1   Motivation: Laffer Curve

Suppose the tax rate is zero. Then the government will not collect any tax revenue because it's not taxing anyone. When it increases its tax rate a little bit, though, it will generate some tax revenue because now it's, you know, actually taxing people.

Now suppose the tax rate is 100%. Because the government is taking away 100 percent of everyone's income, people are not going to bother working, and therefore the government has nothing to tax. So tax revenue is zero.

Point is, the level of tax revenue generated is a nonlinear function of the tax rate that looks like the figure below, which is called the *Laffer curve*.
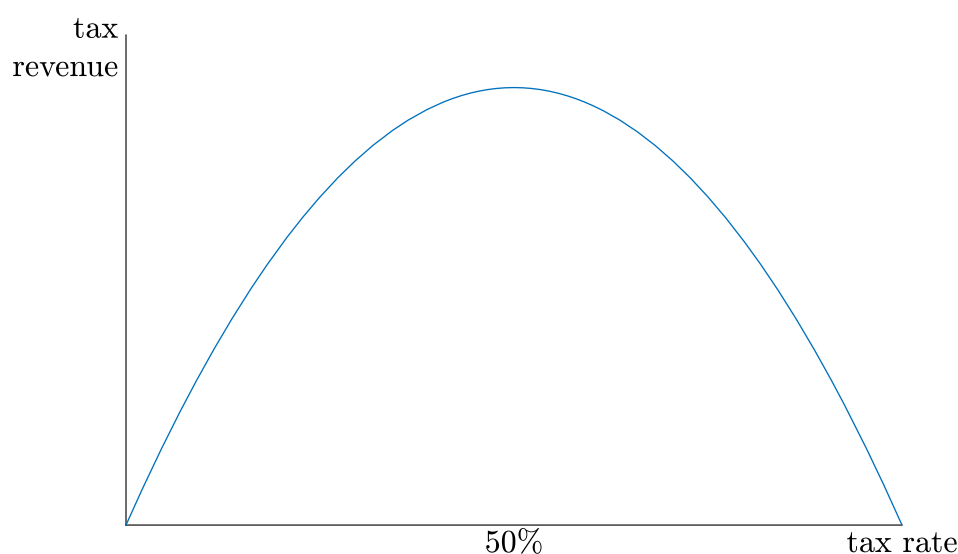

FIGURE 1: When the tax rate is low, more tax revenue can be generated by increasing the tax rate. When the tax rate is too high, however, working will be disincentivized, taxable income will fall, and therefore tax revenue will fall. Tax revenue is maximized at the vertex.

Let r denote tax revenue and t denote the tax rate. The Laffer curve can be captured via regression of form

$$r = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 z + u, \tag{1}$$

where z is just some variable (or variables) that we want to control for. We know $\beta_1 = 0$ because we know that a tax rate of zero implies zero tax revenue. (This is one of those cases where it's okay to not estimate the intercept coefficient.) Because the parabola is upside-down, we also expect $\beta_3 < 0$.

Now suppose we run the regression and come up with the estimate

$$\hat{r} = 500t - 5t^2. \tag{2}$$

The vertex is given by $t = 50$, which means that tax revenue is maximized at a tax rate of 50 percent.[1] If the tax rate is below 50%, then increasing the tax rate can generate more tax revenue. If the tax rate is above 50%, then increasing the tax rate will reduce tax revenue.

The marginal effect tells us how much higher we expect tax revenue to be if the tax rate is a little bit higher. In maths, the marginal effect is given by

$$\frac{\partial \widehat{r}}{\partial t} = 500 - 10t.$$

This equation makes it clear that if the tax rate is below $t = 50$ percent, then the marginal effect of tax rate on tax revenue will be positive. And if the tax rate is above $t = 50$ percent, then the marginal effect of tax rate on tax revenue will be negative.

For example, if a country has a tax rate of $t = 20$ percent, then their marginal effect is

$$\frac{\partial \widehat{r}}{\partial t}(20) = 500 - 10(20) = 300.$$

This says that a country with a tax rate one percentage higher than 20, on average, has tax revenue of about 300 more.

Or if a country has a tax rate of $t = 70$ percent, then their marginal effect is

$$\frac{\partial \widehat{r}}{\partial t}(20) = 500 - 10(70) = -200.$$

This says that a country with a tax rate one percentage higher than 70, on average, has tax revenue of about 200 less.

In general, the **marginal effect** of regressor x_j on dependent variable y is given by

$$\text{ME} \equiv \frac{\partial \widehat{y}}{\partial x_j}. \tag{3}$$

2   Average Marginal Effect (AME)

The marginal effect depends on t. That means it's not possible to capture the entirety of marginal effects of the data in a single number. But we can still use some traditional summary statistics to capture some overall picture of what marginal effects are in the data.

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[1]This value actually seems to be closer to 70 percent in reality, although you should not be lulled into thinking that a 70 percent tax rate is necessarily the socially optimal tax rate. It is of critical importance is how the tax revenue is used. Addressing homelessness in an intelligent way? Great. Building a golden statue of Dear Leader? Not so great.

Suppose we have three countries, shown in the figure below.

- Country 1 has a tax rate of 25 percent and collects 8000 tax revenue (the blue dot). The marginal effect for country 1 is

$$ME_1 = 500 - 10(25) = 250.$$

- Country 2 has a tax rate of 35 percent and collects 13000 tax revenue (the magenta dot). The marginal effect for country 2 is

$$ME_2 = 500 - 10(35) = 150.$$

- Country 3 has a tax rate of 60 percent and collects 11000 tax revenue (the red dot). The marginal effect for country 3 is

$$ME_3 = 500 - 10(60) = -100.$$

Okay, so we have three observations and we've calculated the marginal effect for those three observations. I should not have to convince you then that the average marginal effect is given by

$$AME = \frac{250 + 150 - 100}{3} = 100.$$

It's really just like any other average, except the thing we're averaging is marginal effects, hence its name.
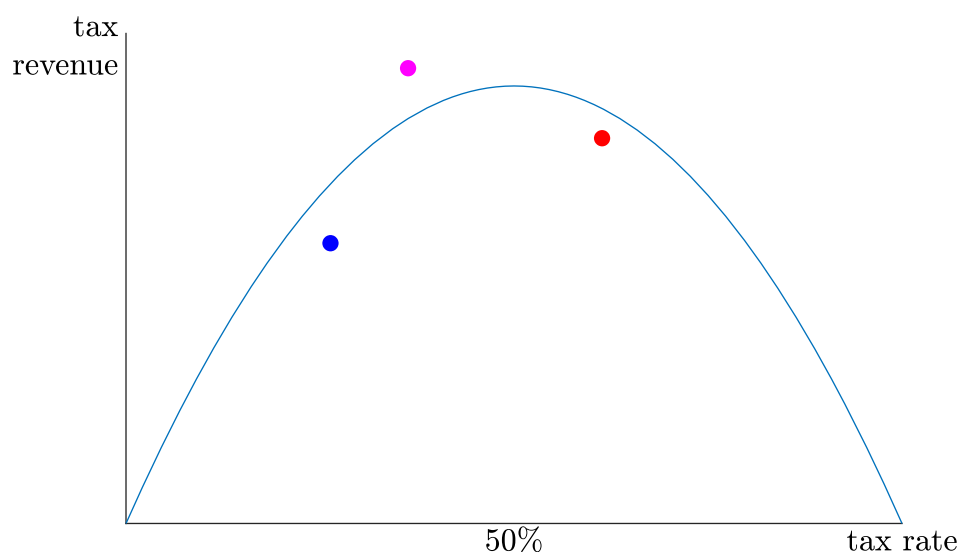
The general definition of the **average marginal effect (AME)** of regressor x_j on y (when there are no interaction terms) is

$$\text{AME} \equiv \frac{1}{n} \sum_{i=1}^{n} \frac{\partial \widehat{y}}{\partial x_j}(x_i) \tag{4}$$

Again in words: evaluate the marginal effect at all n observations, and average them. Geometrically, it is the average slope of the regression for our data, which would be difficult for me to draw and for you to read so I'm not even going to try.

This gives an overall picture of marginal effects in the data set. You can see in the figure that two observations are below 50 percent, and only one observation is above 50 percent, so it shouldn't be too surprising that the average marginal effect is positive.

3   Marginal Effect at the Mean (MEM)

Another way to try to get an overall idea of what marginal effects are on the data set is to evaluate the marginal effect at the mean of the regressor. Here the mean tax rate is

$$\bar{t} = \frac{25 + 35 + 60}{3} = 40.$$

So the marginal effect evaluated at the mean is

$$\text{MEM} = 500 - 10(40) = 100.$$

Oh hey, it's exactly the same as the average marginal effect. This is the case when the regressor of interest is quadratic, but it is not true in general that MEM = AME. The MEM is illustrated in the figure below.
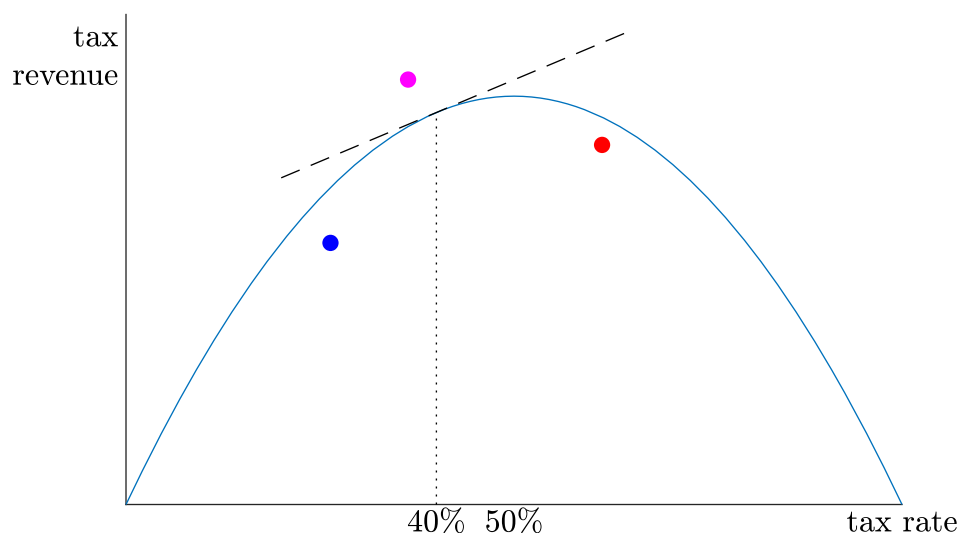
FIGURE 2: The mean tax rate is 40%. Evaluating the marginal effect at this mean value gives the slope shown with the dashed line.

The general definition of the **marginal effect at the mean (MEM)** of regressor x_j on y (when there are no interaction terms) is

$$\text{MEM} \equiv \frac{\partial \widehat{y}}{\partial x_j}(\overline{x}_j), \tag{5}$$

where $\overline{x}_j$ is the mean of regressor x_j.

4   Interaction Terms

Things can get a bit messier when dealing with interaction terms. For instance, you might have something like

$$\widehat{y} = b_1 + b_2 x + b_3 x^2 + \beta_4 xz,$$

which has marginal effect

$$\frac{\partial \widehat{y}}{\partial x} = b_2 + 2b_3 x + \beta_4 z.$$

In this case, you would have to use data for both x and z to calculate the marginal effect or any of its summaries.

For example, if you want to calculate the MEM, you'd plug in both the mean of x and the mean of z, giving

$$\text{MEM} = b_2 + 2b_3\overline{x} + \beta_4\overline{z}.$$

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In other words, we're evaluating the marginal effect at the mean of regressor x and the mean of regressor z.

Likewise, the AME would be

$$\text{AME} = \frac{1}{n} \sum_{i=1}^{n} (b_2 + 2b_3 x_i + \beta_4 z_i),$$

where the AME for each observation i requires plugging in both x_i and z_i to evaluate.

This is here mainly to let you know that in a multivariable regression with interaction terms, marginal effect will depend on other regressors.