ECN 200D – Week 5 Lecture Notes

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There are a bunch of different types of equilibria. We'll be applying a sort of Arrow-Debreu equilibrium concept and comparing it so the social planner's outcome.

1 Simple Endowment Economy

There are two agents—you can think of them as islands. The endowments, which are in terms of only one good and normalized to 1, are given as

$$e_t^1 = \begin{cases} 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases} \qquad e_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ 1 & \text{if } t \text{ is odd} \end{cases}.$$

You can think of t as being seasons; during some seasons, island 1 can produce but island 2 can't; and one period later that is reversed. The good is non-storable. The allocation is $(c_t^1, c_t^2)_{t=0}^{\infty}$, and preferences are given by

$$u_i(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^2).$$

2 Arrow-Debreu Equilibrium

2.1 Introduction

The market opens once. Agents can trade any amount of the good they can by forming contracts, e.g.

> "I, agent 1, promise to pay x units of the good in period 2016 in exchange for y units of the good in 2017. "

An **Arrow-Debreu Equilibrium (ADE)** is a list of prices and allocations,

$$\{\hat{p}_t\}_{t=0}^{\infty} \quad \text{and} \quad \{\hat{c}_t^1, \hat{c}_t^2\},$$

subject to a bunch of constraints. Given equilibrium prices, we must have

$$\begin{aligned}
&\{\hat{c}_t^i\}_{t=0}^{\infty} \in \arg\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) && \textit{(maximized consumption)} \\
&\sum_{t=0}^{\infty} \hat{p}_t c_t^i = \sum_{t=0}^{\infty} \hat{p}_t e_t^i && \textit{(budget constraint)} \\
&\hat{c}_t^1 + \hat{c}_t^2 = \hat{e}_t^1 + \hat{e}_t^2 = 1 && \textit{(market clearing)} \\
&c_t^i \geq 0 && \textit{(nonnegativity)}
\end{aligned}$$

Recall that in this model we are allowing trade across time. Thus, it has to be the case that the value of *total* consumption is equal to the value of *total* endowment. However, in any given period, no one can buy or sell more than 1 unit of the good because that's all there is in that period; hence the market clearing condition.

2.2 Characterization of ADE

Have $\mathscr{L}^i$ be individual i's Lagrangian,

$$\mathscr{L}^i = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) - \lambda^i \left[\sum_{t=0}^{\infty} p_t (c_t^i - e_t^i) \right].$$

That's a hell of a thing. Let's differentiate it with respect to c_t^i and c_{t+1}^i and we get

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t \quad \text{and} \quad \frac{\beta^{t+1}}{c_{t+1}^i} = \lambda^i p_{t+1} \quad \Longrightarrow \quad \beta^t p_t c_t^i = p_{t+1} c_{t+1}^i. \tag{1}$$

Now sum this over both agents and we have

$$\beta^t p_t (c_t^1 + c_t^2) = p_{t+1}(c_{t+1}^i + c_{t+1}^2).$$

But we know that the sum of consumption in any given period has to sum to one, so we can simplify to

$$\beta^t \hat{p}_t = \hat{p}_{t+1}. \tag{2}$$

Oh hey, that looks really useful. And hell, why not just normal $p_0 = 1$, from which we can write $\hat{p}_t = \beta^t$. So we have the sequence of equilibrium prices. Woo. Plugging this into equation (1), we get

$$\beta^t p_t c_t^i = p_{t+1} c_{t+1}^i \quad \Longrightarrow \quad \beta^t p_t c_t^i = \beta^t p_t c_{t+1}^i \quad \Longrightarrow \quad c_t^i = c_{t+1}^i = \hat{c}_i.$$

Hey, now we know that consumption is irrespective of t. That makes sense—in an infinite horizon scenario where each period is the same as the last, there's no reason to expect consumption to change. Consumption is smoothed over each period instead of jerking around from 0 to 1 each time—this is a consequence of the concave utility functions.

But do not be fooled into thinking that $\hat{c}^i = 1/2$. In fact, the individual who receives 1 on the first period has an advantage because they begin without any discounting; whereas the second person begins with a discount of β. Indeed, let's look at the budget constraint with respect to individual 1:

$$\begin{aligned}
\sum_{t=0}^{\infty} \hat{p}_t c_t^1 = \sum_{t=0}^{\infty} \hat{p}_t e_t^1 \quad &\implies \quad \sum_{t=0}^{\infty} \beta^t \hat{c}_t^1 = [1 + 0\beta + 1\beta^1 + 0\beta^3 + 1\beta^4 + \ldots] \\
&\implies \quad \frac{\hat{c}_t^1}{1-\beta} = \frac{1}{1-\beta^2} \\
&\implies \quad \frac{\hat{c}_t^1}{1-\beta} = \frac{1}{(1-\beta)(1+\beta)} \\
&\implies \quad \hat{c}_t^1 = \frac{1}{1+\beta}.
\end{aligned}$$

So there's the allocation sequence for individual 1. It follows that

$$\hat{c}_t^2 = 1 - \frac{1}{1+\beta} = \frac{\beta}{1+\beta}.$$

Yeah, because $\beta < 1$, you can see that individual 2 gets some fraction of what individual 1 gets.

Notice that as β approaches 1, they approach equality of allocation. This makes sense because the effect of discounting is what leads to the discrepancy in allocation in the first place. But $\beta \neq 0$ in actuality. And of $\beta = 0$, then individual 1 gets everything.

So that's the ADE approach—on to the next approach.

3 Social Planner Equilibrium

3.1 The Problem

The benevolent social planner wants to maximize with respect to $\{c_t^1, c_t^2\}$ the objective function

$$\sum_{t=0}^{\infty} \beta^t \left[\alpha \ln(c_t^1) + (1-\alpha)\ln(c_t^2)\right],$$

where $\alpha \in [0,1]$ is the **Pareto weight**. A higher Pareto weight means the planner believes agent 1's utility is more important relative to agent 2's.

The social planner doesn't give a damn about money or prices, but only allocations. Thus, the only constraints the social planner must adhere to are the nonegativity of consumption and endowment limitations, namely,

$$c_t^1 + c_t^2 = 1. \tag{3}$$

3.2 Solving for Allocations

The social planner's Lagrangian function is

$$\mathscr{L}^{SP} = \sum_{t=0}^{\infty} \beta^t \left[\alpha \ln(c_t^1) + (1-\alpha)\ln(c_t^2)\right] - \sum_{t=0}^{\infty} \mu^t (c_t^1 + c_t^2 - 1).$$

With respect to c_t^1 and c_t^2, take the first order conditions to get

$$\frac{\alpha\beta^t}{c_t^1} = \mu^t \quad \text{and} \quad \frac{(1-\alpha)\beta^t}{c_t^2} = \mu^t \quad \Longrightarrow \quad c_t^2 = \frac{1-\alpha}{\alpha} c_1^t.$$

From equation (3), we can deduce that

$$c_t^1 + \frac{1-\alpha}{\alpha} c_1^t = 1 \quad \Longrightarrow \quad \hat{c}^1 = \alpha.$$

Hooray, more consumption smoothing. Similarly, $\hat{c}^2 = 1 - \alpha$. So given α, the **Pareto frontier set** is given by

$$PO(\alpha) = \{c_t^1 = \alpha, c_t^2 = 1 - \alpha\}_{t=0}^{\infty}.$$

That's great and everything, but we still need prices to have a complete description of the equilibrium. You might think that we'll want to pin down α to a specific value in the process. You'd be right.

3.3 The Negishi Method

Suppose the social planner has chosen some arbitrary α. Let the **transfer function**, denoted $t^i(\alpha)$, indicate be the amount that the social planner needs to transfer from agent i, evaluated at prices μ (these are *shadow prices* of the social planner problem), so that the agent can afford the allocation given by α. We want to find the value of α that requires no transfers to take place in order for the agents to afford the social planner's allocations—in other words, we want to find the value of α that satisfies the budget constraints.[1]

So define the transfer function explicitly, and set it equal to zero, with

$$t^i(\alpha) := \sum_{t=0}^{\infty} \mu_t(c_t^i(\alpha) - e_t^i) = 0.$$

It can be shown that $\mu_t = \beta^t$. Focusing on individual 1, we then have

$$t^1(\alpha) := \sum_{t=0}^{\infty} \mu_t(\alpha - e_t^1) := 0 \quad \Longrightarrow \quad t^1(\alpha) := \sum_{t=0}^{\infty} \beta^t(\alpha - e_t^1) := 0.$$

From this it follows that

$$\frac{\alpha}{1-\beta} = \frac{1}{1-\beta^2} \quad \Longrightarrow \quad \hat{\alpha} = \frac{1}{1+\beta}.$$

[1] The social planner ignored the budget constraint. This is how we are re-incorporating it.

Great, so $\hat{\alpha}$ is the one value of α such that $PO(\alpha)$ requires zero transfers and thus satisfies the budget constraints. This also coincides with the result from the Arrow-Debreu equilibrium. More importantly, we have the sequence of prices, namely, $\hat{p}_t = \mu_t$.