ECN 200D—Week 1 Lecture Notes
Intro to Stochastic Models

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1 Stochastic Growth Model

Consider a general growth model,

$$V(k) = \max_{k' \in \Gamma(k)} \left\{ F(k, k') + \beta E\big[V(k')\big] \right\}.$$

This model is deterministic. We will want to consider a similar model in a stochastic environment. Let z_t denote exogenous shocks. We will suppose that z_t is countable in outcomes and, for now, finite. Then we can write the growth model as

$$V(k, z) = \max_{k' \in \Gamma(k,z)} \left\{ F(k, k', z) + \beta E\big[V(k', z')|z\big] \right\}.$$

Since z is random, we'll need to impose some structure on the nature of its randomness. In our case, we will consider Markov randomness.

2 Markov Chains

Definition 1. A stochastic process $\{x_t\}$ is said to have the **Markov property** if for all $k \geq 1$ and all t,

$$\mathbb{P}(x_{t+1}|x_t, x_{t-1}, \ldots, x_{t-k}) = \mathbb{P}(x_{t+1}|x_t).$$

So probabilistically, the outcome of tomorrow's state only depends on today's state, not on the entire history of states. We will assume the Markov property and characterize the process by a **Markov chain**.

A **time-invariant** Markov chain consists of three things.

(a) An n-dimensional state space (i.e. possible realizations of x_t) consisting of canonical vectors e_i, $i = 1, \ldots, n$. For instance, if the third state is realized, then $x_t = e_3$, which has a 1 in the 3rd position:

$$x_t = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(b) An $n \times n$ **transition matrix** P, which records the probabilities of moving from one value of the state to another in one period. For instance, the ijth element P_{ij} is the probability of moving from state i today to state j tomorrow. In other words,

$$P_{ij} = \mathbb{P}(x_{t+1} = e_j | x_t = e_i).$$

(c) An $n \times 1$ initial probability distribution π_0 whose ith element is the probability of being in state i at time zero. So the ith row of π_0 is

$\pi_{0i} = \mathbb{P}(x_0 = e_i)$.

Suppose we are in state $x_t = e_i$. Then the probability that we move to *some* state j tomorrow must sum to 1,

$$\sum_{j=1}^{n} P_{ij} = 1.$$

Otherwise there's some probability that we'd just kind of stop. A matrix P that satisfies this condition is called a **stochastic matrix**. Similarly, the probability that there is *some* initial state must be one:

$$\sum_{i=1}^{n} \pi_{0i} = 1.$$

Otherwise there's some probability that we'd never even start.

Example 1. Let's consider the probability of moving from state 1 to state 2 in *two* periods. Suppose that there are $n = 2$ possible states. We need to go from state 1 to some intermediate state h, and then from h to 2. The relevant probabilities are

$$\begin{aligned}\mathbb{P}(x_{t+2} = e_2 | x_t = e_1) = &\mathbb{P}(x_{t+1} = e_1 | x_t = e_1)\mathbb{P}(x_{t+2} = e_2 | x_{t+1} = e_1) \\ &+ \mathbb{P}(x_{t+1} = e_2 | x_t = e_1)\mathbb{P}(x_{t+2} = e_2 | x_{t+1} = e_2) \\ =& P_{11}P_{12} + P_{12}P_{22}.\end{aligned}$$

The stochastic matrix is

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}. \end{bmatrix} \quad \Longrightarrow \quad P^2 = \begin{bmatrix} P_{11}P_{11} + P_{12}P_{21} & P_{11}P_{12} + P_{12}P_{22} \\ P_{21}P_{11} + P_{22}P_{21} & P_{21}P_{12} + P_{22}P_{22} \end{bmatrix}.$$

Notice that P^2_{12} is exactly what we want. More generally, the probability of moving from i to j in k periods is determined by

$$\mathbb{P}(x_{t+k} = e_j | x_t = e_i) = P_{ij}^k. \qquad \blacksquare$$

Example 2. Let the state variable z_t denote total factor productivity in period t. We want to forecast it. GDP growth y_t can be either in a boom or a bust, where e_1 indicates that we're in a boom, e_2 a recession. The boom state has $y_1 = 1.2$ and the recession state has $y_t = -0.4$.

Let $\bar{y}$ be a 2×1 vector of outcomes for GDP growth,

$$\bar{y} = \begin{bmatrix} 1.2 \\ -0.4 \end{bmatrix}.$$

Then the realization y_t will be given by

$$y_t = \bar{y}' x_t.$$

For instance, if it turns out that we get a boom state, i.e. $x_t = e_1$, then

$$y_t = \begin{bmatrix} 1.2 & -0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1.2.$$

Now notice that

$$\begin{aligned} E[x_{t+1} | x_t = e_i] &= e_1 \mathbb{P}(x_{t+1} = e_1 | x_t = e_i) + e_2 \mathbb{P}(x_{t+1} = e_2 | x_t = e_i) \\ &= e_1 P_{i1} + e_2 P_{i2} \\ &= P' e_i \\ &= \begin{bmatrix} P_{i1} \\ P_{i2} \end{bmatrix} \\ &= P_i'. \end{aligned}$$

It follows that

$$\begin{aligned}
E[y_{t+1}|x_t = e_i] &= E[\bar{y}' x_{t+1}|x_t = e_i] \\
&= \bar{y}' E[x_{t+1}|x_t = e_i] \\
&= \bar{y}' P_i'.
\end{aligned}$$

More generally,

$$E[y_{t+k}|x_t = e_i] = \bar{y}' P^K e_i. \qquad \blacksquare$$

Example 3 (Unconditional Probabilities)**.** Let $\pi_t' = \mathbb{P}(x_t)$ be the $1 \times n$ vector whose ith element is $\mathbb{P}(x_t = e_i)$. Consider $\pi_{1,1}' = \mathbb{P}(x_1 = e_1)$. We can write this as

$$\pi_{1,1}' = \mathbb{P}(x_0 = e_1) P_{11} + \mathbb{P}(x_0 = e_2) P_{21} = [\pi_0' P]_1.$$

More generally, we can write

$$\pi_k' = \pi_0' P^k. \qquad \blacksquare$$

Finally, notice that

$$\pi_{k+1}' = \pi_0' P^{k+1} = [\pi_0' P^k] P = \pi_k' P.$$

3 Stationary Distributions

We have just seen that the unconditional probability distributions evolve according to the law of motion

$$\pi_{t+1}' = \pi_t' P.$$

An unconditional distribution is called **stationary** if it satisfies $\pi_{t+1} = \pi_t$, that is, if the unconditional distribution remains unaltered with the passage of time. This implies that a stationary distribution must satisfy $\pi' = \pi' P$, from which we can write

$$(I - P')\pi = 0.$$

This looks awfully eigenvectory. As long as P is a stochastic matrix, we are guaranteed at least one (unit) eigenvalue and that there is at least one satisfying eigenvector π. However, uniqueness is not guaranteed.

Example 4. Consider the stochastic matrix

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

Suppose $\Pi' = [\pi \;\; 1-\pi]$. Then we have a stationary distribution if

$$[\pi \quad 1-\pi] \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} = [\pi \quad 1-\pi].$$

Multiplying out the left hand side, this condition holds if

$$\begin{aligned} &\pi p + (1-\pi)(1-p) = \pi, \\ &\pi(1-p) + (1-\pi)p = 1-\pi. \end{aligned}$$

Subtract the top line from the bottom line and we get

$$\pi(1-pP) - (1-\pi) = 1 - 2\pi \quad \Longrightarrow \quad 1 - p = 2\pi(1-p).$$

Notice that if $p = 1$, then π can be anything. In this case, we have

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which acts like a "sink" because we always just go to the same state. However, if $p \neq 1$, then we can solve for the unique $\pi = 1/2$. ■

4 Asymptotic Stationarity

Given π_0, we might want to know whether π_t approaches a stationary distribution over time. That is, we want to test whether

$$\lim_{t\to\infty} \pi_0 P^t = \pi_\infty,$$

where π_∞ solves $(1 - P')\pi = 0$. If for all π_0 we have $\pi_0 P^t$ limiting to π_∞, then we say that the Markov chain is **asymptotically stationary** with a **unique invariant distribution**. This will be the case if from every state there exists a positive probability of moving to any other state (in one or more steps). In other words, if there is no situation where it becomes impossible to eventually reach state j from i.

Definition 2. A set E is called an **ergodic set** if

$$\mathbb{P}(x_t \in E | x_{t-1} \in E) = 1,$$

and no proper subset of E has this property.

In other words, we will always go from something in E to something else in E. And furthermore, there is no smaller subset $E' \subset E$ where if we go into E', then we also stay in E'. (Which means we'll end up cycling through *all* elements of E.)

5 Continuous State Spaces

5.1 Discretization

An example of a stochastic process with a continuous state space is

$$\ln(z_{t+1}) = \rho \ln(z_t) + \epsilon_t.$$

If ϵ is i.i.d., then $\ln(z_t)$ follows a Markov process—the conditional expectation will depend only on the last realization of the process.

In terms of computation, it is useful to discretize the continuous states. Pick some extreme values for the process, e.g. three standard deviations from the mean, to set the bounds. Then chop up that interval in to equal-sized parts.

In the case of a Markov chain with finite n, we can conveniently write

$$V(k, z_i) = \max_{k' \in \Gamma(k, z_i)} F(k, k', z_i) + \beta \sum_{j=1}^{n} P_{ij} V(k', z_j).$$

So the expected value is just the nice little sum we're used to seeing.

Example 5. Suppose we have a good state z^h and a bad state z^ℓ. The stochastic matrix is

$$P = \begin{bmatrix} P_{hh} & P_{h\ell} \\ P_{\ell h} & P_{\ell\ell} \end{bmatrix}.$$

Turns out we have *two* Bellman equations—one for each state.

$$\begin{aligned} V(k, z^h) &= \max \left\{ u^h + \beta \big[P_{hh} V(k', z^h) + P_{h\ell} V(k', z^\ell) \big] \right\}, \\ V(k, z^\ell) &= \max \left\{ u^\ell + \beta \big[P_{\ell h} V(k', z^h) + P_{\ell\ell} V(k', z^\ell) \big] \right\}. \end{aligned} \qquad \blacksquare$$

5.2 General Notation

Let S be the **state space** with elements $s \in S$. State transitions are described by the distribution function

$$\Pi(s'|s) = \mathbb{P}(s_{t+1} \leq s' | s_t = s).$$

The transition density, the continuous analogue of P_{ij}, is

$$\pi(s'|s).$$

The initial density is $\pi_0(s)$.

Unconditional distributions evolve according to

$$\pi_t(s_t) = \int_{s_{t-1}} \pi(s_t|s_{t-1})\pi_{t-1}(s_{t-1})\ ds_{t-1}.$$

It's helpful to translate the integrand. The *overall* probability of having state s_t is equal to the probability of having s_t given s_{t-1} times the probability of s_{t-1}, summed (integrated) over all possible values of s_{t-1}. When S is discrete, $\pi(s_j|s_i) = P_{ij}$.

A **stationary** distribution satisfies

$$\pi(s') = \int_s \pi(s'|s)\pi(s)\ ds.$$

Again, translate the integrand. The overall probability of s' is the probability of s' given s times the probability of s, summed (integrated) over all possible values of s. Since π is not indexed by any time, it is an unchanging distribution and thus stationary.

A realization of a stochastic event is $s_t \in S$. The **history** of events up to time t is $s^t = \{s_t, s_{t-1}, \ldots, s_0\}$. The unconditional probability of observing a particular sequence/history s^t is given by $\pi_t(s^t)$. The conditional probability is $\pi_\tau(s^\tau|s^t)$. We will assume that the state in period $t = 0$ is nonstochastic, i.e. $\pi_0(s_0) = 1$ for some s_0.

6 Stochastic Growth Model

6.1 Preferences and Technology

Preferences are expressed as

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\big(c_t(s^t)\big)\pi_t(s^t).$$

We will assume that u is strictly increasing in its arguments, is twice continuously differentiable, strictly concave, and satisfies the Inada condition $\lim_{c\to 0} u_c(c) = \infty$.

Let $z_t(s^t)$ be a stochastic process for total factor productivity shocks. The resource constraint is

$$c_t(s^t) + i_t(s^t) = z_t(s^t)F\big(k_t(s^{t-1}), 1\big),$$

with the law of motion of capital

$$k_{t+1}(s^t) = (1-\delta)k_t(s^{t-1}) + i_t(s^t).$$

The production function will have diminishing marginal products and will satisfy Inada conditions.

7 Aggregate Uncertainty

Let s_t be governed by a Markov process $\{s \in S, \pi(s'|s), \pi_0(s_0)\}$ and have

$$z_t(s^t) = Z\big(z_{t-1}(s^{t-1}), s_t\big).$$

So today's total factor productivity shock depends on yesterday's shock plus today's state. We start a period with z total factor productivity, which is augmented by today's shock s. Then tomorrow's pre-shock factor productivity is $z' = sz$.

For now, let's drop all of the histories from the constraint and law of motion, giving

$$\begin{aligned} c + i &= zsF(k,1), \\ k' &= (1-\delta)k + i. \end{aligned}$$

We can combine the two and rewrite consumption as

$$c = zsF(k, 1) - k' + (1 - \delta)k.$$

Then the recursive planner's problem can be written as

$$V(k, z, s) = \max_{k'} \left\{ u\big(zsF(k, 1) - k' + (1 - \delta)k\big) + \beta \sum_{s'} \pi(s'|s)V(k', z', s') \right\}.$$

We'll want to solve for optimal **policy functions**. In the stochastic case, they are functions of the stochastic state variable, so

$$\begin{aligned} c &= \phi^c(k, z, s), \\ k' &= \phi^k(k, z, s). \end{aligned}$$

The policy functions tell us the new values of the choice variables *purely as a function of the state today.*

Example 6. Let's drop the remaining s terms from the value function, and let's also have $F(k, 1) = k^\alpha$. What we then have is the value function

$$V(k, z) = \max_{k'} \left\{ u\big(zk^\alpha - k' + (1 - \delta)k\big) + \beta E\big[V(k', z')\big|z\big] \right\}.$$

We'll assume that z is a bounded random variable following a first-order Markov process. Also assume that there exists a maximum possible shock to capital so that consumption is always nonnegative. If $\beta < 1$ and shocks are bounded first-order Markov processes, then there exists a unique value function, $k' = \phi^k(k, z)$.

Let's do the typical first order condition rigmarole. With respect to k', we get

$$u'(c) = \beta E\big[V'(k', z')|z\big].$$

Now envelope it with respect to k to get

$$V'(k,z) = u'(c)(\alpha z k^{\alpha-1} + 1 - \delta) \implies V'(k',z') = u'(c')(\alpha z'[k']^{\alpha-1} + 1 - \delta).$$

Combine the FOC and the envelope for the stochastic Euler equation

$$u'(c) = \beta E\left[u'(c')(\alpha z'[k']^{\alpha-1} + 1 - \delta)|z\right]. \qquad \blacksquare$$

Let's focus on the steady state. If z_t, which is Markov, doesn't have a degenerate distribution, then k_t will *not* converge to a single $k' = \phi^k(z,k)$. This is because k is a function of z, and z is stochastic, so k will change with z. Instead, we have an invariant limiting distribution, i.e. there is a unique value of k for each value of z.

One implication is that for sufficiently large t, k should be independent of k_0. As a consequence, averages will converge as $t \to \infty$. The stochastic process for capital stocks is ergodic and will just cycle through.

8 Solving the Policy Function

There are a couple different ways of doing this. We'll focus for now on the *guess and verify* method

We can guess and verify one of two things. We can guess and verify the value function (using the method of undetermined coefficients) and derive the policy function along the way. Or we can guess and verify the policy function directly. Let's do the latter.

Example 7. Let's have log utility and $\delta = 1$. We will guess that $k' = Qzk^\alpha$, where Q is just some number. Apropos the Euler equation we found earlier, we can write

$$\frac{1}{c} = \beta E\left[\frac{\alpha z'[k']^{\alpha-1}}{c'}|z\right].$$

Note that $c = zk^\alpha - k'$. It follows from our guess that

$$c = zk^\alpha - Qzk^\alpha = zk^\alpha[1-Q].$$

Let's use this in the Euler equation.

$$\begin{aligned}
& \frac{1}{zk^\alpha[1-Q]} = \beta E\left[\frac{\alpha z'[k']^{\alpha-1}}{z'[k']^\alpha[1-Q]}|z\right] \\
\implies & \frac{1}{zk^\alpha} = \beta E\left[\alpha[k']^{-1}|z\right] \\
\implies & \frac{1}{zk^\alpha} = \beta E\left[\alpha[Qzk^\alpha]^{-1}|z\right]
\end{aligned}$$

Note that k is already a realized level of capital, and thus it does not depend on z. In fact, nothing remaining in the expectation depends on z. So we have

$$\frac{1}{zk^\alpha} = \beta\alpha[Qzk^\alpha]^{-1} \quad \implies \quad Q = \alpha\beta.$$

Hey, this is the same result as in the deterministic case. Woo. So we have $k' = \alpha\beta zk^\alpha$, and consequently $c = zk^\alpha(1-\alpha\beta)$. ■

If we were to do the method of undetermined, we'd make a guess like

$$V(k,z) = G + B\ln(k) + D\ln(z),$$

then use this in the first order condition and Bellman equation to solve for G, B, and D. See/do the homework.