Game Frames and Strategies

Definition 1. *A* ***game frame*** *consists of the following objects.*

- *Players $I = \{1, 2, \ldots, n\}$, where $n \geq 2$.*
- *For every $i \in I$ there is a set of* ***strategies*** *S_i, e.g. $S_i = \{\text{split}, \text{steal}\}$. The Cartesian product $S = S_1 \times S_2 \times \ldots \times S_n$ is the set of* ***strategy profiles****.*
- *A set of* ***outcomes*** *W.*
- *An outcome function $z : S \to W$.*

In order to have a proper game, we need to include preferences of players, i.e. we know their objectives and know that $\succsim_i$ is defined over W for all i.

Example 1. Consider a game of rock, paper, scissors. The strategies are $S_1 = S_2 = \{R, P, S\}$. Let W_i denote player i winning and D denote a draw. The set of outcomes is $W = \{W_1, W_2, D\}$. We assume that player i's preferences satisfy

$$W_i \succsim D \succsim W_j.$$

The set of strategy profiles is captured by the cells in the table below.

	R	P	S
R	D	W_2	W_1
P	W_1	D	W_2
S	W_2	W_1	D

For now, let's define **rationality** as: a player chooses their most preferred outcome based on their beliefs. We're not making any judgment about whether the beliefs themselves are rational, however—just that they're responding to their beliefs optimally.

We can represent preferences in a few different ways. One is simply by ordering them from best to worst. For instance, top being best,

$$\begin{array}{l} W_2 \\ W_6 \\ W_5 \\ W_4, W_1 \\ W_3. \end{array}$$

In this case, $W_4 \sim W_1$.

We can also order them as a preference relation:

$$W_2 \succ W_6 \succ W_5 \succ W_4 \sim W_1 \succ W_3.$$

Or we can attach ordinal numbers to each outcome, for instance

W_1	W_2	W_3	W_4	W_5	W_6
3	6	2	3	4	5

The larger numbers represent preference, equal numbers indifference. Utilities makes it easy to use a table.

Since $z : S \to W$ and $U_i : W \to \mathbb{R}$, we can compose the two functions into the **payoff function** $\pi_i : S \to \mathbb{R}$. In other words, $\pi_i(s) = u\big(z(s)\big)$.

Example 2. Here are the utilities for player 2 for each outcome.

	c	d	e
a	$\cdot, 3$	$\cdot, 6$	$\cdot, 2$
b	$\cdot, 3$	$\cdot, 4$	$\cdot, 5$

Notice that no matter what player 1 does, player 2 would get a higher payoff if they'd chosen d instead of c. In this case, d strictly dominates c for player 2.

Definition 2. *Strategy $a \in S_i$ is said to* ***strictly dominate*** *strategy b if for any $x \in S_j$, $z(a, x) \succ_i z(b, x)$, or in terms of payoffs,*

$$\pi_i(a, x) > \pi_i(b, x).$$

Definition 3. *Strategy $a \in S_i$ is said to* ***weakly dominate*** *strategy b if*

- *for any $x \in S_j$, $z(a, x) \succsim_i z(b, x)$*
- *there exists $\hat{x} \in S_j$ such that $z(a, \hat{x}) \succ_i z(b, \hat{x})$.*

Or in terms of payoffs,

- *for any $x \in S_j$, $\pi_i(a, x) \geq \pi_i(b, x)$*
- *there exists $\hat{x} \in S_j$ such that $\pi_i(a, \hat{x}) > \pi_i(b, \hat{x})$.*

Definition 4. *Strategy $a \in S_i$ is* ***strictly dominant*** *if it strictly dominates every other $s \in S_i$.*

Definition 5. *Strategy $a \in S_i$ is* ***weakly dominant*** *if for any other $x \in S_i$, either*

- *a weakly dominates x, or*
- *a is equivalent to x.*

Consider the table

	d	e
a	1	2
b	1	2
c	0	2

Strategy a weakly dominates c; it is the superior choice if P2 plays d, but is indifferent if P2 plays e. Strategy a is equivalent to b; no matter what P2 plays, P1 receives the same payoff. So a and b are both weakly dominant strategies.

Note if if you read just "dominant," it is assumed to mean *weakly* dominant.

Second Price Auctions

There are two players. S_i is the set of possible bids of player i. The outcomes is who wins and what they pay: (i, p). If there's a tie, P1 is declared the winner. Given bids (b_1, b_2), the outcome function is

$$z(b_1, b_2) = \begin{cases} (1, b_2) & \text{if } b_1 \geq b_2, \\ (2, b_1) & \text{if } b_1 < b_2. \end{cases}$$

So if P1 has the (weakly) highest bid, then she wins and pays P2's bid. If P2 has the highest bid, then she wins and pays P1's bid.

V_1 is the value of the object to P1; it is the maximum price they'd be willing to pay.

- $(1, p) \succ_1 (1, p')$ iff $p < p'$. In words, P1 would prefer paying less when winning.
- $(1, V_1) \sim_1 (2, p)$ for any p. In words, P1 is indifferent between losing and having to pay their full valuation of the object.
- $(2, p) \sim_1 (2, p')$ for any p, p'. So P1 doesn't care what P2 pays if P1 doesn't even win anyway.

We can represent these preferences with the payoff function

$$\pi_1(b_1, b_2) = \begin{cases} V_1 - b_2 & \text{if } b_1 \geq b_2, \\ 0 & \text{if } b_1 < b_2. \end{cases}$$

Theorem 1. *Bidding the true value is a weakly dominant strategy under these preferences.*

"Proof." Consider all possible actions of P2.

- Start with $b_2 = V_1$. In this case, P1 gets payoff 0 no matter what P1 does.
- Now suppose $b_2 > V_1$. Then P1 could get payoff of zero for bidding less than b_2, in particular, by bidding $b_1 = V_1$. However, if voting greater than or equal to b_2, then P1 gets negative payoff.
- Suppose $b_2 < V_1$. Then P1 could get $V_1 - b_2 > 0$ payoff for bidding greater than or equal to b_2. Bidding less than b_2 gives zero payoff.

So bidding truthfully is weakly dominant—it does no worse than other strategies in any situation; in one case it does better than bidding above b_1; in one case it does better than bidding below b_1.

Suppose that the assumption where $(2, p) \sim_1 (2, p')$ does not hold. In particular, suppose P1 wants P2 to pay as much as possible in the case that P1 loses. Then P1's behavior would change.

Nash Equilibria

Two prisoner's have a choice of confessing or not confessing. If they both confess, they both get nine years in prison. If they both remain silent, they both get one year. If one confesses and the other remains silent, the one who confesses goes free and the one who goes silent gets 11 years.

	C	N
C	$9, 9$	$0, 11$
N	$11, 0$	$1, 1$

We can't say what happens because we don't know preferences.

Definition 6. *Suppose $a \in S_1$ and $b \in S_2$. Strategy profile (a, b) is a **Nash equilibrium** if*

- *$\pi_1(a, b) \geq \pi_1(x, b)$ for all $x \in S_1$,*
- *$\pi_2(a, b) \geq \pi_2(a, y)$ for all $y \in S_2$.*

It's a sort of self-enforcing agreement. Given what P2 is doing, P1 doesn't want to play anything else; and given what P1 is doing, P2 doesn't want to play anything else.

Consider the game below (with payoffs from now on unless stated otherwise):

	D	E	F
A	$1, 0$	$\underline{2}, \underline{3}$	$3, 1$
B	$\underline{3}, 3$	$1, \underline{5}$	$\underline{4}, 4$
C	$\underline{3}, \underline{2}$	$0, 1$	$3, 0$

Strategy profiles (C, D) and (A, E) are Nash equilibria.

We can also talk in terms of a **best response function**. For a strategy of player i, what give player j the highest payoff? Notice that $BR_2(A)$ maps to E,

$$\begin{array}{lll} BR_1(D) = \{B, C\} & BR_1(E) = \{A\} & BR_1(F) = \{B\} \\ BR_2(A) = \{E\} & BR_2(B) = \{E\} & BR_2(C) = \{D\} \end{array}$$

and $BR_1(E)$ maps to A. So (A, E) is a Nash equilibrium. And since $BR_2(C)$ maps to D and $BR_1(D)$ maps to C, (C, D) is another Nash equilibrium.