# Stock Beta

## Definition of Beta

Let $M$ be a **index fund/market portfolio**, i.e. a portfolio of a cross-section of top-performing stocks (e.g. S&P 500). Think of $M$ as reflecting the market as a whole.

Let $r_M$ denote the (random) return of market portfolio $M$. $E[r_M]$ is the expected return on the market portfolio.

A stock's $\boldsymbol{\beta}$ (**beta**) is a quantity that compares $E[r_M]$ against the (random) return on a stock $r_i$, defined as

$$\beta_i \equiv \frac{\mathrm{Cov}(r_i, r_M)}{\mathrm{Var}(r_M)}.$$

## Interpreting Beta

- Because $\mathrm{Cov}(r_M, r_M) = \mathrm{Var}(r_M)$ as a property of covariance, we conclude that the market has $\beta_M = 1$.
- $\beta > 1$ implies that when the stock market did better (worse), on average your stock did *a lot* better (worse).
- $0 \leq \beta < 1$ implies that, on average, your stock did not move as much (or not at all) with the stock market.
- $\beta < 0$ implies that this investment does well when the market does poorly, and vice versa. Like insurance.

  Suppose that investors are risk-averse. Intuitively, we should expect demand for assets with $\beta_i < 0$ to increase since having negatively correlated assets reduces risk of the overall portfolio.

Also notice (or take my word for it) that the formula looks suspiciously similar to the OLS solution of a linear regression. Not a coincidence.

# Best Linear Predictor

## True Beta

Regress $r_i$ on $r_M$ to find the best linear predictor of $r_i$. With the entire population of data at our disposal (and thus precisely knew the true value $\beta$), we can write exactly

$$r_i = \beta_i r_M + c.$$

## Estimating Beta

In reality, we have only a sample and hence must estimate

$$\hat{r}_i = \hat{\beta}_i r_M + \hat{c}.$$

To estimate, use sample analogues of the covariance and variance. Suppose we have $j = 1, \ldots, n$ data points for returns $r_i$ and $r_M$. Then the sample analogues are

$$\widehat{\mathrm{Cov}}(r_i, r_M) = \frac{1}{n-1}\sum_{j=1}^{n}(r_i^j - \bar{r}_i)(r_M^j - \bar{r}_M),$$

$$\widehat{\mathrm{Var}}(r_M) = \frac{1}{n-1}\sum_{j=1}^{n}(r_M^j - \bar{r}_M)^2.$$

Therefore we use the beta estimate

$$\hat{\beta}_i = \frac{\widehat{\mathrm{Cov}}(r_i, r_M)}{\widehat{\mathrm{Var}}(r_M)} = \frac{\frac{1}{n-1}\sum_{j=1}^{n}(r_i^j - \bar{r}_i)(r_M^j - \bar{r}_M)}{\frac{1}{n-1}\sum_{j=1}^{n}(r_M^j - \bar{r}_M)^2}.$$

# Premia

## Risk-Neutrality

When investors are risk-neutral, they must be compensated for possibility of default with a **default premium** in the promised rate – this ensures that the *expected return of the risky asset is the same as that of the safe asset.*

In other words, the default premium is defined to be the promised rate of an asset minus its expected rate. In a risk-neutral world, the expected rate *is* the safe rate, and thus the default premium is $r_{\text{promised}} - r_f$.

## Risk-Aversion

But if investors are risk-averse, they must also be compensated for taking on a risky asset through a **risk premium**. In this world, the *expected return of the risky asset is higher than that of the safe asset.* Intuitively, people hate risk so you have to tempt them with a higher expected return.

The default premium is now given by $r_{\text{promised}} - E[r]$.

## Equity Premium

The **equity premium** is defined as

$$\text{equity premium} \equiv E[r_M] - r_f,$$

which captures the expected return of the market as a whole in excess of the safe return.

# CAPM (Capital Asset Pricing Model)

## CAPM Equation

CAPM is a formula that links expected return on any asset to its best linear predictor based on the market portfolio:

$$E[r_i] = r_f + (E[r_M] - r_f)\beta_i.$$

Intuition: investor is compensated for two things.

- $r_f$ is the return investors get for having to wait to get the return (the *time premium*).
- $[E(r_M) - r_f]\beta_i$ reflects the return investors get for taking on more (or less) risk, relative to the market, as determined by $\beta_i$ (i.e. the *expected risk premium*).

If $\beta_i > 1$, then $i$ is a risky asset relative to the market, and consequently there is a larger expected return. If $\beta_i = 0$, then there is no risk and it is equivalent to the safe return (in expectation). Etc. This is captured by drawing the **security market line** (i.e. plotting the CAPM equation).

CAPM can also be written as

$$E[r_i - r_f] = \beta_i E[r_M - r_f].$$

So the expected premium of the risky asset $i$ over the safe asset is $\beta_i$ times the equity premium.

## Promised Rate of Return Decomposition

With risk-averse agents, we can decompose the promised rate of return into its three constituent elements:

$$\begin{aligned}\text{promised rate} = &\ \text{time premium} \\ &+ \text{risk premium} + \text{default premium} \\ = &\ r_f + \text{risk premium} + (r_{\text{promised}} - E[r]).\end{aligned}$$