Solution 1

Part a. I'll look at the sets in order.

- Consider $A \cup B$. Combining elements from each, you get

$$A \cup B = \{z_2, z_3, z_4, z_5, z_6, z_7, z_8\}.$$

  The only element missing from the union is z_1. We can conclude that $(A \cup B)^c = \{z_1\}$. This is useful because we are told that $\Pr(A \cup B) = 21/24$, so it must be the case that $\Pr\left((A \cup B)^c\right) = \Pr(z_1) = 3/24$.
- Now consider $A \cap C$. Looks like the only element they have in common is z_2, so it must be the case that $\Pr(A \cap C) = \Pr(z_2) = 5/24$.
- Consider $A \cap D$. The only element they share is z_4, so it must be the case that $\Pr(A \cap D) = \Pr(z_4) = 2/24$.
- $B \cap C$ is singleton $\{z_6\}$. It follows that $\Pr(B \cap C) = \Pr(z_6) = 3/24$.
- $B \cap D$ is singleton $\{z_3\}$. It follows that $\Pr(B \cap D) = \Pr(z_3) = 3/24$.

Still gotta find z_5, z_7, and z_8.

Since the elements z_i are all mutually exclusive, we can write

$$\Pr(B) = \Pr(z_3) + \Pr(z_6) + \Pr(z_8),$$
$$\Pr(E) = \Pr(z_7) + \Pr(z_8).$$

This is useful with $\Pr(B)$ because we have numbers for everything except $\Pr(z_8)$, that is,

$$\frac{7}{24} = \frac{3}{24} + \frac{3}{24} + \Pr(z_8) \implies \Pr(z_8) = \frac{1}{24}.$$

Now we can do the same thing with event E, specifically,

$$\frac{2}{24} = \Pr(z_7) + \frac{1}{24} \implies \Pr(z_7) = \frac{1}{24}.$$

Great. Now z_5 is all that is remaining. All probabilities must sum to 1, therefore

$$\frac{3}{24} + \frac{5}{24} + \frac{3}{24} + \frac{2}{24} + \Pr(z_5) + \frac{3}{24} + \frac{1}{24} + \frac{1}{24} = 1 \implies \Pr(z_5) = \frac{6}{24}.$$

Part b. Let's find elements for each union first.

$$A \cup B = \{z_2, z_3, z_4, z_5, z_6, z_7\},$$
$$C \cup D = \{z_2, z_3, z_4, z_6\}.$$

Therefore the intersection of these is the set $(A \cup B) \cap (C \cup D) = \{z_2, z_3, z_4, z_6\}$. Therefore its corresponding probability is

$$\begin{aligned}\Pr\big((A \cup B) \cap (C \cup D)\big) &= \Pr(z_2) + \Pr(z_3) + \Pr(z_4) + \Pr(z_6) \\ &= \frac{5}{24} + \frac{3}{24} + \frac{2}{24} + \frac{3}{24} \\ &= \frac{13}{24}.\end{aligned}$$

Solution 2

We are to find the probability that Individual 1 has the virus, given a positive blood test. So we're thinking in terms of conditional probabilities. Let V_1 denote the event that Individual 1 has the virus, and 1_+ the event that Individual 1 receives a positive test. Then in the maths, we want to find $\Pr(V_1|1_+)$ using the fact that

$$\Pr(V_1|1_+) = \frac{P(V_1 \cap 1_+)}{P(1_+)}.$$

So let's find the sets V_1 and 1_+.

Only one person has the virus, only one person has the defective gene, and it could be the same person with both. I will write the sample space as

$$\begin{aligned}U = \big\{a = (1,1) \quad & b = (1,2) \quad c = (1,3) \\ d = (2,1) \quad & e = (2,2) \quad f = (2,3) \\ g = (3,1) \quad & h = (3,2) \quad i = (3,3)\big\},\end{aligned}$$

where (x, y) means Individual x has the virus, Individual y has the defective gene. With this notation, we can conclude that $V_1 = \{a, b, c\}$ and $1_+ = \{a, b, c, d, g\}$, and furthermore that $V_1 \cap 1_+ = \{a, b, c\}$. Because we are told that each state is equally likely, and because there are nine states, it follows that each state has 1/9 probability. Thus we can conclude that

$$\begin{aligned} \Pr(V_1 \cap 1_+) &= \Pr(a) + \Pr(b) + \Pr(c) &&= \frac{3}{9}, \\ \Pr(1_+) &= \Pr(a) + \Pr(b) + \Pr(c) + \Pr(d) + \Pr(g) &&= \frac{5}{9}. \end{aligned}$$

Now we can plug stuff into the conditional probability formula to get

$$\Pr(V_1|1_+) = \frac{\Pr(V_1 \cap 1_+)}{\Pr(1_+)} = \frac{3/9}{5/9} = \frac{3}{5}.$$

Getting a positive blood test is a correct indicator of having the virus 60% of the time. Kind of a lousy test.

Solution 3

I'll use areas to illustrate the meaning behind the calculations. Initially, we are looking at

economics	math	philosophy	psychology	stats
35/100	22/100	18/100	16/100	9/100

Part a. By ruling out math and statistics, we change the "effective" sample space to

economics	philosophy	psychology
35/69	18/69	16/69

Essentially we are throwing the math and stats people out of consideration, and then taking into account that the remaining probabilities must sum to one, which is where the denominator comes from. Visually, we are accounting for the fact that the size of the sample space under consideration has shrunk, and thus the three remaining majors under consideration take up larger percentages of the now smaller total box. Technically,

$$\Pr(\text{econ} \mid \neg\text{math} \ \& \ \neg\text{stats}) = \frac{\Pr(\text{econ})}{\Pr(\ \neg\text{math} \ \& \ \neg\text{stats})} = \frac{35/100}{(35+18+16)/100} = \frac{35}{69},$$

similarly calculated for the other two.

Part b. By ruling out econ, the sample space under consideration becomes

philosophy	psychology
18/34	16/34

Using the same logic as before, now philosophy takes up 18% of the remaining $18+16 = 34\%$.

Part c. Philosophy gets ruled out and all that remains is

psychology
16/16

Yeah, process of elimination – you rule out every other possibility so of course the only remaining possibility has certain probability.