ECN 200B—First Welfare Theorem Proof

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Preliminary Results

To begin with, we'll start with a lemma and a theorem.[1]

Lemma. *Suppose that $u^i(\cdot)$ is locally nonsatiated and x^* maximizes $u(x)$ subject to $px \leq m$. Then $u(x') \geq u(x^*)$ implies that $px' \geq m$.*

Proof. Suppose there exists some $\hat{x}$ such that $u(\hat{x}) \geq u(x^*)$ and $p\hat{x} < m$. Because preferences are locally nonsatiated, we know that we can construct an ϵ-ball around $\hat{x}$ and inside such a ball exists some x' such that $u(x') > u(\hat{x})$, and furthermore we can choose ϵ so that $px' \leq m$. This means that x' is affordable and is preferred to x^*, meaning that x^* is not actually the maximizer. This is a contradiction and the lemma is established. □

Theorem. *Suppose that x^* maximizes $u(x)$ subject to $px \leq m$. Then $u(x) > u(x^*)$ implies $px > m$.*

Proof. Note that local nonsatiation has not been assumed. But in any case, this follows directly from the definition of a maximizer. If x is preferred to x^* and is also affordable, then x^* isn't even the maximizer. Thus it has to be the case that x is unaffordable. □

First Fundamental Theorem of Welfare Economics

Theorem. *Fix a production economy,*

$$\{I, J, (Y_j)_{j \in J}, (u^i, w^i, s^{i,j})_{i \in I, j \in J}\}.$$

Suppose that all $u^i(\cdot)$ are locally nonsatiated. If (p, x, y) is a competitive equilibrium, then (x, y) is Pareto efficient.

[1] No, I do not know why one qualifies as a lemma and the other a theorem.

The Setup. Suppose that (p, x, y) is a competitive equilibrium but (x, y) is not Pareto efficient. Since it is a competitive equilibrium, we know that

(a) For all i, x^i maximizes $u^i(\tilde{x})$ subject to $p\tilde{x} \leq pw^i + \sum_{j \in J} s^{i,j} p \cdot y^j$.

(b) For any j, y^j maximizes $p \cdot \tilde{y}$ subject to $\tilde{y} \in Y^j$.

(c) $\sum_{i \in I} x^i = \sum_{i \in I} w^i + \sum_{j \in J} y^j$.

And because the allocation is not Pareto efficient, there exists some allocation $(\hat{x}, \hat{y})$ such that

(i) for all i, $\hat{x}^i \in \mathbb{R}^L_+$,

(ii) for all j, $\hat{y}^j \in Y^j$,

(iii) $\sum_{i=1}^I \hat{x}^i = \sum_{i=1}^I w^i + \sum_{j=1}^J \hat{y}^j$,

(iv) for all i, $u^i(\hat{x}^i) \geq u^i(x^i)$,

(v) for some i^*, $u^{i^*}(\hat{x}^{i^*}) > u^i(x^i)$.

The Proof. By the previous theorem, it follows that x^{i^*} must not be affordable, that is,

$$p\hat{x}^{i^*} > pw^i + \sum_{j \in J} s^{i,j} p \cdot y^j.$$

For any of the other individuals, by the previous lemma we must have

$$p\hat{x}^i \geq pw^i + \sum_{j \in J} s^{i,j} p \cdot y^j,$$

and therefore for all i, we have

$$p\hat{x}^i \geq pw^i + \sum_{j \in J} s^{i,j} p \cdot y^j.$$

Summing over all i and taking prices out of the sums, we have

$$p \sum_{i=1}^I \hat{x}^i > p \sum_{i=1}^I w^i + p \sum_{i \in I, j \in J} s^{i,j} \cdot y^j.$$

When we sum over the shares in firm j, each $\sum_{i=1}^I s^{i,j} = 1$, and therefore

$$p \sum_{i=1}^I \hat{x}^i > p \sum_{i=1}^I w^i + p \sum_{j \in J} y^j. \tag{1}$$

Because y^j was assumed to be the (feasible) profit maximizer, it must be the case for any j that $p \cdot \hat{y}^j \leq p \cdot y^j$. So if we sum over all j and take prices out the sums, we get

$$p \cdot \sum_{j=1}^{J} \hat{y}^j \leq p \cdot \sum_{j=1}^{J} y^j.$$

Add the aggregate nominal value of endowments to both sides for

$$p \cdot \sum_{i=1}^{I} w^i + p \cdot \sum_{j=1}^{J} \hat{y}^j \leq p \cdot \sum_{i=1}^{I} w^i + p \cdot \sum_{j=1}^{J} y^j.$$

Combining this with equation (1), it follows that

$$p \cdot \sum_{i=1}^{I} \hat{x}^i > p \cdot \sum_{i=1}^{I} w^i + p \cdot \sum_{j=1}^{J} \hat{y}^j,$$

which contradicts point (iii) above. Thus the competitive equilibrium allocation must be Pareto efficient. □