Exercise 1

T are lump sum taxes and τ^k is the capital income tax rate. Government seeks to finance a fixed amount of government expenditure $\bar{g}$ per capita.

- $\log(c_t) - \theta N_t$ (utility)
- $c_t + \dfrac{b_{t+1}}{R_t} + k_{t+1} = b_t + w_t N_t + (1-\tau_t^k) r_t^k k_t - T_t + (1-\delta)k_t + \pi_t$ (budget)
- $y_t = k_t^\alpha N_t^{1-\alpha}$ (production function)

Part A: Firm's Problem

The firm wants to maximize profit, i.e.

$$\pi_t = k_t^\alpha N_t^{1-\alpha} - w_t N_t - r_t^k k_t.$$

The first order conditions are

$$\begin{aligned} \text{with respect to capital:} \quad & \alpha k_t^{\alpha-1} N_t^{1-\alpha} = r_t^k, \\ \text{with respect to labor:} \quad & (1-\alpha) k_t^\alpha N_t^{-\alpha} = w_t. \end{aligned}$$

Plug this back into the profit function and we have

$$\begin{aligned} \pi_t &= k_t^\alpha N_t^{1-\alpha} - (1-\alpha)k_t^\alpha N_t^{-\alpha} N_t - \alpha k_t^{\alpha-1} N_t^{1-\alpha} k_t \\ &= k_t^\alpha N_t^{1-\alpha} - (1-\alpha)k_t^\alpha N_t^{1-\alpha} - \alpha k_t^\alpha N_t^{1-\alpha} \\ &= 0. \end{aligned}$$

Part B: Household's Problem

- State variables: b_t, k_t, $\mathbf{k}_t$
- Control variables: c_t, b_{t+1}, k_{t+1}, N_t

The Bellman equation, noting that $\pi_t = 0$, is

$$\begin{aligned} V(k_t, \mathbf{k}_t, b_t) = \log(c_t) - \theta N_t + \beta V(k_{t+1}, \mathbf{k}_{t+1}, b_{t+1}) \\ - \lambda_t \left[c_t + \frac{b_{t+1}}{R_t} + k_{t+1} - b_t - w_t N_t - (1-\tau_t^k) r_t^k k_t + T_t - (1-\delta)k_t \right]. \end{aligned}$$

The first order conditions are

$$\text{with respect to } c_t\text{:} \quad \frac{1}{c_t} = \lambda_t,$$

$$\text{with respect to } k_{t+1}\text{:} \quad \beta V_k'(k_{t+1}, \mathbf{k}_{t+1}, b_{t+1}) = \lambda_t,$$

$$\text{with respect to } b_{t+1}\text{:} \quad \beta V_k'(k_{t+1}, \mathbf{k}_{t+1}, b_{t+1}) = \frac{\lambda_t}{R_t},$$

$$\text{with respect to } N_t\text{:} \quad \theta = \lambda_t w.$$

Let's stuff some envelopes. With respect to k_t, we get

$$V_k'(k_t, \mathbf{k}_t, b_t) = \lambda_t[(1-\tau_t^k)r_t^k - 1 - \delta] \quad \Longrightarrow \quad V_k'(k_{t+1}, \mathbf{k}_{t+1}, b_{t+1}) = \frac{1}{c_{t+1}}[(1-\tau_{t+1}^k)r_{t+1}^k + 1 - \delta].$$

With respect to b_t, we get

$$V_b'(k_t, \mathbf{k}_t, b_t) = \lambda_t \quad \Longrightarrow \quad V_b'(k_{t+1}, \mathbf{k}_{t+1}, b_{t+1}) = \frac{1}{c_{t+1}}.$$

Therefore we can rewrite the first order conditions the Euler equations

$$\text{with respect to } k_{t+1}\text{:} \quad \frac{1}{c_t} = \beta \frac{1}{c_{t+1}}[(1-\tau_{t+1}^k)r_{t+1}^k + 1 - \delta],$$

$$\text{with respect to } b_{t+1}\text{:} \quad \frac{1}{c_t} = \frac{\beta R_t}{c_{t+1}},$$

$$\text{with respect to } N_t\text{:} \quad \frac{w_t}{c_t} = \theta.$$

Part B: Steady State Ratios with Lump-Sum Taxes

Suppose that the only kind of taxes are lump-sum, i.e. $\tau^k = 0$.

Capital-Output Ratio. First write

$$\begin{aligned} \frac{k}{Y} &= \frac{k}{k^\alpha N^{1-\alpha}} \\ &= \frac{1}{k^{\alpha-1} N^{1-\alpha}}. \end{aligned}$$

This looks pretty similar to one of the firm's first order conditions, namely, $\alpha k_t^{\alpha-1} N_t^{1-\alpha} = r_t^k$. Substituting in, we have

$$\frac{k}{Y} = \frac{\alpha}{r^k}.$$

Then from the intertemporal Euler equation, we have

$$r^k = \frac{1-\beta(1-\delta)}{\beta}.$$

Therefore the capital-output ratio is

$$\frac{k}{Y} = \frac{\alpha\beta}{1-\beta(1-\delta)},$$

which is unaffected by lump-sum taxes.

Capital-Labor Ratio. Notice that

$$\begin{aligned}
\frac{k}{N} &= \frac{k}{Y}\frac{Y}{N} \\
&= \frac{k}{Y}\frac{k^\alpha N^{1-\alpha}}{N} \\
&= \frac{k}{Y}\left(\frac{k}{N}\right)^\alpha \\
&= \left(\frac{k}{Y}\right)^{1/(1-\alpha)} \\
&= \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{1/(1-\alpha)}.
\end{aligned}$$

So this is not a function of lump-sum taxes either.

Part B: Steady State Ratios with Distortionary Taxes

Capital-Output Ratio. Now τ^k is not equal to zero. We still have

$$\frac{k}{Y} = \frac{\alpha}{r^k}.$$

The intertemporal Euler equation now gives

$$1 = \beta[(1-\tau^k)r^k + 1 - \delta] \quad \Longrightarrow \quad r^k = \frac{1-\beta(1-\delta)}{\beta(1-\tau^k)}$$

So we can write the capital-output ratio as

$$\frac{k}{Y} = \frac{\alpha\beta(1-\tau^k)}{1-\beta(1-\delta)}.$$

Clearly as τ^k increases, the capital-output ratio will decrease. This is unsurprising—you would expect the share of capital to fall if it's taxed.

Consumption. Combine labor demand and labor supply to get

$$(1-\alpha)\left(\frac{k}{N}\right)^{\alpha} = \theta c.$$

It was previously established that

$$\frac{k}{N} = \left(\frac{k}{Y}\right)^{1/(1-\alpha)}.$$

Therefore we can write

$$\begin{aligned} c &= \frac{1-\alpha}{\theta}\left(\frac{k}{Y}\right)^{\alpha/(1-\alpha)} \\ &= \frac{1-\alpha}{\theta}\left(\frac{\alpha\beta(1-\tau^k)}{1-\beta(1-\delta)}\right)^{\alpha/(1-\alpha)}. \end{aligned}$$

Looks like increasing taxes decreases consumption. Again, not a huge surprise.

Capital Stock. In the steady state, the law of motion becomes

$$k = (1-\delta)k + i \quad \Longrightarrow \quad i = \delta k.$$

Furthermore, we can write $\bar{g} = \tau^k r^k k$. Therefore the resource constraint can be written as

$$c + (\delta + \tau^k r^k)k = y.$$

We don't know k and we don't know y, but we do know k/y. So here's the trick we need to solve this:

$$c + (\delta + \tau^k r^k)k = \frac{y}{k}k.$$

Then solving for k, we get

$$c = \left[\frac{y}{k} - (\delta + \tau^k r^k)\right] k$$

$$\implies \quad k = \frac{\frac{1-\alpha}{\theta}\left(\frac{\alpha\beta(1-\tau^k)}{1-\beta(1-\delta)}\right)^{\alpha/(1-\alpha)}}{\left[\frac{1-\beta(1-\delta)}{\alpha\beta(1-\tau^k)} - (\delta + \tau^k r^k)\right]}$$

$$= \frac{\alpha\beta(1-\alpha)(1-\tau^k)\left(\frac{\alpha\beta(1-\tau^k)}{1-\beta(1-\delta)}\right)^{\alpha/(1-\alpha)}}{\theta\left[1-\beta(1-\delta) - \alpha\beta(1-\tau^k)(\delta + \tau^k r^k)\right]}.$$

Okay, that's a lot of stuff and I'm not taking the derivative. But capital is decreasing in τ^k. Intuitively this makes sense—capital is costlier, so you'll divert resources elsewhere.

Output. Alright, we have all of the components for output:

$$y = c + \delta k + \bar{g}$$

$$= \frac{1-\alpha}{\theta}\left(\frac{\alpha\beta(1-\tau^k)}{1-\beta(1-\delta)}\right)^{\alpha/(1-\alpha)} + \delta\frac{\alpha\beta(1-\alpha)(1-\tau^k)\left(\frac{\alpha\beta(1-\tau^k)}{1-\beta(1-\delta)}\right)^{\alpha/(1-\alpha)}}{\theta\left[1-\beta(1-\delta) - \alpha\beta(1-\tau^k)(\delta + \tau^k r^k)\right]} + \tau^k r^k k.$$

Alternatively, take the capital-output ratio,

$$\frac{k}{Y} = \frac{\alpha\beta(1-\tau^k)}{1-\beta(1-\delta)} \quad \implies \quad y = \frac{1-\beta(1-\delta)}{\alpha\beta(1-\tau^k)}k,$$

and now plug in the expression for k to get

$$y = \frac{[1-\beta(1-\delta)](1-\alpha)\left(\frac{\alpha\beta(1-\tau^k)}{1-\beta(1-\delta)}\right)^{\alpha/(1-\alpha)}}{\theta\left[1-\beta(1-\delta) - \alpha\beta(1-\tau^k)(\delta + \tau^k r^k)\right]}.$$

Again, no chance in hell I'm taking the derivative of this thing. But this is also decreasing in τ^k. Since capital is taxed, the household's budget is lessened. Therefore consumption falls. When consumption falls, wages must also fall from the labor Euler equation.

Exercise 2

Let's examine the Friedman Rule. The setup is

- $U\left(C_t, \dfrac{M_t}{P_t}, N_t; Z_t\right)$

- $P_t C_t + Q_t A_t + (1 - Q_t) M_t = A_{t-1} + W_t N_t + \Pi_t$
- $1/Q_t = R_t$ is the gross nominal interest rate
- $C_t = Y_t = A_t N_t^{1-\alpha}$

Part A: Decentralized Equilibrium

I'm confused about where the FOC with respect to bonds comes about if we use the budget constraint as given.

Households maximize lifetime utility subject to their budget constraint.

- state variables: M_{t-1}, B_{t-1}, P_t
- choice variables: C_t, M_t/P_t, N_t, B_t

We are using two Bellman equations. For bonds, leisure, and consumption, use

$$\begin{aligned} V(P_t, M_{t-1}, B_{t-1}) = U\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) &+ \beta E_t\big[V(P_{t+1}, M_t, B_t)\big] \\ &- \lambda_t\big[P_t C_t + Q_t B_t + M_t - B_{t-1} - M_{t-1} - W_t N_t - \Pi_t\big]. \end{aligned}$$

For money, use

$$\begin{aligned} V(P_t, M_{t-1}, B_{t-1}) = U\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) &+ \beta E_t\big[V(P_{t+1}, M_t, B_t)\big] \\ &- \lambda_t\big[P_t C_t + Q_t A_t + (1 - Q_t) M_t - A_{t-1} - W_t N_t - \Pi_t\big]. \end{aligned}$$

The first order conditions are

$$\begin{aligned} \text{with respect to } C_t\text{:} \quad & U'_C\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) = \lambda_t P_t, \\ \text{with respect to } M_t\text{:} \quad & U'_M\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)\frac{1}{P_t} + \beta E_t\big[V'_M(P_{t+1}, M_t, B_t)\big] = \lambda_t(1 - Q), \\ \text{with respect to } B_t\text{:} \quad & \beta E_t\big[V'_B(P_{t+1}, M_t, B_t)\big] = \lambda_t Q_t, \\ \text{with respect to } N_t\text{:} \quad & U'_N\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) = -\lambda_t W_t. \end{aligned}$$

Now throw down some hot envelope action:

$$\text{wrt } M_{t-1}\text{:} \quad V'_M(P_t, M_{t-1}, B_{t-1}) = 0 \quad \Longrightarrow \quad V'_M(P_{t+1}, M_t, B_t) = 0,$$

$$\text{wrt } B_{t-1}\text{:} \quad V'_B(P_t, M_{t-1}, B_{t-1}) = \lambda_t \quad \Longrightarrow \quad V'_B(P_t, M_{t-1}, B_{t-1}) = U'_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, N_{t+1}; Z_{t+1}\right)\frac{1}{P_{t+1}}$$

We get the Euler equations

$$\begin{aligned}
U'_M\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) &= U'_C\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)(1-Q), \\
U'_C\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)\frac{Q_t}{P_t} &= \beta E_t\left[U'_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, N_{t+1}; Z_{t+1}\right)\frac{1}{P_{t+1}}\right], \\
U'_C\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)\frac{W_t}{P_t} &= -U'_N\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right).
\end{aligned}$$

I will try to explain the intuition behind each equation.

- The marginal utility from holding money today and being able to spend it tomorrow—the left-hand size—must equal the marginal utility from consuming today, which is the right-hand side. The term $1 - Q_t$ is the opportunity cost of holding money, so that cost is "gained" by not holding it and instead consuming.
- The left-hand basically says the benefit from buying a bond today, receiving the payment tomorrow, and consuming something with that payment tomorrow, must give the same benefit as just consuming today.
- The disutility from work should equal the benefit of what the wage can purchase.

Part B: Social Planner's Problem

The social planner doesn't give a shit about money or bonds. The only resource constraint for the social planner is that consumption can't exceed output, i.e. $C_t = A_t N_t^{1-\alpha}$. So the Bellman equation is

$$V(P_t, M_{t-1}, B_{t-1}) = U\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) + \beta E_t\big[V(P_{t+1}, M_t, B_t)\big] - \lambda_t\big[C_t - A_t N_t^{1-\alpha}\big].$$

The first order conditions are

$$
\begin{aligned}
\text{with respect to } C_t\text{:} \quad & U_C'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) = \lambda_t, \\
\text{with respect to } M_t\text{:} \quad & U_M'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)\frac{1}{P_t} + \beta E_t[V_M'(P_{t+1}, M_t, B_t)] = 0, \\
\text{with respect to } N_t\text{:} \quad & U_N'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) = -\lambda_t(1-\alpha)A_t N_t^{-\alpha}.
\end{aligned}
$$

The envelope condition with respect to M_{t-1} is

$$
V_M'(P_t, M_{t-1}, B_{t-1}) = 0 \quad \Longrightarrow \quad V_M'(P_{t+1}, M_t, B_t) = 0,
$$

and therefore we have Euler equations of

$$
\begin{aligned}
& U_M'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)\frac{1}{P_t} = 0, \\
& U_N'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) = -U_C'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)(1-\alpha)A_t N_t^{-\alpha}.
\end{aligned}
$$

Part C: Interest Rates and Social Planner's Allocation

Let's compare the Euler equations from the two problems. First, the money equation:

$$
U_M'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)\frac{1}{P_t} = 0 \quad \text{vs} \quad U_M'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right) = U_C'\left(C_t, \frac{M_t}{P_t}, N_t; Z_t\right)(1-Q).
$$

Recall that $1 - Q_t \approx i_t$. Set this equal to zero. Then the money Euler equation is the same in both problems. A nominal interest rate of zero means there are no opportunity costs to holding money and therefore the socially optimal amount of money will be held.

Part D: Friedman Rule and Inflation

The Fisher equation says that, in the steady state,

$$
\pi = i - r.
$$

With the Friedman rule we set $i = 0$, and we know that $r = 1/\beta$. Therefore steady-state inflation is negative, specifically, $-1/\beta$.

Part E: Practicality of the Friedman Rule

An exogenous nominal interest rate leads to price level indeterminacy. More specifically, a policy rule of the form $i_t = 0$ for all t leaves the price level indeterminate in the model. A number of things in the model are functions of the price level, for example money and nominal wage, and hence those become indeterminate as well.

Part F: Interest Rate Rule

The central bank decides to using the interest rate rule

$$i_t = \phi(r_{t-1} + \pi_t),$$

where r_t is the net interest rate (e.g. 0.05 and not 1.05). Iterate this one period forward and we have

$$i_{t+1} = \phi(r_t + \pi_{t+1}),$$

The Fisher equation says that

$$r_t = i_t - E_t[\pi_{t+1}].$$

Solve the updated policy rule for π_{t+1} and plug it into the Fisher equation to obtain

$$r_t = i_t - E_t\left[\frac{1}{\phi} i_{t+1} - r_t\right] \quad \Longrightarrow \quad E_t[i_{t+1}] = \phi i_t.$$

Iterating forward, I get

$$i_t = \frac{1}{\phi^k} E_t[i_{t+k}].$$

Now suppose that $\phi > 1$. **I don't understand why the only stable solution is $\boldsymbol{i_t = 0}$.** Thus we once again get negative steady-state inflation of $-1/\beta$. The result here is the same as if we'd been able to set $i_t = 0$ right away.

Exercise 3

Consider the following setup.

- $E_t[\hat{y}_{t+1}] - \hat{y}_t = \dfrac{1}{\sigma}\left(\hat{i}_t - E_t[\hat{\pi}_{t+1}]\right)$ linearized preference shocks
- $\hat{r}_t = \hat{i}_t - E_t[\hat{\pi}_{t+1}]$ real interest rate
- $\hat{w}_t - \hat{p}_t = \sigma \hat{y}_t + \psi \hat{n}_t$ labor FOC

- $\hat{y}_t = \hat{a}_t + \hat{n}_t$ production
- $\hat{w}_t - \hat{p}_t = \hat{a}_t$ labor hiring rule
- $\hat{a}_{t+1} = \rho_a \hat{a}_t + \epsilon^{\alpha}_{t+1}$ TFP evolution
- $\hat{m}_t - \hat{p}_t = \hat{y}_t - \eta \hat{i}_t + \epsilon^m_t$ money demand

Part A: Real Interest Rate and TFP Shocks

First write the production function as $\hat{n}_t = \hat{y}_t - \hat{a}_t$. Plug this into the labor FOC to get

$$\hat{w}_t - \hat{p}_t = \sigma \hat{y}_t + \psi(\hat{y}_t - \hat{a}_t).$$

Now use the labor hiring rule to write

$$\hat{a}_t = \sigma \hat{y}_t + \psi(\hat{y}_t - \hat{a}_t) \quad \Longrightarrow \quad \hat{y}_t = \frac{1+\psi}{\sigma+\psi}\hat{a}_t.$$

Rewrite the preference shock as

$$\left(E_t[\hat{y}_{t+1}] - \hat{y}_t\right)\sigma = \hat{i}_t - E_t[\hat{\pi}_{t+1}].$$

Then from the interest rate equation, we have

$$\hat{r}_t = E_t[\sigma \hat{y}_{t+1}] - \sigma \hat{y}_t.$$

Plug in the equation derived a few lines ago for $\hat{y}_t$ and $\hat{y}_{t+1}$, giving

$$\hat{r}_t = \sigma \frac{1+\psi}{\sigma+\psi} E_t\left[\hat{a}_{t+1}\right] - \sigma \frac{1+\psi}{\sigma+\psi}\hat{a}_t.$$

Use the TFP evolution for

$$\hat{r}_t = -\sigma \frac{1+\psi}{\sigma+\psi}\left(1-\rho_a\right)\hat{a}_t.$$

There you have it—only TFP shocks will affect $\hat{r}_t$. Monetary shocks therefore have no effect on the real interest rate.

We can do a similar rigmarole for labor. Plug the production function into the labor FOC to get

$$\hat{w}_t - p_t = \sigma(\hat{a}_t + \hat{n}_t) + \psi \hat{n}_t.$$

Set this equal to the labor hiring rule and you get

$$\hat{a}_t = \sigma(\hat{a}_t + \hat{n}_t) + \psi\hat{n}_t \quad \Longrightarrow \quad \hat{n}_t = \frac{1-\sigma}{\sigma+\psi}\hat{a}_t.$$

So labor is also affected only by TFP shocks. Since the output of this economy is contingent upon labor and TFP, we can conclude that monetary shocks have no effect on GDP.

Part B: Interest Rate Rules and Inflation Stability

We have to use a rule that incorporates potential deviations in inflation from steady state in order to have determinancy. One such rule is

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \hat{r}_t,$$

where $\phi_\pi > 1$. Combining this rule with the linearized Fisher equation,

$$\hat{r}_t = \hat{i}_t - E_t[\hat{\pi}_{t+1}],$$

we can see that

$$\phi_\pi \hat{\pi}_t = E_t[\hat{\pi}_{t+1}].$$

When I try to iterate forward, I get

$$\hat{\pi}_t = \frac{1}{\phi_\pi^k} E_t[\hat{\pi}_{t+k}].$$

I don't get why this implies a stable solution requires $\hat{\boldsymbol{\pi}}_t = \mathbf{0}$.

Part C: Money Growth

Take the money demand equation and plug in the equation for $\hat{y}_t$ derived eariler to get

$$\hat{m}_t - \hat{p}_t = \frac{1+\psi}{\sigma+\psi}\hat{a}_t - \eta\hat{i}_t + \epsilon_t^m$$

The interest rate rule guarantees that inflation is at its steady state rate, which in turn implies that prices are at their steady state level (**why?**), i.e. $\hat{p}_t = 0$. The fact that $\hat{\pi}_t = 0$ implies that $E_t[\hat{\pi}_{t+1}] = 0$, and therefore

$$\hat{i}_t = \hat{r}_t = -\sigma\frac{1+\psi}{\sigma+\psi}\left(1-\rho_a\right)\hat{a}_t$$

So we can write

$$\hat{m}_t = \frac{1+\psi}{\sigma+\psi}\hat{a}_t + \eta\sigma\frac{1+\psi}{\sigma+\psi}\left(1-\rho_a\right)\hat{a}_t + \epsilon_t^m$$
$$= \left[1+\eta\sigma\left(1-\rho_a\right)\right]\frac{1+\psi}{\sigma+\psi}\hat{a}_t + \epsilon_t^m.$$

To determine money growth, take the difference of the money demand function. Subtract the following two equations,

$$\hat{m}_t - \hat{p}_t = \hat{y}_t - \eta\hat{i}_t + \epsilon_t^m$$
$$\hat{m}_{t-1} - \hat{p}_{t-1} = \hat{y}_{t-1} - \eta\hat{i}_{t-1} + \epsilon_{t-1}^m,$$

and you end up with

$$(\hat{m}_t - \hat{m}_{t-1}) - (\hat{p}_t - \hat{p}_{t-1}) = (\hat{y}_t - \hat{y}_{t-1}) - \eta(\hat{i}_t - \hat{i}_{t-1}) + \epsilon_t^m - \epsilon_{t-1}^m$$
$$\implies \quad \Delta\hat{m}_t = \Delta\hat{y}_t + \hat{\pi}_t - \eta\Delta\hat{i}_t + \Delta\epsilon_t.$$

The interest rate rule guarantees that inflation is at its steady state level, and thus $\hat{\pi}_t = 0$. So we can write

$$\Delta\hat{m}_t = \Delta\hat{y}_t - \eta\Delta\hat{i}_t + \Delta\epsilon_t^m.$$

Let's come up with equations for some of the Δ terms. First, we have

$$\hat{i}_t = \hat{r}_t = -\sigma\frac{1+\psi}{\sigma+\psi}\left(1-\rho_a\right)\hat{a}_t$$
$$\hat{i}_{t-1} = -\sigma\frac{1+\psi}{\sigma+\psi}\left(1-\rho_a\right)\hat{a}_{t-1},$$

the difference of which gives

$$\eta\Delta\hat{i}_t = -\eta\sigma\frac{1+\psi}{\sigma+\psi}(1-\rho_a)\Delta\hat{a}_t.$$

Great. Now for $\hat{y}_t$, we'll use

$$\hat{y}_t = \frac{1+\psi}{\sigma+\psi}\hat{a}_t$$
$$\hat{y}_{t-1} = \frac{1+\psi}{\sigma+\psi}\hat{a}_{t-1},$$

the difference of which gives

$$\Delta\hat{y}_t = \frac{1+\psi}{\sigma+\psi}\Delta\hat{a}_t.$$

So now we can write the change money as

$$\begin{aligned}
\Delta\hat{m}_t &= \Delta\hat{y}_t - \eta\Delta\hat{i}_t + \Delta\epsilon_t^m \\
&= \frac{1+\psi}{\sigma+\psi}\Delta\hat{a}_t + \eta\sigma\frac{1+\psi}{\sigma+\psi}(1-\rho_a)\Delta\hat{a}_t + \Delta\epsilon_t^m \\
&= [1+\eta\sigma(1-\rho_a)]\frac{1+\psi}{\sigma+\psi}\Delta\hat{a}_t + \Delta\epsilon_t^m.
\end{aligned}$$

This implication is that money growth jumps around with TFP fluctuations as well as money demand fluctuations. In other words, money growth is determined by real shocks.

Part D: Constant Rate of Money Growth

If we assume that money growth is constant, then we can plug in $\Delta\hat{m}_t = 0$ into the above formula, and noting that $\hat{\pi}_t \neq 0$ (**because the interest rate rule isn't being followed anymore?**), we get

$$\hat{\pi}_t = -[1+\eta\sigma(1-\rho_a)]\frac{1+\psi}{\sigma+\psi}\Delta\hat{a}_t + \Delta\epsilon_t^m.$$

So now inflation changes with real shocks. If the central banks wants to be able to control inflation, then constant money growth would be at odds with such an agenda.