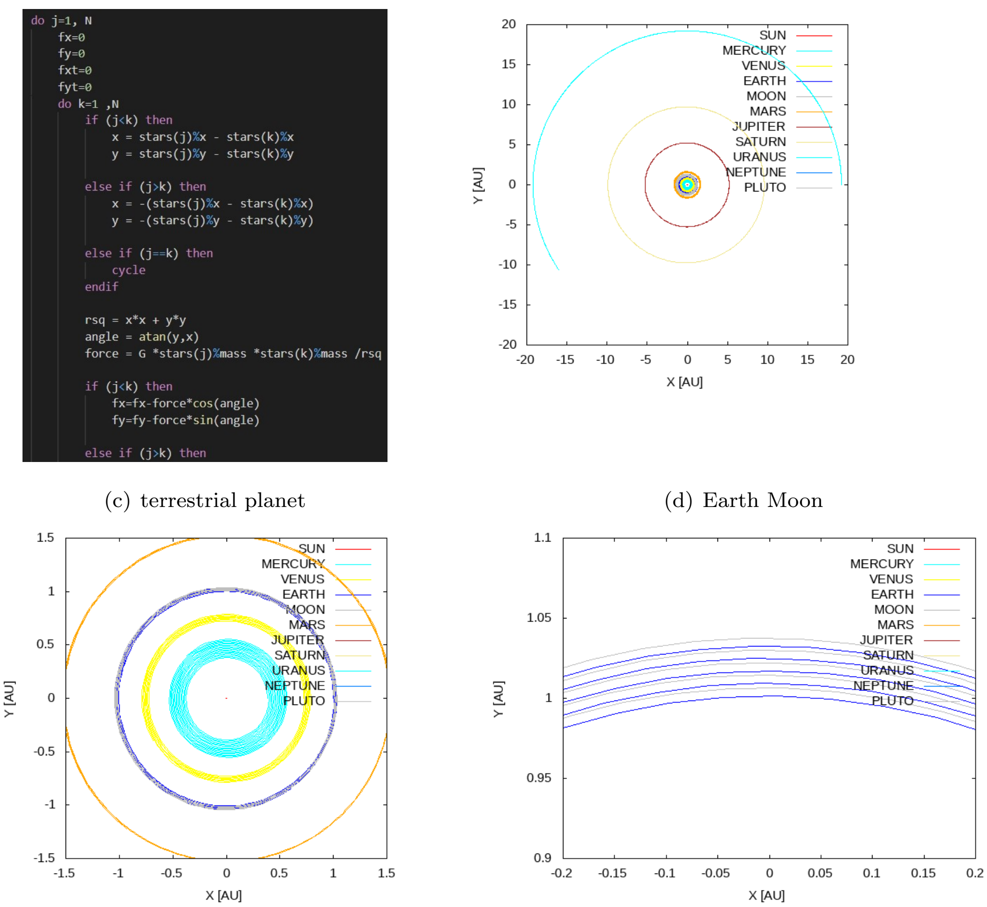
We can see that Fig.6.(a) & (b) show the diverge result, the stars finally go away from the triple system in three different direction (coincidentally their angle seem to be 120°) In Fig.6.(e) & (f), the results start to converge and do **circular motion**, with a binary system inside the triple system and the other star rotation to binary system's center of mass.

I think why this system do circular motion is because : 1. this problem only consider circular obit(eccentricity=0) 2. the initial position of mass3 is far enough to let mass not suffer from too strong gravity force which come from mass1 & mass2.

In Fig.6.(c) & (d) & (g) & (h), there are the results of RK2 method. In program, I set the initial value of each temporary variable to become 0. We can see the reult of RK2 is a little bit converge in 0.1 & 0.01 yr time step.(At least in Fig.6.(f) the stars rotate back to the center at the end, which is compared to Fig.6.(b))

Q4 : Solar System

In this part, I make the update() subroutine in the [physics.f90] have 2 do loops(Fig.5.(a)), the loop can repeat calculate the distance of each star (using $1 <= j <= N$ & $1 <= k <= N$ and simultaneously $j \neq k$)



(a) do loop (b) without PLUTO

(c) terrestrial planet (d) Earth Moon

Figure 5: The trajectory of solar system

But in this simulation, the motion of moon was false...(Fig.5.(d)). I supposed that may I neglect the velocity generated by binary system in setting the initial condition. And MERCURY orbit become larger, I think it cause by the angular momentum lost from doing calculating (discuss in Q2.(c) the conservation of angular momentum and energy)

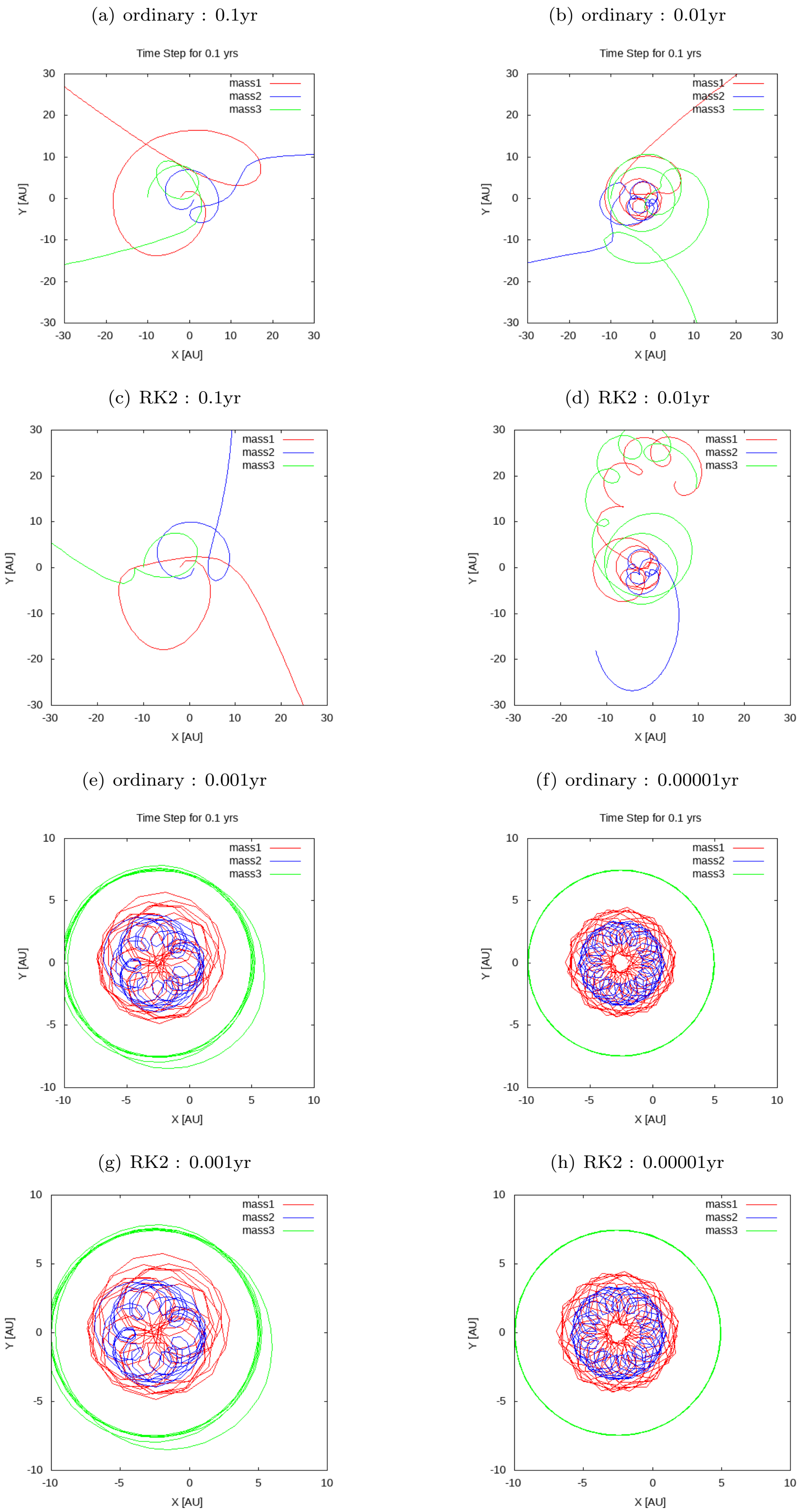


Figure 6: Three Body simulation [ordinary method] in 0.1, 0.01, 0.001, 0.00001 yr time step
. Three Body simulation [RK2] in 0.1, 0.01, 0.001, 0.00001 yr time step