Disclaimer: Chapter 4 is kind of a shit show.

Aggregate Demand

Suppose there are I consumers with rational preference relations $\succsim_i$ and Walrasian demand function $x_i(p, w_i)$. Then **aggregate demand** can be written as

$$x(p, w_1, ..., w_I) = \sum_{i=1}^{I} x_i(p, w_i).$$

So aggregate demand relies on prices but specific wealth levels of each consumer. We would like to know when we can write aggregate demand as $x\left(p, \sum_i w_i\right)$ so that it depends only on aggregate wealth $\sum_i w_i$.

If we can, then aggregate demand must be identical for any two distributions of the same total amount of wealth. Which is to say, for $(w_1, ..., w_I) \neq (w'_1, ..., w'_I)$ where $\sum_i w_i = \sum_i w'i$, we need to have $\sum_i x_i(p, w_i) = \sum_i x_i(p, w'_i)$. In other words, after wealth is redistributed, we need the exact same overall amount of good ℓ to be demanded in aggregate, for every ℓ. This means that the summed change of each $x_{\ell i}(p, w_i)$ must be zero after some change in wealth distribution (but not wealth total):

$$\sum_i \frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} dw_i = 0 \text{ for every } \ell.$$

Okay, so when can that be true? If and only if the coefficients of the different dw_i are equal for every ℓ and any two individuals i, j. That is, when

$$\frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} = \frac{\partial x_{\ell j}(p, w_j)}{\partial w_j}$$

This follows from the requirement that $\sum_i dw_i = 0$.

So as wealth is changed for each person i, the demand for good ℓ will change at the same rate – the wealth expansion paths are parallel. There are two notable cases where this can be true: where consumers have identical homothetical preferences and where consumers have preferences that are quasilinear with respect to the same good. More generally,

Proposition 1. *A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that preferences admit indirect utility functions of the Gorman form with the coefficients on w_i the same for every consumer i. That is,*

$$v_i(p, w_i) = a_i(p) + b(p)w_i.$$

This is a pretty damn restrictive condition on preferences. Maybe we can loosen this restriction if we consider aggregate demand function that depend on more than just total wealth, for example, perhaps the variance of wealth or even the entire statistical distribution of wealth.

Consider the case where individual i's wealth is generated by some process that can be described as a function of prices p and aggregate wealth w, that is, $w_i = (p, w)$. For instance, a government program might base an individual's taxes (and thus final wealth position) on his wage rate and the total wealth of society. We call a family of such functions, $\big(w_1(p, w), ..., (w_I(p, w)\big)$ with $\sum_i w_i(p, w) = w$ for all (p, w) a **wealth distribution rule**. It is a fancy way of saying we might take some wealth from person i and give it to person j with no overall loss in total w. In such a case, we can always write aggregate demand as a function $x(p, w) = \sum_i x_i\big(p, w_i(p, w)\big)$.

The Weak Axiom

Continuity, homogeneity of degree zero, and Walras' law all hold for the aggregate demand function $x(p, w_1, ..., w_I) = \sum_i x_i(p, w_i)$. It sure would be nice is WARP would hold for aggregate demand, no?

Assume that there's a wealth redistribution rule so that we can have $x(p, w) = \sum_i x_i\big(p, w_i(p, w)\big)$. In fact, let's assume a more specific distribution rule where the relative wealth of consumers is fixed and independent of prices. So $w_i(p, w) = \alpha_i w$ for any w. Therefore, we have aggregate demand function

$$x(p, w) = \sum_i x_i(p, \alpha_i w).$$

Definition 1. *The aggregate demand function $x(p, w)$ satisfies the **weak axiom** if $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$ implies $p' \cdot x(p, w) > w'$ for any (p, w) and (p', w').*

Recall that $x(p, w)$ satisfies the weak axiom if and only if it satisfies the law of demand for compensated price changes. That is, for any compensated price change from an initial situation (p, w) to a new price-wealth pair $(p', w') = \big(p', p' \cdot x(p, w)\big)$, we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0,$$

with strict inequality whenever $x(p, w) \neq x(p', w')$. If a price-wealth change was a compensated one for every consumer i, then because each individual demand satisfies the weak axiom, we would have

$$(p' - p) \cdot [x_i(p', \alpha_i w') - x_i(p, \alpha_i w)] \leq 0$$

Adding over all i gives precisely

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0.$$

So aggregate demand satisfies the weak axiom for any price-wealth change that is compensated for every consumer.

But a price-wealth change that is compensated in the aggregate need not be compensated for every individual – wealth might be adjusted upwards but all of that new wealth could, perhaps, only go to one consumer, in which case we would no longer have $\alpha_i w' = p \cdot x_i(p, \alpha_i w)$ for some or all i. Then individual wealth effects might throw off consumer i's individual compensated law of demand, thus throwing off the aggregate law of demand and failing the weak axiom.

So a property as basic as the weak axiom cannot be expected to hold generally for aggregate demand. Suppose instead, however, that the law of demand holds at the individual level for *uncompensated* price changes.

Proposition 2. *The individual demand function $x_i(p, w)$ satisfies the* ***uncompensated law of demand*** *property if*

$$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0$$

for any p, p' and w_i, with strict inequality if $x_i(p', w_i) \neq x_i(p, w_i)$. the analogous definition applies to the aggregate demand function $x(p, w)$.

Proposition 3. *If $x_i(p, w_i)$ satisfies the uncompensated law of demand property, then $D_p(x, w_i)$ is negative semidefinite. That is, for all dp, we have*

$$dp \cdot D_p x_i(p, w_i) dp \leq 0.$$

Proposition 4. *If $D_p x_i(p, w_i)$ is negative semidefinite for all p, then $x_i(p, w_i)$ satisfies the uncompensated law of demand property.*

Proposition 5. *If every consumer's Walrasian demand function $x_i(p, w)$ satisfies the uncompensated law of demand property, then so does the aggregate demand $x(p, w) = \sum_i x_i(p, \alpha_i w)$. As a consequence, the aggregate demand $x(p, w)$ satisfies the weak axiom.*

Proposition 6. *If $\succsim_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand property.*

Prepare yourself: this is where the section goes to shit.

Proposition 7. *Suppose that $\succsim_i$ is defined on the consumption set $X = \mathbb{R}^L_+$ and is representable by a twice continuously differentiable concave function $u_i(\cdot)$. If*

$$-\frac{x_i \cdot D^2 u_i(x_i) x_i}{x_i \cdot \nabla u_i(x_i)} < 4 \quad \text{for all} \quad x_i,$$

then $x_i(p, w)$ satisfies the unrestricted law of demand property.

What the fuck is that supposed to mean?

Proposition 8. *Suppose that all consumers have identical preferences $\succsim$ defined on $\mathbb{R}^L_+$, with individual demand functions denoted $\tilde{x}(p, w)$, and that individual wealth is uniformly distributed on an interval $[0, \overline{w}]$. Then the aggregate demand function,*

$$x(p) = \int_0^{\overline{w}} \tilde{x}(p, w)\, dw,$$

satisfies the unrestricted law of demand property.

Note that the strong axiom is a rather strong assumption, so the chances that a real economy satisfies it is virtually zero.

The Representative Consumer

We might want to treat the aggregate demand function as if it were generated by a fictional **representative consumer** whose preferences can be used as a measure of aggregate societal welfare. So when can we?

Suppose we are using a wealth distribution rule, that $\sum_i w_i(p, w) = w$ for all (p, w), and that every $w_i(\cdot, \cdot)$ is continuous and homogeneous of degree one. Then aggregate demand takes the conventional demand form, $x(p, w) = \sum_i x_i\big(p, w_i(p, w)\big)$, where $x(p, w)$ also is continuous, homogeneous of degree zero, and satisfies Walras' law.

Definition 2. *A* ***positive representative consumer*** *exists if there is a rational preference relation $\succsim$ on $\mathbb{R}^L_+$ such that the aggregate demand function $x(p, w)$ is precisely the Walrasian demand function generated by this preference relation. That is, $x(p, w) \succ x$ whenever $x \neq x(p, w)$ and $p \cdot x \leq w$.*

A positive representative consumer can be thought of as a fictional individual whose utility maximization problem when facing society's budget set would generate the economy's aggregate demand function. The positive representative consumer must exist if we want to treat aggregate demand as we would an individual demand function.

Definition 3. *A* ***social welfare function*** *is a function $W : \mathbb{R}^I \to \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, ..., u_I)$ of utility levels for the I consumers in the economy.*

The idea is that $W(u_1, ..., u_I)$ expresses society's judgments on how individual utilities have to be compared to produce an ordering of possible social outcomes. Oh hey, let's also just go ahead and assume that social welfare functions are increasing, concave, and differentiable, because we are rapidly approaching the part of economics where we just add whatever assumptions make our lives easier without further deliberation. (Rant over.)

Now let's hypothesize that some benevolent central planner wants to redistribute wealth in order to maximize social welfare. She is essentially solving

$$\arg\max_{w_1,...,w_I} W\big(v_1(p, w_1), ..., v_I(p, w_I)\big) \quad \text{s.t.} \sum_{i=1}^{I} w_i \leq w,$$

where $v_i(p, w)$ is consumer i's indirect utility function. The solution of this problem defines a social indirect utility function $v(p, w)$.

Proposition 9. *Suppose that for each level of prices p and aggregate wealth w, the wealth distribution $\big(w_1(p, w), ..., w_I(p, w)\big)$ solves the social welfare problem. Then the value function $v(p, w)$ of the social welfare problem is an indirect utility function of a positive representative consumer for the aggregate demand function $x(p, w) = \sum_i x_i\big(p, w_i(p, w)\big)$.*

Definition 4. *The positive representative consumer $\succsim$ for the aggregate demand $x(p, w) = \sum_i x_i\big(p, w_i(p, w)\big)$ is a* ***normative representative consumer*** *relative to the social welfare function $W(\cdot)$ if for every (p, w), the distribution of wealth $\big(w_1(p, w), ..., w_I(p, w)\big)$ solves the social welfare problem and, therefore, the value function of the social welfare function is an indirect utility function for $\succsim$.*

It is *not true* that whenever aggregate demand can be generated by a positive representative consumer, that this representative consumer's preferences have normative content. It may even be the case that a positive representative consumer exists but there is no social welfare function that leads to a normative representative consumer.