Geometric Distribution

A geometric distribution is the number of trials until one success in a series of Bernoulli trials. (Or number of failures on the right!)

$$P(X=k) = (1-p)^{k-1}p \qquad P(X=k) = (1-p)^k p$$

$$E(X) = \frac{1}{p} \qquad E[X] = \frac{1-p}{p}$$

$$var(X) = \frac{1-p}{p^2} \qquad var(X) = \frac{1-p}{p^2}$$

Negative Binomial Distribution

Let X denote the the number of trials needed until getting the r^{th} success. A negative binomial distribution, denoted $NB(r,p)$, is the probability of the number of trials undertaken until the rth success.

$$NB(r,p) = P(X=k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$$E(X) = \frac{r}{p} \qquad var(X) = \frac{r(1-p)}{p^2}$$

Poisson Distribution

Let λ be the parameter which indicates the average number of events in the given time interval. The range of a Poisson Distribution is $\{0, 1, 2, ...\}$. The Poisson Distribution is given by

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad E(X) = \lambda, \quad var(X) = \lambda.$$

Suppose $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$ and that X, Y are independent. Then we have

(a) $E(X+Y) = E(X) + E(Y) = \lambda + \mu$,

(b) $X + Y \sim Poisson(\lambda + \mu) = P(X+Y=z)$

$$= e^{\lambda - \mu} \sum_{y=0}^{z} \frac{\lambda^{z-y}}{(z-y)!} \frac{\mu^y}{y!}.$$

Exponential Distribution

The exponential distribution is commonly used to model waiting times between occurrences of rare events. Given a rate parameter λ, we have for $x \geq 0$,

$$f_X(x) = \lambda e^{-\lambda x} \qquad F_X(x) = 1 - e^{-\lambda x},$$

$$E(X) = \frac{1}{\lambda}, \qquad var(X) = \frac{1}{\lambda^2}.$$

Gamma Distribution

Let Y denote the sum of r number of $exponential(\lambda)$ random variables. Then Y denotes the number of time until the r^{th} event.

$$f(y) = e^{-\lambda y} \frac{(\lambda y)^{y-1}}{(r-1)!}.$$

Uniform Distribution

$$f(x) = \frac{1}{b-a} \qquad F(x) = \frac{x-a}{b-a}$$

$$E[X] = \frac{1}{2}(a+b) \qquad var(X) = \frac{1}{12}(b-a)^2$$

Normal Distribution

$$\text{CLT: } \frac{(\overline{X} - \mu)}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$X \sim N(\mu, \sigma^2) \implies X = \mu + \sigma Z$$

$$X \sim N(\mu, \sigma^2) \implies \alpha X \sim N(\alpha\mu, \alpha^2\sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Bivariate Normal Distribution

If $X \sim N(0,1)$, $Y \sim N(0,1)$, $corr(X,Y) = \rho$, then

$$Y|X = x \sim N(\rho x, 1 - \rho^2)$$

$$X|Y = y \sim N(\rho y, 1 - \rho^2)$$

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad \text{for } X, Y \sim N(0,1)$$

$$Y = \rho X + (1 - \rho^2) Z$$

$$x = r\cos(\theta), y = r\sin(\theta), \theta = \arctan\left(\frac{y}{x}\right)$$

Expectation

$$E[X] = \sum_x x P(X=x)$$

$$E[X] = \int_x x P(X=x)\, dx$$

$$E[X+Y] = E[X] + E[Y]$$

$$E[XY] = E[X]E[Y], \quad \text{for } X, Y \text{ independent}$$

$$E[XY] = \sum_x \sum_y xy P(X=x, Y=y)$$

Variance

(a) $var(c) = 0$, where c is a constant

(b) $var(X) = E(X^2) - E(X)^2$

(c) $var(aX) = a^2 \cdot var(X)$, where a is a constant

(d) $var(X+Y) = var(X) + var(Y)$, independent X, Y

(e) $var(X+Y) = var(X) + var(Y) + 2cov(X,Y)$

Covariance

(a) $cov(X,X) = var(X)$

(b) $cov(X,c) = 0$, where c is a constant

(c) $cov(X,Y) = E(XY) - E(X)E(Y)$

(d) $cov(X,Y) = 0$ for independent X, Y

(e) $cov(X,Y) = cov(Y,X)$

(f) $cov(aX,Y) = a \cdot cov(X,Y)$

(g) $cov(X, Y+Z) = cov(X,Y) + cov(X,Z)$

Correlation

(a) $corr(X,Y) = \dfrac{cov(X,Y)}{\sqrt{var(X)var(Y)}}$

(b) $corr(X,X) = 1$

(c) $corr(X,Y) = corr(Y,X)$

(d) X, Y independent $\Rightarrow corr(X,Y) = 0$

(e) $-1 \leq corr(X,Y) \leq 1$

(f) $corr(aX+b, Y) = corr(X,Y)$ for $a > 0$, $-corr(X,Y)$ for $a < 0$

Conditional Expectation

(a) $E[X|Y=y] = \sum_{\text{all } x} xP(X=x|Y=y)$

(b) $E[X|Y=y] = \int_x x f_{X|Y}(x,y)\, dx$

(c) $f_Y(y) = \int_{\text{over } x} f(x,y)\, dx$

(d) $E[E(Y|X)] = E[Y]$

(e) $E[X+Y|Z] = E[X|Z] + E[Y|Z]$

(f) $E[Y] = \sum_x E[Y|X=x]P(X=x)$

(g) $E[g(X)Y|X] = g(X)E[Y|X]$

(h) $var(X) = E\Big(var(\ X \mid Y\ )\Big) + var\Big(E[\ X \mid Y\ ]\Big)$

Strategy: separate into independent and dependent parts.

Moment Generating Functions

(a) $M_X(t) = E\left[e^{tX}\right] = \sum_x e^{tx} p_X(x)$

(b) $M_X(t) = E\left[e^{tX}\right] = \int_x e^{tx} f_X(x)\, dx$

(c) $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

(d) $\frac{d}{dt}[M_X(t)] = E[X]$

(e) $\frac{d^2}{dt^2}[M_X(t)] = E[X^2]$

(f) $Bernoulli(p):\ 1-p+pe^t$

(g) $Binomial(n,p):\ (1-p+pe^t)^n$

(h) $Geometric(p):\ \dfrac{pe^t}{1-(1-p)e^t}$ for $t < -\ln(1-p)$

(i) $Poisson(\lambda):\ e^{\lambda(e^t-1)}$

(j) $Exponential(\lambda):\ \left(1-\dfrac{t}{\lambda}\right)^{-1}$

(k) $Gamma(r,\lambda):\ \left(1-\dfrac{t}{\lambda}\right)^{-r}$

(l) $NB(r,p):\ \dfrac{(1-p)^r}{(1-pe^t)^r}$

(m) $N(\mu,\sigma^2):\ e^{t\mu+\frac{1}{2}\sigma^2 t^2}$

Joint, Marginal, Conditional Probabilities

$$P(X=x, Y=y) = P(x,y)$$

$$P(X=x) = \sum_{\text{all } y} P(X=x, Y=y)$$

$$= \sum_{\text{all } y} P(X=x|Y=y)P(Y=y)$$

$$f(x,y) = \int_x \int_{y(x)} f(x,y)\, dxdy = 1$$

$$f_X(x) = \int_{y(x)} f(x,y) dy$$

$$f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)}$$

Change of Variables

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}$$

Geometric Series

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

Poisson Approximation

$Binomial(n,p)$ with large n and small p is approximated by $Poisson(np)$.

Chebyshev's Inequality

$$P(|X-\mu| \geq \epsilon) \leq \frac{var(X)}{\epsilon^2}$$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Markov's Inequality

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Law of Large Numbers

Let X_i be i.i.d, and let $\overline{X}_n = (X_1 + ... + X_n)/n$ represent the average. Then

$$P(|\bar{X}_n - \mu| > \epsilon) \to 0 \quad \text{as } n \to \infty.$$

Conditional Probability

(a) $P(AB) = P(A|B)P(B) = P(B|A)P(A)$

(b) $P(A|B) = \dfrac{P(AB)}{P(B)}$

(c) $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

Series

$$e^x = \sum_{n=0}^{n} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$