ECN 200B—Externalities Part 2

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1 The Setup

Let's show how a public good can break down the first fundamental theorem of welfare economics. Suppose there are $L+1$ commodities. There are $I \geq 2$ individuals whose preferences satisfy $u^i : \mathbb{R}_+^{L+1} \to \mathbb{R}$, written as $u^i(x^i, y)$, where $x^i \in \mathbb{R}_+^L$ and $y \in \mathbb{R}_+$. Let $w^i \in \mathbb{R}_+^L$ be the endowments of good $\ell = 1, \ldots L$. Commodity $L+1$ has to be produced according to the technology $F : \mathbb{R}_+^L \to R_+$, where $F(X) = Y$ and X denotes the inputs of the first L commodities.

There is only one firm, and individual i owns $s^i \in [0,1]$ of that firm's stock. The first L commodities are private and the $L+1$th commodity is public, i.e. nonrival and nonexclusive.

Let y_i be the number of units of good $L+1$ purchased by individual i at at price of q. Since good $L+1$ is public, individual i's utility function is

$$u^i\left(x^i, \sum_{k=1}^{I} y^k\right).$$

We will be looking for a **Nash-Walrasian equilibrium**. It is Walrasian with respect to utility maximization and market clearing; it is Nashian because everyone responds to what everyone else is doing.

Definition 1. A competitive equilibrium consists of $\big(p, q, (\bar{x}^i, \bar{y}^i)_{i=1}^I, \bar{X}, \bar{Y}\big)$ such that

(a) *Individual Rationality:* $(\bar{x}^i, \bar{y}^i)$ solves

$$\max_{x^i, y^i} \left\{ u^i\left(x^i, y^i + \sum_{k\neq i}^{I} \bar{y}^i\right)\right\} \qquad \text{subject to} \qquad px^i + qy^i \leq pw^i + s^i[q\bar{Y} - p\bar{X}].$$

(b) *Profit Maximization:* $(\bar{X}, \bar{Y})$ solves

$$\max_{X,Y} qY - pX \quad \text{such that} \quad F(X) = Y.$$

(c) *Market Clearing:* $\sum_{i=1}^{I} x^i + \bar{X} = \sum_{i=1}^{I} w^i$ and $\sum_{i=1}^{I} y^i = \bar{Y}$.

Suppose that $u^i \in C^2$ and $F \in C^2$. Also assume the usual properties, e.g. u^i is quasiconcave, $u^i_y > 0$, F is convex, etc.Suppose that $(p, q, \bar{x}, \bar{y}, \bar{X}, \bar{Y})$ is a competitive equilibrium and $(\bar{x}, \bar{y}, \bar{X}, \bar{Y})$ is Pareto efficient. Let's find the contradiction needed to show that the FFToWE breaks down.

2 The Characterization

Thanks to our lovely set of assumptions, we know we're working with interior solutions. It follows from individual rationality that

$$Du^i \left(\bar{x}^i, \bar{y}^i + \sum_{k \neq i}^{I} \bar{y}^k \right) = \lambda^i(p, q). \tag{1}$$

The (p, q) term indicates that when taking the derivative with respect to x^i, we should use the price p; when with respect to y^i, we should use the price q.

It follows from profit maximization, after taking the first order condition with respect to X that

$$p = qDF(\bar{X}). \tag{2}$$

Since $(\bar{x}, \bar{y}, \bar{X}, \bar{Y})$ is Pareto efficient, it must solve the "don't screw anyone over" constraint characterization, i.e.

$$\max_{x,y,X,Y} \left\{ u^1 \left(x^1, \sum_{i=2}^{I} y^i \right) \right\} \quad \text{such that} \quad u^2 \left(x^2, \sum_{i=1}^{2} y^i \right) \geq u^2 \left(\bar{x}^2, \sum_{i=1}^{2} \bar{y}^i \right),$$

along with $F(X) = Y$, $\sum_{i=1}^{I} y^i = Y$, and

$$\sum_{i=1}^{I} x^i + X = \sum_{i=1}^{I} w^i.$$

We may as well use market clearing for y^i to write

$$\max_{x,X,Y} \left\{ u^1 \left(x^1, Y \right) \right\} \quad \text{such that} \quad u^2 \left(x^2, Y \right) \geq u^2 \left(\bar{x}^2, \bar{Y} \right),$$

along with $F(X) = Y$ and

$$\sum_{i=1}^{I} x^i + X = \sum_{i=1}^{I} w^i.$$

3 The Lagrangian

Since it is essentially a constant, let's define $\bar{V}^i = u^i(\bar{x}^i, \bar{Y})$. Then we'll be using the Lagrangian

$$\mathcal{L} = u^1(x^1, Y) - \sum_{i \neq 1}^{I} \mu^i \left[\bar{V}^i - u^i(x^i, Y)\right] - \delta \left[\sum_{i=1}^{I} x^i + X - \sum_{i=1}^{I} w^i\right] + \epsilon[F(X) - Y].$$

We'll need to take the first order conditions with respect to each x_i, X, and Y. It's not as gross as it sounds.

(a) $\mu^i D_{x^i} u^i(x^i, Y) = \delta$

(b) $\delta = \epsilon DF(X)$

(c) $\sum_{i=1}^{I} \mu^i u_Y^i(x^i, Y) = \epsilon$, where $\mu^1 = 1$.

Consider equation (a) with $i = 1$. Since $\mu^i = 1$, we have

$$D_{x^1} u^1(\bar{x}^1, \bar{Y}) = \delta.$$

And from equation (1), we have

$$D_{x^1} u^1(\bar{x}^1, \bar{Y}) = \lambda^1 p.$$

It follows that p is a scalar multiple of δ. So let's just normalize p by dividing it with λ^1. Then $p = \delta$. When we can write equations (1) and (a) as, respectively,

$$D_{x^i} u^i(\bar{x}^i, \bar{Y}) = \lambda^i \delta = \frac{\delta}{\mu^i},$$

from which it follows that $\mu^i = 1/\lambda^i$.

We can use this with equation (c) to write

$$\sum_{i=1}^{I} \mu^i u_Y^i(x^i, Y) = \sum_{i=1}^{I} \frac{1}{\lambda^i} u_Y^i(x^i, Y) = \epsilon.$$

From equation (1), we can write

$$\frac{1}{\lambda_i} u_Y^i(\bar{x}^i, \bar{Y}) = q,$$

and therefore

$$\sum_{i=1}^{I} \frac{1}{\lambda^i} u_Y^i(x^i, Y) = \sum_{i=1}^{I} q = Iq = \epsilon.$$

And then from equation (b), we have

$$eDF(\bar{X}) = \delta = p = IqDF(\bar{X}).$$

From equation (2), we have

$$p = qDF(\bar{X}).$$

Therefore, $IqDF(\bar{X}) = qDF(\bar{X})$. We can't have $DF(\bar{X}) = 0$ because then $p = 0$. So it must be the case that $Iq = q$. We have assumed that $I \geq 1$, so then it must be the case that $q = 0$. But we can't have $q = 0$ because then $p = 0$ as well. So it cannot be the case that the competitive equilibrium is Pareto efficient.