Expected Utility Theory

Let C be the set of all possible outcomes. We'll be assuming that the number of possible outcomes is finite. Furthermore, probabilities of outcomes are objectively known.

Definition 1. *A **simple lottery** L is a list $L = (p_1, ..., p_n)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$. Each p_n is interpreted as the probability of outcome n occurring.*

A simple lottery can be represented geometrically as a point in the $N-1$ **dimensional simplex**,

$$\Delta = \{p \in \mathbb{R}^N_+ : p_1 + ... + p_N = 1\}.$$

Each vertex of the simplex is where some $p_n = 1$, i.e. n occurs with certainty.

Definition 2. *Given K simple lotteries, $L_k = (p^k_1, ..., p^k_N)$, where $k = 1, ..., K$, and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the **compound lottery** $(L_1, ..., L_K : \alpha_1, ..., \alpha_K)$ is the risky alternative the yields the simple lottery L_k with probability α_k for $k = 1, ..., K$.*

Yeah, that's an assful. What it says is, there are a bunch of different lotteries, each with a different probability of outcome n occurring. We're randomizing over which lottery is used. So the probability that we get L_k is α_k.

We can calculate a corresponding **reduced lottery** $L = (p_1, ..., p_N)$, where the probability of outcome n occurring is

$$p_n = \alpha_1 p^1_n + ... + \alpha_K p^K_n.$$

So the reduced lottery L of any compound lottery can be obtained by vector addition:

$$L = \alpha_1 L_1 + ... + \alpha_K L_K \in \Delta.$$

We now rely on a **consequentialist** premise: only the reduced lottery over final outcomes is of relevance to the decision maker. Which is to say, it doesn't matter if the probabilities arise from a simple lottery or some absurdly complex compound lottery – what matters is the probabilities and only the probabilities.

So let's just have $\mathscr{L}$ be the set of all simple lotteries over the set of outcomes C. The decision maker has a rational preference relation $\succeq$ on $\mathscr{L}$, which is complete and transitive. Note that the rationality assumption here is stronger than in the case of certainty – the more complex the alternatives, the heavier a burden carried by the rationality postulates.

Definition 3. *The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ is **continuous** if for any $L, L', L'' \in \mathscr{L}$, the sets*

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L''\} \subseteq [0,1],$$
$$\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha)L'\} \subseteq [0,1],$$

are closed.

This is not an intuitive definition. What it means is, small changes in probabilities do not change the nature of the ordering between the two lotteries. The point is, if you have a good, mediocre, and bad outcome, there is some sufficiently small probability of the bad outcome such that mixing it with the good outcome is still better than the mediocre outcome.

Definition 4. *The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ satisfies the **independent axiom** if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$, we have*

$$L \succsim L' \text{ if and only if } \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

What this means is, we can mix two lotteries with a third one, by the same proportion, and the ordering of the mixture remains the same – it is independent of the third lottery.

Definition 5. *The utility function $U : \mathscr{L} \to \mathbb{R}$ has an **expected utility form** if there is an assignment of numbers $(u_1, ..., u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, ..., p_N) \in \mathscr{L}$, we have*

$$U(L) = u_1 p_1 + ... + u_N p_N.$$

*A utility function $U : \mathscr{L} \to \mathbb{R}$ with the expected utility form is called a **von Neumann-Morgenstern expected utility function**.*

The expression $U(L) = \sum_n u_n p_n$ is a general form for a linear function in the probabilities $(p_1, ..., p_N)$.

Definition 6. *A utility function $U : \mathscr{L} \to \mathbb{R}$ has an expected utility form if and only if it is **linear**, that is, if and only if it satisfies the property that*

$$U\left(\sum_{k=1}^{K} \alpha_K L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, ..., K$, and probabilities $(\alpha_1, ..., \alpha_k) \geq 0$, $\sum_k \alpha_k = 1$.

Expected utility is a cardinal property of utility functions defined on the space of lotteries and are only preserved under certain transformations.

Proposition 1. *Suppose that $U : \mathscr{L} \to \mathbb{R}$ is a von Neumann-Morgenstern expected utility function for the preference relation $\succsim$ on $\mathscr{L}$. Then $\tilde{U} : \mathscr{L} \to \mathbb{R}$ is another von Neumann-Morgenstern utility function for $\succsim$ if and only if there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathscr{L}$.*

Proposition 2. *If the preference relation $\succsim$ on $\mathscr{L}$ is represented by a utility function $U(\cdot)$ that has the expected utility form, then $\succsim$ satisfies the independence axiom.*

Expected Utility Theorem

Proposition 3 (Expected Utility Theorem)**.** *Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, ..., N$ in such a manner that for any two lotteries $L = (p_1, ..., p_n)$ and $L' = (p'_1, ..., p'_N)$, we have*

$$L \succsim L' \quad \textit{if and only if} \quad \sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n.$$

One advantage of the expected utility theorem is technical – it is convenient for analysis. Another advantage is normative – it might provide a guide to action, i.e. how to think when faced with uncertainty.

That said, the theory seems to run into some difficulties, especially in the more extreme cases.