ECN 200B—Externalities Part 2

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1 The Setup

Let's show how a public good can break down the first fundamental theorem of welfare economics. Suppose there are $L+1$ commodities. There are $I \geq 2$ individuals whose preferences satisfy $u^i : \mathbb{R}_+^{L+1} \to \mathbb{R}$, written as $u^i(x^i, y)$, where $x^i \in \mathbb{R}_+^L$ and $y \in \mathbb{R}_+$. Let $w^i \in \mathbb{R}_+^L$ be the endowments of good $\ell = 1, \ldots L$. Commodity $L+1$ has to be produced according to the technology $F : \mathbb{R}_+^L \to R_+$, where $F(X) = Y$ and X denotes the inputs of the first L commodities.

There is only one firm, and individual i owns $s^i \in [0,1]$ of that firm's stock. The first L commodities are private and the $L+1$th commodity is public, i.e. nonrival and nonexclusive.

Let y_i be the number of units of good $L+1$ purchased by individual i at at price of q. Since good $L+1$ is public, individual i's utility function is

$$u^i \left(x^i, \sum_{k=1}^{I} y^k \right).$$

We will be looking for a **Nash-Walrasian equilibrium**. It is Walrasian with respect to utility maximization and market clearing; it is Nashian because everyone responds to what everyone else is doing.

Definition 1. A competitive equilibrium consists of $\left(p, q, (\bar{x}^i, \bar{y}^i)_{i=1}^I, \bar{X}, \bar{Y}\right)$ such that

(a) *Individual Rationality:* $(\bar{x}^i, \bar{y}^i)$ solves

$$\max_{x^i, y^i} \left\{ u^i \left(x^i, y^i + \sum_{k \neq i}^{I} \bar{y}^i \right) \right\} \qquad \text{subject to} \qquad px^i + qy^i \leq pw^i + s^i[q\bar{Y} - p\bar{X}].$$

(b) *Profit Maximization:* $(\bar{X}, \bar{Y})$ solves

$$\max_{X,Y} qY - pX \quad \text{such that} \quad F(X) = Y.$$

(c) *Market Clearing:* $\sum_{i=1}^{I} x^i + \bar{X} = \sum_{i=1}^{I} w^i$ and $\sum_{i=1}^{I} y^i = \bar{Y}$.

Suppose that $u^i \in C^2$ and $F \in C^2$. Also assume the usual properties, e.g. u^i is quasiconcave, $u_y^i > 0$, F is convex, etc.Suppose that $(p, q, \bar{x}, \bar{y}, \bar{X}, \bar{Y})$ is a competitive equilibrium and $(\bar{x}, \bar{y}, \bar{X}, \bar{Y})$ is Pareto efficient. Let's find the contradiction needed to show that the FFToWE breaks down.

2 The Characterization

Thanks to our lovely set of assumptions, we know we're working with interior solutions. It follows from individual rationality that

$$Du^i \left(\bar{x}^i, \bar{y}^i + \sum_{k \neq i}^{I} \bar{y}^k \right) = \lambda^i(p, q). \tag{1}$$

The (p, q) term indicates that when taking the derivative with respect to x^i, we should use the price p; when with respect to y^i, we should use the price q.

It follows from profit maximization, after taking the first order condition with respect to X that

$$p = qDF(\bar{X}). \tag{2}$$

Since $(\bar{x}, \bar{y}, \bar{X}, \bar{Y})$ is Pareto efficient, it must solve the "don't screw anyone over" constraint characterization, i.e.

$$\max_{x,y,X,Y} \left\{ u^1 \left(x^1, \sum_{i=2}^{I} y^i \right) \right\} \quad \text{such that} \quad u^2 \left(x^2, \sum_{i=1}^{2} y^i \right) \geq u^2 \left(\bar{x}^2, \sum_{i=1}^{2} \bar{y}^i \right),$$

along with $F(X) = Y$, $\sum_{i=1}^{I} y^i = Y$, and

$$\sum_{i=1}^{I} x^i + X = \sum_{i=1}^{I} w^i.$$

We may as well use market clearing for y^i to write

$$\max_{x,X,Y} \left\{ u^1 \left(x^1, Y \right) \right\} \quad \text{such that} \quad u^2 \left(x^2, Y \right) \geq u^2 \left(\bar{x}^2, \bar{Y} \right),$$

along with $F(X) = Y$ and

$$\sum_{i=1}^{I} x^i + X = \sum_{i=1}^{I} w^i.$$

3 The Lagrangian

Since it is essentially a constant, let's define $\bar{V}^i = u^i(\bar{x}^i, \bar{Y})$. Then we'll be using the Lagrangian

$$\mathcal{L} = u^1(x^1, Y) - \sum_{i \neq 1}^{I} \mu^i \left[\bar{V}^i - u^i(x^i, Y)\right] - \delta \left[\sum_{i=1}^{I} x^i + X - \sum_{i=1}^{I} w^i\right] + \epsilon[F(X) - Y].$$

We'll need to take the first order conditions with respect to each x_i, X, and Y. It's not as gross as it sounds.

(a) $\mu^i D_{x^i} u^i(x^i, Y) = \delta$

(b) $\delta = \epsilon DF(X)$

(c) $\sum_{i=1}^{I} \mu^i u_Y^i(x^i, Y) = \epsilon$, where $\mu^1 = 1$.

Consider equation (a) with $i = 1$. Since $\mu^i = 1$, we have

$$D_{x^1} u^1(\bar{x}^1, \bar{Y}) = \delta.$$

And from equation (1), we have

$$D_{x^1} u^1(\bar{x}^1, \bar{Y}) = \lambda^1 p.$$

It follows that p is a scalar multiple of δ. So let's just normalize p by dividing it with λ^1. Then $p = \delta$. When we can write equations (1) and (a) as, respectively,

$$D_{x^i} u^i(\bar{x}^i, \bar{Y}) = \lambda^i \delta = \frac{\delta}{\mu^i},$$

from which it follows that $\mu^i = 1/\lambda^i$.

We can use this with equation (c) to write

$$\sum_{i=1}^{I} \mu^i u_Y^i(x^i, Y) = \sum_{i=1}^{I} \frac{1}{\lambda^i} u_Y^i(x^i, Y) = \epsilon.$$

From equation (1), we can write

$$\frac{1}{\lambda_i}u^i_Y(\bar{x}^i,\bar{Y}) = q,$$

and therefore

$$\sum_{i=1}^{I}\frac{1}{\lambda^i}u^i_Y(x^i,Y) = \sum_{i=1}^{I} q = Iq = \epsilon.$$

And then from equation (b), we have

$$eDF(\bar{X}) = \delta = p = IqDF(\bar{X}).$$

From equation (2), we have

$$p = qDF(\bar{X}).$$

Therefore, $IqDF(\bar{X}) = qDF(\bar{X})$. We can't have $DF(\bar{X}) = 0$ because then $p = 0$. So it must be the case that $Iq = q$. We have assumed that $I \geq 1$, so then it must be the case that $q = 0$. But we can't have $q = 0$ because then $p = 0$ as well. So it cannot be the case that the competitive equilibrium is Pareto efficient.

4 Lindahl Equilibrium

In order to re-establish the FFToWE, let's suppose that each individual i pays for an individualized share q^i for the public good Y, that is, individual i pays q^iY. Our goal is to find how much of the public good Y to produce, how much of share each individual i pays for the public good, all while each individual is maximizing their constrained utility, the firm is maximizing profit, and markets clear. This is captured in the following definition.

Definition 2. A **Lindahl equilibrium** is $\big(p, q, (x^i, q^i)_{i=1}^I, \bar{X}, \bar{Y}\big)$ such that

i. $(\bar{x}^i, \bar{Y})$ solves

$$\max_{x^i,Y} u^i(x^i,Y) \quad \text{subject to} \quad px^i + q^iY \leq pw^i + s^i[q\bar{Y} - p\bar{X}]$$

ii. $(\bar{Y}, \bar{X})$ solves

$$\max_{X,Y} qY - pX \quad \text{subject to} \quad F(X) = Y$$

iii. $\sum_{i=1}^I x^i + \bar{X} = \sum_{i=1}^I w^i$ and $\sum_{q=1}^I q^i = q$.

Theorem 1. *Suppose each u^i is locally nonsatiated. If $\big(p, q, (\bar{x}^i, \bar{y}^i)_{i=1}^I, \bar{X}, \bar{Y}\big)$ is a Lindahl equilibrium, then $(\bar{x}^i, \bar{X}, \bar{Y})$ is Pareto efficient.*

Proof. Suppose otherwise, i.e. assume it is not Pareto efficient. Then there exists some (x^i, X, Y) such that

(a) $\sum_{i=1}^I x^i + X = \sum_{i=1}^I w^i$ and $F(X) = Y$.

(b) For all i, $u^i(x^i, Y) \geq u^i(\bar{x}^i, \bar{Y})$,

(c) Some i is strictly better off. Without loss of generality, let's just assume this $i = 1$. So $u^1(x^1, Y) > u^1(\bar{x}^1, \bar{Y})$.

From equilibrium condition (i), $(\bar{x}^i, \bar{Y})$ is chosen at nominal wealth $pw^1 + s^1[q\bar{Y} - p\bar{X}]$. But we know that $u^1(x^1, Y)$ is preferred to $(\bar{x}^1, \bar{Y})$, so it must be the case that (x^1, Y) is not affordable at that same nominal wealth, i.e.

$$px^1 + q^1 Y > pw^1 + s^1[q\bar{Y} - p\bar{X}]. \tag{3}$$

Because every other individual is no worse off under (x^i, X, Y), and because preferences are locally nonsatiated, it must be the case that they are spending no less than they were before, i.e.

$$px^i + q^i Y \geq pw^i + s^i[q\bar{Y} - p\bar{X}]. \tag{4}$$

So if we add up equation (3) with each equation in (4), we end up with

$$\begin{aligned}
& \sum_{i=1}^I px^i + \sum_{i=1}^I q^i Y > \sum_{i=1}^I w^i + \sum_{i=1}^I s^i[q\bar{Y} - p\bar{X}] \\
\implies \quad & \sum_{i=1}^I px^i + qY > p\sum_{i=1}^I w^i + [q\bar{Y} - p\bar{X}] \\
\implies \quad & p\sum_{i=1}^I [x^i - w^i] + qY > q\bar{Y} - p\bar{X}.
\end{aligned}$$

From condition (a), we know that $\sum_{i=1}^I [x^i - w^i] = -X$. Therefore we have

$$-pX + qY > q\bar{Y} - p\bar{X}.$$

But this implies that $(\bar{X}, \bar{Y})$ is not profit maximizing, which is a contradiction of the equilibrium conditions. Therefore $(\bar{x}^i, \bar{X}, \bar{Y})$ is indeed Pareto efficient. Woo. $\square$