

1 Linearity in Parameters

The OLS estimation technique requires that our model be *linear in parameters*. What this means is, each β term must appear essentially as a constant: we cannot have β_1^2 or $\log(\beta_1)$ or $\beta_1\beta_2$, for instance. This is because the OLS technique is only able to solve explicitly for each β if they appear in a linear fashion.

However, this does not necessitate that the model be linear in *variables*. There is no reason why we can't specify a model of the form

$$y = \beta_1 + \beta_2 \log(x) + u$$

if we think it is useful to do so.¹ And it certainly might be useful to do so. Consider the relationship between healthcare expenditure and life expectancy. We would expect more healthcare expenditure to be correlated with higher life expectancy, but at a diminishing rate (since there is a natural limit to life expectancy that medical treatment cannot overcome). So it wouldn't make sense to impose a linear relationship on healthcare expenditure and life expectancy; instead, use a log to capture diminishing returns to healthcare expenditure.

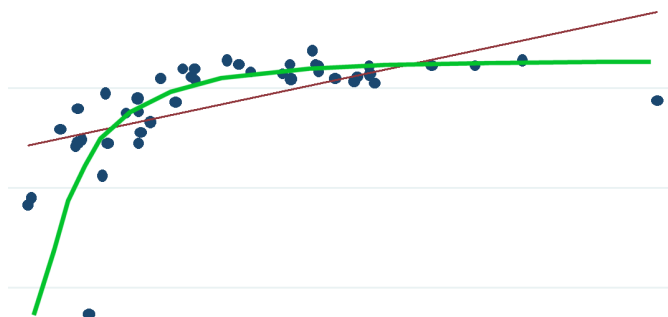


FIGURE 1: Specifying a logarithmic relationship (green) generates a better fit of the data compared to a linear relationship (red).

2 Functional Forms

There are an infinite number of ways we could specify such a model. I focus on three due to their salient economic interpretations. It will be assumed throughout that the zero conditional mean assumption holds; this implies that x and u are uncorrelated. In

¹If you'd like, you can define $v \equiv \log(x)$ and rewrite the model as $y = \beta_1 + \beta_2 v + u$, from which it is obvious that the model exhibits the same form as that with which we are familiar.

practice, this means that when we change x , there is no change in u on average. This is useful because we take derivatives, for which the zero conditional mean assumption implies that $du/dx = 0$.

2.1 Linear-Linear Regression

A **linear-linear** regression is of the form

$$y = \beta_1 + \beta_2 x + u. \quad (1)$$

It is named as such because the dependent variable is linear (it's simply y) and the regressor variable is also linear (it's simply x). The interpretation is that an increase in x by one unit is associated with a change in y of β units. This is the regression we've been focusing on.

2.2 Linear-Log Regression

A **linear-log** regression is of the form

$$y = \beta_1 + \beta_2 \log(x) + u. \quad (2)$$

Although $\log(x)$ may do a better job of capturing the data or a salient economic phenomenon, we are not interested in how a change in $\log(x)$ will affect y ; we are interested in how a change in x will affect y . We'll have to do a little work to squeeze out that information, but it's not so bad. First, we can take the derivative of both sides with respect to x , which yields

$$\frac{dy}{dx} = \frac{\beta_2}{x}.$$

Now multiply both sides by dx , and then multiply the right-hand side by $100/100$. Moving around the 100 factors, we end up with

$$dy = \frac{\beta_2}{100} \left(\frac{dx}{x} \times 100 \right).$$

Notice that the term in parentheses is the percentage change in x .

Although calculus operations are in terms of infinitesimally small changes dy and dx , the equation still constitutes a valid approximation for small changes Δy and Δx . In practice, we will typically consider $\% \Delta x = 1$. The interpretation in words is: the change in

the level of y equals $\beta_2/100$ times the percentage change in x ,

$$\Delta y \approx \frac{\beta_2}{100} \times \% \Delta x. \quad (3)$$

2.3 Log-Linear Regression (Semi-Elasticity)

A **log-linear** regression is of the form

$$\log(y) = \beta_1 + \beta_2 x + u. \quad (4)$$

To interpret, start by taking the derivative of both sides with respect to x , which yields

$$\frac{d \log(y)}{dx} = \beta_2.$$

In order to squeeze y out of the left-hand side, we will appeal to the chain rule of calculus.

In particular, we can write

$$\frac{d \log(y)}{dy} \frac{dy}{dx} = \beta_2.$$

We know that $d \log(y)/dy = 1/y$, so let's make that substitution. Let's also multiply both sides by $100 \times dx$, which yields

$$\frac{dy}{y} \times 100 = (100 \times \beta_2) dx.$$

In words: the percentage change in y equals $100 \times \beta_2$ times the change in the level of x ,

$$\% \Delta y \approx 100 \beta_2 \times \Delta x. \quad (5)$$

In this form, coefficient β_2 is referred to as the **semi-elasticity** of y with respect to x .

2.4 Log-Log Regression (Elasticity)

A **log-log** regression is of the form

$$\log(y) = \beta_1 + \beta_2 \log(x) + u. \quad (6)$$

To interpret, take the derivative of both sides with respect to x , which gives

$$\frac{d \log(y)}{dx} = \frac{\beta_2}{x}.$$

Use the chain rule again on the right-hand side so that

$$\frac{d \log(y)}{dy} \frac{dy}{dx} = \frac{\beta_2}{x}.$$

We know that $d \log(y)/dy = 1/y$, so let's make that substitution. Also multiply both sides by dx and both sides by 100. Doing so yields

$$\frac{dy}{y} \times 100 = \beta_2 \left(\frac{dx}{x} \times 100 \right).$$

In words: the percentage change in y is equal to β_2 times the percentage change in x ,

$$\% \Delta y \approx \beta_2 \times \% \Delta x. \quad (7)$$

In this form, coefficient β_2 is referred to as the **elasticity** of y with respect to x , which you hopefully remember from a microeconomics course.

3 Summary

Again, there are a multitude of other functional forms we could consider, e.g. quadratic forms, that are useful in certain contexts. Those will be discussed later as they become pertinent. But for now, here is a table of the four functional forms introduced here.

Model	Dependent Variable	Regressor	Interpretation of β_2
linear	y	x	$\Delta y = \beta_2 \times \Delta x$
linear-log	y	$\log(x)$	$\Delta y \approx \frac{\beta_2}{100} \times \% \Delta x$
log-linear (semi-elasticity)	$\log(y)$	x	$\% \Delta y \approx 100 \beta_2 \times \Delta x$
log-log (elasticity)	$\log(y)$	$\log(x)$	$\% \Delta y \approx \beta_2 \times \% \Delta x$

4 Life Expectancy and Healthcare Expenditure

Consider data with three variables: country, life expectancy at birth, and healthcare spending per-capita in 2015. As can be asceratined from the Stata output on the next page, the linear-linear regression

$$\text{lifeexpect} = \beta_1 + \beta_2 \text{hcspending} + u$$

gives goodness-of-fit measure $R^2 = 0.363$. The linear-log regression as suggested earlier,

$$\text{lifeexpect} = \beta_1 + \beta_2 \log(\text{hcspending}) + u,$$

gives goodness-of-fit measure $R^2 = 0.542$, implying a better fit. The interpretation of the linear-log model is that a 1% increase in healthcare spending is associated with, on average, an increase in life expectancy by about 0.0465 years.

There's an **important caveat** in comparing the R^2 of different models: it is *not* meaningful to compare the R^2 of models that have different dependent variables! If the regressors are different but the dependent variables are the same, then it is fine to compare R^2 .

```
. regress lifeexpect hcspending
```

Source	SS	df	MS	Number of obs	=	44
				F(1, 42)	=	23.93
Model	394.963176	1	394.963176	Prob > F	=	0.0000
Residual	693.098579	42	16.5023471	R-squared	=	0.3630
				Adj R-squared	=	0.3478
Total	1088.06175	43	25.3037617	Root MSE	=	4.0623

lifeexpect	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
hcspending	.0014559	.0002976	4.89	0.000	.0008553	.0020565
_cons	73.98101	1.148863	64.39	0.000	71.66251	76.29951

```
. gen loghc = log(hcspending)
```

```
. regress lifeexpect loghc
```

Source	SS	df	MS	Number of obs	=	44
				F(1, 42)	=	49.67
Model	589.550424	1	589.550424	Prob > F	=	0.0000
Residual	498.511331	42	11.8693174	R-squared	=	0.5418
				Adj R-squared	=	0.5309
Total	1088.06175	43	25.3037617	Root MSE	=	3.4452

lifeexpect	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
loghc	4.646454	.6592863	7.05	0.000	3.31596	5.976947
_cons	42.30245	5.19564	8.14	0.000	31.81723	52.78768