ECN 200B—Arrow-Debreu Proof

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Theorem (Arrow-Debreu Theorem)**.** *Fix an exchange economy. If u^i is continuous, strictly monotone, and strictly quasiconcave; and if each $w^i \gg 0$; then there exists a competitive equilibrium.*

The Setup. Normalize prices to the simplex Δ. We will refer to the interior of the simplex as

$$\Delta^o = \Delta \cap \mathbb{R}^L_{++} = \left\{ p \in \mathbb{R}^L_{++} \,\middle|\, \sum_{\ell=1}^{L} p_\ell = 1 \right\},$$

and the boundary of the simplex as

$$\Delta^\partial = \Delta \setminus \Delta^o.$$

For any i and any $p \in \Delta^o$, let $x^i(p)$ maximize individual i's utility subject to $p \cdot x \leq p \cdot w^i$. Define the excess aggregate demand in the usual way,

$$z(p) = \sum_{i=1}^{I} x^i(p) - w^i.$$

Properties of Excess Demand.

Claim 1. *$z(p)$ is a continuous function.* Since utility is assumed to be strictly quasiconcave, it follows that $x^i(p)$ is a function, and therefore so is

$z(p)$. Because utility is also assumed to be continuous, it follows that $x^i(p)$ is continuous[1], and therefore $z(p)$ is continuous as well. //

Claim 2. *For any* $p \in \Delta^o$, $p \cdot z(p) = 0$. Since utility is strictly monotone, it is locally nonsatiated, and therefore the budget constraint is an equality, that is, $x^i(p) = w^i(p)$ for all i. It follows that $p \cdot [x^i(p) - w^i(p)] = 0$ for all i. And therefore

$$p \cdot \sum_{i=1}^{I} [x^i(p) - w^i] = p \cdot z(p) = 0. \qquad //$$

One problem arises when $p \in \Delta^\partial$, however. Specifically, if $p_L = 0$, then $x^i(p)_L$ will be undefined—strongly monotone preferences and strictly positive endowments means everyone will want to buy an infinite number of good L. So we'll need to consider boundary prices and interior prices separately, but not *too* separately. I'll explain in a bit.

The Gamma Correspondence. I'll illustrate the idea before defining anything. Suppose $p \in \Delta^o$, and $z(p) = (3, 1, 1, -1)$. Let's solve

$$\max_{\delta_1, \delta_2, \delta_3, \delta_4} 3\delta_1 + 1\delta_2 + 1\delta_3 - 1\delta_4$$

such that $\sum_{j=1}^{4} \delta_j = 1$. Clearly we would put all of the weight on δ_1 because it is the largest positive number; that is, have $\delta_1 = 1$, the other $\delta_{j \neq 1} = 0$. So the idea is, whichever good has the highest excess demand, we should set its price to 1 and every other price to zero. Let $\Gamma(p)$ be the maximizing set of δ_j, in this case, $\Gamma(p) = [1, 0, 0, 0]$.

Now suppose that $p \in \Delta^\partial$. In this case, excess demand is not defined. What we'll do is take all possible vectors $\gamma \in \Delta$ that satisfy $p \cdot \gamma = 0$. For instance, if $p = (.5, .5, 0, 0)$, then $\delta_1 = \delta_2 = 0$, and $\delta_3 + \delta_4 = 1$, so $\delta = (0, 0, .5, .5)$ would work, or $(0, 0, .1, .9)$.

[1] By the Theorem of the Maximum, which states that an optimized function is continuous as its parameter changes, in this case p, under certain conditions.

Thus the correspondence we'll be working with is

$$\Gamma(p) = \begin{cases} \arg\max_{\gamma\in\Delta} z(p)\cdot\gamma & \text{if } p \in \Delta^0, \\ \{\gamma\in\Delta | p\cdot\gamma = 0\} & \text{if } p\in\Delta^\partial. \end{cases}$$

An important point to note here is that if $p \in \Delta^o$ and $z(p) \neq (0,\ldots,0)$, then $\Gamma(p) \subseteq \Delta^\partial$. And thus by the contrapositive, if $\Delta(p) \not\subseteq \Delta^\partial$ and $p \in \Delta^*$, then $z(p) = (0,\ldots,0)$ and we are done.

Eventually we want to invoke Kakutani's theorem on $\Gamma(p)$, but of course $\Gamma(p)$ must satisfy certain conditions to justify doing so. In the interest of time, we will take for granted a few different properties, although each can be shown.

(a) We need Γ to map $\Delta \twoheadrightarrow \Delta$. This condition is rather obvious since each γ is taken from Δ.

(b) Γ is nonempty, compact, and convex-valued.

(c) Γ is upper-hemicontinuous for $p \in \Delta^o$.

Let's show upper-hemicontinuity. Consider a sequence of prices $(p_n)_{n=1}^\infty \in \Delta^o$ that converges to positive prices in all except one $\bar{p}_L = 0$, that is,

$$\big(p_{1(n)},\ldots,p_{L-1(n)},p_{L(n)}\big) \to (\bar{p}_1,\ldots,\bar{p}_{L-1},0) = \bar{\delta} \in \Delta^\partial.$$

Let $\delta_n \in \Gamma(p_n)$ for any n. Our question is this: if p_n converges to $\bar{p}$, will the correspondence converge to some $\gamma \in \Gamma(\bar{p})$?

We know that $z_L(p_n) \to \infty$. We also know that $p_{\ell(n)} > 0$ and $\bar{p}_\ell > 0$ for any $\ell \neq L$. Thus, for large enough n, we'll have

$$z_L(p_n) > z_\ell(p_n).$$

Which means for large enough n, we'll have $\Gamma(p_n) = \{0,0,\ldots,0,1\}$. Therefore $\gamma_n \to (0,0,\ldots,0,1) = \bar{\gamma}$. Furthermore, $\bar{p}\cdot\bar{\gamma} = 0$. So $\bar{\gamma} \in \Gamma(\bar{p})$. Hemicontinuity is established.

This is good—even though we're treating boundary prices and interior prices with separate correspondences, the case correspondence $\Gamma(p)$ is still upper-hemicontinuous.

The Fixed Point. Okay great, so we can apply Kakutani's theorem—there exists some $p^* \in \Delta$ such that $p^* \in \Gamma(p^*)$. What can we say about p^*?

The most important thing we can say is that it's not in the boundary. To see why, suppose $p^* = (1, 1, 0)$. Then we'll have $\delta = (0, 0, 1) \in \Gamma(p^*)$. But then if we take $\Gamma(\delta)$, we'll get, among other choices, $\Gamma(\delta) = (.5, .5, 0) \notin \Gamma(p^*)$.

More generally, suppose p^* has $p^*_L = 0$. Then the correspondence $\Gamma(p^*)$ will consist of vectors with $\delta_L > 0$. So when we plug δ back into the correspondence with $\Gamma(\delta)$, it will return vectors with $\tilde{\delta}_L = 0$, which cannot be in $\Gamma(p^*)$ because $\Gamma(p^*)$ consists of vectors with $\delta_L > 0$.

Since we know there exists some fixed point p^*, it follows that it must be in the interior. Furthermore, we know that if $p^* \in \Delta^o$ and $z(p) \neq (0, \ldots, 0)$, then $\Gamma(p^*) \subseteq \Delta^\partial$. By the contrapositive, the fact that $\Gamma(p^*) \nsubseteq \Delta^\partial$ means that $z(p^*) = (0, \ldots, 0)$. And thus we know that the equilibrium exists and has strictly positive prices on all commodities.