Solution 1a

Move through each information set and see if the assessment implies a rational choice.

Start with information set $\{x, y\}$ of Player 2. Their strategy is to always play f. They believe there is a 1/3 chance that node x is the one that is reached, in which case they would get a payoff of 3. They believe there is a 2/3 chance that node y is the one that is reached, in which case they would also get a payoff of 3. Thus the their expected payoff for f is

$$\frac{1}{3}[3] + \frac{2}{3}[3] = 3.$$

If they decided to play g instead, their payoff would be

$$\frac{1}{3}[6] + \frac{2}{3}[0] = 2.$$

So, given Player 2's beliefs, their choice to play f over g is sequentially rational.

Now consider information set $\{w, z\}$ of Player 1. Their strategy is to always play h. They believe there is a 1/2 probability that node w is the one that is reached, in which case h gives them a payoff of 3. They believe there is a 1/2 probability that node z is the one that is reached, in which case h gives them a payoff of 0. So the expected payoff is h is

$$\frac{1}{2}[3] + \frac{1}{2}[0] = 1.5.$$

If they decided to play k instead, then their payoff would be

$$\frac{1}{2}[1] + \frac{1}{2}[1] = 1.$$

So, given Player 1's beliefs, their choice to play h over k is sequentially rational.

Now consider Player 2's choice of d or e. They choose to play e, after which Player 1 plays h, giving a payoff of 2. If they chose instead to play d, they'd get 1. So Player 2's choice of playing e over h is sequentially rational.

Okay, finally, let's look at Player 1's choice at the root between a, b, and c.

- Strategy a is played with probability 1/8. In this case, the node $x \in \{x, y\}$ of Player 2 is reached, and their strategy in that information set is to always play f. So we end up at the outcome $[0, 3]$.
- Strategy b is played with probability 3/8. In this case, the node $y \in \{x, y\}$ of Player 2 is reached, and their strategy in that information set is to always play f. So we end up at the outcome $[0, 3]$.

- Strategy c is played with probability 4/8. In this case, Player 2 then plays e. Then the node $z \in \{w, z\}$ of Player 1 is reached. They choose to play h for outcome $[0, 2]$.

So Player 1's expected payoff is

$$\frac{1}{8}[0] + \frac{3}{8}[0] + \frac{4}{8}[0] = 0.$$

But Player 1 could do better than this. Notice that if Player 1 chooses to play k instead of h, they'd get payoff of $1 > 0$ in the case of c, which is an improvement. (This also implies that Player 1 should always play c instead of mixing over a and b, which both give zero.) Therefore this assessment is not sequentially rational for Player 1.

Solution 1b

The first step is to find the probabilities $P_{\text{root},\sigma}(x)$ for all nodes x contained within nontrivial information sets, translated to English as "the probability that x is reached from the root of the tree if σ is implemented." There are four nodes found within nontrivial information sets: x, y, w, and z.

- The probability of node x being reached from the root is simply 1/8, that is, the probability that Player 1 plays a.
- The probability of node y being reached from the root is simply 3/8, that is, the probability that Player 1 plays a.
- The probability of node w being reached is the 4/8 probability Player 1 plays c, times the 3/4 probability that Player 2 plays d, so 3/8.
- The probability of node z being reached is the 4/8 probability Player 1 plays c, times the 1/4 probability that Player 2 plays d, so 1/8.

Now we do Bayesian updating to specify μ. This is very easy to do if everything is in terms of common denominators. Since reaching y is three times as likely as reaching x, Player 2's beliefs should reflect that. Those are the only two nodes within the information set, so it must be the case that $\mu(x) = 1/4$ and $\mu(y) = 3/4$. Similarly, it must be the case that $\mu(w) = 3/4$ and $\mu(z) = 1/4$.

But I'll be more rigorous about it. First, find the probabilities of reaching the nontrivial information sets.

- The probability of reaching $\{x, y\}$ is the probability that Player 1 plays a plus the probability that Player 1 plays b, so 4/8.

- The probability of reaching $\{w, z\}$ is the probability that Player 1 plays c (because all of Player 2's subsequent choices will lead to $\{w, z\}$), that is, 4/8.

Now apply the typical updating mechanics to get

$$\mu(x) = \frac{1/8}{4/8} = \frac{1}{4} \qquad \mu(y) = \frac{3/8}{4/8} = \frac{3}{4} \qquad \mu(w) = \frac{3/8}{4/8} = \frac{3}{4} \qquad \mu(z) = \frac{1/8}{4/8} = \frac{1}{4}.$$

Oh hey, that's what I said it would be earlier. Good. Great. Grand. Wonderful.

So the system of beliefs satisfying Bayesian updating at reached information sets is

$$\mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{array} \right).$$

Solution 2

If the game is simple enough, it might be easiest to find all of the Nash equlibria and then see which ones are WSE. I will do this approach first.

Nash Equilibrium Approach. To that end, the strategic-form of the game is

	E	F
AC	$\underline{1.5}, \underline{0.5}$	$\underline{1}, \underline{0.5}$
AD	$1, \underline{1}$	$\underline{1}, \underline{1}$
BC	$\underline{1.5}, \underline{1}$	$0.5, 0$
BD	$1, \underline{1.5}$	$0.5, 0.5$

So we have four NE to check: (AC, E), (AC, F), (AD, F), and (BC, E). We can represent these as behavioral strategies

$$(AC, E) = \left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right),$$

$$(AC, F) = \left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right),$$

$$(AD, F) = \left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right),$$

$$(BC, E) = \left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right).$$

$\boldsymbol{(AC, E)}$**.** I'll start from the bottom.

- Node x is not reached.
- Node y is reached. Therefore Bayesian updating requires that

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

- E and F are both rational for Player 2 when Player 1 plays C.
- Player 1 can choose C over D regardless of what Player 2 does.
- Player 1 playing A gives payoff 1; Playing B gives payoff 1 as well. So A is rational.

Sequential rationality checks out. This qualifies as a WSE.

$\boldsymbol{(AC, F)}$**.** I'll start from the bottom.

- Node x is not reached.
- Node y is reached. Therefore Bayesian updating requires that

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

- E and F are both rational for Player 2 when Player 1 plays C.
- Player 1 can choose C over D no matter what Player 2 does at node y.
- Player 1 playing A gives payoff 1; Playing B gives payoff 1 as well. So A is rational.

Sequential rationality checks out. This qualifies as a WSE, too. In fact, I just copied and pasted this from the previous one.

$(\boldsymbol{AD}, \boldsymbol{F})$. I'll start from the bottom.

- Node x is not reached.
- Node y is not reached either. This means there are no Bayesian restrictions upon the beliefs of Player 2. So we can write

$$\mu = \begin{pmatrix} x & y \\ p & 1-p \end{pmatrix}.$$

But it still has to be the case for Player 2 that playing F is rational. This will be the case when

$$F: \quad p(0) + (1-p)(0) \geq p(2) + (1-p)(0) \quad : E,$$

which requires that $p = 0$. Hence we require

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

- Playing A is rational for Player 1 because $1 > 0$.
- Playing D is rational for Player 1 because $1 = 1$.

And so another WSE has been found.

$(\boldsymbol{BC}, \boldsymbol{E})$. I'll start from the bottom.

- Node x is reached.
- Node y is also reached. This means we must place Bayesian restrictions upon the beliefs of Player 2, in this case,

$$\mu = \begin{pmatrix} x & y \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

which are determined entirely by Nature.

- Given the beliefs in the last step, Playing E is rational because

$$F: \quad \frac{1}{2}(0) + \frac{1}{2}(0) \leq \frac{1}{2}(2) + \frac{1}{2}(0) \quad : E,$$

- Player 1 choosing B is rational because $1 = 1$.
- Player 1 choosing C is rational because $2 > 1$.

And so another WSE has been found.

From Scratch Approach. The other way is to just work your way through and pick up the WSE as you go.

- Let's start by trying AC. Then x is not reached but y is, so

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

  Since Player 2 is thus convinced they are in node y, they will choose to play either E or F because they both give payoff zero.

  Suppose they pick E. Then A is rational because it gives payoff 1 compared to $BE \to 1$. C is rational because $CE \to 2$ compared to $D \to 1$. Therefore (AC, E) is a WSE.

  Now suppose they pick F. Then A is rational because it gives payoff 1 compared to $BF \to 0$. C is rational because $CF \to 1$ compared to $D \to 1$. Therefore (AC, F) is a WSE.

- Now let's try AD. Neither x nor y are reached, so Bayes places no restrictions on belief. Thus

$$\mu = \begin{pmatrix} x & y \\ p & 1-p \end{pmatrix}.$$

  Is A rational? Yes, it is always rational because it weakly dominates B. Is D rational? Well, D is only rational when F is played. So we need to ask when F is rational. This will depend on the beliefs of Player 2. Specifically, given beliefs p and $1-p$, the expected payoffs must be such that

$$E: \quad p(2) + (1-p)0 \leq p(0) + (1-p)0 \quad : F$$

  This implies that F is rational only when $p = 0$. So

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

  Therefore (AD, F) is a WSE.

- Let's now try BC. Since x and y are both reached,

$$\mu = \begin{pmatrix} x & y \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

For B to be rational, it must be that E is chosen. C is always rational because C weakly dominates D. So when can Player 2 choose E? Always, because E weakly dominates F.

We have (BC, E) as another WSE.

- Let's try BD. Because x is reached but y isn't,

$$\mu = \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}.$$

This means Player 2 should choose E. This makes B a rational choice compared to A because the payoffs are both 1. But this makes C the rational choice because $CE \to 2$ compared to $D \to 1$. So BD cannot be part of a WSE.