ECN 200D – Week 2 Lecture Notes

William M Volckmann II

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These notes will briefly discuss Poisson processes and bargaining. Then it will begin to introduce the Mortensen-Pissarides model of labor markets.

1 Poisson Processes

Let $N_t(\omega)$ denote the number of occurrences of the event ω up to time t. The set $N = \{N_t, t \geq 0\}$ is an **arrival process**. We will be focusing on the **Poisson process**.

(a) Any jump in N will be of size 1.

(b) For any $t, s \geq 0$, the difference $N_{t+s} - N_t$ is independent of u for any $u < t$. (*Poisson is memoryless.*)

(c) For any t, t', the difference $N_{t+s} - N_t = N_{t'+s} - N_{t'}$. (*Poisson is time invariant.*)

Recall that the probability mass function for a Poisson distributed random variable is

$$P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

It turns out that λ is the **arrival rate**, that is, it tells us how many times an event will happen per unit time.

Considering $E[N_t]$, we can calculate it as

$$\begin{aligned} E[N_t] &= \sum_{k=0}^{\infty} kP(N_k = k) \\ &= \sum_{k=0}^{\infty} k\frac{e^{-\lambda t}(\lambda t)^k}{k!}. \end{aligned}$$

Since the first term will evaluate to zero, we can start the sum at $k = 1$ instead, giving

$$e^{-\lambda t}\sum_{k=1}^{\infty} k\frac{(\lambda t)^k}{k!}.$$

We have $k/[k(k-1)(k-2)\ldots]$, which we can simplify to

$$e^{-\lambda t}\sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}.$$

We can shift the index back down to zero, but we have to compensate by shifting the k terms in the summand up by one, giving

$$e^{-\lambda t}\sum_{k=0}^{\infty} \frac{(\lambda t)^{k+1}}{k!} = e^{-\lambda t}(\lambda t)\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}.$$

The sum is the Taylor series expansion for $e^{\lambda t}$, so we can finally simplify to

$$e^{-\lambda t}(\lambda t)e^{\lambda t} = \lambda t.$$

Great, so $E[N_t] = \lambda t$. This should be intuitive. If we something happens, on average,λ times in 1 unit of time, then we should expect it to happen $t\lambda$ times in t units of time.

2 Bargaining

We'll be considering a game between two players – a firm and worker. The idea is that firms have machines, workers have the skill and labor to use the machines. Alone they're unproductive, but together they can produce output. The question is, how is the output divided between the two? In order words, they need to bargain over a wage.

If they make an agreement, then their payoffs will be in some set S. In particular, there will be some function $f(S, d) \to S$ that gives the solution to the bargaining problem. In other words, the function looks at the set of agreements payoffs S and the disagreement payoffs and assigns an "optimal" agreement. The particular agreement assigned will depend on the bargaining power of the players. Let $\theta \in [0, 1]$ be player 1's **bargaining power**. (Then $1 - \theta$ is player 2's bargaining power.)

We will consider two axiomatic approaches to bargaining.

2.1 Nash Bargaining

The axioms John Nash decided would be reasonable for a bargaining equilibrium are the following.

(a) *Irrelevance to equivalent utility representations.* The set S represents preferences, which we'll usually be able to represent as utility functions. Utility functions, of course, can be subjected to positive monotone transformations without altering the underlying preferences. Thus, we want the same solutions to emerge with any utility function that represents the preferences.

(b) *Independence of Irrelevant Alternatives.* Suppose that $f(S, d)$ maps to S', where $S' \subset S$. Then it should be the case that the solution of $f(S, d) = f(S', d)$. In other words, none of the agreements in $S \setminus S'$ were a solution, so we shouldn't expect the solution to change if those agreements become infeasible.

(c) *Pareto Efficiency.*

The collection of these axioms give rise to **Nash bargaining**.

There is a rather long and ugly proof, but it turns out that the solution for the bargaining problem is actually the Stone-Geary function

$$f_\theta(S, d) = \arg\max_{s \in S} (s_1 - d_1)^\theta (s_2 - d_2)^{1-\theta}.$$

The coordinate $s = (s_1, s_2)$ contains two points in the agreement set for player 1 and 2, respectively. Player 1 moving to s_1 generates a surplus of $s_1 - d_1$, and player 2 moving to s_2 generates a surplus of $s_2 - d_2$. Finding the agreement points that maximize these surpluses, weighted by the relative bargaining power, is what solves the bargaining problem.

Example 1. We have a pie! Um, it's a pie of size 1. We want to choose what fraction q of the pie will go to player 1. Then player 2 will get fraction $1 - q$. If no agreement is made, then they get d_1 and d_2 fractions, respectively.

We can write the Nash bargaining problem as

$$\arg\max_{q \in [0,1]} (q - d_1)^\theta (1 - q - d_2)^{1-\theta}.$$

Since we are only interested in the maximizing argument q, we can take a monotonic transformation of the objective function without altering the maximizers. So hey, why not take a logarithm of it? Then we're solving

$$\arg\max_{q \in [0,1]} \theta \log(q - d_1) + (1 - \theta) \log(1 - q - d_2).$$

It's a function only of θ, so we can take the derivative to find the critical point. This first-order condition gives

$$\frac{\theta}{q - d_1} - \frac{1 - \theta}{1 - q - d_2} = 0 \quad \Longrightarrow \quad \frac{\theta}{q - d_1} = \frac{1 - \theta}{1 - q - d_2}.$$

Do some algebra and you'll find that

$$q = d_1 + \theta(1 - d_1 - d_2).$$

This is unsurprising, since this is indeed the solution to the Stone-Geary utility maximization problem. You can think of it like this. Player 1 automatically gets fraction d_1 of the pie, player 2 automatically gets fraction d_2 of the pie. Then, of the remaining pie, player 1 gets θ of it. Similarly, player 2 gets

$$1 - q = d_2 + (1 - \theta)(1 - d_1 - d_2).$$ ■

Example 2. Now let's suppose that a firm meets a worker. If they agree to work, then they'll produce output of p. The worker can also just sit at home and collect an unemployment payment of z. So they'll need to bargain over some wage w for the worker. Let θ be the bargaining power of the worker.

If the job is accepted, then the worker gets w; if not, then z. So the surplus generated by the worker would be $w - z$. If the job is accepted, then the firm produces p; if not, then the firm still has that w in wage payments. So the surplus generated by the firm would be $p - w$. Thus, the bargaining problem is

$$\arg\max_w (w - z)^\theta (p - w)^{1-\theta}.$$

Again, let's log it up to instead solve

$$\arg\max_w \theta \log(w - z) + (1 - \theta)\log(p - w).$$

It's a function of one variable w, so just take the derivative with respect to w and set it equal to zero to find the critical point. As you might have expected, the solution is

$$w = z + \theta(p - z).$$ ■

2.2 Kalai Bargaining

Add one axiom to Nash bargaining and we have **Kalai bargaining**.

(d) *Monotonicity.* If $S \subseteq S'$, then $f_j(S', d) \geq f_j(S, d)$, for players $j = 1, 2$. In other words, if we are considering a larger set of possible agreements, then no one can be worse off.

I find this to be rather questionable since choice paralysis is a well-studied phenomenon. But whatever. The nice thing about incorporating this extra axiom is that the solution becomes even simpler, in particular,

$$f_\theta^k(S, d) = \arg\max_{s \in S}(s_1 - d_1) \quad \text{s.t.} \quad s_1 - d_1 = \frac{\theta}{1-\theta}(s_2 - d_2).$$

The takeaway is that we now have proportionate solutions to the bargaining situation.

Example 3. Let's split that pie! Sounds sexy. The problem we want to solve is

$$\max_q (q - d_1) \quad \text{s.t.} \quad q - d_1 = \frac{\theta}{1-\theta}(1 - q - d_2).$$

Well ah, the constraint already gives us everything we need. Just solve it for q to find that $q = d_1 + \theta(1 - d_1 - d_2)$. This is the same solution given by the Nash bargaining problem! Indeed, for simple models like this, the two will give the same results. ■

3 Mortensen-Pissarides Model

3.1 The Environment

(a) Time is continuous an infinite in horizon.

(b) The labor force is normalized to 1.

(c) u denotes the proportion of unemployed workers. (Also the number of unemployed workers because of the previous point.)

(d) v is the number of vacancies.

(e) Workers receive z per unit of time when unemployed.

(f) There exist infinitely many firms that could potentially open a vacancy.

(g) Each firm has one vacancy.

(h) While the vacancy is unfilled, firms pay cost $p \cdot c$, where p is the amount of good that would have been produced were the vacancy filled. Think of c as a recruitment cost. We multiply them together to capture the idea that more productive jobs have higher recruitment costs.

(i) The rate of job destruction is given by λ, the arrival rate of a Poisson process.

(j) After λ hits, the worker goes back to unemployment, and the firm goes back to the pool of firms that could potentially open a vacancy.

3.2 The Matching Function

We will be using a **matching function** $m(u, v) \to m$ that describes the amount of **matches** made, i.e. jobs formed, within a period. Once a match is formed, any surplus is split between the firm and the worker according to Nash bargaining, where β denotes the bargaining power of the worker.

(a) $m(u, v)$ is increasing in both arguments. The intuition is that if there are more unemployed workers, then firms will have an easier time finding the right worker; and if there are more vacancies, then workers will have an easier time finding the right job.

(b) $m(u, v)$ is concave. This isn't really important for us.

(c) $m(u, v)$ is homogeneous of degree one, which is to say, $m(\lambda u, \lambda v) = \lambda m(u, v)$ for $\lambda > 0$.[1]

Define $\theta = v/u$. This represents **market tightness**. If the number of vacancies v is relatively large compared to the number of job seekers u, then the labor market is considered *tight*. On the other hand, when the number of vacancies is relatively small compared to the number of job seekers, then the labor market is considered *slack*.[2]

Now let's derive some useful objects from the matching function. Suppose this period begins with 100 vacancies and 10 matches are made. Then the proportion of filled vacancies is 10/100, which is the (Poisson) arrival rate of workers to a firm. More generally, the arrival rate of workers to firms is $m(u, v)/v$. Since $m(u, v)$ is homogeneous of degree one, we can write

$$\frac{1}{v} m(u, v) = m\left(\frac{u}{v}, 1\right) = m\left(\frac{1}{\theta}, 1\right) = q(\theta).$$

We can similarly define the (Poisson) arrival rate of jobs to workers as

$$\frac{1}{u} m(u, v) = m(1, \theta) = \theta m\left(\frac{1}{\theta}, 1\right) = \theta q(\theta).$$

Notice that $q'(\theta) < 0$. This is because increasing θ decreases the first argument of m, and m is increasing in both arguments. Thus, the negative derivative. Intuitively, this means that when there are more vacancies, job seekers are more likely to find a job elsewhere. Alternatively, when u becomes smaller, there are fewer workers from which to match, so $q(\theta)$ would become smaller.

On the other hand, $d[\theta q(\theta)]/d\theta > 0$. We can write $\theta q(\theta) = m(1, \theta)$. Since $m(1, \theta)$ is increasing in both arguments, the positivity of the derivative is

[1] It was said in lecture to exhibit constant returns to scale, but I find that terminology awkward since we are not even dealing with returns.

[2] I find it helpful to think of workers as the "commodity." If it's hard to find a worker, then the market is tight; and vice versa.

clear. This should be intuitive. When v becomes larger, there are more job openings posted, and thus workers will have an easier time landing jobs. Similarly, when u becomes smaller, there are less people looking at the existing pool of vacancies, and thus it becomes easier for them to land those jobs.