Exercise 1

- Preferences: $u(C_t, N_t) = \log(C_t) - \gamma N_t$.
- Technology: $Y_t = F(Z_t, K_t, N_t) = Z_t K_t^\alpha N_t^{1-\alpha}$.
- Depreciation: $\delta = 1$.
- TFP: Z follows a Markov chain over the set $Z = \{z_1, \ldots, z_N\}$ with probabilities p_{ij}.

Part 1: The Bellman Equation

The Bellman equation is

$$V(K_t, Z_t) = \max_{C_t, N_t, K_{t+1}} \log(C_t) - \gamma N_t + \beta E_t \left[V(K_{t+1}, Z_{t+1}) \right],$$

subject to the constraint $C_t + K_{t+1} = Z_t K_t^\alpha N_t^{1-\alpha}$.

Notice that C_t can never be zero because of the logarithm. This implies that hours worked N_t can never be zero. The marginal (dis)utility from labor is $-\gamma$, which means any constraint on labor hours might as well be binding.[1]

Now I'm going to rewrite the Bellman equation as

$$V(K_t, Z_t) = \max_{C_t, N_t, K_{t+1}} \log(C_t) - \gamma N_t + \beta E_t \left[V(K_{t+1}, Z_{t+1}) \right] - \lambda_t [C_t + K_{t+1} - Z_t K_t^\alpha N_t^{1-\alpha}].$$

Part 2: First-Order Conditions

Intratemporal Euler Equation. We get this by taking the first order conditions with respect to C_t and N_t since they are in the same period. We get

$$\frac{1}{C_t} = \lambda_t,$$

$$\gamma = \lambda_t (1-\alpha) Z_t K_t^\alpha N_t^{-\alpha},$$

from which the intratemporal Euler equation emerges as

$$\frac{1}{C_t} = \frac{\gamma}{(1-\alpha) Z_t K_t^\alpha N_t^{-\alpha}}. \tag{1}$$

[1] Yeah, I don't get it either.

Intertemporal Euler Equation. Take the first order conditions with respect to C_t and K_{t+1}. We get

$$\frac{1}{C_t} = \lambda_t,$$

$$\beta E_t \left[V'(K_{t+1}, Z_{t+1})\right] = \lambda_t,$$

from which it follows that

$$\frac{1}{C_t} = \beta E_t \left[V'(K_{t+1}, Z_{t+1})\right].$$

Now take the derivative with respect to k_t to get the envelope condition,

$$\begin{aligned}
V'(K_t, Z_t) &= \lambda \alpha Z_t K_t^{\alpha-1} N_t^{1-\alpha} \\
&= \frac{1}{C_t} \alpha Z_t K_t^{\alpha-1} N_t^{1-\alpha} \\
\implies \quad V'(K_{t+1}, Z_{t+1}) &= \frac{1}{C_{t+1}} \alpha Z_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha}
\end{aligned}$$

Therefore the intertemporal Euler equation becomes

$$\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} \alpha Z_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha}\right]. \tag{2}$$

Part 3: Guess and Verify

Our conjecture is that $K' = BZK^\alpha N^{1-\alpha}$, and therefore also that $C = (1-B)ZK^\alpha N^{1-\alpha}$. Plugging these into the intertemporal Euler equation, we get

$$\begin{aligned}
\frac{1}{(1-B)Z_t K_t^\alpha N_t^{1-\alpha}} &= \beta E_t \left[\frac{\alpha K_{t+1}^{-1} Z_{t+1} K_{t+1}^\alpha N_{t+1}^{1-\alpha}}{(1-B)Z_{t+1} K_{t+1}^\alpha N_{t+1}^{1-\alpha}}\right] \\
\implies \quad \frac{1}{(1-B)Z_t K_t^\alpha N_t^{1-\alpha}} &= \beta E_t \left[\frac{\alpha}{(1-B)BZ_t K_t^\alpha N_t^{1-\alpha}}\right] \\
\implies \quad B = \alpha\beta. &
\end{aligned}$$

We have found $K' = \alpha\beta ZK^\alpha N^{1-\alpha}$ and $C = (1-\alpha\beta)ZK^\alpha N^{1-\alpha}$.

Part 4: Policy Functions

Use the previous result, specifically for C, with the intratemporal Euler equation to get

$$\frac{1}{(1-\alpha\beta)Z_t K_t^{\alpha} N_t^{1-\alpha}} = \frac{\gamma}{(1-\alpha)Z_t K_t^{\alpha} N_t^{-\alpha}} \quad \Longrightarrow \quad \frac{1-\alpha}{\gamma(1-\alpha\beta)} = N_t.$$

We can now solve for the policy function,

$$K' = \alpha\beta Z K^{\alpha} \left[\frac{1-\alpha}{\gamma(1-\alpha\beta)}\right]^{1-\alpha},$$

$$C = (1-\alpha\beta) Z K^{\alpha} \left[\frac{1-\alpha}{\gamma(1-\alpha\beta)}\right]^{1-\alpha}.$$

Part 5: Discussion

Notice that consumption and investment are both increasing in Z. Labor supply, on the other hand, is not a function of Z.

Exercise 2

Have $N_t = 1$. Explain how to solve with value function iteration.

- Preferences: $u(C_t) = \log(C_t) - \gamma$
- Technology: $Y_t = F(Z_t, K_t, N_t) = Z_t K_t^{\alpha}$.
- Depreciation: $\delta = 1$.
- TFP: Z follows a Markov chain over the set $Z = \{z_1, \ldots, z_N\}$ with probabilities p_{ij}.
- Loop through until you find successive value functions that are "close enough."

The MATLAB operation is roughly

1. Discretize the possible realizations for any random variables
2. Specify the Markov process
3. Set up the value function. Each row corresponds to one combination of state variables (e.g. z_1 and k_1), and columns refer to choices, e.g. k_{t+1}.

On paper, you would exploit the contraction mapping theorem.

1. Now make a guess for the value function v_0. Usually $v_0 = 0$ is an easy starting point. Then solve
$$v_1(Z,K) = \max_{C,K'} u(C) - \gamma + \beta E[v_0(Z',K')].$$
Plug in your optimizing values of C and K' and you get $v_1(Z,K)$.
2. Now solve
$$v_2(Z,K) = \max_{C,K'} u(C) + \beta E[v_1(Z',K')].$$
Plug in your optimizing values of C and K' and you will have a complete expression for $v_2(Z,K)$. Wash, rinse, repeat, until $v_{n+1} - v(n) < \epsilon$ for some threshold ϵ you've chosen.

Exercise 3

- Preferences: $\log(C_t) + \dfrac{\theta}{1-\eta}(L_t^{1-\eta} - 1)$. Note that L is leisure.
- Households own the capital stock K_t and make investments I_t.
- Law of Motion: $K_{t+1} = (1-\delta)K_t + I_t$.
- Labor Supply: $N_t + L_t = 1$.
- Labor earns a wage of w_t. Profits are π_t.
- r^k is the is the return on capital. It relates to the real interest rate because $r_t = r_t^k + 1 - \delta$.
- Production: $Y_t = Z_t K_t^{\alpha}(X_t N_t)^{1-\alpha}$.
- $X_t/X_{t-1} = \gamma$. This then gives the trend growth rate of "effective labor."
- Technology be Markov: $\log(Z_{t+1}) = \rho \log(Z_t) + \epsilon_{t+1}$.

Part 1: Detrending

X_t is the trend factor that we will need to adjust for. Z doesn't grow over time because it's just a set of shocks; and N doesn't grow because it's fixed at $N = 1$. Everything else, however, trends.

Let's first adjust Y_t so that we can focus on its trend. To do so, divide by X_t. What you get is the de-trended production function,

$$\begin{aligned}
\frac{Y_t}{X_t} &= Z_t K_t^\alpha \frac{X_t^{1-\alpha}}{X_t} N_t^{1-\alpha} \\
&= Z_t K_t^\alpha X_t^{-\alpha} N_t^{1-\alpha} \\
&= Z_t \left[\frac{K_t}{Z_t}\right]^\alpha N_t^{1-\alpha} \\
\implies \quad y_t &= Z_t k_t^\alpha N_t^{1-\alpha}.
\end{aligned}$$

The law of motion turns into

$$\begin{aligned}
\frac{K_{t+1}}{X_t} &= (1-\delta)\frac{K_t}{X_t} + \frac{I_t}{X_t} \\
\implies \quad \frac{X_{t+1}}{X_{t+1}}\frac{K_{t+1}}{X_t} &= (1-\delta)\frac{K_t}{X_t} + \frac{I_t}{X_t} \\
\implies \quad \gamma k_{t+1} &= (1-\delta)k_t + i_t.
\end{aligned}$$

Detrending utility doesn't work out so nicely. So for now just take it for granted that

$$U(c, L) = \ln(c) + \frac{\theta}{1-\eta}(L^{1-\eta} - 1).$$

Until further notice, just assume that we can change C_t to c_t and keep the rest the same.

Part 2: Household and Firm Problems

Household's Problem. The state variables in period t are the shock Z_t, the individual's capital stock k_t, and the aggregate capital stock $\mathbf{k}_t$. The household chooses how much to consume, how much of their capital stock to supply, and how much leisure to indulge in. This gives the Bellman equation

$$V(Z_t, k_t, \mathbf{k}_t) = \max_{c_t, k_{t+1}^s, L_t^s} \ln(c_t) + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) + \beta E_t[V(Z_{t+1}, k_{t+1}, \mathbf{k}_{t+1})].$$

The detrended resource constraint is

$$c_t + \gamma k_{t+1}^s = w_t(1 - L_t^s) + r_t k_t^s + \pi.$$

Firm's Problem. They're profit maximizers, of course.

$$\max_{N_t^d, k_t^d} Z_t k_t^\alpha N_t^{1-\alpha} - w_t N_t^d - r_t^k k_t^d.$$

Part 3: First Order Conditions

Firm FOC. The two first order conditions are

$$(1-\alpha) Z_t k_t^\alpha N_t^{-\alpha} = w_t,$$
$$\alpha Z_t k_t^{\alpha-1} N_t^{1-\alpha} = r_t^k.$$

Plug these into the profit function and we get

$$\begin{aligned}
\pi &= Z_t k_t^\alpha N_t^{1-\alpha} - \left[(1-\alpha) Z_t k_t^\alpha N_t^{-\alpha}\right] N_t^d - \left[\alpha Z_t k_t^{\alpha-1} N_t^{1-\alpha}\right] k_t^d \\
&= Z_t k_t^\alpha N_t^{1-\alpha} - \left[(1-\alpha) Z_t k_t^\alpha N_t^{1-\alpha}\right] - \left[\alpha Z_t k_t^\alpha N_t^{1-\alpha}\right] \\
&= Z_t k_t^\alpha N_t^{1-\alpha} - Z_t k_t^\alpha N_t^{1-\alpha} \\
&= 0.
\end{aligned}$$

Household FOC. I'm going to use the Bellman equation

$$\begin{aligned}
V(Z_t, k_t, \mathbf{k}_t) = \max_{c_t, k_{t+1}^s, L_t^s} &\ln(c_t) + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) + \beta E_t[V(Z_{t+1}, k_{t+1}, \mathbf{k}_{t+1})] \\
&- \lambda_t \left[c_t + \gamma k_{t+1}^s - w_t(1 - L_t^s) - r_t k_t^s\right].
\end{aligned}$$

Let's first find the intratemporal Euler equation with c_t and L_t done gives

$$\frac{1}{c_t} = \lambda_t,$$
$$\theta L_t^{-\eta} = \lambda_t w_t.$$

From this we can derive the labor supply function,

$$\frac{1}{c_t} = \frac{\theta}{w} L_t^{-\eta} \quad \Longrightarrow \quad L_t^s = \left(\frac{\theta}{w} c_t\right)^{1/\eta}.$$

Now to find the intertemporal Euler equation, use c_t and k_{t+1}. We already have the first. The second gives

$$\beta E_t[V_k'(Z_{t+1}, k_{t+1}, \mathbf{k}_{t+1})] = \lambda_t \gamma.$$

Combining this with the c_t condition, we gets us some

$$\frac{1}{\gamma}\beta E_t[V_k'(Z_{t+1}, k_{t+1}, \mathbf{k}_{t+1})] = \frac{1}{c_t}.$$

Now envelope it up. You'll get

$$\begin{aligned} V_k'(Z_t, k_t, \mathbf{k}_t) &= \lambda_t r_t \\ &= \frac{1}{c_t} r_t \\ \implies \quad V_k'(Z_{t+1}, k_{t+1}, \mathbf{k}_{t+1}) &= \frac{1}{c_{t+1}} r_{t+1} \end{aligned}$$

Combine this with the FOC above and we get the intertemporal Euler equation,

$$\beta E_t\left[\frac{1}{c_{t+1}} r_{t+1}\right] = \frac{\gamma}{c_t}.$$

Part 4: RCE

A **recursive competitive equilibrium** consists of

- a value function, $V(k^s, Z, \mathbf{k})$,
- a capital decision rule, $k_s' = k_s'(k^s, \mathbf{k}, Z)$,
- a consumption decision rule, $c = c(k^s, \mathbf{k}, Z)$,
- a labor supply rule, $N(k_s, \mathbf{k}, Z)$,
- a law of motion for the aggregate capital stock $\mathbf{k}' = g(\mathbf{k}, Z)$,
- prices $w(\mathbf{k}, Z)$ and $r^k(\mathbf{k}, Z)$,

such that

- the value function and decision rules solve the household's problem, given the price and law of motion for $\mathbf{k}$,
- the firm's optimality conditions are satisfied,

- all markets clear:

$$k_d = k_s = \mathbf{k},$$

$$g(\mathbf{k}, Z) = k_s(\mathbf{k}, \mathbf{k}, Z),$$

$$N_d = N_s = N,$$

$$y_t = c_t + i_t.$$

Part 5: Zero Profits (Already Showed)

Part 6: Social Planner Equivalence

A planner doesn't give a damn about prices. So the planner's problem is solve

$$V(Z_t, k_t, \mathbf{k}_t) = \max_{c_t, k^s_{t+1}, L^s_t} \ln(c_t) + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) + \beta E_t[V(Z_{t+1}, k_{t+1}, \mathbf{k}_{t+1})].$$

subject to markets clearing and the law of motion of capital holding. The two constraints are

$$C_t + I_t = Z_t K_t^{\alpha}(X_t N_t)^{1-\alpha},$$

$$K_{t+1} = (1-\delta)K_t + I_t.$$

The detrending is the same, so

$$c_t + i_t = Z_t k_t^{\alpha} N_t^{1-\alpha},$$

$$\gamma k_{t+1} = (1-\delta)k_t + i_t.$$

We can then rewrite the Bellman equation as

$$\begin{aligned} V(Z_t, k_t, \mathbf{k}_t) = \max_{c_t, k^s_{t+1}, L^s_t} & \ln(c_t) + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) + \beta E_t[V(Z_{t+1}, k_{t+1}, \mathbf{k}_{t+1})] \\ & - \lambda_t \left[c_t + \gamma k_{t+1} - Z_t k_t^{\alpha}(1-L_t)^{1-\alpha} - (1-\delta)k_t\right]. \end{aligned}$$

The intratemporal Euler equation is going to be

$$\frac{1}{c_t} = \frac{\theta L_t^{1-\eta}}{(1-\alpha) Z_t k_t^{\alpha} N_t^{\alpha}},$$

which is the same once you recognize that the denominator of the right hand side is w.

The intertemporal Euler equation will be

$$\frac{\gamma}{c_t} = \beta E_t \left[\frac{\alpha Z_{t+1} k_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta}{c_{t+1}} \right],$$

which is the same once you recognize that $\alpha Z_{t+1} k_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} = r_{t+1}^k$.

Exercise 4

Part 1: Linearize the Decentralized RBC Model

Here's the list of things we need to linearize:

- $y_t = Z_t k_t^{\alpha} N_t^{1-\alpha}$ (production function)
- $c_t + i_t = Z_t k_t^{\alpha} N_t^{1-\alpha}$ (resource constraint)
- $\gamma k_{t+1} = (1-\delta) k_t + i_t$ (law of motion)
- $N_t + L_t = 1$ (labor constraint)
- $\log(Z_{t+1}) = \rho \log(Z_t) + \epsilon_{t+1}$ (shock process)
- $\alpha Z_t k_t^{\alpha-1} N_t^{1-\alpha} = r_t^k$ (capital demand)
- $(1-\alpha) Z_t k_t^{\alpha} N_t^{-\alpha} = w_t$ (labor demand)
- $\dfrac{1}{c_t} = \dfrac{\theta}{w} L_t^{-\eta}$ (intratemporal Euler equation)
- $\dfrac{\gamma}{c_t} = \beta E_t \left[\dfrac{r_{t+1}}{c_{t+1}} \right]$ (intertemporal Euler equation)
- $r_t = r_t^k + 1 - \delta$ (definition of r_t)

Okaaay, so we have to linearize this one at a time.

- $\hat{y}_t = \hat{z}_t + \alpha \hat{k}_t + (1-\alpha) \hat{N}_t$ (production function)
- $\hat{y}_t = \dfrac{c}{y} \hat{c}_t + \dfrac{i}{y} \hat{i}_t$ (resource constraint)
- $\gamma \hat{k}_{t+1} = (1-\delta) \hat{k}_t + \dfrac{i}{k} \hat{i}_t$ (law of motion)
- $N \hat{N}_t = -L \hat{L}_t$ (labor constraint)
- $\hat{Z}_{t+1} = \rho \hat{Z}_t + \epsilon_{t+1}$ (shock process)
- $\hat{r}_t^k = \hat{Z}_t + (1-\alpha)(\hat{N}_t - \hat{k}_t)$ (capital demand)
- $\hat{w}_t = \hat{Z}_t + \alpha[\hat{k}_t - \hat{N}_t]$ (labor demand)
- $\hat{c}_t - \hat{w}_t = \eta \hat{L}_t$ (intratemporal Euler equation)
- $\hat{c}_t = E_t[\hat{c}_{t+1} - \hat{r}_{t+1}]$ (intertemporal Euler equation)
- $r \hat{r}_t = (r - 1 + \delta) \hat{r}_t^k$ (definition of r_t)

Part 2: Derive the Steady State

We want to solve for i/y, c/y, i/k, N, L, r^k, and r.

- The intertemporal Euler equation gives $r = \gamma/\beta$.
- The definition of r_t gives $r^k = \gamma/\beta - 1 + \delta$.
- Plug the above into the capital demand equation and use the production function to get $\alpha/r^k = k/y$. Now solve the law of motion to get $i/k = \gamma - 1 + \delta$.
- It follows that
$$\frac{i}{y} = \frac{i}{k}\frac{k}{y} = (\gamma - 1 + \delta)\frac{\alpha}{r^k} = \alpha\frac{\gamma - 1 + \delta}{\gamma/\beta - 1 + \delta}.$$
- Because $c + i = y$, it follows that $c/y + i/y = 1$. Therefore
$$\frac{c}{y} = 1 - \frac{i}{y} = 1 - (\gamma - 1 + \delta)\frac{\alpha}{r^k} = 1 - \alpha\frac{\gamma - 1 + \delta}{\gamma/\beta - 1 + \delta}.$$
- Solve the labor demand equation for $(1-\alpha)y/N = w$. Plug this into the intertemporal Euler equation for
$$\frac{1}{c} = \frac{\theta N}{(1-\alpha)y}L^{-\eta} \quad \Longrightarrow \quad \frac{1-\alpha}{\theta}\frac{y}{c} = L^{-\eta} - L^{1-\eta}.$$
Unfortunately L cannot be solved for explicitly. But once you find it, then you also have $N = 1 - L$.

Part 3: Calibration

Calibration is essentially the process of choosing the "deep" parameters of the model: δ, β, η, θ, ρ, and so forth. We want to make the model's predictions to match certain observations from the data. We're interested in the business cycle properties of the model, and do not want to cheat by choosing parameter values that help the model deliver good business cycle implications. Instead, choose parameter values such that the long-run implications of the model matches long-run average observations from the data. Common values are

- $\beta = 0.99$
- $\alpha = 1/3$, which is the capital share
- $\delta = 0.025$, matching the empirical K/Y ratio
- $\gamma = 1.004$ to match long-run trends in GDP growth
- $\eta = 1$
- $\theta = 0.20$ to match steady-state percentage of hours worked
- $\rho = 0.979$ as the persistence of a shock.

Part 4: Linearized System

We want to set up $AE[\hat{x}_{t+1} \quad \hat{\omega}_{t+1}]' = B[\hat{x}_t \quad \hat{\omega}_t]'$. Here's how I like to set it up.

A =

$\hat{k}_{t+1}$	$\hat{Z}_{t+1}$	$\hat{c}_{t+1}$	$\hat{i}_{t+1}$	$\hat{y}_{t+1}$	$\hat{w}_{t+1}$	$\hat{r}^k_{t+1}$	$\hat{N}_{t+1}$	$\hat{L}_{t+1}$	$\hat{r}_{t+1}$	
0	0	0	0	0	0	0	0	0	0	production function
0	0	0	0	0	0	0	0	0	0	resource constraint
γ	0	0	0	0	0	0	0	0	0	law of motion
0	0	0	0	0	0	0	0	0	0	labor constraint
0	1	0	0	0	0	0	0	0	0	shock
0	0	0	0	0	0	0	0	0	0	capital demand
0	0	0	0	0	0	0	0	0	0	labor demand
0	0	0	0	0	0	0	0	0	0	intratemporal Euler
0	0	1	0	0	0	0	0	0	-1	intertemporal Euler
0	0	0	0	0	0	0	0	0	0	equation for r

B =

$\hat{k}_t$	$\hat{Z}_t$	$\hat{c}_t$	$\hat{i}_t$	$\hat{y}_t$	$\hat{w}_t$	$\hat{r}^k_t$	$\hat{N}_t$	$\hat{L}_t$	$\hat{r}_t$	
α	1	0	0	-1	0	0	$1-\alpha$	0	0	production function
0	0	c/y	i/y	-1	0	0	0	0	0	resource constraint
$1-\delta$	0	0	i/k	0	0	0	0	0	0	law of motion
0	0	0	0	0	0	0	N	L	0	labor constraint
0	ρ	0	0	0	0	0	0	0	0	shock
$\alpha-1$	1	0	0	0	0	-1	$1-\alpha$	0	0	capital demand
α	1	0	0	0	-1	0	$-\alpha$	0	0	labor demand
0	0	1	0	0	-1	0	$-\eta$	0	0	intratemporal Euler
0	0	1	0	0	0	0	0	0	0	intertemporal Euler
0	0	0	0	0	0	$r-1+\delta$	0	0	$-r$	equation for r

Part 5: Blanchard-Kahn

Premultiply both sides by B^{-1}. Let $C = B^{-1}$. We can decompose C into $P\Lambda P^{-1} = C$, where Λ is a diagonal matrix with each eigenvalue and P is the corresponding eigenvector matrix.

The Blanchard-Kahn condition states that all jump variables must have eigenvalues greater than 1, whereas state variables have eigenvalues of less than one. If the Blanchard-Kahn conditions are not satisfied, then either

- there are too many unstable eigenvalues (i.e. are greater than 1), in which case there is an explosive solution, or
- there are too many stable eigenvalues (i.e. are less than 1), in which case there are multiple equilibria.

If the Blanchard-Kahn conditions hold, and the transversality condition is met, then we can pin down the policy functions F and P.

Exercise 5

Shit happens and we find out that the solution is

$$\hat{\omega}_t = \mathbf{F}\hat{x}_t,$$

$$\hat{x}_{t+1} = \mathbf{P}\hat{x}_t,$$

where

$$\mathbf{F} = \begin{bmatrix} 0.5220 & 0.6036 \\ 0.4513 & 0.7523 \\ -0.0401 & 0.0664 \\ -0.3539 & 0.7433 \\ 0.0708 & -0.1487 \\ -1.4695 & 4.7863 \\ 0.0973 & 1.4956 \\ -0.9027 & 1.4956 \end{bmatrix} \begin{Bmatrix} c \\ w \\ r \\ N \\ L \\ i \\ y \\ r^k \end{Bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0.9287 & 0.1383 \\ 0 & 0.9790 \end{bmatrix} \begin{Bmatrix} k \\ Z \end{Bmatrix}.$$

Part 1: Initial Shock

There's a 1% shock to $\hat{Z}_0$. What is the initial effect on consumption, investment, capital stock, and hours worked?

This period's capital stock is unaffected because k_0 is given before this period's shock. For consumption, we can write $\hat{c} = F_k^c \hat{k}_0 + F_z^c \hat{Z}_0$, which evaluates as

$$\hat{c}_0 = 0.5220 \times 0 + 0.6036 \times 1 = 0.6036.$$

For investment, we have

$$\hat{i}_0 = -1.4695 \times 0 + 4.7863 \times 1 = 4.7863.$$

Then for hours worked, we'll get

$$\hat{N}_0 = -0.3539 \times 0 + 0.7433 \times 1 = 0.7433.$$

Part 2: Subsequent Periods

In the second period, the shock is now

$$\hat{Z}_1 = 0 \times \hat{k}_0 + 0.9790 \times \hat{Z}_0 = 0.9790.$$

The capital stock becomes

$$\hat{k}_1 = 0.9287 \times 0 + 0.1383 \times 1 = 0.1383.$$

And now we can calculate the change in consumption, investment, and hours worked:

$$\hat{c}_1 = 0.5220 \times 0.1383 + 0.6036 \times 0.9790 \approx 0.66,$$

$$\hat{i}_1 = -1.4695 \times 0.1383 + 4.7863 \times 0.9790 \approx 4.48,$$

$$\hat{N}_1 = -0.3539 \times 0.1383 + 0.7433 \times 0.9790 \approx 0.68.$$

And so on and so forth. Here's some MATLAB code.

```
%Set number of periods and persistence
n = 40;
rho = 0.9790;

%Declare policy functions
F = [0.5220 0.6036; 0.4513 0.7523; -0.0401 0.0664; -0.3539 0.7433; ...
    0.0708 -0.1487; -1.4695 4.7863; 0.0973 1.4956; -0.9027 1.4956];
P = [0.9287 0.1383; 0 rho];

%Set initial conditions
Z(1)=1;
k(1)=0;

%Calculate impulse repsonse for n periods
for t=1:n
    c(t) = F(1,1) * k(t) + F(1,2) * Z(t);
    i(t) = F(6,1) * k(t) + F(6,2) * Z(t);
    N(t) = F(4,1) * k(t) + F(4,2) * Z(t);

    if t<n
    k(t+1) = P(1,1) * k(t) + P(1,2) * Z(t);
    Z(t+1) = P(2,1) * k(t) + P(2,2) * Z(t);
    end
end

%plot graphs
kplot = plot(k);
hold on;
cplot = plot(c);
iplot = plot(i);
Nplot = plot(N);
legend([kplot, cplot, iplot, Nplot],{'capital', 'consumption',...
    'investment', 'labor'});
zero = refline(0,0);
set(zero,'LineStyle','--', 'color', 'k');
hold off;
```

Part 3: Intuition

The marginal product of labor is $(1-\alpha)Zk^{\alpha}N^{-\alpha}$, so you can see how a positive TFP shock will increase MPL. Same thing with MPK. This increases investment and hours worked.

TFP will gradually move back to its steady-state value, which means investment and hours worked will converge back to their steady states. Capital will have a concave shape, reflecting the path of investment. Consumption will initially increase but only slowly converges back to its steady-state value.

Labor hours increase initially, as mentioned, but will decrease and will actually dip a little below steady state for a while before converging back.

Part 4: Performance

The RBC model can explain about 75% of the fluctuations in the data.[2] There are some dubious predictions, however. First, consumption is too smooth and investment is too volatile.

One issue in the model is that it requires large and persistent TFP shocks in order to closely match the data. The magnitude in particular does not match data.

Similarly, the model requires high elasticity of labor supply in order to generate internal propagation within the model. The standard model works mostly through the *intensive margin*—how many hours per worker. Real labor markets, on the other hand, adjust via the *extensive margin*—how many workers, e.g. employment.

Indivisible labor can resolve much of the need for large, persistent TFP shocks and for excessive intensive margin adjustment. Capital utilization can resolve the excessive volatility of investment.

[2] Of course, an epistemologically nuanced individual does not confuse the ability to explain something with a true explanation.