ECN 200D—Week 8 Lecture Notes

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We've seen the how the Arrow-Debreu equilibrium coincides with the social planner's solution, assuming the first welfare theorem holds. Let's briefly examine the sequential markets equilibrium as well.

1 Sequential Markets Equilibrium

1.1 The Household's Problem

The households aim to solve

$$\max_{\{c_t, i_t, x_{t+1}, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

(a) for all t, $c_t + i_t = w_t n_t + r_t k_t + \pi_t$, *(budget constraint)*

(b) $x_{t+1} = (1-\delta)x_t + i_t$ *(law of motion of capital)*,

(c) $c_t, x_{t+1} \geq 0$, $n_t \in [0,1]$, $k_t \in [0, x_t]$, x_0 given.

Thankfully, we can do pretty much all of the same simplifications we did in the Arrow-Debreu equilibrium: $k_t = x_t$, $n_t = 1$, $i_t = k_{t+1} - (1-\delta)k_t$, and

$\pi_t = 0$. Then we can rewrite the problem as

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

(a) $c_t + k_{t+1} = w_t + r_t k_t + (1-\delta) k_t$,

(b) $c_t, k_{t+1} \geq 0$, k_0 given.

1.2 The Firm's Problem

$$\text{for any } t, \max_{\{k_t, n_t\}} F(k_t, n_t) - w_t n_t - r_t k_t.$$

1.3 Defining the Equilibrium

Definition 1. A **sequential markets equilibrium** for the neoclassical growth model is a list of prices $\{w_t, r_t\}_{t=0}^{\infty}$ and allocations for the household $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and firms $\{k_t^d, n_t^d\}_{t=0}^{\infty}$ such that

(a) given prices, $\{c_t, k_{t+1}^s\}_{t=0}^{\infty}$ solves the household's problem,

(b) given prices, $\{k_t^d, n_t^d\}_{t=0}^{\infty}$ solves the firm's problem,

(c) for any t, $k_t^s = k_t^d = k_t$ and $n_t^s = n_t^d = 1$,

(d) $F(k_t, 1) = c_t + i_t$.

If you go through and derive the first order conditions, you will find that they're exactly the same as in the Arrow-Debreu equilibrium, which in turn makes them the same as those found in the social planner's problem. So again, we may as well just solve the relatively easy social planner's problem, no? This, of course, it assuming that the first welfare theorem holds.

2 Recursive Competitive Equilibrium

But um, what if the first welfare theorem does not hold? The link between Arrow-Debreu and the social planner breaks. In this situation, one approach is to solve the *recursive competitive equilibrium*. The word "recursive" should get you thinking about dynamic programming—in particular, about Bellman equations, value functions, state and control variables.

There is a little bit of a conceptual hurdle we must overcome first, however. Specifically, we need to recognize that k refers to the representative agent's available capital, and not aggregate available capital. So let K refer to the aggregate level of capital. We need to differentiate between the two because the agent's choice of capital should not have any effect on market prices—it is a price taking economy—whereas the aggregate capital K most certainly should. The usage will be elucidated upon in the material that follows.

2.1 The Firm's Problem

A representative firm wants to solve

$$\max_{n_t^d, k_t^d} F(k^d, n^d) - wn^d - rk^d.$$

But we'll need to be more specific about what w and r are. It might be temping to write $w = F_n(k, 1)$ since we are still assuming constant returns to scale, but this is problematic—it suggests that the wage is a function of this single representative firm's capital k. In actuality, the wage is going to be determined by the aggregate capital in the economy, K. Same thing with the rental price of capital. So we have

$$w = F_n(K, 1), \quad r = F_k(K, 1).$$

2.2 The Household's Problem

The representative household has the Bellman equation

$$V(k, K) = \max_{c,k'} u(c) + \beta V(k', K').$$

In a typical Bellman equation, everything should either be a state variable, control variable, or a parameter. The aggregate capital K', however, is something else entirely—it is the aggregate capital next period as "chosen" by the entire economy. This makes it a novel object for our analysis.

The constraints of the Bellman equation are

(a) $c + k' = w + (r + 1 - \delta)k,$ *(budget constraint)*

(b) $w = w(K) = F_n(K, 1),$

(c) $r = r(K) = F_k(K, 1),$

(d) $K' = H(K).$

The function $H(K)$ is the agent's **rational expectation** about K', which means that unsettling term in the Bellman equation is now in terms of the state variable K.

2.3 Defining the Equilibrium

A **recursive competitive equilibrium** consists of

(a) the household value function $V : \mathbb{R}^2_+ \to \mathbb{R}$,

(b) policy functions $c, g : \mathbb{R}^2_+ \to \mathbb{R}_+$,

(c) pricing functions $w, r : \mathbb{R}_+ \to R_+$,

(d) an aggregate law of motion for capital, $H : \mathbb{R}_+ \to R_+$ such that

 (i) given w, r, and H, V solves the household's Bellman equation with associated policy functions c and g,

(ii) $w = w(K) = F_n(K, 1)$,

(iii) $r = r(K) = F_k(K, 1)$,

(iv) $H(K) = K' = g(K, K)$,

(v) $c + K' = F(K, 1) + (1 - \delta)K$.

Note that because the measure of households is 1, c is both aggregate consumption and individual consumption. Also, the policy function that says how a representative agent should accumulate capital should apply to the economy as a whole, and thus to K as well.

2.4 Example

This example will illustrate what happens if we goof up and use k where we should have used K. The model will be the same basic environment as in the neoclassical growth model, but there will now be a government that taxes a fraction $\tau \in [0, 1]$ of the household's income and then returns the tax as a lump sum.

The household value function is still

$$V(k, K) = \max_{c,k'} u(c) + \beta V(k', K'),$$

now subject to the constraints

(a) $c + k' = (1 - \delta)k + [w + rk](1 - \tau) + T$,

(b) $K' = H(K)$,

(c) $w = F_n(K, 1)$,

(d) $r = F_k(K, 1)$,

(e) $T = \tau[F_n(K, 1) + F_k(K, 1)k] = \tau[F(K, 1)]$.

The last equality follows from Euler's theorem. Note that the lump sum payment T is both the aggregate and individual lump sum payment because the measure of the households is 1. Pedantically, each agent receives $T/1$.

Since T is not a parameter nor is it a state or control variable, we'll use the expression with τ.

2.4.1 Ignoring the Distinction Between k and K

If we do ignore the distinction between k and K, then the budget constraint can be written as

$$\begin{aligned}
c + k' &= (1-\delta)k + [w + rk](1-\tau) + T \\
&= (1-\delta)k + [F_n(k,1) + F_k(k,1)k](1-\tau) + \tau[F_n(k,1) + F_k(k,1)k] \\
&= (1-\delta)k + F_n(k,1) + F_k(k,1)k \\
&= (1-\delta)k + F(k,1) \\
&= f(k).
\end{aligned}$$

You'll find a first order condition of

$$u'\big(f(k) - k'\big) = \beta u'\big(f(k') - k''\big) f(k'),$$

which might make you all warm and fuzzy inside since it's the same first order condition as in the other problems. One might expect that result—I mean, all we're doing differently is taking money away from the households and giving it right back. But it turns out that ***this is wrong!***