ECN 200D—Endogenous Job Destruction

Adapted from Athanasios Geromichalos's lectures
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1 The Setup

Let's add some nuance to the McCall setup. We'll have the same model except when λ hits a match, the new productivity of that match is given by px, where p is **general productivity** and x is **idiosyncratic productivity**. We will suppose that $x \sim G$ where $\text{supp}(G) = [0, 1]$. In practice we could have the support of G going to any finite maximum, but working between zero and one is nice so let's just normalize it to that.

The idea is that the worker's idiosyncratic productivity will change every now and then. If the productivity is hit with a negative enough shock, then the match will end and the job is destroyed. In the previous model, the worker was working at full capacity $x = 1$; when λ hit, the idiosyncratic productivity fell to $x = 0$ and the match was ended. Now there is a possibility of a negative shock to idiosyncratic productivity, but the worker might remain productive enough to maintain the match.

We will assume that when a new match is created that the idiosyncratic productivity is $x = 1$. Every hit of λ requires the worker and the firm to decide whether to continue the match and furthermore they must renegotiate the wage. Thus, wage can be written $w(x)$.

As in the previous model, job destruction will be equal to job creation in equilibrium. Job creation is nothing new; simply multiply the job arrival rate by the number of unemployed. The job destruction side will be a bit different, however. We will conjecture that there is a **reservation productivity** R, below which the job will end.[1] The parameter λ is the arrival rate of an idiosyntratic productivity shock, and because G is a cumulative distribution function, there is a $G(R)$ probability that the shock is bad enough to end the

[1]This doesn't mean the worker is fired by the firm. In fact, it means they mutually agree to end the working relationship.

relationship. Thus,

$$\theta q(\theta)u = (1-u)\lambda G(R).$$

Solving for u, we have the Beveridge curve

$$u = \frac{\lambda G(R)}{\lambda G(R) + \theta q(\theta)}. \tag{1}$$

2 Value Function of Firm Search

First, notice that the value of the firm having the worker depends on how productive the worker actually is, so we can write $J(x)$. Again, we will rely on the proposed reservation productivity R, which must be met in order for the match to take place. In other words, the value to a firm of having a worker with productivity R satisfies $J(R) = 0$.

The value function of a firm looking to fill a vacancy is

$$rV = -pc + q(\theta)\big(J(1) - V\big). \tag{2}$$

The cost of search pc is lost, but with arrival rate $q(\theta)$, the firm gets a maximally productive worker with $x = 1$, this switching their state to $J(1)$ from V.

It is still the case that $V = 0$ in equilibrium due to free entry of firms into the labor search market. Therefore we can rewrite equation (2) as

$$J(1) = \frac{pc}{q(\theta)}. \tag{3}$$

This equation tells us the expected cost of opening a vacancy. This is because $q(\theta)$ is the arrival rate of a job, and its inverse give the expected length of time for an arrival.

The value function of a firm having a job is more complicated that what we have seen so far. In a general sense, it is given by

$$rJ(x) = px - w(x) + \lambda \int_0^1 J(s)\, dG(s) - \lambda J(x).$$

The firm receives the production of px, and they lose the wage $w(x)$. Then with the arrival rate of the idiosyncratic production shock, they gain the expected value of $J(s)$ and lose what they had, $J(x)$.

We can do a bit better than this by exploiting our assumption that the reservation productivity is a thing. In particular, if s is below the reservation productivity, then the match is destroyed and the firm goes back into the mass of firms with vacancies for no value. Thus, we can write

$$\begin{aligned} rJ(x) &= px - w(x) + \lambda \int_R^1 J(s)\, dG(s) + \lambda \int_0^R 0\, dG(s) - \lambda J(x) \\ &= px - w(x) + \lambda \int_R^1 J(s)\, dG(s) - \lambda J(x). \end{aligned} \tag{4}$$

3 Worker Value Functions

The value of being employed to a worker is going to depend on x since the wage depends on x, so we can write $W(x)$. Keeping in mind that a worker starts a new job with $x = 1$, the value to a worker of being unemployed is

$$rU = z + \theta q(\theta)\big(W(1) - U\big). \tag{5}$$

The value to a worker of having a job with idiosyncratic productivity x, in unsimplified form, is given by

$$rW(x) = w(x) + \lambda \int_R^1 W(s)\, dG(s) + \lambda \int_0^R U\, dG(s) - \lambda W(x).$$

The worker receives a wage of $w(x)$. Then we consider the arrival of an idiosyncratic productivity shock. If the shock is above the reservation productivity, then the worker considers the expected value of that continuation. On the other hand, if the shock is below the reservation productivity, then the worker goes to unemployment. In either case, the worker loses $W(x)$. The minor simplification is to note that U is not a function of productivity shocks, so it can be taken out of the integral. Then the integral simply evaluates to $G(R)$ because G is a CDF. So we can write the function as

$$rW(x) = w(x) + \lambda \int_R^1 W(s)\, dG(s) + \lambda G(R)U - \lambda W(x). \tag{6}$$

4 Bargaining

Buckle up, because this is not going to be pretty.

For any $x \geq R$, the Nash bargaining problem is to solve

$$\arg\max_{w(x)} [W(x) - U]^{\beta} J(x)^{1-\beta}.$$

As usual, log it up to get

$$\arg\max_{w(x)} \beta \log[W(x) - U] + (1-\beta)\log[J(x)].$$

The first order condition gives

$$\begin{aligned} & \frac{\beta}{W(x) - U} \frac{dW(x)}{dw(x)} + \frac{1-\beta}{J(x)} \frac{dJ(x)}{dw(x)} = 0 \\ \implies \quad & \frac{\beta}{W(x) - U} \frac{1}{r+\lambda} - \frac{1-\beta}{J(x)} \frac{1}{r+\lambda} = 0 \end{aligned}$$

From this we can conclude that

$$\beta J(x) = (1-\beta)\big(W(x) - U\big) \tag{7}$$

Equation (7) will prove important in a few steps. But we're not quite there yet. Solve equation (6) for $W(x)$ and equation (4) for $J(x)$ to get

$$\begin{aligned} & \beta \left[px - w(x) + \lambda \int_R^1 J(s)\, dG(s) \right] \\ = (1-\beta) & \left[w(x) + \lambda \int_R^1 W(s)\, dG(s) + \lambda G(R)U - rU - \lambda U \right]. \end{aligned}$$

Notice that $\lambda G(R)U - \lambda U$ can be written as

$$\lambda \int_0^R U\, dG(s) - \lambda \int_0^1 U\, dG(s) = -\lambda \int_R^1 U\, dG(s).$$

Using that result, brute force your way through the algebra to get

$$w(x) = \beta px + (1-\beta)rU + \lambda \int_R^1 \beta J(s) - (1-\beta)\big(W(s) - U\big)\, dG(s).$$

This doesn't look good, but there is one very nice cancellation hiding in plain sight. The integrand, according to equation (7), is equal to zero.

So at this point we have

$$w(x) = \beta p x + (1-\beta) r U.$$

But hey, don't we have an equation for exactly rU? Yes. Yes we do. Equation (5), to be exact. So let's use that bad boy to get

$$w(x) = \beta p x + (1-\beta) z + (1-\beta)\theta q(\theta)[W(1) - U].$$

Let's use equation (7) again. In particular, it follows that

$$\beta J(1) = (1-\beta)\big(W(1) - U\big),$$

which we can use to write

$$w(x) = \beta p x + (1-\beta) z + \theta q(\theta) \beta J(1).$$

Now use equation (3) to write

$$\begin{aligned} w(x) &= \beta p x + (1-\beta) z + \theta q(\theta) \beta \frac{pc}{q(\theta)} \\ &= \beta p x + (1-\beta) z + \theta \beta p c. \end{aligned}$$

Hot damn, now let's just rewrite this a little bit more to get

$$\begin{aligned} w(x) &= (1-\beta) z + \beta p (x + c\theta) && (8) \\ &= z + \beta (p x - z + p c \theta). && (9) \end{aligned}$$

Finally! Woo.

Equation (8) is the **wage curve**. It is usually easiest to work with in this form, but equation (9) might be more intuitive as far as interpretation goes. The wage is the worker's outside offer plus the proportion of the surplus and ended search cost weighted by the worker's bargaining power. The only difference here compared to the exogenous job destruction model is that the surplus depends on x because production px depends on x.

5 Value Function of Filled Vacancy

We need to solve for u, v, w, and R. We have the wage curve. We also have the Beveridge curve. We still need to derive the job creation and job destruction curves.

Continuing where we left of, we can write the value to a firm of having a filled vacancy as

$$(r+\lambda)J(x) = px - w(x) + \lambda \int_R^1 J(s)\, dG(s).$$

We can substitute the wage curve $w(x) = (1-\beta)z + \beta p(x + c\theta)$, which we can write as

$$(r+\lambda)J(x) = (1-\beta)px - (1-\beta)z - \beta pc\theta + \lambda \int_R^1 J(s)\, dG(s).$$

Doesn't look like much, but we can see that is it linear in x and, in fact, upward sloping. Now recall that $J(R) = 0$. By plugging R into the above, we get

$$(r+\lambda)J(R) = 0 = (1-\beta)pR - (1-\beta)z - \beta pc\theta + \lambda \int_R^1 J(s)\, dG(s). \tag{10}$$

Because $J(R) = 0$, we can take $(r+\lambda)[J(x) - J(R)] = (r+\lambda)J(x)$, giving

$$(r+\lambda)J(x) = (1-\beta)p(x-R) \quad \Longrightarrow \quad J(x) = \frac{(1-\beta)p(x-R)}{r+\lambda}. \tag{11}$$

We'd determined in the previous lecture that

$$J(1) = \frac{pc}{q(\theta)}.$$

Using this along with having $x = 1$ in equation (11), we get

$$J(1) = \frac{(1-\beta)p(1-R)}{r+\lambda} = \frac{pc}{q(\theta)} \quad \Longrightarrow \quad \frac{(1-\beta)(1-R)}{r+\lambda} = \frac{c}{q(\theta)}. \tag{12}$$

This here is the **job creation curve**, at least as given in class. But I look at this thing and think, why the hell can't I just solve it for R? If you do so, you end up with

$$R = 1 - \frac{r+\lambda}{1-\beta}\frac{c}{q(\theta)}.$$

R as a function of θ is downward sloping, which you can see by just taking the derivative.

The intuition behind the job creation curve is this. When θ is increased, there are more vacancies relative to unemployed workers.

Now let's evaluate equation (11) at $x = s$ and plug that into equation (10), and then divide by $(1-\beta)p$ to get

$$R = \frac{z}{p} + \frac{\beta c\theta}{(1-\beta)} - \frac{\lambda}{r+\lambda}\left[\int_R^1 s\, dG(s) - \int_R^1 R\, dG(s)\right]. \tag{13}$$

This here be the **job destruction curve**. It is positive (take my word for it). It is also upward sloping. Let's consider R to be $R(\theta)$ and differentiate it. What we end up with is

$$R'(\theta) = \frac{\beta c}{1-\beta} + \frac{\lambda}{r+\lambda}\left[\int_R^1 g(s)\, ds\right] > 0.$$

The intuition behind the job destruction curve is this. When θ rises, that means workers have more outside options – there are more vacancies relative to unemployed workers. This means a lower level of productivity from the worker becomes more tolerable – it's harder to replace them, so let's not destroy the job.

So we have an upward sloping job destruction curve and a downward sloping job creation curve. The intersection? We want it. Figure 1 has it. Alright, so we've pinned down R^*, θ^*, u^*, and $w^*(x)$. Success!

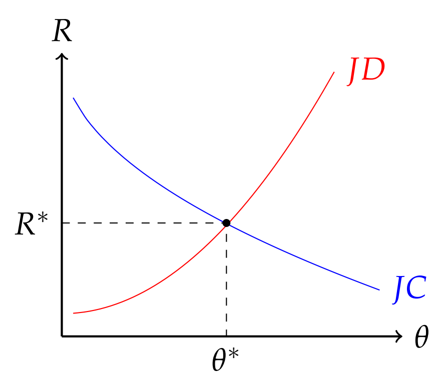

Figure 1: The unique reservation productivity and market tightness in equilibrium.

6 Welfare Properties

Okay, that's all great and everything. Let's go back to the exogenous destruction model and think about any externalities that this model might exhibit. In other words, are there any inefficiencies inherent in this search process? To analyze this, we want to compare the results we've found that emerged from the decentralized economy, and compare them to what a benevolent (i.e. optimizing) social planner would have done instead.

6.1 Social Planner Problem

Dynamic programming! Yeah. What the social planner wants to choose is the number of vacancies v that are allowed in the labor market – this is the control variable. The state variable will be the level of unemployment u. Whatever we do, we must satisfy the law of motion of unemployment,

$$\dot{u} = (1-u)\lambda - u\theta q(\theta).$$

Two things of note. First, e^{-rt} is the discounting factor. Second, it's easier to work with θ than to work with v, so we can instead use $\theta u = v$. That said, the social planner is trying to solve

$$\int_0^\infty e^{-rt}\left[(1-u)p + uz - pcu\theta\right]\, dt.$$

The term $(1-u)p$ the payoff from total worker output; the term uz is the payoff from the unemployed; and $pcu\theta = pcv$ is the cost of searching.

6.2 The Hamiltonian

This is a bit of a recap from mini-macro. Let x be the state variable and c be the control variable. We want to maximize

$$\int_0^\infty e^{-rt}u\big(x(t), c(t)\big)\, dt \quad \text{s.t.} \quad \dot{x}(t) = \gamma\big(x(t), c(t)\big),$$

Define the **Hamiltonian** to be

$$H = e^{-rt}u(\cdot) + \mu(t)\gamma(\cdot).$$

We'll need to satisfy the first order conditions

$$\frac{\partial H}{\partial c} = 0, \quad \frac{\partial H}{\partial x} = -\dot{\mu}.$$

Now let's try to apply this to our problem. The Hamiltonian is

$$H = e^{-rt}\left[(1-u)p + uz - pcu\theta\right] + \mu\left[(1-u)\lambda - u\theta q(\theta)\right].$$

6.2.1 First Order Condition 1

We want $\partial H/\partial\theta = 0$. Differentiate this bad boy with respect to θ to get

$$\begin{aligned}
\frac{\partial H}{\partial \theta} &= -pcue^{-rt} - \mu[uq(\theta) + u\theta q'(\theta)] = 0 \\
\implies \quad &= pce^{-rt} + \mu q(\theta)\left[1 + \theta\frac{q'(\theta)}{q(\theta)}\right] = 0 \\
\implies \quad &= pce^{-rt} + \mu q(\theta)\left[1 - \eta(\theta)\right] = 0,
\end{aligned}$$

where $\eta(\theta) = -\theta q'(\theta)/q(\theta)$.[2] Solve this for μ and we have

$$\mu(t) = -\frac{pce^{-rt}}{q(\theta)\left[1-\eta(\theta)\right]}. \tag{14}$$

Differentiate with respect to t and we have

$$\dot{\mu} = \frac{pcre^{-rt}}{q(\theta)\left[1-\eta(\theta)\right]}. \tag{15}$$

6.2.2 First Order Condition 2

Now differentiate with respect to the state variable to get

$$\begin{aligned}
\frac{\partial H}{\partial u} &= e^{-rt}(-p + z - pc\theta) + \mu[-\lambda - \theta q(\theta)] = -\dot{\mu} \\
\implies \quad & e^{-rt}(p - z + pc\theta) + \mu[\lambda + \theta q(\theta)] = \dot{\mu}.
\end{aligned} \tag{16}$$

Let's plug equations (14) and (15) into (16) to get

$$\begin{aligned}
& e^{-rt}(p - z + pc\theta) - \left(\frac{pce^{-rt}}{q(\theta)\left[1-\eta(\theta)\right]}\right)\left[\lambda + \theta q(\theta)\right] = \frac{pcre^{-rt}}{q(\theta)\left[1-\eta(\theta)\right]} \\
\implies \quad & (p - z + pc\theta) - \left(\frac{pc}{q(\theta)\left[1-\eta(\theta)\right]}\right)\left[\lambda + \theta q(\theta)\right] = \frac{pcr}{q(\theta)\left[1-\eta(\theta)\right]}.
\end{aligned}$$

[2] This actually comes from a homework problem. So, um, see that.

Now multiply both sides by $1 - \eta(\theta)$ and we have

$$[1-\eta(\theta)](p-z) + [1-\eta(\theta)]pc\theta - \frac{pc}{q(\theta)}[\lambda + \theta q(\theta)] = \frac{pcr}{q(\theta)}.$$

Finally, solve the equation for $[1-\eta(\theta)](p-z)$ and we have

$$[1-\eta(\theta)](p-z) = \frac{pcr}{q(\theta)} - \frac{[1-\eta(\theta)]\theta q(\theta)pc}{q(\theta)} + \frac{pc}{q(\theta)}[\lambda + \theta q(\theta)].$$

After some simplification, we get

$$[1-\eta(\theta)](p-z) = \frac{pc\,[r+\lambda+\theta q(\theta)\eta(\theta)]}{q(\theta)}. \tag{17}$$

6.3 The Hosios Condition

Going back to the decentralized problem, by equating the wage curve to the job creation curve, that is,

$$z + \beta(p - z + \theta pc) = p - (r+\lambda)\frac{pc}{q(\theta)},$$

a little algebra gives

$$(1-\beta)(p-z) = \frac{pc[r+\lambda+\beta\theta q(\theta)]}{q(\theta)}. \tag{18}$$

Compare this to the result in equation (17). The implication is that the decentralized market only achieves the optimal outcome if and only if $\beta = \eta(\theta)$. This condition is referred to as the **Hosios condition**. It is unlikely to be met, however. Sure, they're both in the interval $[0,1]$, but really, come on. The parameter β is just some exogenously given thing, whereas $\eta(\theta)$ is endogenous. It's just not gonna happen.

Comparing the two is still useful, however. In particular, suppose that $\beta > \eta(\theta)$. Then $1-\beta < 1-\eta(\theta)$. Ideally, the firm should be getting a surplus share of $\eta(\theta)$, but they're only getting $1-\beta$. Since firm payoff is too low, it means firm entry into the labor market is too low. Therefore unemployment is inefficiently high.

Now suppose the opposite case, in which $1-\beta > 1-\eta(\theta)$. This means that firms are getting a larger payoff than they should, so too many firms have entered the labor market. This means unemployment is inefficiently low. Sounds a bit unintuitive – it can be socially suboptimal to have *more* employed workers.

7 Solution Summary

i. Use the law of motion of unemployment at the steady state to derive the Beveridge curve.

ii. Because the value of searching for a worker $V = 0$ in equilibrium, it follows from the equation for rV that $J(1) = pc/q(\theta)$.

iii. Assume there exists some reservation level of productivity. Rewrite $rJ(x)$ by breaking the integral into two parts: before R and after R.

iv. Break up $rW(x)$ in the same way.

v. Do Nash bargaining. Plus (iii) and (iv) into the bargaining condition.

vi. Rewrite $G(R) - 1$ as $-\int_R^1 dG(s)$ and simplify a bunch.

vii. Plug in rU into (vi).

viii. Evaluate Nash condition at $x = 1$ and plug that into (vii).

ix. Substitute $J(1)$ into (viii) and simplify to get the wage curve,

$$w(x) = (1 - \beta)z + \beta p(x + c\theta).$$

x. Plug the wage curve into $(r + \lambda)J(x)$ and evaluate at $J(R)$. (Recall that $J(R) = 0$.) Then solve for $(r + \lambda)[J(x) - J(R)] = (r + \lambda)J(x)$. End up with

$$J(x) = \frac{(1 - \beta)p(x - R)}{r + \lambda}.$$

xi. Evaluate (x) at $J(1)$ and set it equal to $J(1) = pc/q(\theta)$. Cancel out the p terms and we have the job creation curve.

xii. Evaluate (x) at $x = s$ and plug into $(r + \lambda)J(R) = 0 = \ldots$. Divide by $(1 - \beta)p$. This monstrosity is the job descrubtion curve.

xiii. JD is upward sloping, JC is downward sloping, and their intersection marks R^* and θ^*. Plug θ^* into the Beveridge curve to get u^*.