Neural Networks Derivatives

Wendrel Moraes

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Artificial Neural Network

$$C = \sum_{i=0} (a_i^L\text{-}y_i)^2$$

$$a_i^l = \sigma\left(z_i^l\right)$$

$$z_i^l = \sum_{j=0} W_{ij}^{l\text{-}1} \cdot a_j^{l\text{-}1} + B_i^l$$

$$
\begin{aligned}
\sigma(x) &= \frac{1}{1+e^{-x}} \\
\sigma'(x) &= \sigma(x)(1-\sigma(x)) \\
\frac{\partial C}{\partial W^1_{00}} &= \frac{\partial C}{\partial a^2_0}\frac{\partial a^2_0}{\partial z^2_0}\frac{\partial z^2_0}{\partial W^1_{00}} \\
&= 2(a^2_0 - y_0)\sigma'(z^2_0)(a^1_0) \\
\frac{\partial C}{\partial B^2_1} &= \frac{\partial C}{\partial a^2_1}\frac{\partial a^2_1}{\partial z^2_1}\frac{\partial z^2_1}{\partial B^2_1} \\
&= 2(a^2_1 - y_1)\sigma'(z^2_1)(1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C}{\partial W^0_{11}} &= \frac{\partial C}{\partial a^2_0}\frac{\partial a^2_0}{\partial z^2_0}\frac{\partial z^2_0}{\partial a^1_1}\frac{\partial a^1_1}{\partial z^1_1}\frac{\partial z^1_1}{\partial W^0_{11}} \\
&+ \frac{\partial C}{\partial a^2_1}\frac{\partial a^2_1}{\partial z^2_1}\frac{\partial z^2_1}{\partial a^1_1}\frac{\partial a^1_1}{\partial z^1_1}\frac{\partial z^1_1}{\partial W^0_{11}} \\
&= \left(\frac{\partial a^1_1}{\partial z^1_1}\frac{\partial z^1_1}{\partial W^0_{11}}\right)\left(\frac{\partial C}{\partial a^2_0}\frac{\partial a^2_0}{\partial z^2_0}\frac{\partial z^2_0}{\partial a^1_1} + \frac{\partial C}{\partial a^2_1}\frac{\partial a^2_1}{\partial z^2_1}\frac{\partial z^2_1}{\partial a^1_1}\right) \\
&= \sigma'(z^1_1)(a^0_1)\left(2(a^2_0 - y_0)\sigma'(z^2_0)W^1_{01} + 2(a^2_1 - y_1)\sigma'(z^2_1)W^1_{11}\right) \\
\frac{\partial C}{\partial W^0_{11}} &= \frac{\partial a^1_1}{\partial W^0_{11}} \cdot \frac{\partial C}{\partial a^1_1} \\
\frac{\partial C}{\partial a^1_1} &= \sum_{j=0}^{1} 2(a^2_j - y_j)\sigma'(z^2_j)W^1_{j1} \\
\frac{\partial C}{\partial W^0_{11}} &= \sigma'(z^1_1)(a^0_1)\frac{\partial C}{\partial a^1_1} \\
\frac{\partial C}{\partial B^1_1} &= \sigma'(z^1_1)(1)\frac{\partial C}{\partial a^1_1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C}{\partial a^l_i} &= \sum_{j=0}^{n^{l+1}} \sigma'(z^{l+1}_j)W^l_{ji}\frac{\partial C}{\partial a^{l+1}_j} \\
\frac{\partial C}{\partial W^l_{ji}} &= \sigma'(z^{l+1}_j)(a^l_i)\frac{\partial C}{\partial a^{l+1}_j} \\
\frac{\partial C}{\partial B^l_i} &= \sigma'(z^l_i)(1)\frac{\partial C}{\partial a^l_i}
\end{aligned}
$$

Gated Recurrent Neuron

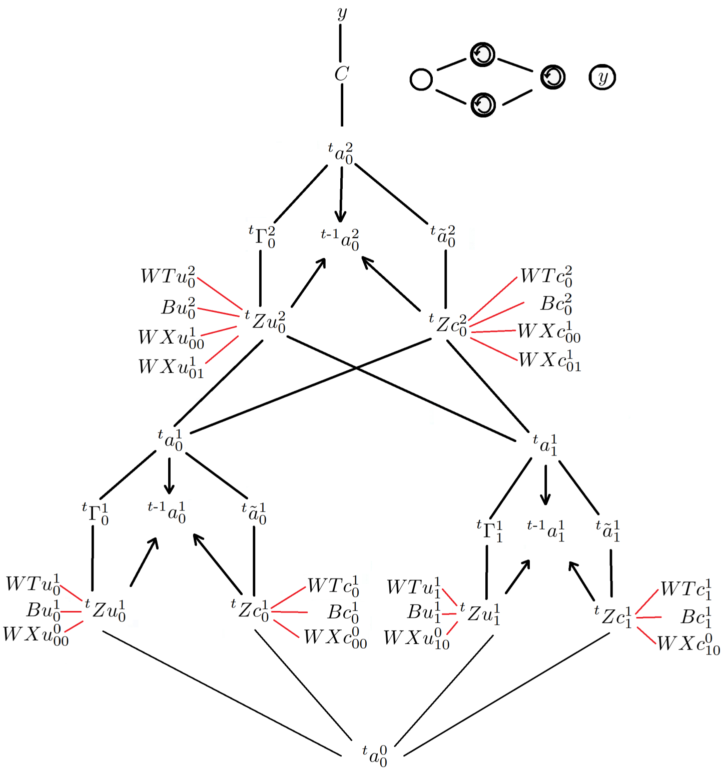
$$^{t}a_i^l = {}^{t}\Gamma_i^l \cdot {}^{t}\tilde{a}_i^l + \left(1\text{-}{}^{t}\Gamma_i^l\right) {}^{t\text{-}1}a_i^l$$

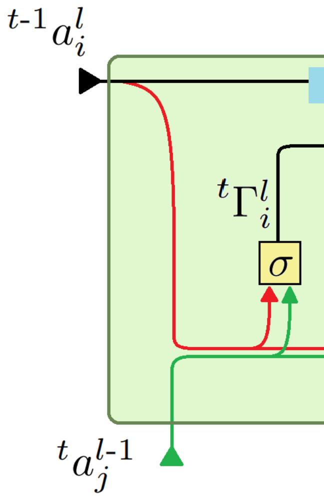
$$^{t}\Gamma_i^l = \sigma\left({}^{t}Zu_i^l\right)$$

$$^{t}\tilde{a}_i^l = \tanh\left({}^{t}Zc_i^l\right)$$

$$^{t}Zu_i^l = WTu_i^l \cdot {}^{t\text{-}1}a_i^l + \sum_{j=0} WXu_{ij}^{l\text{-}1} \cdot {}^{t}a_j^{l\text{-}1} + Bu_i^l$$

$$^{t}Zc_i^l = WTc_i^l \cdot {}^{t\text{-}1}a_i^l + \sum_{j=0} WXc_{ij}^{l\text{-}1} \cdot {}^{t}a_j^{l\text{-}1} + Bc_i^l$$





GRU Derivatives

$$\frac{\partial C}{\partial WTc_0^2} = \frac{\partial C}{\partial^t a_0^2}\frac{\partial^t a_0^2}{\partial^t ac_0^2}\frac{\partial^t ac_0^2}{\partial^t Zc_0^2}\frac{\partial^t Zc_0^2}{\partial WTc_0^2}$$

$$= \underbrace{2({}^t a_0^2 - y_0)}\cdot {}^t\Gamma_0^2 \cdot tanh'({}^t Zc_0^2)\cdot {}^{t\text{-}1}a_0^2$$

$$= \frac{\partial C}{\partial^t a_0^2}\cdot {}^t\Gamma_0^2 \cdot tanh'({}^t Zc_0^2)\cdot {}^{t\text{-}1}a_0^2$$

$$\frac{\partial C}{\partial WTu_0^2} = \frac{\partial C}{\partial^t a_0^2}\frac{\partial^t a_0^2}{\partial^t \Gamma_0^2}\frac{\partial^t \Gamma_0^2}{\partial^t Zu_0^2}\frac{\partial^t Zu_0^2}{\partial WTu_0^2}$$

$$= \frac{\partial C}{\partial^t a_0^2}\cdot ({}^t ac_0^2 - {}^{t\text{-}1}a_0^2)\cdot \sigma'({}^t Zu_0^2)\cdot {}^{t\text{-}1}a_0^2$$

$$\frac{\partial C}{\partial Bc_0^2} = \frac{\partial C}{\partial^t a_0^2}\frac{\partial^t a_0^2}{\partial^t ac_0^2}\frac{\partial^t ac_0^2}{\partial^t Zc_0^2}\frac{\partial^t Zc_0^2}{\partial Bc_0^2}$$

$$= \frac{\partial C}{\partial^t a_0^2}\cdot {}^t\Gamma_0^2 \cdot tanh'({}^t Zc_0^2)\cdot (1)$$

$$\frac{\partial C}{\partial Bu_0^2} = \frac{\partial C}{\partial^t a_0^2}\frac{\partial^t a_0^2}{\partial^t \Gamma_0^2}\frac{\partial^t \Gamma_0^2}{\partial^t Zu_0^2}\frac{\partial^t Zu_0^2}{\partial Bu_0^2}$$

$$= \frac{\partial C}{\partial^t a_0^2}\cdot ({}^t ac_0^2 - {}^{t\text{-}1}a_0^2)\cdot \sigma'({}^t Zu_0^2)\cdot (1)$$

$$\frac{\partial C}{\partial WXu_{ji}^{l\text{-}1}} = \frac{\partial C}{\partial^t a_j^l}\cdot ({}^t ac_j^l - {}^{t\text{-}1}a_j^l)\cdot \sigma'({}^t Zu_j^l)\cdot {}^t a_i^{l\text{-}1}$$

$$\frac{\partial C}{\partial WXc_{ji}^{l\text{-}1}} = \frac{\partial C}{\partial^t a_j^l}\cdot {}^t\Gamma_j^l \cdot tanh'({}^t Zc_j^l)\cdot {}^t a_i^{l\text{-}1}$$

$$\frac{\partial C}{\partial^t a_i^{l\text{-}1}} = \sum_{j=0}^{n^l}\left(\frac{\partial C}{\partial^t a_j^l}\frac{\partial^t a_j^l}{\partial^t ac_j^l}\frac{\partial^t ac_j^l}{\partial^t Zc_j^l}\frac{\partial^t Zc_j^l}{\partial^t a_i^{l\text{-}1}} + \frac{\partial C}{\partial^t a_j^l}\frac{\partial^t a_j^l}{\partial^t \Gamma_j^l}\frac{\partial^t \Gamma_j^l}{\partial^t Zu_j^l}\frac{\partial^t Zu_j^l}{\partial^t a_i^{l\text{-}1}}\right)$$

$$\frac{\partial C}{\partial^t a_i^{l\text{-}1}} = \frac{\partial C}{\partial^t a_j^l}\sum_{j=0}^{n^l}({}^t\Gamma_j^l \cdot tanh'({}^t Zc_j^l)\cdot WXc_{ji}^{l\text{-}1}$$
$$+({}^t ac_j^l - {}^{t\text{-}1}a_j^l)\cdot \sigma'({}^t Zu_j^l)\cdot WXu_{ji}^{l\text{-}1})$$

$$\frac{\partial C}{\partial^{t\text{-}1} a_j^l} = \frac{\partial C}{\partial^t a_j^l}\left(\frac{\partial^t a_j^l}{\partial^{t\text{-}1} a_j^l} + \frac{\partial^t a_j^l}{\partial^t ac_j^l}\frac{\partial^t ac_j^l}{\partial^t Zc_j^l}\frac{\partial^t Zc_j^l}{\partial^{t\text{-}1} a_j^l} + \frac{\partial^t a_j^l}{\partial^t \Gamma_j^l}\frac{\partial^t \Gamma_j^l}{\partial^t Zu_j^l}\frac{\partial^t Zu_j^l}{\partial^{t\text{-}1} a_j^l}\right)$$

$$\frac{\partial C}{\partial^{t\text{-}1} a_j^l} = \frac{\partial C}{\partial^t a_j^l}\left((1 - {}^t\Gamma_j^l) + {}^t\Gamma_j^l \cdot tanh'({}^t Zc_j^l)\cdot WTc_j^l\right)$$
$$+ ({}^t ac_j^l - {}^{t\text{-}1}a_j^l)\cdot \sigma'({}^t Zu_j^l)\cdot WTu_j^l$$

$$\frac{\partial C}{\partial Bc_j^l} = \frac{\partial C}{\partial^t a_j^l} \cdot {}^t\Gamma_j^l \cdot tanh'({}^tZc_j^l) \cdot (1)$$

$$\frac{\partial C}{\partial Bu_j^l} = \frac{\partial C}{\partial^t a_j^l} \cdot ({}^tac_j^l - {}^{t\text{-}1}a_j^l) \cdot \sigma'({}^tZu_j^l) \cdot (1)$$

$$\frac{\partial C}{\partial WTc_j^l} = \frac{\partial C}{\partial^t a_j^l} \cdot {}^t\Gamma_j^l \cdot tanh'({}^tZc_j^l) \cdot {}^{t\text{-}1}a_j^l$$

$$\frac{\partial C}{\partial WTu_j^l} = \frac{\partial C}{\partial^t a_j^l} \cdot ({}^tac_j^l - {}^{t\text{-}1}a_j^l) \cdot \sigma'({}^tZu_j^l) \cdot {}^{t\text{-}1}a_j^l$$

$$\frac{\partial C}{\partial^t a_i^{l\text{-}1}} = \frac{\partial C}{\partial^t a_j^l} \sum_{j=0}^{n^l} ({}^t\Gamma_j^l \cdot tanh'({}^tZc_j^l) \cdot WXc_{ji}^{l\text{-}1}$$
$$+({}^tac_j^l - {}^{t\text{-}1}a_j^l) \cdot \sigma'({}^tZu_j^l) \cdot WXu_{ji}^{l\text{-}1})$$

$$\frac{\partial C}{\partial^{t\text{-}1} a_j^l} = \frac{\partial C}{\partial^t a_j^l}((1 - {}^t\Gamma_j^l) + {}^t\Gamma_j^l \cdot tanh'({}^tZc_j^l) \cdot WTc_j^l)$$
$$+ ({}^tac_j^l - {}^{t\text{-}1}a_j^l) \cdot \sigma'({}^tZu_j^l) \cdot WTu_j^l)$$