

Perpetuities and Annuities

Let C be the initial cash flow, r be the cost of capital, and g be the growth rate of cash flow.

$$\begin{aligned}\text{PV for period 1 payment:} & \quad \frac{C}{1+r} \\ \text{PV for period 2 payment:} & \quad \frac{C}{(1+r)} \frac{(1+g)}{(1+r)} \\ \text{PV for period 3 payment:} & \quad \frac{C}{(1+r)} \frac{(1+g)^2}{(1+r)^2} \\ \text{PV for period } T \text{ payment:} & \quad \frac{C}{(1+r)} \frac{(1+g)^{T-1}}{(1+r)^{T-1}}\end{aligned}$$

Sum this all up and we get

$$PV = \frac{C}{1+r} + \frac{C}{1+r} \frac{(1+g)}{(1+r)} + \frac{C}{1+r} \frac{(1+g)^2}{(1+r)^2} + \dots + \frac{C}{1+r} \frac{(1+g)^{T-1}}{(1+r)^{T-1}}.$$

If we multiply both sides by $(1+g)/(1+r)$, then

$$PV \frac{(1+g)}{(1+r)} = \frac{C}{1+r} \frac{(1+g)}{(1+r)} + \frac{C}{1+r} \frac{(1+g)^2}{(1+r)^2} + \frac{C}{1+r} \frac{(1+g)^3}{(1+r)^3} + \dots + \frac{C}{1+r} \frac{(1+g)^T}{(1+r)^T}.$$

Now subtract the two. All terms cancel out on the RHS except for the first and last. This yields

$$PV - \frac{(1+g)}{(1+r)} PV = \frac{C}{1+r} - \frac{C}{1+r} \frac{(1+g)^T}{(1+r)^T}.$$

Solve this for PV and you get the formula for a growing annuity,

$$PV = \frac{C}{r-g} \left[1 - \frac{(1+g)^T}{(1+r)^T} \right]. \quad (1)$$

Note that if there is no cash flow growth, then $g = 0$ and we have the constant cash flow annuity.

By definition of a perpetuity, we take $T \rightarrow \infty$. For PV to take on a finite value, we require the second term to go to zero in the limit. This will happen as long as the geometric ratio satisfies

$$\frac{(1+g)}{(1+r)} < 1 \implies g < r.$$

So, having made that assumption and having taken the limit, we now have

$$PV \left[1 - \frac{(1+g)}{(1+r)} \right] = \frac{C}{1+r}.$$

Write the bracketed term with common denominator, then solve for PV for the result,

$$PV = \frac{C}{r-g}. \quad (2)$$

Again, if $g = 0$ then we have the constant cash flow perpetuity formula.