Exercise 1

We are dealing with

$$[\pi^*, \ \ 1-\pi^*] = [\pi^*, \ \ 1-\pi^*]\begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}.$$

When is there a unique stationary distribution?

Breaking out the matrix, we have

$$\begin{aligned} 1-\pi^* &= \pi^*(1-p_{11}) + (1-\pi^*)p_{22} \\ \pi^* &= \pi^* p_{11} + (1-\pi^*)(1-p_{22}). \end{aligned}$$

Subtract the second line from the first line to get

$$\begin{aligned} 1-2\pi^* &= \pi^*(1-2p_{11}) + (1-\pi^*)(2p_{22}-1) \\ &= \pi^*(1-2p_{11}-2p_{22}+1) + 2p_{22} - 1 \\ \implies \quad (2-p_{11}-p_{22})\pi^* &= 1-p_{22}. \end{aligned}$$

It follows that

$$\pi^* = \frac{1-p_{22}}{2-p_{11}-p_{22}}.$$

As long as the denominator is greater than zero, we have uniqueness, i.e. if $p_{11}+p_{22} < 2$.

Exercise 2

Note that $\bar{y}^2$ is another 2×1 matrix even though that flies in the face of matrix multiplication. That could have been stated somewhere.

We know that $y_{t+1} = \bar{y}' x_{t+1}$, and therefore

$$\begin{aligned} E[y_{t+1}|x_t] &= \bar{y}' E[x_{t+1}|x_t] \\ &= \bar{y}' e_1 \times P(x_{t+1} = e_1|x_t) + \bar{y}' e_2 \times P(x_{t+1} = e_2|x_t) \\ &= 1 \times P(x_{t+1} = e_1|x_t) + 5 \times P(x_{t+1} = e_2|x_t) \\ &= [1 \quad 5] \begin{bmatrix} P(x_{t+1} = e_1|x_t) \\ P(x_{t+1} = e_2|x_t) \end{bmatrix} \\ &= [1 \quad 5] \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} x_t \\ &= \bar{y}' P' x_t. \end{aligned}$$

Therefore we want to solve

$$[1 \quad 5] \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [1.8 \quad 3.4],$$

which gives rise to the system of equations

$$p_{11} + 5p_{12} = 1.8,$$

$$p_{21} + 5p_{22} = 3.4.$$

We also need to think about

$$
\begin{aligned}
E[y_{t+1}^2|x_t = e_i] &= E[\bar{y}'x_{t+1}\bar{y}'x_{t+1}|x_t] \\
&= \bar{y}'e_1\bar{y}'e_1 \times P(x_{t+1} = e_1|x_t) + \bar{y}'e_2\bar{y}'e_2 \times P(x_{t+1} = e_2|x_t) \\
&= 1 \times P(x_{t+1} = e_1|x_t) + 25 \times P(x_{t+1} = e_2|x_t) \\
&= [1 \quad 25]' \begin{bmatrix} P(x_{t+1} = e_1|x_t) \\ P(x_{t+1} = e_2|x_t) \end{bmatrix} \\
&= \bar{y}^{2'} \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} x_t \\
&= \bar{y}^{2'} P' x_t.
\end{aligned}
$$

Therefore we want to solve

$$
[1 \quad 25] \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [5.8 \quad 15.4],
$$

which gives rise to the system of equations

$$
\begin{aligned}
p_{11} + 25p_{12} &= 5.8, \\
p_{11} + 5p_{12} &= 1.8, \\
p_{21} + 25p_{22} &= 15.4, \\
p_{21} + 5p_{22} &= 3.4.
\end{aligned}
$$

Solving the first two equations gives $p_{21} = 1/5$ and $p_{11} = 4/5$; they sum to 1, so that's a good sign. The second two equations give $p_{22} = 3/5$ and $p_{12} = 2/5$; these also sum to 1. Therefore

$$
P = \begin{bmatrix} 4/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix}.
$$

Exercise 3

(a) The stationary distribution satisfies

$$[\pi \quad 1-\pi] = [\pi \quad 1-\pi] \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \quad \Longrightarrow \quad \pi^* = \frac{6}{13}.$$

(b) $P(y_1 = 2|y_0 = 1) = p_{12} = 0.7.$

(c) The conditional probability of observing $y_2 = 2$ is

$$\begin{aligned} P(y_2 = 2|y_0 = 1) &= P(y_1 = 1|y_0 = 1)P(y_2 = 2|y_1 = 1) + P(y_1 = 2|y_0 = 1)P(y_2 = 2|y_1 = 2) \\ &= P_{11}P_{12} + P_{12}P_{22} \\ &= (0.3)(0.7) + (0.7)(0.4) \\ &= .49. \end{aligned}$$

The conditional probability of observing $y_2 = 1$ is

$$\begin{aligned} P(y_2 = 1|y_0 = 1) &= P(y_1 = 1|y_0 = 1)P(y_2 = 1|y_1 = 1) + P(y_1 = 2|y_0 = 1)P(y_2 = 1|y_1 = 2) \\ &= P_{12}P_{21} + P_{11}P_{11} \\ &= (0.7)(0.6) + (0.3)(0.3) \\ &= .51. \end{aligned}$$

(d) The squared Markov matrix is

$$P^2 = \begin{bmatrix} 0.51 & 0.49 \\ 0.42 & 0.58 \end{bmatrix}.$$

Element P_{11}^2 is the probability of having the first state; element P_{12}^2 is the probability of having the second state.

Exercise 4

(a) The Bellman equation, including budget constraint and law of motion of capital, is

$$V(K, z) = \ln(C) + \beta E[V(K', z')] - \lambda_t[C + K' - zF(K, L)].$$

The state variables are K and z. The control variables are C, K', and L.

(b) The first order conditions are, with respect to C and K',

$$\begin{aligned} \frac{1}{C} &= \lambda_t, \\ \beta E[V_K'(K', z')] &= \lambda_t, \\ \implies \quad \frac{1}{C} &= \beta E[V_K'(K', z')]. \end{aligned} \tag{1}$$

The envelope condition with respect to k gives

$$\begin{aligned} V_K'(K, z) &= \lambda_t z F_K(K, L) \\ &= \frac{1}{C} z F_K(K, L) \\ \implies \quad V_K'(K', z') &= \frac{z' F_K(K', L')}{C'}. \end{aligned}$$

Combining this with equation (1), we get the Euler equation

$$\frac{1}{C} = \beta E\left[\frac{z' F_K(K', L')}{C'}\right].$$

Using the functional form $F(K, L) = K^\alpha L^{1-\alpha}$, we have

$$\frac{1}{C} = \alpha\beta E\left[\frac{z'[K']^{\alpha-1}[L']^{1-\alpha}}{C'}\right].$$

(c) There is no utility for leisure, so we can have $L = 1$. Then we can write $C = zK^\alpha - K'$, and therefore we can write the Bellman equation as

$$V(K, z) = \ln(zK^\alpha - K') + \beta E[V(K', z')].$$

Substituting in our guess for the value function, we get

$$V(K,z) = \ln(zK^\alpha - K') + \beta E[G + B\ln(k') + D\ln(z')].$$

Take the first order condition with respect to k' and we end up with

$$\frac{1}{zK^\alpha - K'} = \beta E\left[\frac{B}{K'}\right] = \frac{\beta B}{K'} = \quad\Longrightarrow\quad K' = \frac{\beta B z K^\alpha}{1+\beta B}.$$

The expectations operator above disappears because K' is a choice variable and B is just some constant. Keeping in mind that $\ln(z') = \rho\ln(z) + \epsilon'$, and assuming that $E[\epsilon_t] =$ for all t, we have

$$\begin{aligned}
G + B\ln(K) + D\ln(z) &= \log\left(zK^\alpha - \frac{\beta B z K^\alpha}{1+\beta B}\right) + \beta E\left[G + B\ln\left(\frac{\beta B z K^\alpha}{1+\beta B}\right) + D\ln(z')\right] \\
&= \log\left(zK^\alpha - \frac{\beta B z K^\alpha}{1+\beta B}\right) + \beta\left[G + B\ln\left(\frac{\beta B z K^\alpha}{1+\beta B}\right)\right] + \beta D E[\ln(z')] \\
&= \log\left(\frac{zK^\alpha}{1+\beta B}\right) + \beta G + \beta B\ln\left(\frac{\beta B z K^\alpha}{1+\beta B}\right) + \beta D E[\rho\ln(z) + \epsilon'] \\
&= \alpha\log(K) - \log(1+\beta B) + \beta G + \beta B\ln(\beta B) + \alpha\beta B\log(K) \\
&\quad + \beta B\ln(z) - \beta B\ln(1+\beta B) + \beta D\rho\ln(z).
\end{aligned}$$

It follows that $B\log(K) = (\alpha + \alpha\beta B)\log(K)$, and therefore $B = \alpha + \alpha\beta B$, and consequently

$$B = \frac{\alpha}{1-\alpha\beta}.$$

Furthermore, it must be the case that $D\ln(z) = (\beta B + D\beta\rho)\ln(z)$, from which it follows that $D = \beta B + D\beta\rho$, and consequently

$$\begin{aligned}
D &= \frac{\beta B}{1-\beta\rho} \\
&= \frac{\alpha\beta}{(1-\beta\rho)(1-\alpha\beta)}.
\end{aligned}$$

Finally, G. We have

$$
\begin{aligned}
G &= \beta G + \beta B \ln(\beta B) - \beta B \ln(1+\beta B) - \log(1+\beta B) \\
\implies \quad G(1-\beta) &= \frac{\alpha\beta}{1-\alpha\beta} \ln\left(\frac{\alpha\beta}{1-\alpha\beta}\right) + \frac{\alpha\beta}{1-\alpha\beta} \ln\left(1-\alpha\beta\right) + \log\left(1-\alpha\beta\right) \\
\implies \quad G &= \frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right].
\end{aligned}
$$

Well, that was tedious.

We found earlier that

$$K' = \frac{\beta B z K^{\alpha}}{1+\beta B}.$$

Now that we know what B is, we can solve for the policy function,

$$
\begin{aligned}
\phi^K(K,z) &= \frac{\beta B z K^{\alpha}}{1+\beta B} \\
&= \frac{\beta\left(\frac{\alpha}{1-\alpha\beta}\right) z K^{\alpha}}{1+\beta\left(\frac{\alpha}{1-\alpha\beta}\right)} \\
&= \alpha\beta z K^{\alpha}.
\end{aligned}
$$