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Three-step general discrete-time Zhang neural network design and application to time-variant matrix inversion

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ABSTRACT

In recent decades, neural networks methods have widely applied to many science and engineering fields. Zhang neural network (ZNN) as a special type of recurrent neural networks was proposed by Zhang et al, which has been applied to different time-variant problems solving. ZNNs are usually used to process the continuous-time signal in time-variant systems as continuous-time ZNN (CTZNN) models. Since digit devices and computers are widely applied in science and engineering, it is necessary to develop a general method to discretize CTZNN models to discrete-time ZNN (DTZNN) models. In previous work, Euler forward difference and two types of three-step Zhang et al. discretization (ZeaD) formulas were applied to discretize CTZNN models. In this paper, a three-step general ZeaD formula based on Taylor expansion is designed to approximate the first-order derivative of the target point, and discretize CTZNN models for time-variant matrix inversion. In comparison, the two types of three-step ZeaD formulas are the special cases of the proposed general ZeaD formula. For the situation of the time derivative of objective matrix unknown, two formulas of estimating the derivative are provided, and two other corresponding three-step general DTZNN models are proposed. Theoretical analyses present the stability and convergence of the three general DTZNN models for time-variant matrix inversion. The numerical experiment results substantiate the efficacy and superiority of the three proposed general DTZNN models for time-variant matrix inversion with the theoretical steady-state residual errors, comparing with those of Newton iteration and one-step DTZNN model. In addition, by comparing the numerical results of the two general DTZNN models for the situation of the time derivative of objective matrix unknown with the one of the derivative known, the steady-state residual errors of the formers are slightly bigger than the latter.

1. Introduction

In recent decades, with the remarkable feature such as distributed-storage and high-speed parallel-processing capability, convenience of hardware implementations and excellent performance in large-scale online applications, neural networks and neural-dynamics methods have widely applied to many science and engineering fields [1–8]. Due to the in-depth research in neural networks, the neural-dynamics approach based on recurrent neural networks is one of the important methods for solving time-variant problems [9,10]. In order to guarantee exponential convergence of the model, Zhang neural network (ZNN) as a special type of recurrent neural networks was proposed by Zhang et al., which has been applied to time-variant equation and inequality solving [11], optimization [12–14], and matrix inversion [15,16]. ZNNs are usually used to process the continuous-time signal in time-variant systems as continuous-time ZNN (CTZNN) models. Since digit devices and computers are widely applied in science and engineering, which receive and operate on signals in digital form, it is the main trend that most continuous-time methods will be discretized [17]. For the reasons, it is necessary to develop a general method to discretize CTZNN models to discrete-time ZNN (DTZNN) models for various real-world applications. For example, DTZNN models for time-variant matrix inverse are applied to robot manipulator, such as motion generation, tracking control and optimal control [18–20].

Neural dynamics of CTZNN models is usually presented as differential equations describing how the states change with respect to time. Numerical differentiation is the key to discretization of CTZNN models. Numerical differentiation, which describes methods for estimating the derivative of a mathematical function, is widely used to solve ordinary differential equations (ODEs) and partial differential equations (PDEs) in numerical analysis and engineering applications [21]. So far, a number of different meth-

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Table 1
Comparisons of numerical differentiation formulas in DTZNN models.

	Formula	n-step	General-form	Accuracy
This paper	Eq. (11)	3	Yes	High
[21,39]	$\dot{f}_k \doteq \frac{f_{k+1}-f_k}{\tau}$	1	No	Low
[11,12,40,42]	$\dot{f}_k \doteq \frac{2f_{k+1}-3f_k+2f_{k-1}-f_{k-2}}{2\tau}$	3	No	High
[12,41]	$\dot{f}_k \doteq \frac{6f_{k+1}-3f_k-2f_{k-1}-f_{k-2}}{10\tau}$	3	No	High

ods have been developed to construct useful difference formulas for first-order derivative approximation, such as undetermined coefficients [22,23], the finite difference [22,24–26], the regularization [27], the polynomial interpolation [28,29] and Richardson extrapolation [29]. In this paper, numerical differentiation is used to discretize CTZNN model. At present numerical differentiation formulas are mostly the finite difference approximation of derivatives, which cannot be used to the discretization of continuous-time models [24,30].

Matrix inversion is an essential topic of many solution in the different science and engineering fields, such as robotics control [31], machine learning [32], and optimization [33]. Generally speaking, the conventional methods for matrix inversion were designed for constant matrices [34–36]. However, many practical systems are time-variant, the corresponding matrices may also be time-variant. Thus, it is completely necessary to develop the corresponding methods for online time-variant matrix inversion. For time-variant matrix inversion, Getz and Marsden [37] proposed an explicit dynamic method which is convergent exponentially when the design parameter is sufficiently large and the initial value is sufficiently close to the theoretical one. Zhang and Ge [38] proposed a continuous-time ZNN (CTZNN) model for time-variant matrix inversion depicted in implicit dynamics, which is global exponential convergent. Subsequently an explicit CTZNN model was proposed by Guo and Zhang as a synthesis of the two above methods [39]. In order to implement ZNN in digit device, DTZNN models for time-variant matrix inversion [21,39–42] are proposed as the discretization results of CTZNN models by exploiting numerical differentiation techniques.

One-step DTZNN models are proposed by exploiting Euler forward difference formula to dicretize the CTZNN models [21,39]. In order to obtain better accuracy, a number of numerical differentiation formulas were proposed using Taylor-type numerical differentiation techniques by Zhang et al to discretize CTZNN models, which are called Zhang et al discretization (ZeaD) formulas. The corresponding discrete-time models are called multistep DTZNN models [11,12,40–42]. At present, most of multistep DTZNN models are three-step, of which the ZeaD formulas are presented as

$$\dot{f}_k \doteq \frac{2f_{k+1}-3f_k+2f_{k-1}-f_{k-2}}{2\tau}$$

or

$$\dot{f}_k \doteq \frac{6f_{k+1}-3f_k-2f_{k-1}-f_{k-2}}{10\tau},$$

where symbol $\doteq$ denotes the computational assignment operation. The details are shown and compared in Table 1.

Note that there perhaps are many ZeaD formulas of this kind. Thus, the problem is whether there exists a general ZeaD formula to generalize all the present three-step ZeaD formulas.

Remark 1. The general ZeaD formula is different from the general numerical differentiation formulas in references [22,24–26]. The methods in the references are effective to compute the derivatives of a function at specified points within its domain. However, the numerical differentiation formulas of them cannot be used to discretize continuous-time models for neglect of stability and convergence of the systems.

In this study, a three-step general ZeaD formula is designed to approximate the first-order derivative. Then, a three-step general DTZNN model is proposed for time-variant matrix inversion. For the situation of the derivative of the time-variant matrix $\dot{A}_k$ unknown, two other DTZNN models are proposed. By theoretical analyses and numerical experiments verification, the three proposed general DTZNN models are stable and convergent with order $O(\tau^3)$.

The rest of this paper is organized into five sections. Section 2 shows the problem formulation of time-variant matrix inversion and the formulation of CTZNN model, one-step and three-step general DTZNN models. In Section 3, a three-step general ZeaD formula is designed based on the Taylor expansion. Two formulas of estimation of $\dot{A}_k$ are established for the situation of $\dot{A}_k$ unknown, and three corresponding DTZNN models are proposed. The proposed DTZNN models for time-variant matrix inversion are investigated and analyzed in Section 4. Section 5 provides two illustrative numerical examples to substantiate the efficacy and superiority of the proposed three-step general DTZNN models for time-variant matrix inversion. Section 6 concludes this paper with final remarks. Before ending this introductory section, it is worth pointing out the main contributions of this paper as follows.

1) This paper provides a three-step general ZeaD formula design method based on bilinear transform. By considering the consistency and convergence, the ZeaD formula can be used not only to compute the derivatives of the target points, but also to discretize continuous-time models. However, the general numerical differentiation formula in references [22,24–26] cannot be applied to discretize continuous-time models. Besides, the three-step ZeaD formulas in up-to-date references [11,12,40,41] are just special cases of such a general formula.
2) In this paper, a three-step general DTZNN model for time-variant matrix inversion is proposed by exploiting the general ZeaD formula to discretize the CTZNN model. The stability and convergence of the DTZNN model are proved theoretically. Besides, for the situation of $\dot{A}_k$ unknown, two formulas of estimating $\dot{A}_k$ are provided, and two other three-step general DTZNN models are proposed, by which the actual application scope is further expanded.
3) The numerical experiment results substantiate the efficacy and superiority of the three proposed three-step general DTZNN models for time-variant matrix inversion with the error of order $O(\tau^3)$, comparing with those of Newton iteration and one-step DTZNN model.

2. Problem formulation, ZeaD formula and ZNN models

The problem formulation of time-variant matrix inversion, CTZNN and one-step DTZNN models are presented in this section. Then, general ZeaD formulas, DTZNN models and the objectives of them are presented.

2.1. Problem formulation

In this paper, the problem of time-variant matrix inversion is considered:

$$A(t)X(t) = I, \tag{1}$$

where, at any time instant t, $A(t) \in \mathbb{R}^{m\times m}$ is a nonsingular continuously differentiable time-variant coefficient matrix. $I \in \mathbb{R}^{m\times m}$ is the identity matrix, and $X(t) \in \mathbb{R}^{m\times m}$ is the time-variant unknown matrix to be obtained in our study.

2.2. CTZNN model

CTZNN model for time-variant matrix inversion [39] is presented as

$$\dot{X}(t) = -X(t)\dot{A}(t)X(t) - \mu X(t)(A(t)X(t) - I), \tag{2}$$

where $X(t)$ is the state matrix corresponding to time-variant theoretical inverse $A^{-1}(t)$. $\dot{X}(t)$ is the time derivative of $X(t)$ and $\dot{A}(t)$ is the time derivative of $A(t)$. In addition, μ is a positive real number used to scale the convergence rate of (2). As presented in [37,39], the state matrix $X(t)$ of the Getz–Marsden dynamic system (2) can converge exponentially to $A^{-1}(t)$ when the value of μ is sufficiently large, and $X(0)$ is sufficiently close to $A^{-1}(0)$.

For the convenience, Eq. (2) is rewritten as

$$\dot{X}(t) = -G(X(t), t), \tag{3}$$

where $G(X(t), t) = X(t)\dot{A}(t)X(t) + \mu X(t)(A(t)X(t) - I)$.

2.3. One-step DTZNN model and Newton iteration

Euler forward difference formula is presented as

$$\dot{f}_k \doteq \frac{f_{k+1} - f_k}{\tau}, \tag{4}$$

where f_k and f_{k+1} indicate the present and next samples of $f(t)$ at time $t_k = k\tau$ and $t_{k+1} = (k+1)\tau$ respectively, and τ is the sampling interval or sampling period.

Thus, the one-step DTZNN model is established by using (4) directly to discretize CTZNN model (3):

$$\begin{aligned} X_{k+1} &\doteq X_k - \tau G_k \\ &\doteq X_k - \tau X_k\dot{A}_kX_k - hX_k(A_kX_k - I), \end{aligned} \tag{5}$$

where $G_k = G(X_k, k)$ is the discretization result of $G(X(t), t)$ at time $t_k = k\tau$, and $h = \tau\lambda$ denotes the step size. Eq. (5) is termed as one-step DTZNN model for time-variant matrix inversion.

Moreover, Newton iteration is a classical algorithm in numerical analysis, which is usually used to solve numerical dynamic problem [20,45]. Solving (1) by Newton iteration yields

$$X_{k+1} \doteq -X_k(A_kX_k - I). \tag{6}$$

Evidently, Newton iteration (6) is actually a special case of the one-step DTZNN model (5) when $h = 1$ and the time-derivative term $\dot{A}_k$ is omitted.

2.4. General ZeaD formula

The n-step general ZeaD formula is presented as

$$\dot{f}_k = \frac{1}{\tau}\left(\sum_{i=1}^{n+1} a_i f_{k-n+i}\right) + O(\tau^p), \tag{7}$$

where n is the step of ZeaD formula (7). In addition, $a_i \in \mathbb{R}$ is the coefficient with $i = 1, 2, \ldots, n+1$, and error term $O(\tau^p)$ is the truncation error. Eq. (7) is termed as n-step pth-order ZeaD formula. In addition, a_{n+1} and a_1 must satisfy $a_{n+1} \neq 0$ and $a_1 \neq 0$ to guarantee (7) to be n-step ZeaD formula.

Thus, the three-step general ZeaD formula is

$$\dot{f}_k = \frac{a_4 f_{k+1} + a_3 f_k + a_2 f_{k-1} + a_1 f_{k-2}}{\tau} + O(\tau^p), \tag{8}$$

where the coefficient a_i (with $i = 1, 2, 3, 4$) is determined in Section 3, and also is the objective of general ZeaD formula and three-step general DTZNN model.

2.5. General DTZNN models for time-variant matrix inversion

In this paper, the three-step general ZeaD formula is applied to discretize CTZNN model (3), and the three-step general DTZNN model is obtained as

$$X_{k+1} = \frac{-a_3X_k - a_2X_{k-1} - a_1X_{k-2} - \tau G_k}{a_4} + O(\tau^{p+1}), \tag{9}$$

where $O(\tau^{p+1})$ is the truncation error.

Eq. (9) is the DTZNN model in the situation of $\dot{A}_k$ is known. However, in certain real-world application, it may be difficult to obtain the value of $\dot{A}_k$ directly. Thus, it is worth investigating the DTZNN model with $\dot{A}_k$ unknown. In this situation, the present $\dot{A}_k$ can be estimated by the present and past samples A_i with $i = k$, $k-1$, ... Therefore, the general formula for estimation of $\dot{A}_k$ is given by

$$\dot{A}_k = \frac{1}{\tau}\left(\sum_{i=1}^{n} \alpha_i A_{k-n+i}\right) + O(\tau^{p_1}), \tag{10}$$

where p_1 is the order of (10) and $O(\tau^{p_1})$ is the truncation error of estimating $\dot{A}_k$. The optimal estimation of $\dot{A}_k$ is to achieve the following two goals. The first is to find (10) to estimate $\dot{A}_k$ by less samples A_i. The second is that (10) should be enough accuracy to satisfy the requirement of the three-step general DTZNN model. In order to make sure that the truncation error of (10) does not degrade the truncation error of (9), the order p_1 should meet the condition $p_1 \geq p$. In this paper, p_1 is set as $p_1 = p$ or $p_1 = p + 1$. Thus, for this situation, two other general DTZNN models can be obtained.

3. Three-step general DTZNN model design

In this section, a general ZeaD formula is designed, and a three-step general DTZNN model is proposed for time-variant matrix inversion by using the general ZeaD formula to discretize the CTZNN model. For the situation of $\dot{A}_k$ unknown, two other general models are proposed.

3.1. Three-step general ZeaD formula design

For the readability of the paper and the integrity of the structure, the important achievements of general DTZNN model design are placed in the following theorems and corollaries.

Theorem 1. *A three-step general ZeaD formula is proposed as*

$$\dot{f}_k \doteq \frac{(-a_1 + \frac{1}{2})f_{k+1} + 3a_1 f_k - (3a_1 + \frac{1}{2})f_{k-1} + a_1 f_{k-2}}{\tau}, \tag{11}$$

where $a_1 < 0$, which is convergent with the truncation error of $O(\tau^2)$.

Proof. Based on four results of n-step method presented in Appendix, Eq. (11) is convergent when it satisfies both consistency and zero-stability. Note that three-step ZeaD formula (11) is obtained by (8) satisfying both consistency and zero-stability.

At first, let (8) satisfy the consistency. The pth-order Taylor expansion [29] is presented as

$$f_{k+1} = f_k + \tau\dot{f}_k + \frac{\tau^2}{2}\ddot{f}_k + \cdots + \frac{\tau^p}{p!}f_k^{(p)} + O(\tau^{p+1}). \tag{12}$$

In order to get the three-step general ZeaD formula with the truncation error of $O(\tau^2)$, the Taylor expansions of f_{k+1}, f_{k-1} and f_{k-2} are given by

$$f_{k+1} = f_k + \tau\dot{f}_k + \frac{\tau^2}{2}\ddot{f}_k + \frac{\tau^3}{6}f_k^{(3)} + O(\tau^4), \tag{13}$$

$$f_{k-1} = f_k - \tau\dot{f}_k + \frac{\tau^2}{2}\ddot{f}_k - \frac{\tau^3}{6}f_k^{(3)} + O(\tau^4), \tag{14}$$

$$f_{k-2} = f_k - 2\tau\dot{f}_k + 2\tau^2\ddot{f}_k - \frac{4\tau^3}{3}f_k^{(3)} + O(\tau^4). \tag{15}$$

By substituting (13) through (15) into (8), we obtain

$$b_0 f_k + b_1\tau\dot{f}_k + b_2\tau^2\ddot{f}_k + b_3\tau^3 f_k^{(3)} + O(\tau^4) = 0, \tag{16}$$

where $b_0 = a_4 + a_3 + a_2 + a_1, b_1 = a_4 - a_2 - 2a_1 - 1, b_2 = a_4/2 + a_2/2 + 2a_1, b_3 = a_4/6 - a_2/6 - 4a_1/3$.

By assuming (8) with the truncation error of $O(\tau^3)$, the conditions below must be satisfied:

$$\begin{cases} b_0 = a_4 + a_3 + a_2 + a_1 = 0 \\ b_1 = a_4 - a_2 - 2a_1 - 1 = 0 \\ b_2 = \dfrac{a_4}{2} + \dfrac{a_2}{2} + 2a_1 = 0 \\ b_3 = \dfrac{a_4}{6} - \dfrac{a_2}{6} - \dfrac{4a_1}{3} = 0. \end{cases} \tag{17}$$

By solving (17), the solution is obtained as $a_4 = 5/12,\ a_3 = 1/4,\ a_2 = -3/4,\ a_1 = 1/12$. Thus, the characteristic equation of (8) is

$$\rho(\gamma) = 5\gamma^3 + 3\gamma^2 - 9\gamma + 1 = 0, \tag{18}$$

the roots of which are 1, $(-4-\sqrt{21})/5$ and $(-4+\sqrt{21})/5$. It does not satisfy the zero-stability. Therefore, three-step ZeaD formula (8) has no solution when the truncation error is $O(\tau^3)$.

By assuming (8) with the truncation error of $O(\tau^2)$, the conditions below must be satisfied:

$$\begin{cases} b_0 = a_4 + a_3 + a_2 + a_1 = 0 \\ b_1 = a_4 - a_2 - 2a_1 - 1 = 0 \\ b_2 = \dfrac{a_4}{2} + \dfrac{a_2}{2} + 2a_1 = 0. \end{cases} \tag{19}$$

The solution of (19) is

$$\begin{cases} a_4 = -a_1 + \dfrac{1}{2} \\ a_3 = 3a_1 \\ a_2 = -3a_1 - \dfrac{1}{2}. \end{cases} \tag{20}$$

By substituting (20) into (17), b_3 is given by

$$b_3 = -a_1 + \frac{1}{6}. \tag{21}$$

Secondly, let (8) satisfy the zero-stability. Eq. (8) can be rewritten as

$$a_4 f_{k+1} + a_3 f_k + a_2 f_{k-1} + a_1 f_{k-2} = \tau\dot{f}_k. \tag{22}$$

The characteristic equation of (22) is

$$\rho(\gamma) = a_4\gamma^3 + a_3\gamma^2 + a_2\gamma + a_1 = 0. \tag{23}$$

By bilinear transformation $\gamma = (1+\omega\tau/2)/(1-\omega\tau/2)$ [43], we obtain

$$c_3\left(\frac{\omega\tau}{2}\right)^3 + c_2\left(\frac{\omega\tau}{2}\right)^2 + c_1\left(\frac{\omega\tau}{2}\right) + c_0 = 0, \tag{24}$$

where $c_0 = a_4 + a_3 + a_2 + a_1,\ c_1 = 3a_4 + a_3 - a_2 - 3a_1,\ c_2 = 3a_4 - a_3 - a_2 + 3a_1,\ c_3 = a_4 - a_3 + a_2 - a_1$. By substituting (20) into (24), we have

$$\left(-8a_1\left(\frac{\omega\tau}{2}\right)^2 + 2\left(\frac{\omega\tau}{2}\right) + 2\right)\left(\frac{\omega\tau}{2}\right) = 0. \tag{25}$$

Based on Routh's stability criterion [44], we can obtain

$$a_1 < 0. \tag{26}$$

Therefore, the three-step general ZeaD formula is presented as

$$\dot{f}_k = \frac{(-a_1+\frac{1}{2})f_{k+1} + 3a_1 f_k + (-3a_1-\frac{1}{2})f_{k-1} + a_1 f_{k-2}}{\tau} + \left(a_1 - \frac{1}{6}\right)\tau^2 f_k^{(3)} + O(\tau^3) \tag{27}$$

or

$$\dot{f}_k = \frac{(-a_1+\frac{1}{2})f_{k+1} + 3a_1 f_k + (-3a_1-\frac{1}{2})f_{k-1} + a_1 f_{k-2}}{\tau} + O(\tau^2), \tag{28}$$

where $a_1 < 0$. The term of the truncation error of (27) is $R(\tau) = (a_1 - 1/6)\tau^2 f_k^{(3)} + O(\tau^3) = O(\tau^2)$, and the value of which becomes smaller when a_1 approaches toward 0. In addition, the ZeaD formulas with different values of a_1 are listed in Appendix. Thus, the proof is completed. □

Corollary 1. *The truncation error of (11) becomes smaller when a_1 approaches toward 0.*

Proof. The proof is known from the truncation error of (27). □

3.2. Estimation of $\dot{A}_k$

Theorem 2. *For the situation of $\dot{A}(t)$ unknown, the estimation of $\dot{A}(t)$ for three-step DTZNN model is formulated as*

$$\dot{A}_k = \frac{3A_k - 4A_{k-1} + A_{k-2}}{4\tau} + O(\tau^2) \tag{29}$$

or

$$\dot{A}_k = \frac{11A_k - 18A_{k-1} + 9A_{k-2} - 2A_{k-3}}{6\tau} + O(\tau^3). \tag{30}$$

Proof. The proof is presented in Appendix. □

3.3. Three-step general DTZNN models

Based on the three-step general ZeaD formula, the three-step general DTZNN model can be obtained by exploiting (11) to discretize CTZNN model (3), and is presented as

$$X_{k+1} \doteq \frac{-3a_1X_k + (3a_1+\frac{1}{2})X_{k-1} - a_1X_{k-2} - \tau G_k}{-a_1 + \frac{1}{2}}, \tag{31}$$

where $a_1 < 0$.

For the situation of $\dot{A}_k$ unknown, the three-step general DTZNN model for $\dot{A}_k$ estimated by (29) is proposed as

$$X_{k+1} \doteq \frac{-3a_1X_k + (3a_1+\frac{1}{2})X_{k-1} - a_1X_{k-2} - \tau\hat{G}_k}{-a_1 + \frac{1}{2}}, \tag{32}$$

where $\hat{G}_k = X_k(3A_k - 4A_{k-1} + A_{k-2})X_k/(4\tau) + \mu X_k(A_kX_k - I)$ and $a_1 < 0$.

The three-step DTZNN model for $\dot{A}_k$ estimated by (30) is obtained as

$$X_{k+1} \doteq \frac{-3a_1X_k + (3a_1+\frac{1}{2})X_{k-1} - a_1X_{k-2} - \tau\tilde{G}_k}{-a_1 + \frac{1}{2}}, \tag{33}$$

where $\tilde{G}_k = X_k(11A_k - 18A_{k-1} + 9A_{k-2} - 2A_{k-3})X_k/(6\tau) + \mu X_k(A_kX_k - I)$ and $a_1 < 0$.

4. Theoretical analyses and results

In this section, the stability and convergence of the general DTZNN models are presented in Theorems 3 through 7. The effectiveness of them is proved by theoretical analyses.

Theorem 3. *The three-step general DTZNN model (31) for time-variant matrix inversion is zero-stable and convergent with the truncation error of order $O(\tau^3)$.*

Proof. In view of (27), Eq. (11) can be rewritten as

$$\dot{f}_k = \frac{(-a_1 + \frac{1}{2})f_{k+1} + 3a_1 f_k - (3a_1 + \frac{1}{2})f_{k-1} + a_1 f_{k-2}}{\tau} + O(\tau^2). \tag{34}$$

By using (34) to discretize CTZNN model (3), three-step DTZNN model (31) can be rewritten as

$$X_{k+1} = \frac{-3a_1 X_k + (3a_1 + \frac{1}{2})X_{k-1} - a_1 X_{k-2} - \tau G_k}{-a_1 + \frac{1}{2}} + O(\tau^3), \tag{35}$$

where the truncation error is $O(\tau^3)$. Moreover, according to (35) and the design processes of (11) and (31), it is known that three-step DTZNN model (31) is zero-stable and convergent with the truncation error of order $O(\tau^3)$. Thus, the proof is completed. □

Theorem 4. *The three-step general DTZNN model (32) for time-variant matrix inversion is zero-stable and convergent with the truncation error of order $O(\tau^3)$.*

Proof. In view of (32) and Theorem 3, the following equation is obtained as

$$\begin{aligned} X_{k+1} =& \frac{-3a_1 X_k + (3a_1 + \frac{1}{2})X_{k-1} - a_1 X_{k-2} - \tau \hat{G}_k}{-a_1 + \frac{1}{2}} + O(\tau^3) \\ =& \frac{-3a_1 X_k + (3a_1 + \frac{1}{2})X_{k-1} - a_1 X_{k-2}}{-a_1 + \frac{1}{2}} \\ &- \frac{\tau}{-a_1 + \frac{1}{2}}(X_k(\dot{A}_k + O(\tau^2))X_k + \mu X_k(A_k X_k - I)) + O(\tau^3) \\ =& \frac{-3a_1 X_k + (3a_1 + \frac{1}{2})X_{k-1} - a_1 X_{k-2} - \tau G_k}{-a_1 + \frac{1}{2}} \\ &- \frac{\tau}{-a_1 + \frac{1}{2}}(X_k O(\tau^2) X_k) + O(\tau^3) \\ =& \frac{-3a_1 X_k + (3a_1 + \frac{1}{2})X_{k-1} - a_1 X_{k-2} - \tau G_k}{-a_1 + \frac{1}{2}} + O(\tau^3). \end{aligned} \tag{36}$$

The result of (36) proves that the truncation error of DTZNN model (32) is the same order as the one of DTZNN model (31).

Moreover, according to (36) and the design processes of (11) and (32), it is known that three-step DTZNN model (32) is zero-stable and convergent with the truncation error of order $O(\tau^3)$. Thus, the proof is completed. □

Theorem 5. *The three-step general DTZNN model (33) for time-variant matrix inversion is zero-stable and convergent with the truncation error of order $O(\tau^3)$.*

Proof. The proof is similar to that of Theorem 4. □

Theorem 6. *For three-step general DTZNN model (31) to solve time-variant matrix inversion, the steady-state residual error $\lim_{k\to\infty} \|E_{k+1}\|_F = \lim_{k\to\infty} \|A_{k+1}X_{k+1} - I\|_F$ is of order $O(\tau^3)$, with symbol $\|\cdot\|_F$ being the Frobenius norm of a matrix.*

Proof. According to (35) and Theorem 3, $X_{k+1} = X^*_{k+1} + O(\tau^3)$ with X^*_{k+1} being the theoretical value (i.e., $X^*_{k+1} = A^{-1}_{k+1}$). As A_{k+1} is uniformly bounded, the following result is obtained:

$$\begin{aligned} \|E_{k+1}\|_F =& \|A_{k+1}(X^*_{k+1} + O(\tau^3)) - I\|_F \\ =& \|A_{k+1}O(\tau^3)\|_F = O(\tau^3). \end{aligned} \tag{37}$$

Therefore, $\lim_{k\to\infty} \|E_{k+1}\|_F = O(\tau^3)$. Thus, the proof is completed. □

Theorem 7. *For three-step general DTZNN models (32) and (33) to solve time-variant matrix inversion, the steady-state residual errors are both of order $O(\tau^3)$.*

Proof. Based on Theorems 4 through 6, it can be readily generalized and similarly proved that the steady-state residual errors of the proposed DTZNN models (32) and (33) are both of order $O(\tau^3)$. Thus, the proof is completed. □

5. Numerical experiments and comparisons

In this section, numerical experiments based on two time-variant matrices are provided to verify the efficacy of the proposed general DTZNN models for the time-variant matrix inversion.

Example 1. The first example for time-variant matrix inversion is a two-dimensional time-variant matrix:

$$A_k = \begin{bmatrix} \cos(t_k) & \sin(t_k) \\ -\sin(t_k) & \cos(t_k) \end{bmatrix}. \tag{38}$$

It is not difficult to know that the time-variant theoretical inverse is

$$X^*_k = A^{-1}_k = \begin{bmatrix} \cos(t_k) & -\sin(t_k) \\ \sin(t_k) & \cos(t_k) \end{bmatrix}. \tag{39}$$

Starting with $X(0) = [0.2, 0; 0, 0.2]$, with the parameters set as $h = 0.5$, $\tau = 0.01$ and $t_k \in [0, 10]$ s, the corresponding numerical results of three-step DTZNN model (31) with $a_1 = -0.25$ to solve for the time-variant matrix inverse of A_k in (38) are shown in Fig. 1. Specifically, the element trajectories of the state X_k are shown in Fig. 1(a), from which we could observe that the solution of DTZNN model (31) converges to the theoretical time-variant solution. Fig. 1(b) shows that the residual error of (31) is convergent. These results substantiate the efficacy of three-step general DTZNN model (31) for the time-variant matrix inversion.

In order to display the advantages of three-step DTZNN model (31) over Newton iteration (6) and one-step DTZNN model (5) to solve for the time-variant matrix inverse, numerical experiment is performed and numerical results are illustrated in Fig. 2, of which the parameters $X(0)$, h, and a_1 set as the same value as Fig. 1. In Fig. 2(a), the maximum steady-state residual errors (MSSREs) of Newton iteration (6), one-step DTZNN model (5) and three-step DTZNN model (31) are of order 10^{-1}, 10^{-2} and 10^{-3} respectively when $\tau = 0.1$. The MSSREs are of order 10^{-2}, 10^{-4} and 10^{-6} respectively when $\tau = 0.01$, which are shown in Fig. 2(b). These results substantiate that three-step general DTZNN model (31) is superior to Newton iteration (6) and one-step DTZNN (5) for the time-variant matrix inversion.

Fig. 3 shows the numerical results of three-step DTZNN model (31) with different values of a_1. In the numerical experiment, the parameters are set the same as Fig. 1, except a_1. Fig. 3(a) shows the residual errors of three-step DTZNN model (31) with $a_1 = -0.1$, $a_1 = -0.3$ or $a_1 = -0.6$. The residual error of (31) with $a_1 = -0.1$ comes into steady state at about $k = 20$, and the MSSRE is about 8×10^{-7}. The residual errors of (31) with $a_1 = -0.3$ and $a_1 = -0.6$ come into steady state at about $k = 60$ and $k = 150$, and the MSSREs are about 10^{-6} and 2×10^{-7}, respectively. The results substantiate that three-step DTZNN model (31) with $a_1 = -0.1$ is superior to (31) with $a_1 = -0.3$ and $a_1 = -0.6$ over the convergence rate and the steady-state residual error. Fig. 3(b) shows the relation between the MSSRE of DTZNN model (31) and the coefficient of the truncation error (i.e., $-a_1 + 1/6$). The MSSRE and the coefficient are linear. When the coefficient of the truncation error turns bigger, the smaller the MSSRE becomes.

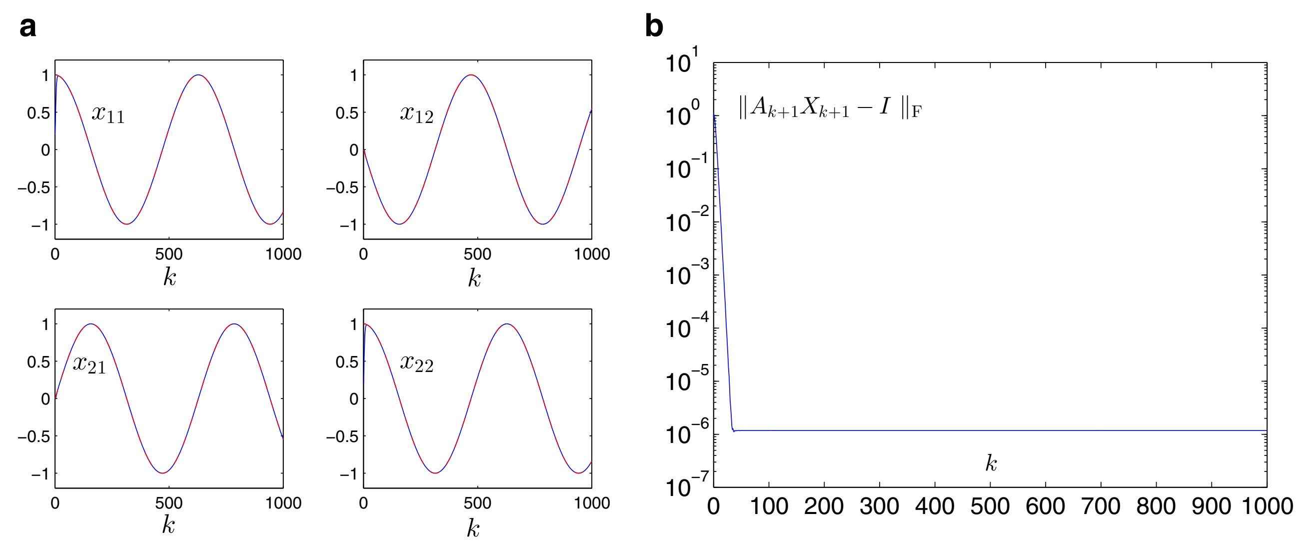

Fig. 1. Numerical results of three-step general DTZNN model (31): (a) state trajectories of (31); (b) residual error of (31).

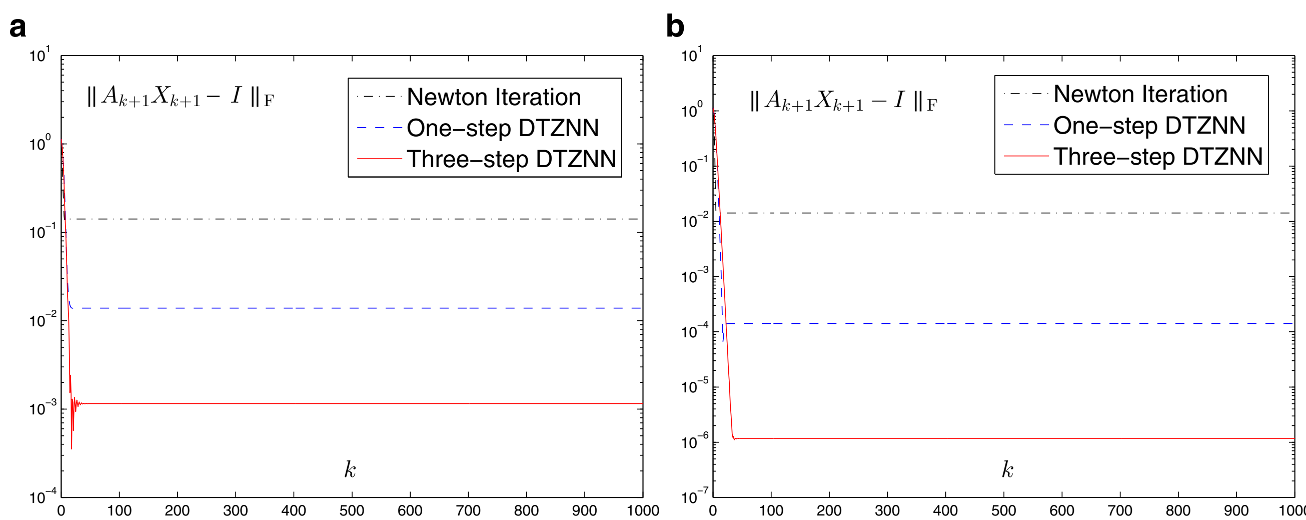

Fig. 2. Residual errors of Newton iteration (6), one-step DTZNN model (5) and three-step DTZNN model (31) with different values of τ: (a) $\tau = 0.1$; (b) $\tau = 0.01$.

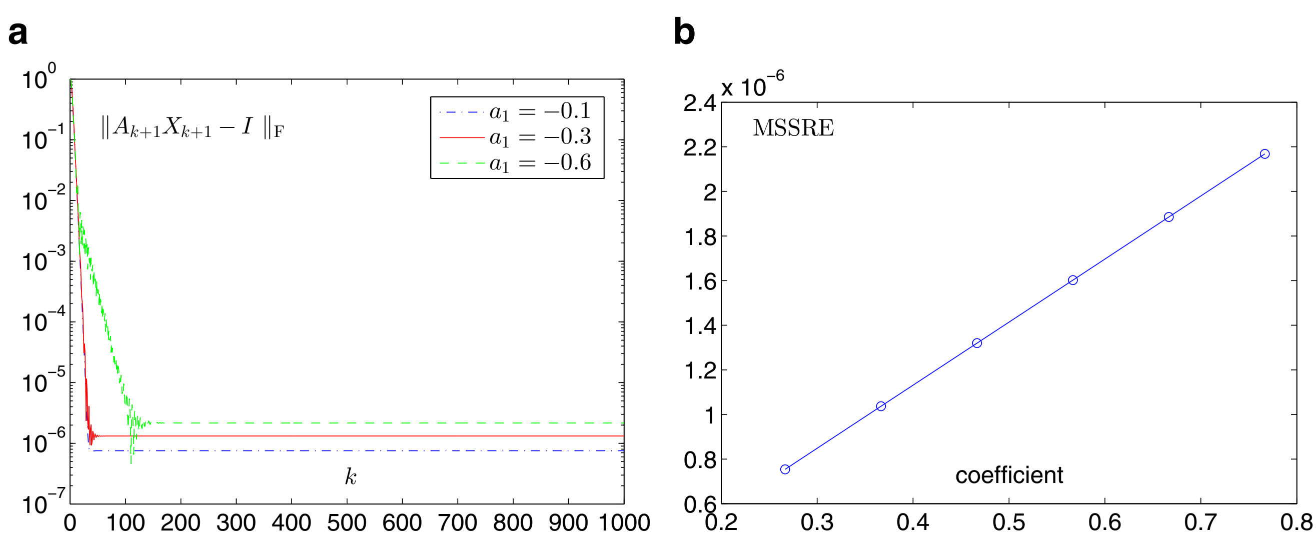

Fig. 3. Residual errors of (31) with different values of a_1: (a) residual errors; (b) relation between MSSRE and coefficient of the truncation error.

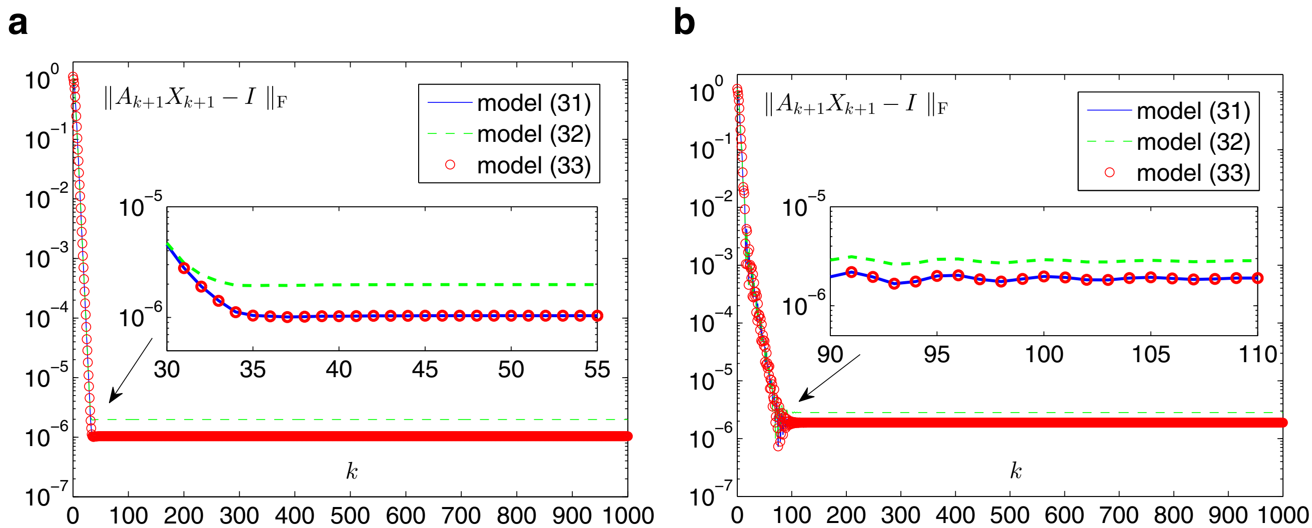

Fig. 4. Residual errors of (31), (32) and (33) with different values of a_1: (a) with $a_1 = -0.2$; (b) with $a_1 = -0.5$.

Table 2
MSSREs of (31), (32) and (33) with different values of a_1.

a_1	Coefficient	Model (31)	Model (32)	Model (33)
−0.1	0.2667	7.5409×10^{-7}	1.6967×10^{-6}	7.5422×10^{-7}
−0.2	0.3667	1.0369×10^{-6}	1.9795×10^{-6}	1.0370×10^{-6}
−0.3	0.4667	1.3197×10^{-6}	2.2622×10^{-6}	1.3198×10^{-6}
−0.4	0.5667	1.6024×10^{-6}	2.5450×10^{-6}	1.6026×10^{-6}
−0.5	0.6667	1.8852×10^{-6}	2.8278×10^{-6}	1.8853×10^{-6}
−0.6	0.7667	2.1680×10^{-6}	3.1106×10^{-6}	2.1681×10^{-6}

Table 3
MSSREs of (31) with different values of a_1, h and τ.

a_1	h	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.001$
	0.2	2.3162×10^{-3}	2.5895×10^{-6}	2.5927×10^{-9}
−0.2	0.4	1.2561×10^{-3}	1.2959×10^{-6}	1.2964×10^{-9}
	0.6	8.5144×10^{-4}	8.6411×10^{-7}	8.6241×10^{-10}
	0.2	2.9477×10^{-3}	3.2957×10^{-6}	3.2998×10^{-9}
−0.3	0.4	1.5986×10^{-3}	1.6494×10^{-6}	1.6499×10^{-9}
	0.6	1.0836×10^{-3}	1.0998×10^{-6}	1.0999×10^{-9}
	0.2	3.5793×10^{-3}	4.0019×10^{-6}	4.0069×10^{-9}
−0.4	0.4	1.9411×10^{-3}	2.0028×10^{-6}	2.0035×10^{-9}
	0.6	1.3158×10^{-3}	1.3354×10^{-6}	1.3356×10^{-9}

In order to investigate the influence of the estimating $\dot{A}_k$ on the residual error, another numerical experiment is carried out with DTZNN models (31), (32) and (33) as shown in Fig. 4. Fig. 4(a) shows the trajectories of DTZNN models (31), (32) and (33) for solving the time-variant matrix inversion of (38) with $a_1 = -0.2$. The trajectory of (33) almost coincides with the trajectory of (31), and the both MSSREs are about 10^{-6} which are of order $O(\tau^3)$. The MSSRE of (32) is about 2×10^{-6} which is slightly larger than the MSSREs of (31) and (33), and is also of order $O(\tau^3)$. Similar situation appears in Fig. 4(b) with $a_1 = -0.5$. Furthermore, more detailed data on a_1, the coefficient of the truncation error and the MSSREs are shown in Table 2. The numerical results of Fig. 4 and Table 2 substantiate that the three proposed DTZNN models (31), (32) and (33) are stable and convergent to the theoretical solution with the truncation error of order $O(\tau^3)$. In contrast, model (33) is more accurate in solving the inverse of time-variant matrix, and the cost is that it needs one more addition operation and one more storage unit for A_{k-3} than model (32). For further investigation, DTZNN model (31) is used to solve for the time-variant matrix inverse of A_k in (38) with $a_1 = -0.2$, $h = 0.4$ and different values of τ as shown in Fig. 5. The numerical results substantiate that DTZNN model (31) is convergent to the theoretical solution with the theoretical truncation error, and the MSSRE is of order $O(\tau^3)$ for $\tau = 0.1$, 0.01, 0.001, and 0.0001. In addition, more data of the numerical experiment are listed in Table 3. Thus, the following results are obtained as shown in Table 3.

1) For the proposed DTZNN model (31), the MSSRE is of order $O(\tau^3)$ for different values of a_1 and h.
2) When h and τ remain invariant, and a_1 turns smaller, the MSSRE of (31) becomes smaller.
3) When a_1 and τ remain invariant, and step size h turns bigger, the MSSRE of (31) becomes smaller.

Example 2. In this example, a three-dimensional time-variant rotation matrix A_k is considered as

$$A_k = \begin{bmatrix} c_y c_z & c_y s_z & -s_y \\ s_x s_y c_z - c_x s_z & s_x s_y s_z + c_x c_y & s_x c_y \\ c_x s_y c_z + s_x s_z & c_x s_y s_z - s_x c_z & c_x c_y \end{bmatrix} \tag{40}$$

where $s_x = \sin(\theta_1 t_k)$, $s_y = \sin(\theta_2 t_k)$, $s_z = \sin(\theta_3 t_k)$, $c_x = \cos(\theta_1 t_k)$, $c_y = \cos(\theta_2 t_k)$, $c_z = \cos(\theta_3 t_k)$. Parameters θ_1, θ_2 and θ_3 are the angular frequencies of rotation around the X, Y and Z axes, respectively. For convenience, they are set as $\theta_1 = 0.3$, $\theta_2 = 1$ and $\theta_3 = 0.5$. The proposed DTZNN models (31), (32) and (33) with different values of h, τ and a_1 are used to solve for the time-variant matrix inverse of A_k in (40), and the corresponding numerical results are shown as Figs. 6 and 7.

As Fig. 6 shown, the numerical results illustrate that DTZNN models (31), (32) and (33) are stable and convergent with the steady-state residual errors for the time-variant matrix inverse of A_k in (40), and the MSSREs are of order $O(\tau^3)$. Furthermore, in Fig. 6(a), when the value of a_1 turns smaller for DTZNN model (31), the MSSRE becomes smaller. Fig. 6(b) shows that the steady-state residual error of DTZNN model (33) almost coincides with that of (31), and the steady-state residual error of DTZNN model (32) is slightly larger than that of (31). In order to investigate the influence of h and τ on the residual error, another numerical experiment is performed with DTZNN model (31) when parameters set as $a_1 = -0.2$ and different values of h and τ to solve for the time-variant matrix inverse of A_k in (40), and the numerical results are shown as Fig. 7. Specifically, the steady-state residual errors all are of order $O(\tau^3)$ for $\tau = 0.01$ and $\tau = 0.001$ as shown in Fig. 7(a) through (d). By comparing Fig. 7(a) with (c), or by compar-

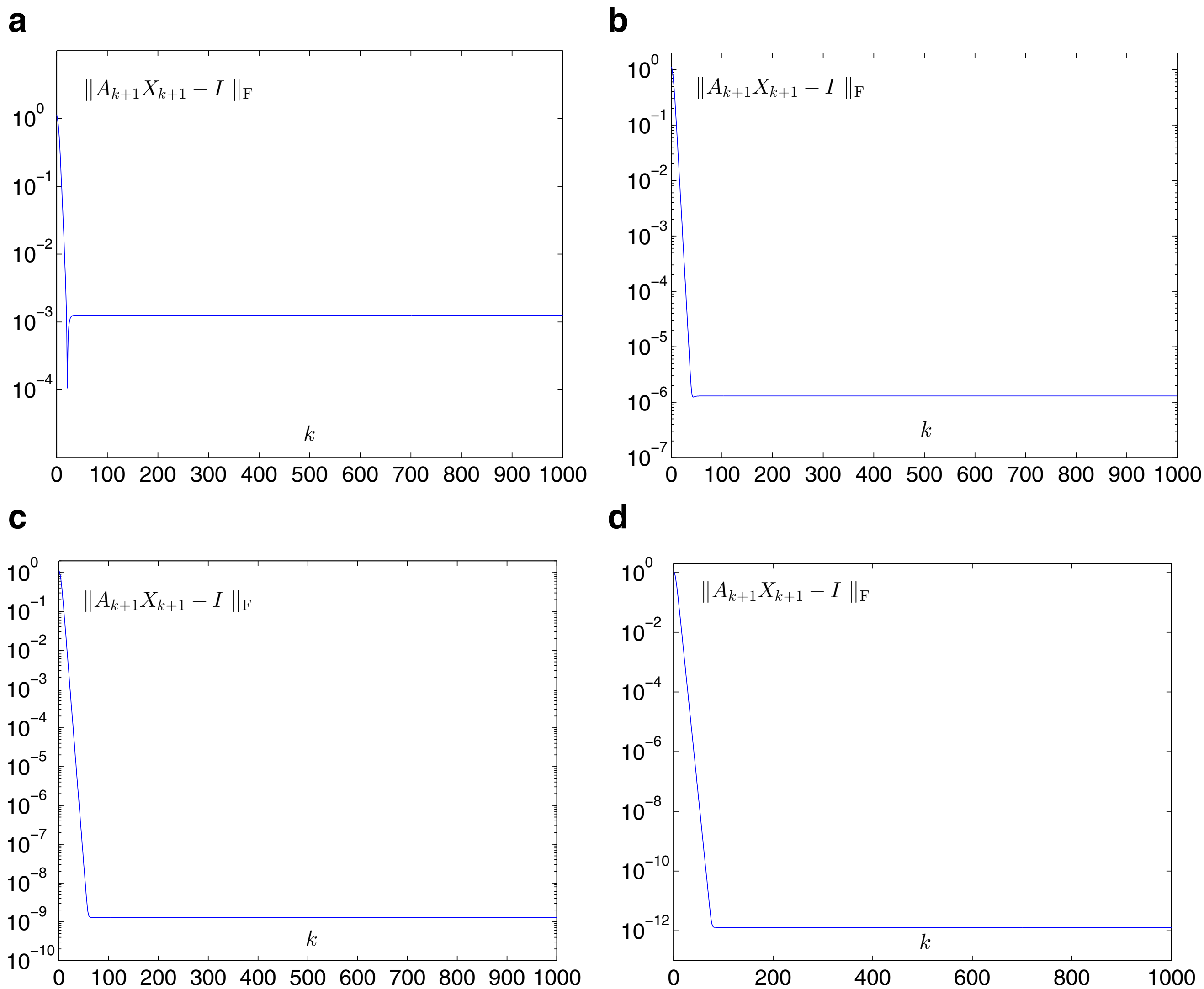

Fig. 5. Residual errors of (31) with different values of τ: (a) with $\tau = 0.1$; (b) with $\tau = 0.01$; (c) with $\tau = 0.001$; (d) with $\tau = 0.0001$.

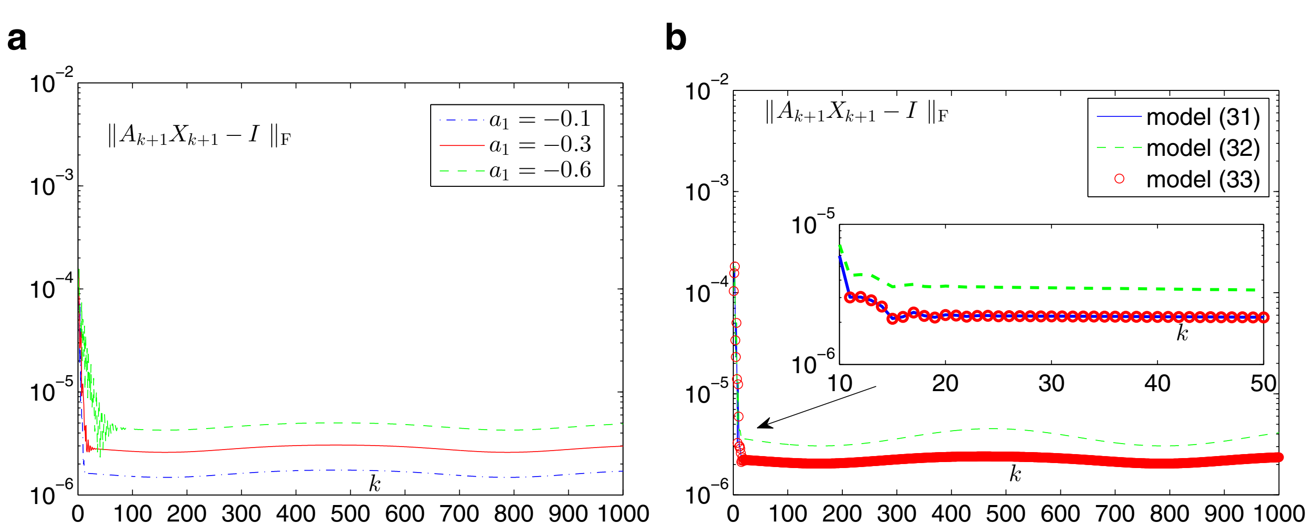

Fig. 6. Residual errors of (31), (32) and (33) with different values of a_1: (a) (31) with $a_1 = -0.1$, -0.3 and -0.6; (b) (31), (32) and (33) with $a_1 = -0.2$.

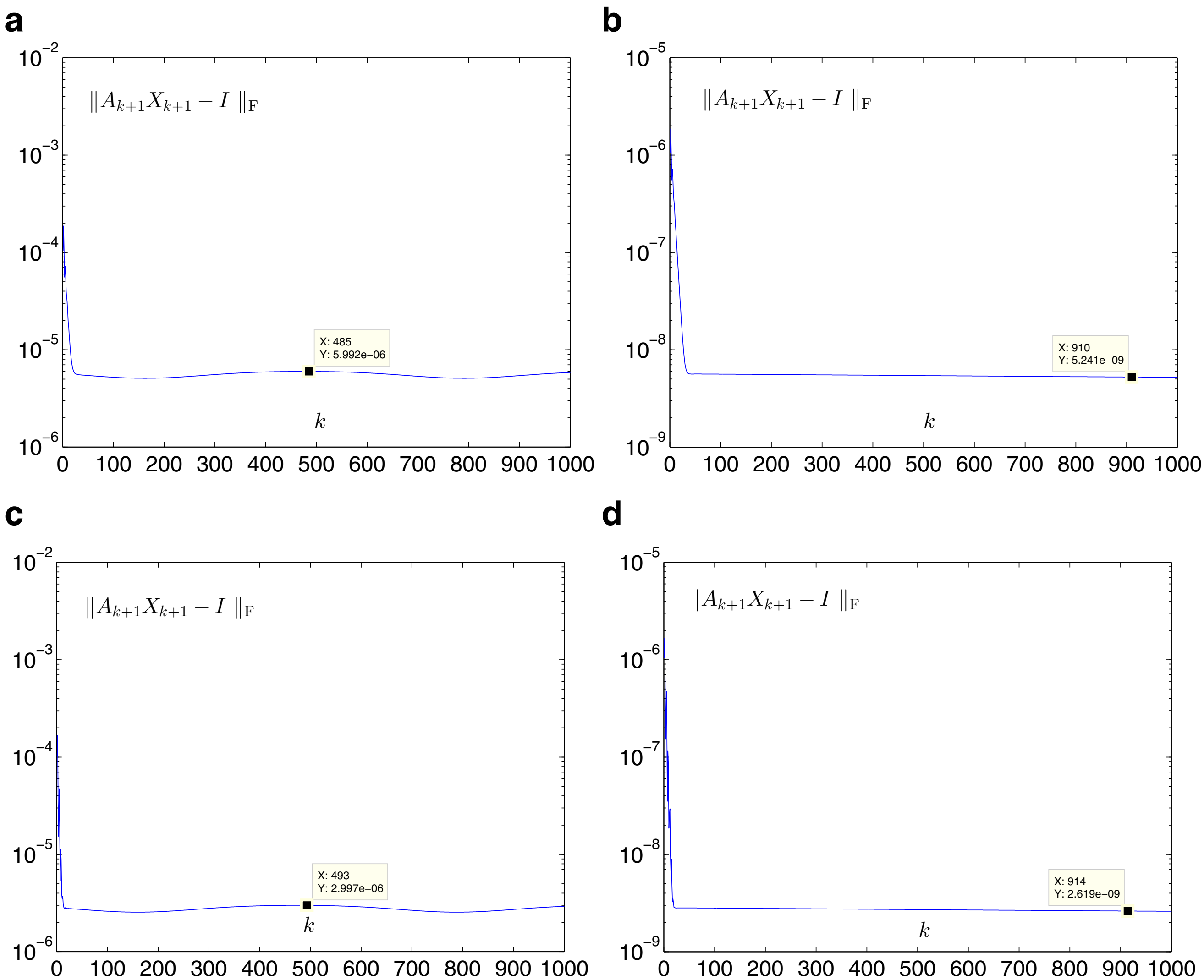

Fig. 7. Residual errors of (31) with different values of h and τ: (a) with $h = 0.2$ and $\tau = 0.01$; (b) with $h = 0.2$ and $\tau = 0.001$; (c) with $h = 0.4$ and $\tau = 0.01$; (b) with $h = 0.4$ and $\tau = 0.001$.

ing Fig. 7(b) with (d), it is concluded that the steady-state residual error becomes smaller when h turns bigger in the case of DTZNN model (31) with the same values of a_1 and τ.

6. Conclusion

In this paper, general ZeaD formula (11) has been designed for the first time to approximate the first-order derivative based on Taylor expansion and convergence definitions of discrete-time model. For the situation of $\dot{A}_k$ unknown, Eqs. (29) and (30) have been provided to estimate $\dot{A}_k$ with the truncation error $O(\tau^2)$ and $O(\tau^3)$, respectively. Therefore, three-step general DTZNN models (31), (32) and (33) have been obtained by using ZeaD formulas (11) to discretize CTZNN model (2) in the situations of $\dot{A}_k$ known and unknown. The theoretical analyses and results have presented the stability and convergence of the three-step general DTZNN models for time-variant matrix inversion. Numerical results have shown that proposed DTZNN models (31), (32) and (33) for time-variant matrix inversion are convergent to the theoretical solution with the MSSREs of order $O(\tau^3)$, and superior to one-step DTZNN model (5) with the MSSRE of order $O(\tau^2)$. Moreover, for the situation of $\dot{A}_k$ unknown, the MSSRE of DTZNN model (33) is almost the same as that of DTZNN model (31), and the MSSRE of DTZNN model (32) is slightly bigger than that of DTZNN model (31). In contrast, model (33) is more accurate in solving the inverse of time-variant matrix, and the cost is that it needs one more addition operation and one more storage unit for A_{k-3} than model (32). Besides, the future research directions can be the investigations of applying three-step general DTZNN models to other time-variant problems solving, and developing some different multi-step general ZeaD formulas to discretize CTZNN models.

Acknowledgments

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Appendix

A1. Results of n-step method

We have the following four results for an n-step method [30,46].

Result 1: An n-step method $\sum_{i=1}^{n+1} \alpha_i x_{k-n+i} = \tau \sum_{i=1}^{n+1} \beta_i \psi_{k-n+i}$ can be checked for zero-stability by determining the roots of its characteristic polynomial $\rho(\gamma) = \sum_{i=1}^{n+1} \alpha_i \gamma^{i-1}$. If all roots denoted by γ of the polynomial $\rho(\gamma)$ satisfy $|\gamma| \le 1$ with $|\gamma| = 1$ being simple, then the corresponding n-step method is zero-stable (i.e., has zero-stability).

Result 2: An n-step method is said to be consistent (i.e., has consistency) of order p if the truncation error for the exact solution is of order $O(\tau^{p+1})$ where $p > 0$.

Result 3: An n-step method is convergent, i.e., $x_{[t/\tau]} \to x^*(t)$, for all $t \in [0, t_f]$, as $\tau \to 0$, if and only if the method is zero-stable and consistent. That is, zero-stability plus consistency means convergence, which is also known as Dahlquist equivalence theorem.

Result 4: A zero-stable consistent n-step method converges with the order of its truncation error.

A2. Collection of three-step ZeaD formulas

The collection of three-step ZeaD formulas with different values of a_1 is shown in Table A1.

Table A1
Collection of three-step ZeaD formulas with different values of a_1.

a_1	Formula
$a_1 < 0$	$\dot{f}_k \doteq \frac{(-a_1+1/2)f_{k+1}+3a_1 f_k-(3a_1+1/2)f_{k-1}+a_1 f_{k-2}}{\tau}$
$-1/16$	$\dot{f}_k \doteq \frac{9f_{k+1}-3f_k-5f_{k-1}-f_{k-2}}{16\tau}$
-0.1	$\dot{f}_k \doteq \frac{6f_{k+1}-3f_k-2f_{k-1}-f_{k-2}}{10\tau}$
-0.2	$\dot{f}_k \doteq \frac{7f_{k+1}-6f_k+f_{k-1}-2f_{k-2}}{10\tau}$
-0.25	$\dot{f}_k \doteq \frac{3f_{k+1}-3f_k+f_{k-1}-f_{k-2}}{4\tau}$
-0.3	$\dot{f}_k \doteq \frac{8f_{k+1}-9f_k+4f_{k-1}-3f_{k-2}}{10\tau}$
-0.4	$\dot{f}_k \doteq \frac{9f_{k+1}-12f_k+7f_{k-1}-4f_{k-2}}{10\tau}$
-0.5	$\dot{f}_k \doteq \frac{2f_{k+1}-3f_k+2f_{k-1}-f_{k-2}}{2\tau}$
-0.6	$\dot{f}_k \doteq \frac{11f_{k+1}-18f_k+13f_{k-1}-6f_{k-2}}{10\tau}$

A3. Proof of Theorem 3

According to (10), the following formula is used to find the estimation of $\dot{A}_k$ with the truncation error of $O(\tau^3)$,

$$\dot{A}_k = \frac{\alpha_5 A_k + \alpha_4 A_{k-1} + \alpha_3 A_{k-2} + \alpha_2 A_{k-3} + \alpha_1 A_{k-4}}{\tau} + O(\tau^3). \quad (41)$$

The general formula corresponding to (41) is

$$\dot{f}_k = \frac{\alpha_5 f_k + \alpha_4 f_{k-1} + \alpha_3 f_{k-2} + \alpha_2 f_{k-3} + \alpha_1 f_{k-4}}{\tau} + O(\tau^3). \quad (42)$$

The Taylor expansions of f_{k-3} and f_{k-4} are

$$f_{k-3} = f_k - 3\tau\dot{f}_k + \frac{9}{2}\tau^2\ddot{f}_k - \frac{9}{2}\tau^3 f_k^{(3)} + O(\tau^4) \quad (43)$$

and

$$f_{k-4} = f_k - 4\tau\dot{f}_k + 8\tau^2\ddot{f}_k - \frac{32}{3}\tau^3 f_k^{(3)} + O(\tau^4). \quad (44)$$

By substituting (14), (15), (43) and (44) into (42), we obtain

$$d_4 f_k - d_3\tau\dot{f}_k + \frac{d_2}{2}\tau^2\ddot{f}_k - \frac{d_1}{6}\tau^3 f_k^{(3)} = O(\tau^4) \quad (45)$$

where $d_4 = \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1$, $d_3 = \alpha_4 + 2\alpha_3 + 3\alpha_2 + 4\alpha_1 + 1$, $d_2 = \alpha_4 + 4\alpha_3 + 9\alpha_2 + 16\alpha_1$, $d_1 = \alpha_4 + 8\alpha_3 + 27\alpha_2 + 64\alpha_1$. By solving (45), we have

$$\begin{cases} d_4 = \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 = 0 \\ d_3 = \alpha_4 + 2\alpha_3 + 3\alpha_2 + 4\alpha_1 + 1 = 0 \\ d_2 = \alpha_4 + 4\alpha_3 + 9\alpha_2 + 16\alpha_1 = 0 \\ d_1 = \alpha_4 + 8\alpha_3 + 27\alpha_2 + 64\alpha_1 = 0. \end{cases} \quad (46)$$

The solution of (46) is

$$\begin{cases} \alpha_5 = \alpha_1 + \frac{11}{6} \\ \alpha_4 = -4\alpha_1 - 3 \\ \alpha_3 = 6\alpha_1 + \frac{3}{2} \\ \alpha_2 = -4\alpha_1 - \frac{1}{3}. \end{cases} \quad (47)$$

When $\alpha_1 = 0$, the solution of (45) turns to be

$$\begin{cases} \alpha_5 = \frac{11}{6} \\ \alpha_4 = -3 \\ \alpha_3 = \frac{3}{2} \\ \alpha_2 = -\frac{1}{3}. \end{cases} \quad (48)$$

Thus, the formula with consistency is

$$\dot{f}_k = \frac{11f_k - 18f_{k-1} + 9f_{k-2} - 2f_{k-3}}{6\tau} + O(\tau^3). \quad (49)$$

Then, we verify the zero-stability of (49). The right characteristic polynomial of (49) is

$$\rho(\gamma) = 11\gamma^3 - 18\gamma^2 + 9\gamma - 2 = 0, \quad (50)$$

the characteristic roots of which are $\gamma = 1$, $(7+\sqrt{39})/22$ and $(7-\sqrt{39})/22$, and satisfy the zero-stable condition. Therefore, Eq. (49) is consistent and zero-stable. That is to say, Eq. (49) is convergent. Applying (49) to estimate $\dot{A}_k$, we can obtain

$$\dot{A}_k = \frac{11A_k - 18A_{k-1} + 9A_{k-2} - 2A_{k-3}}{6\tau} + O(\tau^3). \quad (51)$$

If the truncation error of (41) is $O(\tau^2)$, then (41) turns to be

$$\dot{A}_k = \frac{\alpha_5 A_k + \alpha_4 A_{k-1} + \alpha_3 A_{k-2} + \alpha_2 A_{k-3} + \alpha_1 A_{k-4}}{\tau} + O(\tau^2). \quad (52)$$

By solving (45), we obtain

$$\begin{cases} d_4 = \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 = 0 \\ d_3 = \alpha_4 + 2\alpha_3 + 3\alpha_2 + 4\alpha_1 + 1 = 0 \\ d_2 = \alpha_4 + 4\alpha_3 + 9\alpha_2 + 16\alpha_1 = 0. \end{cases} \quad (53)$$

The solution of (53) is

$$\begin{cases} \alpha_5 = -\alpha_2 + 13\alpha_1 + \frac{3}{2} \\ \alpha_4 = 3\alpha_2 - 24\alpha_1 - 2 \\ \alpha_3 = -3\alpha_2 + 10\alpha_1 + \frac{1}{2}. \end{cases} \quad (54)$$

When $\alpha_2 = 0$ and $\alpha_1 = 0$, the solution is

$$\begin{cases} \alpha_5 = \frac{3}{2} \\ \alpha_4 = -2 \\ \alpha_3 = \frac{1}{2}. \end{cases} \quad (55)$$

Thus, the formula with consistency is

$$\dot{f}_k = \frac{3f_k - 4f_{k-1} + f_{k-2}}{4\tau} + O(\tau^2). \quad (56)$$

Next, we verify the zero-stability of (56). The right characteristic polynomial of (56) is

$$\rho(\gamma) = 3\gamma^2 - 4\gamma + 1 = 0, \quad (57)$$

the characteristic roots of which are $\gamma = 1$ and $1/3$, and satisfy the zero-stability. Therefore, Eq. (56) is both consistent and zero-stable. That is to say, Eq. (56) is convergent. Applying (56) to estimate $\dot{A}_k$, we have

$$\dot{A}_k = \frac{3A_k - 4A_{k-1} + A_{k-2}}{4\tau} + O(\tau^2). \quad (58)$$

Thus, this theorem is proved.

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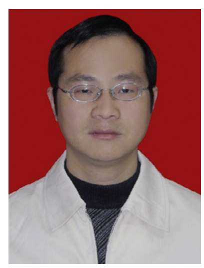

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