Phys 129L, Section 5B

Task 1: Z_G: $\prod_i \sum_{n_i} e^{-\beta(n_i\varepsilon_i - \mu n_i)}$

$\beta = \frac{1}{k_B T}$ Pauli Exclusion: $\rightarrow \sum_{n_i=0,1} e^{-\beta(n_i \varepsilon_i - \mu n_i)}$

$= 1 + e^{-\beta(\varepsilon_i - \mu)}$

$$\rightarrow \boxed{Z_G = \prod_i \left(1 + e^{-\beta(\varepsilon_i - \mu)}\right)}$$

Task 2: a. n_0 = bosons in ground state
n_ε = bosons in excited state
N = total bosons.

The particles are indistinguishable so the microstates are: $(n_0, n_\varepsilon) = (N,0), (N-1,1) \ldots (0,N)$ which add to $N+1$ microstates.

b. $E_{microstate} = n_\varepsilon \varepsilon$: $Z_C = \sum_{n_\varepsilon=0}^{N} (e^{-\beta\varepsilon})^{n_\varepsilon}$ ← Geom. series. ← probability of each microstate

$$= \boxed{\frac{1 - e^{-(N+1)\beta\varepsilon}}{1 - e^{-\beta\varepsilon}}}$$ ← normalization term

$$P(n_\varepsilon) = \frac{e^{-\beta n_\varepsilon \varepsilon}}{Z_C}$$

c. $\langle n_\varepsilon \rangle_C = \sum_{n_\varepsilon=0}^{N} n_\varepsilon P(n_\varepsilon) = \sum_{n_\varepsilon=0}^{N} n_\varepsilon \frac{e^{-\beta n_\varepsilon \varepsilon}}{Z_C}$

$\langle n_0 \rangle_C = N - \langle n_\varepsilon \rangle_C$

d. $Z_Q = \prod_{j=0} \frac{1}{1 - e^{-\beta\varepsilon_j}} = \frac{1}{(1 - e^{-\beta\varepsilon})}$

e. $\langle n_\varepsilon \rangle_Q = \frac{1}{e^{\beta\varepsilon} - 1}$ $\quad \langle n_0 \rangle_Q = N - \langle n_\varepsilon \rangle_Q$

f. $Z \to \Omega_G = \prod \frac{1}{1-e^{\beta(\mu-\varepsilon_j)}} = \frac{1}{(1-e^{\beta\mu})(1-e^{\beta(\mu-\varepsilon)})}$

Must have $\mu < \varepsilon$

g. $\langle n \rangle = k_B T \frac{\partial}{\partial \mu} \ln(\Omega_G)$ $\quad \ln(\Omega_G) = -\ln(1-e^{\beta\mu}) - \ln(1-e^{\beta(\mu-\varepsilon)})$

$\to \langle n \rangle = \frac{1}{e^{-\beta\mu}-1} + \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$

h. $\langle n_0 \rangle_G = \frac{1}{e^{-\beta\mu}-1}$ $\quad N = 10^5$