

# Dynamic Programming – Infinite Horizon

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*These notes borrow heavily from Stokey and Lucas, re-written in a way that I find easier to follow. That sometimes means more exposition, more explanation, worked-out examples, and added (occasionally silly) comments. Also probably some added typos and other mistakes.*

## 1 Infinite Horizon

Let's begin by considering the Cobb-Douglas case where  $f(k) = k^\alpha$  and  $u(c) = \ln(c)$ . Recall the law of motion of capital from the finite horizon problem,

$$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha. \quad (1)$$

As  $T$  becomes very large, then because  $0 < \alpha\beta < 1$ , it follows that the coefficient in front of  $k_t^\alpha$  will become very close to  $\alpha\beta$  for most of the sequence. And at this point, we might wonder why we cannot just take the limit as  $T$  goes to infinity as the solution for the infinite horizon problem. Doing so would give an solution of

$$k_{t+1} = \alpha\beta k_t^\alpha.$$

Well, it turns out we *can* just take the limit for the infinite horizon problem, and this is true in general. To prove it, one must first establish that interchanging max and lim operators is kosher, and this is actually rather difficult to show.

So nuts to that. Let's try a different approach. The limit suggests that there is a fixed rule that is used in every period. In this example, the rule seems like it should be to take a fixed proportion of your available capital to the next period, in particular,  $\alpha\beta$  of it. More generally, we can conjecture that there is a fixed “savings function” or **policy function** of the form

$$k_{t+1} = g(k_t).$$

This is fairly intuitive. The social planner problem looks the same from any given period; the only meaningful difference is the capital stock in each particular period.

But finding that savings function could be challenging. There isn't any general way we can do it from looking at first order and boundary conditions. The change of variable  $z_t = k_t/k_{t-1}^\alpha$  is specific to the example. So we need a new approach entirely.

Let's try thinking in terms of only *today* and *tomorrow*, in particular, of choosing consumption today  $c_0$  and tomorrow's beginning-of-period capital  $k_1$ . Let  $v(\cdot)$  be the **value function**, supposing one exists, that gives the value of the maximized objective function; it's not unlike an indirect utility function, as far as interpretation goes. So  $v(k_0)$  is the level of utility we'd get from solving the planner's problem, for instance. Or, given some beginning-of-period capital stock  $k_1$  in period 1, the best that could be done from that point on is  $v(k_1)$ . Using this, we can rewrite the social planner's problem as

$$\begin{aligned} & \max_{c_0, k_1} [U(c_0) + \beta v(k_1)] \\ \text{s.t. } & c_0 + k_1 \leq f(k_0), \\ & c_0, k_1 \geq 0, \quad k_0 > 0 \text{ given.} \end{aligned}$$

Of course, we don't actually know what  $v(\cdot)$  is. All we really know about it is that it solves the maximization problem. That is,

$$v(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \{U(f(k_0) - k_1) + \beta v(k_1)\}.$$

Now that we are dealing with only two indices, we can adopt the more convenient  $k$  for current and  $k'$  for next-period. Also, recall our conjecture (which is actually true) that there exists a time-invariant policy rule  $k' = g(k)$  that is followed in optimum. Thus, we can actually write the maximization problem as

$$v(k) = U(f(k) - g(k)) + \beta v(g(k)) \quad \text{s.t.} \quad 0 \leq g(k) \leq f(k).$$

Notice that we are able to drop the max operator since  $g(k)$  is defined to be the optimal policy rule – the fact that we are using it means we will attain the maximum.

So we have one equation with an unknown function  $v(\cdot)$ , which we call a **functional equation**. Optimizing such problems via functional equations is what constitutes **dynamic programming**. This will be our bread and butter.

Let's suppose that  $v(\cdot)$  is differentiable and that there is an interior solution. Then the first order condition (differentiate with respect to  $g(k)$  and set equal to zero) is

$$U'_{g(k)}(f(k) - g(k)) = \beta v'_{g(k)}(g(k)). \quad (2)$$

To find the envelope condition, differentiate with respect to  $k$  to get

$$v'_k(k) = U'_{f(k)}(f(k) - g(k))f'_k(k) - U'_{g(k)}(f(k) - g(k))g'_k(k) + \beta v'_{g(k)}(g(k))g'_k(k).$$

Notice that equation (2) implies that the latter two terms sum to zero. Thus, the envelope condition is

$$v'(k) = U'_{f(k)}(f(k) - g(k))f'_k(k). \tag{3}$$