Problem 1: Monopolistic Competition in NK Models

- There is a continuum of firms indexed over $(0, 1)$.
- Each firm produces a variety of good.
- Consumers have constant elasticity of substitution between different goods, denoted ϵ.
- Because there is a continuum of different prices and constant elasticity of substitution, aggregate consumption is given by a CES aggregate of all of the individual varieties.

Consumers maximize

$$\max_{C_t, N_t, B_t} E_t \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi} \right) Z_t \right]$$

subject to

$$\int_0^1 p_t(j) c_t(j)\, dj + Q_t B_t = B_{t-1} + W_t N_t + \pi_t,$$

where total consumption is a CES aggregate over product varieties, i.e.

$$C_t = \left[\int_0^1 c_t(j)^{\frac{\epsilon-1}{\epsilon}}\, dj \right]^{\frac{\epsilon}{\epsilon-1}}$$

The aggregate price index is defined as

$$P_t = \left[\int_0^1 p_t(j)^{1-\epsilon}\, dj \right]^{\frac{1}{1-\epsilon}}.$$

Part A: Price Stickiness and Perfect Competition

With perfect competition, price equals marginal cost. There is no need to set costs in such a scenario; they adjust automatically and instantaneously. In order to have some sort of price stickiness, we have to be in a scenario in which firms actually have to set their prices in the first place—the stickiness reflecting the inability to do so.

Part B: Household Demand Curve

We want to choose demand for product j, that is, choose $c_t(j)$ in order to maximize total consumption subject to total expenditure $\int_0^1 p_t(j) c_t(j)\, dj = X_t$. We can do this by setting up the Lagrangian,

$$\mathcal{L} = \left[\int_0^1 c_t(j)^{\frac{\epsilon-1}{\epsilon}} \, dj\right]^{\frac{\epsilon}{\epsilon-1}} - \lambda_t \left[\int_0^1 p_t(j)c_t(j) \, dj - X_t\right].$$

With respect to $c_t(j)$, we get the first order condition

$$C_t^{\frac{1}{\epsilon}} c_t(j)^{-\frac{1}{\epsilon}} = \lambda_t p_t(j).$$

This is true for any good, say good k as well. So we have

$$C_t^{\frac{1}{\epsilon}} c_t(k)^{-1/\epsilon} = \lambda_t p_t(k).$$

Combining the two, it follows that

$$\left[\frac{c_t(j)}{c_t(k)}\right]^{-1/\epsilon} = \frac{p_t(j)}{p_t(k)} \quad \Longrightarrow \quad c_t(j) = \left[\frac{p_t(k)}{p_t(j)}\right]^{\epsilon} c_t(k).$$

Plug this into the definition of X_t and you get

$$\begin{aligned} X_t &= \int_0^1 p_t(j) \left[\frac{p_t(k)}{p_t(j)}\right]^{\epsilon} c_t(k) \, dj \\ &= p_t(k)^{\epsilon} c_t(k) \int_0^1 p_t(j)^{1-\epsilon} \, dj. \end{aligned}$$

Now notice that

$$P_t^{1-\epsilon} = \int_0^1 p_t(j)^{1-\epsilon} \, dj.$$

So we can continue rewriting the definition of X_t to get

$$\begin{aligned} & X_t = p_t(k)^{\epsilon} c_t(k) P_t^{1-\epsilon} \\ \Longrightarrow \quad & c_t(k) = \frac{X_t}{P_t}\left[\frac{P_t}{p_t(k)}\right]^{\epsilon} \end{aligned}$$

What's true of k is true of j, so we can replace j into the above equation. Let's plug this expression for $c_t(j)$ into the definition of C_t to get

$$
\begin{aligned}
C_t &= \left[\int_0^1 \left(\frac{X_t}{P_t}\left[\frac{P_t}{p_t(j)}\right]^{\epsilon}\right)^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}} \\
&= \left(\left[\frac{X_t}{P_t}\right]^{\frac{\epsilon-1}{\epsilon}} P_t^{\epsilon-1}\left[\int_0^1 p_t(j)^{1-\epsilon}\, dj\right]\right)^{\frac{\epsilon}{\epsilon-1}} \\
&= \left[\frac{X_t}{P_t}\right] P_t^{\epsilon}\left[\int_0^1 p_t(j)^{1-\epsilon}\, dj\right]^{\frac{\epsilon}{\epsilon-1}}.
\end{aligned}
$$

The last factor, using the definition of P_t, is

$$
\left[\int_0^1 p_t(j)^{1-\epsilon}\, dj\right]^{\frac{\epsilon}{\epsilon-1}} = P_t^{-\epsilon}.
$$

So continuing where we left off with C_t, we now have

$$
\begin{aligned}
C_t &= \left[\frac{X_t}{P_t}\right] P_t^{\epsilon} P_t^{-\epsilon} \\
\Longrightarrow \quad X_t &= P_t C_t.
\end{aligned}
$$

Plug this into the equation we had for $c_t(j)$ and we get

$$
\begin{aligned}
c_t(j) &= \frac{P_t C_t}{P_t}\left[\frac{P_t}{p_t(j)}\right]^{\epsilon} \\
&= C_t\left[\frac{P_t}{p_t(j)}\right]^{\epsilon}.
\end{aligned}
$$

Part C: Expenditure and Price Index

We want to show that the Lagrange multiplier on $X_t = \int_0^1 p_t(j)c_t(j)\ dj$ is related to the aggregate price index. From the first-order condition and the expression we just derived for

$c_t(j)$, we have

$$\begin{aligned}
\lambda_t &= \frac{1}{p_t(j)} C_t^{\frac{1}{\epsilon}} c_t(j)^{-1/\epsilon} \\
&= \frac{1}{p_t(j)} C_t^{\frac{1}{\epsilon}} \left(C_t \left[\frac{P_t}{p_t(j)} \right]^{\epsilon} \right)^{-1/\epsilon} \\
&= \frac{1}{P_t}.
\end{aligned}$$

The Lagrange multiplier is the shadow price of relaxing the budget constraint. It tells us how much the objective function—in this case, consumption—changes if we have one extra dollar to spend. It buys $1/P_t$ units of consumption. As it should—typically $1/P_t$ will be the real price of consumption.

Part D: Household's First-Order Conditions

Since $X_t = P_t C_t$, and the expenditure constraint is $\int_0^1 p_t(j) c_t(j)\ dj = X_t$, we can substitute this into the budget constraint and write

$$P_t C_t + Q_t B_t = B_{t-1} + W_t N_t + \pi_t.$$

Thus we can write the problem as

$$\begin{aligned}
V(P_t, B_{t-1}) = & \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi} \right) Z_t + E\big[V(P_{t+1}, B_t)\big] \\
& - \lambda_t [P_t C_t + Q_t B_t - B_{t-1} - W_t N_t - \pi_t],
\end{aligned}$$

which gives first-order conditions

$$\begin{aligned}
\text{with respect to } C_t : \qquad & C_t^{-\sigma} Z_t = \lambda_t P_t, \\
\text{with respect to } N_t : \qquad & N_t^{\psi} Z_t = \lambda_t W_t, \\
\text{with respect to } B_t : \qquad & \beta E_t \left[V_B'(P_{t+1}, B_t) \right] = \lambda_t Q_t.
\end{aligned}$$

Using the envelope condition with B_{t-1}, we get

$$V_B'(P_t, B_{t-1}) = \lambda_t \quad \Longrightarrow \quad V_B'(P_{t+1}, B_t) = \lambda_{t+1},$$

and therefore the third FOC can become

$$\beta E\left[\lambda_{t+1}\right] = \lambda_t Q_t.$$

Or, if you'd prefer the Euler equations (which I would),

$$\begin{aligned}
\beta E_t\left[\frac{C_{t+1}^{-\sigma} Z_{t+1}}{P_{t+1}}\right] &= \frac{Q_t}{P_t} C_t^{-\sigma} Z_t, \\
N_t^{\psi} &= \frac{W_t}{P_t} C_t^{-\sigma}.
\end{aligned}$$

Problem 2: The NK Phillips Curve

Each firm chooses a price p_t^* that maximizes their expected discounted future profits. They have to take the future into account because they might not be able to adjust their price in future periods. The probability of not being able to reset the price in a given period is θ.

Let ϕ be the real marginal cost and let $y_t(j)$ be the output of firm j. Therefore firms are facing the problem

$$\max E_t\left[\sum_{s=0}^{\infty} \theta^s \left(\beta^s \frac{\lambda_{t+s}}{\lambda_t}\right)\left(\frac{p_t(j)}{P_{t+s}} y_{t+s}(j) - \phi_{t+s} y_{t+s}(j)\right)\right],$$

subject to the constraints

- $c_t(j) = \left[\frac{p_t(j)}{P_t}\right]^{-\epsilon} C_t,$ (demand for good j)
- $c_t(j) = y_t(j)$ (firm market clearing)
- $C_t = Y_t$ (aggregate market clearing)

Part A: Optimal Pricing Behavior

Doing this as a Lagrangian is a pain in the ass. In fact, I'm not even sure how to write it because the sums are kind of weird. So I'm just going to substitute things instead. I think it's worth noting that these conditions are done under the assumption that the price will not be able to change. Which means that if we update the demand by s periods, we'll have

$$c_{t+s}(j) = \left[\frac{p_t(j)}{P_{t+s}}\right]^{-\epsilon} C_{t+s}.$$

Then the objective function can be written as

$$\implies \quad \max E_t\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s\frac{\lambda_{t+s}}{\lambda_t}\right)\left(\frac{p_t(j)}{P_{t+s}}c_{t+s}(j)-\phi_{t+s}c_{t+s}(j)\right)\right]$$

$$\implies \quad \max E_t\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s\frac{\lambda_{t+s}}{\lambda_t}\right)\left(\frac{p_t(j)}{P_{t+s}}\left[\frac{p_t(j)}{P_{t+s}}\right]^{-\epsilon}C_{t+s}-\phi_{t+s}\left[\frac{p_t(j)}{P_{t+s}}\right]^{-\epsilon}C_{t+s}\right)\right].$$

Now take the first order condition with respect to $p_t(j)$ and we get

$$E_t\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s\frac{\lambda_{t+s}}{\lambda_t}\right)\left((1-\epsilon)p_t(j)^{-\epsilon}\left[\frac{1}{P_{t+s}}\right]^{1-\epsilon}C_{t+s}+\epsilon\phi_{t+s}p_t(j)^{-\epsilon-1}\left[\frac{1}{P_{t+s}}\right]^{-\epsilon}C_{t+s}\right)\right]=0..$$

Divide everything by $1-\epsilon$ to get

$$E_t\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s\frac{\lambda_{t+s}}{\lambda_t}\right)\left(p_t(j)^{-\epsilon}\left[\frac{1}{P_{t+s}}\right]^{1-\epsilon}C_{t+s}-\frac{\epsilon}{\epsilon-1}\phi_{t+s}p_t(j)^{-\epsilon-1}\left[\frac{1}{P_{t+s}}\right]^{-\epsilon}C_{t+s}\right)\right]=0.$$

Now make the price ratios conform to the demand functions:

$$E_t\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s\frac{\lambda_{t+s}}{\lambda_t}\right)\left(\frac{1}{P_{t+s}}\left[\frac{p_t(j)}{P_{t+s}}\right]^{-\epsilon}C_{t+s}-\frac{\epsilon}{\epsilon-1}\phi_{t+s}p_t(j)^{-1}\left[\frac{p_t(j)}{P_{t+s}}\right]^{-\epsilon}C_{t+s}\right)\right]=0.$$

Now replace the demand functions with c_{t+s} and therefore y_{t+s} and we have

$$E_t\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s\frac{\lambda_{t+s}}{\lambda_t}\right)\left(\frac{1}{P_{t+s}}y_{t+s}(j)-\frac{\epsilon}{\epsilon-1}\phi_{t+s}p_t(j)^{-1}y_{t+s}(j)\right)\right]=0.$$

Multiply everything by $p_t(j)$, which is now the optimal price p_t^* as the result of the first order condition, and we get

$$E_t\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s\frac{\lambda_{t+s}}{\lambda_t}\right)\left(\frac{p_t^*}{P_{t+s}}y_{t+s}(j)-\frac{\epsilon}{\epsilon-1}\phi_{t+s}y_{t+s}(j)\right)\right]=0.$$

Part B: Huh?

Let's try to digest the meaning of this monstrosity. Suppose for example that $\theta \to 0$ as limiting behavior[1], which means that firms can reset prices every period. Then every period

[1]This allows us to say that $\theta^s \to 0$ when $s = 0$. Otherwise it would be undefined 'cuz 0^0 isn't a thing.

except for period $s = 0$ will evaluate to zero and we'll be left with

$$\frac{p_t^*}{P_t} y_t(j) - \frac{\epsilon}{\epsilon - 1} \phi_t y_t(j) = 0 \quad \Longrightarrow \quad p_t^* = \frac{\epsilon}{\epsilon - 1} \phi_t P_t.$$

Since ϕ_t is the real marginal cost, it means that $\phi_t P_t$ is the nominal marginal cost. And hence the optimal price is a markup $\epsilon/(\epsilon - 1)$ over the nominal marginal cost.

Now if we introduce price stickiness with $\theta > 0$, then we're making a similar comparison over time periods in expectation—the markup still exists and is still the same number.

Part C: Aggregate Prices

Recall that the aggregate price index is defined as

$$P_t = \left[\int_0^1 p_t(j)^{1-\epsilon} \, dj \right]^{\frac{1}{1-\epsilon}}.$$

It follows that

$$P_t^{1-\epsilon} = \int_0^1 p_t(j)^{1-\epsilon} \, dj.$$

Because θ is the probability of not being able to change price, it follows that θ is the proportion, on average, of firms that will not be able to change prices. So in period t, a proportion θ of firms will be stuck with $p_{t-1}(j)$ prices, whereas the remainder will have been able to reset and thus have prices $p_t^*(j)$. Therefore we can write

$$P_t^{1-\epsilon} = \int_0^\theta p_{t-1}(j)^{1-\epsilon} \, dj + \int_\theta^1 p_t^*(j)^{1-\epsilon} \, dj.$$

Since all firms are identical, we should have $p_t^*(j) = p_t^*(k)$ for all j, k. In other words, it's just some constant. So we can take it out of the integral and simply evaluate over the bounds, in which case we get

$$P_t^{1-\epsilon} = \int_0^\theta p_{t-1}(j)^{1-\epsilon} \, dj + (1 - \theta)(p_t^*)^{1-\epsilon}.$$

First term is essentially the proportion θ of last period's price level, to the power of $1 - \epsilon$. So we can write

$$P_t^{1-\epsilon} = \theta P_{t-1}^{1-\epsilon} \, dj + (1 - \theta)(p_t^*)^{1-\epsilon}.$$

The conclusion is that the aggregate price level in period t is based on two things. The proportion θ is inherited from last period's price level because they were unable to reset prices. The remaining proportion $1 - \theta$ were able to reset their prices (to the same thing) and so this proportion reflects the optimal prices.

Part D: Linearizing Expected Inflation

We want to linearize the condition in part A,

$$E_t\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s\frac{\lambda_{t+s}}{\lambda_t}\right)\left(\frac{p_t^*}{P_{t+s}}y_{t+s}(j)-\frac{\epsilon}{\epsilon-1}\phi_{t+s}y_{t+s}(j)\right)\right]=0,$$

and in part C,

$$P_t^{1-\epsilon}=\theta P_{t-1}^{1-\epsilon}+(1-\theta)(p_t^*)^{1-\epsilon}.$$

Linearizing Part C. This is fairly straightforward:

$$\begin{aligned}&(1-\epsilon)P^{1-\epsilon}\hat{P}_t=\theta(1-\epsilon)P^{1-\epsilon}\hat{P}_{t-1}+(1-\epsilon)(1-\theta)(p_t^*)^{1-\epsilon}\hat{p}_t^*\\ \implies\quad &P^{1-\epsilon}\hat{P}_t=\theta P^{1-\epsilon}\hat{P}_{t-1}+(1-\theta)(p^*)^{1-\epsilon}\hat{p}_t^*.\end{aligned}$$

In the steady state everyone will have been able to reset their prices, and therefore $p_t^*=P$. So now we can write

$$\hat{P}_t=\theta\hat{P}_{t-1}+(1-\theta)\hat{p}_t^*.$$

Now subtract a $\hat{P}_{t-1}$ from both sides and we have

$$\hat{P}_t-\hat{P}_{t-1}=(1-\theta)(\hat{p}_t^*-\hat{P}_{t-1})\quad\implies\quad\pi_t=(1-\theta)(\hat{p}_t^*-\hat{P}_{t-1}).$$

Linearizing Part A. Let's define

$$Q_{t+s}=\left(\frac{p_t^*}{P_{t+s}}y_{t+s}(j)-\frac{\epsilon}{\epsilon-1}\phi_{t+s}y_{t+s}(j)\right).$$

Note that in the steady state, again, all prices will have been reset, and therefore we can express all prices as p_t^* if we'd like. Therefore the steady-state condition becomes

$$\sum_{s=0}^{\infty}\theta^s\beta^s\left(y(j)-\frac{\epsilon}{\epsilon-1}\phi y(j)\right)=\sum_{s=0}^{\infty}\theta^s\beta^sQ=0.$$

Since θ and β are both nonzero, it must be the case that $Q=0$.

This will be useful later, so let's get it out of the way now while we're on the topic. Notice that $Q=0$ implies that $\phi=(\epsilon-1)/\epsilon$. In other words, the steady-state real marginal cost is the inverse of the markup cost.

Okay, since $Q = 0$, the linearization with respect to the stochastic discount factor will always be zero because we'll always be multiplying it by $Q = 0$. So we really only have to focus on the stuff in the parentheses, which will be multiplied by $\theta^s\beta^s$. Then the linearization becomes

$$\sum_{s=0}^{\infty}\theta^s\beta^s y(j)E_t\left[\hat{p}_t^* - \hat{P}_{t+s} + \hat{y}(j)_{t+s} - \hat{\phi}_{t+s} - \hat{y}(j)_{t+s}\right] = \sum_{s=0}^{\infty}\theta^s\beta^s y(j)E_t\left[\hat{p}_t^* - \hat{P}_{t+s} - \hat{\phi}_{t+s}\right] = 0$$

$$\implies \quad \sum_{s=0}^{\infty}(\theta\beta)^s\hat{p}_t^* = \sum_{s=0}^{\infty}(\theta\beta)^s E_t\left[\hat{P}_{t+s} + \hat{\phi}_{t+s}\right],$$

where the $y(j)$ drops out because it isn't indexed by s and hence we can take it out of the sum and cancel. It turns out that the above equation is actually the forward-solution to

$$\hat{p}_t^* = \beta\theta E_t[\hat{p}_{t+1}^*] + (1-\beta\theta)(\hat{\psi}_t + \hat{P}_t).$$

No, I have no damn idea how we're supposed to know that. Anyway, we found earlier that $\pi_t = (1-\theta)(\hat{p}_t^* - \hat{P}_{t-1})$, so let's use this to remove the price terms. In particular we can substitute

$$\hat{p}_t^* = \frac{\pi_t}{1-\theta} + \hat{P}_{t-1} \quad \text{and} \quad \hat{p}_{t+1}^* = \frac{\pi_{t+1}}{1-\theta} + \hat{P}_t.$$

Then we can write

$$\frac{\pi_t}{1-\theta} + \hat{P}_{t-1} = \beta\theta E_t\left[\frac{\pi_{t+1}}{1-\theta} + \hat{P}_t\right] + (1-\beta\theta)(\hat{\psi}_t + \hat{P}_t)$$

$$\implies \quad \pi_t = \beta E_t[\pi_{t+1}] + \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{\phi}_t.$$

Part E: Phillips Curve and Degree of Price Stickiness

We define $\tilde{\kappa}$ to be

$$\tilde{\kappa} = \frac{(1-\theta)(1-\beta\theta)}{\theta}.$$

Using the calculus, it is easy enough to show that

$$\frac{\partial\tilde{\kappa}}{\partial\theta} = \frac{\theta^2 - 1}{\theta^2} < 0.$$

So decreasing price stickiness through decreasing θ causes an increase in $\tilde{\kappa}$. Consequently, current inflation is more affected by $\hat{\phi}_t$, i.e. deviations from the steady state marginal cost, which in turn reflects the magnitude of the output gap (as will be shown later).

Part F: Linearizing Aggregate Output

Aggregate hours worked can be written as

$$N_t = \int_0^1 N_t(j)\, dj.$$

Firms product output according to $y_t(j) = A_t N_t(j)$. We want to show that $\hat{y}_t = \hat{a}_t + \hat{n}_t + d_t$, where d_t reflects price dispersion and wen can be shown to be zero.

First solve for $N_t(j)$ and plug it into the integral. Doing so gives

$$N_t = \int_0^1 \frac{y_t(j)}{A_t}\, dj.$$

From the previous problem we showed that in equilibrium we'll have $y_t(j) = c_t(j)$, and furthermore

$$c_t(j) = C_t \left[\frac{P_t}{p_t(j)}\right]^{\epsilon}.$$

Since it must be the case that $C_t = Y_t$ in equilibrium, this allows us to write the integral as

$$N_t = \frac{Y_t}{A_t} P_t^{\epsilon} \int_0^1 \left[\frac{1}{p_t(j)}\right]^{\epsilon} dj.$$

So now let's linearize. Doing so gives

$$\begin{aligned} N_t \hat{n}_t = {} & \frac{Y}{A} P^{\epsilon} \left(\int_0^1 \left[\frac{1}{p_t(j)}\right]^{\epsilon} dj \right) \hat{y}_t - \frac{Y}{A} P^{\epsilon} \left(\int_0^1 \left[\frac{1}{p_t(j)}\right]^{\epsilon} dj \right) \hat{a}_t \\ & - \epsilon \frac{Y}{A} P^{\epsilon} \left(\int_0^1 \left[\frac{1}{p_t(j)}\right]^{\epsilon} dj \right) \hat{P}_t - \epsilon \frac{Y}{A} P^{\epsilon} \left(\int_0^1 \left[\frac{1}{p_t(j)}\right]^{\epsilon-1} dj \right) \hat{p}_t(j). \end{aligned}$$

A lot of those terms are just steady-state values of N. So we can write

$$\begin{aligned} & N\hat{n}_t = N\hat{y}_t - N\hat{a}_t - \epsilon N \hat{P}_t - \epsilon N \left(\int_0^1 \left[\frac{1}{p_t(j)}\right]^{\epsilon-1} dj \right) \hat{p}_t(j) \\ \implies \quad & \hat{n} = \hat{y}_t - \hat{a}_t - d_t, \end{aligned}$$

where $d_t = \epsilon[\hat{P}_t - \hat{p}_t(j)]$. **I am very unsure about this price dispersion term.**

Part G: Real Marginal Cost and the Output Gap

Take it for granted that the firm's labor demand is, in linearized form, $\hat{w}_t - \hat{a}_t = \hat{\phi}_t$, and that GDP in the classical model is

$$\hat{y}_t^n = \frac{1+\psi}{\sigma+\psi}\hat{a}_t.$$

We want to show that real marginal cost is related to the output gap, and consequently the Phillips curve we found in part D can be expressed in terms of the output gap.

In the first problem, we showed that

$$C_t^{-\sigma} Z_t = \lambda_t P_t \quad \text{and} \quad N_t^{\psi} Z_t = \lambda_t W_t.$$

Combine the two for the intratemporal Euler equation

$$C_t^{-\sigma}\frac{W_t}{P_t} = N_t^{\psi} \quad \Longrightarrow \quad C_t^{-\sigma} w_t = N_t^{\psi},$$

where w_t is the real wage. We also know that the real marginal cost is $\phi_t = w_t/A_t$. So replace w_t in the above Euler equation and you get

$$C_t^{-\sigma}\phi_t A_t = N_t^{\psi}.$$

Recall that $C_t = Y_t$ in equilibrium. So when we linearize, we get

$$-\sigma Y^{-\sigma}\phi A\hat{y}_t + Y^{-\sigma}\phi A\hat{\phi}_t + Y^{-\sigma}\phi\frac{A}{P}\hat{a}_t = \psi N^{\psi}\hat{n}_t$$

N in the steady state is equal to the common LHS term, so we can write

$$-\sigma\hat{y}_t + \hat{a}_t + \hat{\phi}_t = \psi\hat{n}_t.$$

We found in part F that $\hat{n}_t = \hat{y}_t - \hat{a}_t$, since $d_t = 0$. Plug this in the previous equation for

$$-\sigma\hat{y}_t + \hat{a}_t + \hat{\phi}_t = \psi(\hat{y}_t - \hat{a}_t) \quad \Longrightarrow \quad \hat{\theta}_t = \hat{y}_t(\sigma+\psi) - A_t(1+\psi).$$

Finally, use the expression for the natural rate of output, $\hat{y}_t^n$, to replace $\hat{a}_t$, which gives

$$\hat{\theta}_t = \hat{y}_t(\sigma+\psi) - \frac{\sigma+\psi}{1+\psi}\hat{y}_t^n(1+\psi) \quad \Longrightarrow \quad \hat{\phi}_t = (\hat{y}_t - \hat{y}_t^n)(\sigma+\psi).$$

Define $\tilde{y}_t = \hat{y}_t - \hat{y}_t^n$ to be the output gap. We can see here that the real marginal cost relates to the output gap. If the output gap is zero, then $\hat{\phi}_t = 0$, indicating that the marginal cost is at its steady state level, namely, $\phi_t = \epsilon/(\epsilon - 1)$.

If the output gap is positive, then the real marginal cost is above its steady state value and therefore the markup is below what the firm would like; so in some sense, the economy is less distorted.

In part D, we found that

$$\pi_t = \beta E_t(\pi_{t+1}) + \tilde{\kappa}\hat{\phi}_t.$$

We can now substitute in for $\hat{\phi}_t$ to relate inflation to the output gap, namely,

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa \tilde{y}_t,$$

where

$$\kappa = \frac{(1-\theta)(1-\beta\theta)(\sigma+\psi)}{\theta}.$$

A larger current output gap implies higher current inflation; and vice versa.

Part H: Inflation and Future Expected Output Gaps

Take the equation we just derived and substitute forward. What we get is

$$\pi_t = E_t\left[\kappa \sum_{i=0}^{T} \beta^i \tilde{y}_{t+i} + \beta^{T+1}\pi_{t+T+1}\right].$$

If we take the limit, the inflation term will drop to zero. Then we have

$$\pi_t = \kappa E_t\left[\sum_{i=0}^{\infty} \beta^i \tilde{y}_{t+i}\right].$$

The idea is that the output gap is a function of current and future differences in the actual real interest rate and the natural real interest rate, in particular,

$$\tilde{y}_t = -\frac{1}{\sigma}\sum_{i=0}^{\infty} E_t[\hat{r}_{t+i} - \hat{r}^n_{t+1}].$$

The natural rate $\hat{r}^n_t$ is determined entirely by demand and supply shocks. But in a world with price stickiness, $\hat{r}_t$ can be set by the central bank. By specifying a rule that targets the natural rate, plus deviations from steady-state inflation to prevent price indeterminacy, the bank can entirely close the output gap (and expected output gaps). By doing so, they can also achieve steady state inflation—this is the *divine coincidence*.