# Phys 129L, Section SA

1.
$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2) \quad ; \hbar = 1$$

a.
$$U_1 = \sqrt{\frac{p_x^2}{2m}},\ U_2 = \sqrt{\frac{p_y^2}{2m}},\ V_1 = \sqrt{\frac{m\omega^2 x^2}{2}},\ \sqrt{V_2 = \frac{m\omega^2 y^2}{2}}$$

$$\rightarrow H = U_1^2 + U_2^2 + V_1^2 + V_2^2 = E$$

4-D hypersphere: radius $r = \sqrt{E}$

$$\rightarrow V = \frac{\pi^2}{2} r^4 = \frac{\pi^2 E^2}{2}$$

$$(U_1, U_2, V_1, V_2) \rightarrow \quad J = \begin{pmatrix} \partial V_{1x} & \partial V_{1y} & \partial V_{1p_x} & \partial V_{1p_y} \\ \partial V_{2x} & \partial V_{2y} & \partial V_{2p_x} & \partial V_{2p_y} \\ \partial U_{1x} & \partial U_{1y} & \partial U_{1p_x} & \partial U_{1p_y} \\ \partial U_{2x} & \partial U_{2y} & \partial U_{2p_x} & \partial U_{1p_y} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{m\omega^2}{2}} & & 0 & \\ & \sqrt{\frac{m\omega^2}{2}} & & \\ 0 & & \sqrt{\frac{1}{2m}} & \\ & & & \sqrt{\frac{1}{2m}} \end{pmatrix} \rightarrow |J| = \left(\frac{m\omega^2}{2}\right)\left(\frac{1}{2m}\right) = \frac{\omega^2}{4}$$

$$\rightarrow \Omega(E) = |J^{-1}| V = \frac{4}{\omega^2}\frac{\pi^2 E^2}{2} = 2\frac{\pi^2 E^2}{\omega^2}$$

$$\rightarrow g(E) = \frac{d\Omega(E)}{dE} = \boxed{\frac{4\pi^2 E}{\omega^2}}$$

1\. b. $Z(\beta) = \int_0^\infty g(E) e^{-\beta E} dE$

$= \frac{4\pi^2}{\omega^2} \int_0^\infty E e^{-\beta E} dE$

| D | I |
|---|---|
| $E$ + | $e^{-\beta E}$ |
| 1 − | $-\frac{1}{\beta} e^{-\beta E}$ |
| 0 | $\frac{1}{\beta^2} e^{-\beta E}$ |

$= \frac{4\pi^2}{\omega^2}\left(-\frac{E}{\beta} e^{-\beta E} - \frac{1}{\beta^2} e^{-\beta E}\Big|_0^\infty\right)$

$= \frac{4\pi^2}{\omega^2}\left((0-0) + \left(0 + \frac{1}{\beta^2}\right)\right) = \boxed{\left(\frac{2\pi}{\omega\beta}\right)^2}$

c. $H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2) + \lambda(x^2+y^2)^2$

$= \frac{p_x^2+p_y^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \lambda r^4$

$\Omega_p = \int d^2p = \int_0^{p_{max}} 2\pi p\, dp$ , $H \le E$

$\rightarrow \frac{p^2}{2m} \le E - \frac{1}{2}m\omega^2 r^2 - \lambda r^4$

$\rightarrow p_{max} = \left(2m\left(E - \frac{1}{2}m\omega^2 r^2 - \lambda r^4\right)\right)^{1/2}$

$\rightarrow \int_0^{p_{max}} 2\pi p\, dp = 2\pi m\left(E - \frac{1}{2}m\omega^2 r^2 - \lambda r^4\right)$

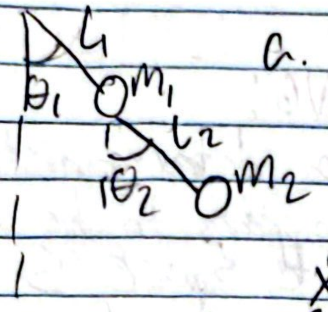
$\Omega(E) = \int \Omega_p d^2x = \int_0^{r_{max}} 2\pi r \cdot \Omega_p \cdot dr = 4\pi^2 m \int_0^{r_{max}} r\left(E - \frac{1}{2}m\omega^2 r^2 - \lambda r^4\right) dr$

$r_{max}$: $E - \frac{1}{2}m\omega^2 r_{max}^2 - \lambda r_{max}^4 = 0$ $\rightarrow$ Quadratic Formula

$\rightarrow r_{max} = \sqrt{\frac{-\frac{1}{2}m\omega^2 + \sqrt{4m^2\omega^4 + 4(E\lambda)}}{+2\lambda}} \propto E^{1/4}$

$\boxed{\Rightarrow \Omega(E) \propto E^{1/2} + E^{3/4} + E^{5/4}}$

Task 2




a. $x_1 = L_1\sin\theta_1$ ; $y_1 = -L_1\cos\theta_1$

$x_2 = x_1 + L_2\sin\theta_2 = L_1\sin\theta_1 + L_2\sin\theta_2$

$y_2 = y_1 - L_2\cos\theta_2 = -L_1\cos\theta_1 - L_2\cos\theta_2$

$\dot{x}_1 = L_1\cos\theta_1\dot{\theta}_1$ , $\dot{y}_1 = L_1\sin\theta_1\dot{\theta}_1$

$\dot{x}_2 = \dot{x}_1 + L_2\cos\theta_2\dot{\theta}_2$

$\dot{y}_2 = \dot{y}_1 + L_2\sin\theta_2\dot{\theta}_2$

$\dot{x}_2 = L_1\cos\theta_1\dot{\theta}_1 + L_2\cos\theta_2\dot{\theta}_2$ ; $\dot{y}_2 = L_1\sin\theta_1\dot{\theta}_1 + L_2\sin\theta_2\dot{\theta}_2$

$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$

$V = -m_1 g L_1\cos\theta_1 - m_2 g(L_1\cos\theta_1 + L_2\cos\theta_2)$

$$\boxed{\mathcal{L} = T - V = \frac{1}{2}(a_1\dot{\theta}_1^2 + 2a_2\dot{\theta}_1\dot{\theta}_2 + a_3\dot{\theta}_2^2) - V(\theta_1, \theta_2)}$$

$p_i = \frac{\partial\mathcal{L}}{\partial\dot{\theta}_i}$ = Generalized momentum

$\rightarrow p_1 = \frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = a_1\dot{\theta}_1 + a_2\dot{\theta}_2$

$\rightarrow p_2 = \frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = a_3\dot{\theta}_2 + a_2\dot{\theta}_1$

as a matrix: $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}^{-1}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$

$$\rightarrow \boxed{H = T + V = p_1\dot{\theta}_1 + p_2\dot{\theta}_2 - \mathcal{L}}$$

$$= \boxed{\frac{1}{2}\begin{bmatrix} p_1 & p_2 \end{bmatrix}\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}^{-1}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + V(\theta_1, \theta_2)}$$