ECN 200B—Rationalizing Data

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1 Individual Problem

Suppose we have some data, $(p_t, x_t)_{t=1}^T$. We say that the utility function $u : \mathbb{R}_+^L \to \mathbb{R}$ **rationalizes** the data if for any t,

$$x_t = \arg\max_x \{u(x) : p_t x \leq p_t x_t\}.$$

In other words, the the utility function has to be one such that each x in the data set maximizes the utility function subject to the budget constraint.

Theorem 1. *The exists a continuous, strictly concave, strongly monotone utility function $u(\cdot)$ that rationalizes the data if and only if there exists number $(\lambda_t, V_t)_{t=1}^T$, $\lambda > 0$, such that for any t, t',*

$$V_t' \leq V_t + \lambda_t p_t (x_{t'} - x_t), \tag{1}$$

with strict inequality if $x_t \neq x_{t'}$.

Proof. We will be assuming that $u(\cdot)$ is smooth just for ease.

Only if. Have $V_t = u(x_t)$. From the first order condition, there exists a Lagrange multiplier $\lambda_t > 0$ such that $Du(x_t) = \lambda_t p_t$. When we're dealing with concavity, it means that the first order linear approximation overstates the function, so for any x and any t,

$$u(x) \leq u(x_t) + Du(x_t)(x - x_t),$$

which satisfies equation (1).

If. Suppose for any t, λ_t satisfies (1) and that $x_t \neq x_{t'}$ so that the inequality is strict. For each t, define

$$u_t(x) = V_t + \lambda_t p_t(x - x_t).$$

This function is linear and thus is concave. Notice that $u_t(x_t) = V_t$. Also define $u(x)$ to be the *lower envelope* of the set of $u_t(x)$ functions, i.e.

$$u(x) = \min\{u_1(x), \ldots, u_T(x)\}.$$

This is our candidate for rationalization. Let's try it.

Pick some arbitrary t and x such that $p_t x \leq p_t x_t$, i.e. x is affordable in period t. By the way $u(x)$ is constructed, it follows that

$$\begin{aligned} u(x) &\leq u_t(x) \\ &= V_t + \lambda_t p_t(x - x_t) \\ &\leq V_t \qquad (2) \\ &= u_t(x_t), \end{aligned}$$

where line (2) follows from $p_t x \leq p_t x_t$, which implies that $p_t(x - x_t) < 0$.

Also notice that $u(x_t) \leq u_t(x_t)$ by construction. Suppose for a moment that $u(x_t) < u_t(x_t)$. Then it follows from the way $u(x)$ is defined that there is some t' such that $u(x_t) = u_{t'}(x_t) < u_t(x_t)$. Therefore from the definition of $u_t(x_t)$, we have

$$u_{t'}(x_t) = V_{t'} + \lambda_{t'} p_{t'}(x_t - x_{t'}) < V_t.$$

But this violates the Afriat condition of

$$V_{t'} + \lambda_{t'} p_{t'}(x_t - x_{t'}) \geq V_t.$$

So it must be the case that $u(x_t) = u_t(x_t)$. $\square$

2 Market Problem

Suppose we have data $\left(p_t, (w_t^i)_{i=1}^I\right)_{t=1}^T$. A profile $(u^i : \mathbb{R}_+^L \to \mathbb{R})_{i=1}^I$ is said to **rationalize** the data if for every t, p_t is a competitive equilibrium price vector for the economy $\{I, (u^i, w_t^i)_{t=1}^T\}$. The previous theorem tested for individual rationality, but now we are testing a more restrictive set of conditions—we need individual rationality to hold but we also need the market to clear.

Theorem 2. *The set utility functions $(u^i : \mathbb{R}^L_+ \to \mathbb{R})_{i=1}^I$ rationalizes the data if and only if there exists $\big((x_t^i \in \mathbb{R}^L_+)_{i=1}^I\big)_{t=1}^T$ such that for all t, $\big(p_t, (x_t^i)_{t=1}^T\big)$ is a competitive equilibrium of $\{I, (u^i, w_t^i)_{t=1}^T\}$.*

So the Walrasian demands, derived from the utility functions, must satisfy the utility maximization problem for each agent and clear the market.

Theorem 3. *There exists a profile of continuous, strictly concave, strongly monotone utility functions $(u^i : \mathbb{R}^L_+ \to \mathbb{R})_{i=1}^I$ that rationalize the data if and only if there exists $\{(x_t^i \in \mathbb{R}^L_+)_{i=1}^I\}_{t=1}^T$ and numbers $\{(\lambda_t^i, V_t^i)_{i=1}^I\}_{t=1}^T$ such that*

- ***(a)*** *for any i, t, t', $V_{t'}^i \leq V_t^i + \lambda_t^i p_t (x_{t'}^i - x_t^i)$, with strict inequality when $x_t^i \neq x_{t'}^i$,*
- ***(b)*** *$\sum_{i=1}^I x_t^i = \sum_{i=1}^I w_t^i$ for all t.*

Theorem 4. *There exists a nontaugological (i.e. falsifiable), finite set of conditions such that*

- ***(a)*** *it includes only the data set,*
- ***(b)*** *a data set is rationalizable by a profile of continuous, strictly concave, strongly monotone utility functions if and only if it satisfies the conditions.*

The test is necessary and sufficient, and is also the strongest of such tests. But there's a problem. "Nobody knows what the fucking test is." As the size of the problem increases, the time required to solve the test grows exponentially.

Consider two Edgeworth boxes—actually rectangles. One is a very tall and skinny rectangle, the other short and wide. The tall one has a flat price vector, the short one has a steeper price vector. What matters is what when you superimpose the two, the two price vectors meet outside of the Edgeworth boxes, which means WARP would fail. Therefore the text can actually fail in principle and isn't tautological.