ECN 200D: Week 9 Lecture Notes
Lucas Trees and Cash

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1 The Lucas Trees Model (CAPM)

1.1 The Setup

The **Lucas Trees model** is a capital asset pricing model (CAPM). In it, there exist trees that produce fruit of quantity d_t in period t, where $d_t \in \{d_1, \ldots, d_N\}$ possible states of production. Typically we will be assuming a Markov process, i.e.

$$P(d_{t+1} = d_j | d_t = d_i) = \pi_{ij}.$$

The supply of trees T is fixed and exogenous, so we may as well just normalize the supply of trees to $T = 1$. Fruit from the tree is nonstorable. An agent has a share s of the fruit that the tree produces.

1.2 The Bellman Equation

The Bellman equation for the representative agent is given by

$$V_i(s) = \max_{c_i, s'} \left\{ u(c_i) + \beta \sum_{j=1}^{N} \pi_{ij} V_j(s') \right\}.$$

Given that the current state of the world is i, the agent wants to choose how much to consume and how much of a share of the tree to have tomorrow based on the expected value of tomorrow's level of production.

Let ψ_i denote the price of a unit of stock when the state of the world is i. Then the agent's budget constraint is can be written as

$$c_i + \psi_i s' = (\psi_i + d_i)s.$$

In other words, the amount consumed today and the amount of tomorrow's shares purchased today must equal the value of today's shares plus the value of today's fruit, i.e. today's nominal wealth.

1.3 First-Order Conditions

Plugging the budget constraint into the Bellman equation gives

$$V_i(s) = \max_{s'} \left\{ u\big([\psi_i + d_i]s - \psi s'\big) + \beta \sum_{j=1}^{N} \pi_{ij} V_j(s') \right\}.$$

Then the first-order condition is

$$u'(c_i)\psi_i = \beta \sum_{j=1}^{N} \pi_{ij} V_j'(s'). \tag{1}$$

Substitute the policy function $s' = g(s)$ and take the first-order condition of

$$V_i(s) = u\big([\psi_i + d_i]s - \psi_i g(s)\big) + \beta \sum_{j=1}^{N} \pi_{ij} V_j\big(g(s)\big)$$

with respect to s to derive the envelope condition

$$V_i'(s) = u'(c_i)(\psi_i + d_i). \tag{2}$$

Updating the envelope condition by a period, change the state of the world to j, and in combination with the first-order condition we get

$$u'(c_i)\psi_i = \beta \sum_{j=1}^{N} \pi_{ij} u'(c'_j)(\psi'_j + d_j). \tag{3}$$

In equilibrium we will have supply equaling demand. There's a fixed supply $s = s' = s'' = \ldots = 1$. It follows from the budget constraint that $c_i = d_i$ $c'_j = d_j$. Therefore we can write condition (3) as

$$u'(d_i)\psi_i = \beta \sum_{j=1}^{N} \pi_{ij} u'(d_j)(\psi'_j + d_j). \tag{4}$$

This condition hold for all $i = 1, \ldots N$, so we have N conditions.

1.4 Special Case: $N = 1$

When $N = 1$, the model is not stochastic—we have $d_t = d$ in all cases. So we can rewrite the asset pricing formula in equation (3) as

$$u'(d)\psi = \beta u'(d)(\psi' + d) \quad \Longrightarrow \quad \psi' = \frac{\psi}{\beta} - d.$$

This sequence follows an "explosive path," so the only admissible solution is the steady state solution where

$$\psi = \frac{\psi}{\beta} - d \quad \Longrightarrow \quad \psi^* = \frac{\beta d}{1 - \beta}.$$

So the price of the asset is the value of the discount stream of dividends, i.e. its "fundamental value."

1.5 General Case

Okay, now let's consider the steady state of the general case,

$$u'(d_i)\psi_i = \beta \sum_{j=1}^{N} \pi_{ij} u'(d_j)(\psi_j + d_j),$$

where $i = 1, \ldots, N$. We can write the system as the matrix

$$U\psi = \beta\pi U(\psi + d),$$

where π is the **Markov matrix** in which the (ij)th entry is π_{ij}. Doing some matrix algebra, we can solve for

$$\psi^* = (U - \beta\pi U)^{-1}\beta\pi U' d.$$

This is, without question, something for a computer to solve.

2 Monetary Theory: The Basics

2.1 The Fisher equation

In general, asset price is equal to its discounted payment, i.e.

$$\text{asset price} = \frac{\text{payment}}{(1+r)^n}. \tag{5}$$

We'll be dealing in one period intervals, so we'll have $n = 1$ from now on.

Let p_t be the price of an asset that has a nominal interest rate of i. The real price of the asset is the goods you're giving up, e.g. the real price $1/p_t$. The payment in real terms is $(1+i)/p_{t+1}$. So from equation (5),

$$\frac{1}{p_t} = \frac{(1+i)/p_{t+1}}{1+r} \implies \frac{p_{t+1}}{p_t} = \frac{1+i}{1+r}. \tag{6}$$

The rate of inflation is the rate of growth in the price level, so

$$\pi_t = \frac{p_{t+1} - p_t}{p_t} \implies \frac{p_{t+1}}{p_t} = 1 + \pi_t. \tag{7}$$

Plug in equation (6) and we get

$$\frac{1+i}{1+r} = 1 + \pi_t.$$

Solving for i, we get the Fisher equation,

$$i = r + \pi_t + r\pi_t. \tag{8}$$

2.2 Inflation and Money Growth

Claim. *In a steady state monetary model where supply of money grows at a constant rate μ, $\pi = \mu$.*

Proof. The money growth rate implies that $M_{t+1} = (1+\mu)M_t$. In a steady state, real variables do not change, and therefore the level of real balances M_t/p_t is constant. It follows that

$$\frac{M_t}{p_t} = \frac{M_{t+1}}{p_{t+1}} \implies \frac{p_{t+1}}{p_t} = \frac{M_{t+1}}{M_t} = 1 + \mu.$$

From equation (refinflation), it follows that

$$1 + \mu = 1 + \pi \implies \mu = \pi. \qquad \square$$

2.3 Illiquid Real Interest Rate

Claim. *The real interest rate of a fully illiquid bond in a standard (monetary) model is given by $r = 1/\beta - 1$.*

Think of a world where agents can buy a one period real discount bond which gives you one unit of the numeraire good tomorrow. What price ψ are you willing to pay? Well, given your discount rate of β, you'll be willing to pay $\beta \cdot 1$. It follows from equation (5) that

$$\beta = \frac{1}{1+r} \quad \Longrightarrow \quad r = \frac{1}{\beta} - 1.$$

2.4 The Friedman Rule

Claim. *The rate of money growth is $\mu = \beta - 1 < 0$ when the nominal interest rate $i = 0$.*

Proof. From the Fisher equation, we have $1 + i = (1 + r)(1 + \pi)$. From the previous results, we know that $1 + r = 1/\beta$ and $1 + \pi = 1 + \mu$. It follows that

$$1 + i = \frac{1}{\beta}(1 + \mu) \quad \Longrightarrow \quad \mu = \beta(1 + i) - 1.$$

Since the assumption is $i = 0$, the result follows. □

3 Cash in Advance (CIA) Model

3.1 The Setup

We will have measure 1 of homogeneous agents in an endowment economy with nonstorable endowment $e_t = e$. The central bank prints money and the money supply grows at the rate of μ, so

$$M_{t+1} = (1 + \mu)M_t.$$

We will assume that $\mu \geq \beta - 1$, which does allow for negative μ. The quantity of money printed T_t will be transferred to each agent—and since the measure of agents is 1, it means each individual receives T_t.

This model is a little bit on the goofy side. The agent *must have money on hand* in order to buy and thus consume the good, even though they have the exact same good as their endowment. Furthermore, they must choose how much money to carry with them from the previous period. So if they chose to not carry any cash from last period to this period, and if the central bank prints no cash, then the agent can't buy, and therefore cannot consume, anything—even though they have a positive endowment of the thing they want to consume!

3.2 The Bellman Equation

Let m denote the choice of a representative agent of how much money to hold. The Bellman equation for the agent is

$$V(m) = \max_{c,m'} \{u(c) + \beta V(m')\}, \tag{9}$$

subject to the budget constraint

$$pc + m' = pe + m + T, \tag{10}$$

and the cash in advance constraint,

$$pc \leq m + T. \tag{11}$$

The CIA constraint says that the agent cannot spend more on consumption than money they have with them.

When we combine the budget constraint and the CIA constraint, we end up with $m' \geq pe$, or better yet in real terms,

$$e \leq \frac{m'}{p}.$$

This seems sensible—we've already established that the agent cannot eat their endowment, so the only way to get utility from it is to sell it for money and then eventually use the money for consumption. This means there's no reason to not sell all of the endowment and have at least m' in cash available for next period.

Now evidently if we take the Lagrange multipliers for the problem, then we can rewrite the Bellman equation as

$$V(m) = \max_{m'} \left\{ u\left(e + \frac{m + T - m'}{p}\right) + \lambda\left(\frac{m'}{p} - e\right) + \beta V(m') \right\}.$$

I guess this is the way of incorporating the CIA constraint into the Bellman equation itself. I don't really have any intuition beyond that. In practical terms, we can at least say that

$$\lambda\left(\frac{m'}{p} - e\right) = 0$$

functions as a typical complementary slackness condition.

3.3 First-Order Conditions

With respect to m', we have

$$\frac{u'(c)}{p} - \frac{\lambda}{p} = \beta V'(m').$$

The envelope condition (remember that we can just ignore the chain-rule effects and keep the direct effect) gives

$$V'(m) = \frac{u'(c)}{p} \implies V'(m') = \frac{u'(c')}{p'}.$$

Combine the first-order condition and the envelope condition to get

$$\frac{u'(c)}{p} - \frac{\lambda}{p} = \beta \frac{u'(c')}{p'} \implies u'(c) - \lambda = \beta u'(c') \frac{p}{p'}. \tag{12}$$

3.4 Equilibrium Conditions

M is the money supply m in the money demand. So in equilibrium, $M = m$. We also have that $M' = M + T$. These two conditions allow us to rewrite the budget constraint as

$$pc = pe. \tag{13}$$

So the amount of money spent on consumption equals the amount of money earned from selling the endowment. That is, $c = c' = \ldots = e$. Given how silly this model is, this result should not come as a surprise.

Furthermore, notice that

$$\frac{p}{p'} = \frac{1/p'}{1/p} = \frac{M'/p'}{M(1+\mu)/p} = \frac{z'}{(1+\mu)z},$$

where $z_t = M_t/p_t$ is real money balances. In the steady state, every real variable is constant, so $z' = z$. Therefore

$$\frac{p}{p'} = \frac{1}{1+\mu}.$$

This allows us to rewrite equation (12) as

$$u'(c)\left[1 - \frac{\beta}{1+\mu}\right] = \lambda. \tag{14}$$

Furthermore, we can rewrite the complementary slackness condition as

$$\lambda\left[(1+\mu)z - e\right] \geq 0. \tag{15}$$

3.5 Money Growth Rates

In this weird model, the only reason to hold money is so you can buy your endowment. How how much money will you hold? It will depend on μ, and in particular, whether the Friedman rule is adhered to or not.

Case 1: $\boldsymbol{\mu = \beta - 1}$. This is the case where the Friedman rule is adhered to, i.e. when $i = 0$. From equation (14), this would imply that $\lambda = 0$, so complementary slackness is satisfied. Thus we know that $m' > pe$.[1]

Case 2: $\boldsymbol{\mu > \beta - 1}$. It follows that

$$1 - \frac{\beta}{1+\mu} > 0.$$

From equation (14), it follows that $\lambda > 0$. This in turn implies from complementary slackness that $pe = m'$. In words, you should carry the exact amount of money you need to buy stuff and not one penny more.

4 Cash Goods vs. Credit Goods

4.1 The Setup

Suppose the endowment is coconuts. The coconuts can be converted into two goods, say, coconut juice and coconut candy, which have the same price. So in this model, there are two consumption goods arising from the same endowment. Here's the quirk of the model. If you want to consume c_1, then you have to purchase it using money. The market for c_2, however, does not require cash.

We'll be making the following assumptions about utility. First, both unmixed partials are strictly negative, i.e. $u_{11} < 0$ and $u_{22} < 0$. Furthermore, we will ensure an interior solution (positive consumption of both goods) by imposing an Inada condition of

$$\lim_{c_1 \to 0} c_1(c_1, c_2) = \infty = \lim_{c_2 \to 0} c_2(c_1, c_2).$$

[1]In lecture he said this implies a strict inequality, but I do not see why that is necessarily true—there is nothing preventing both factors in the complementary slackness condition from being zero. (Although it's rare in practice to actually have them both zero.)

In other words, if you have practically no c_i, then consuming a little bit more c_i will increase your utility dramatically.

4.2 The Bellman Equation

The Bellman equation for the representative agent is

$$V(m) = \max_{c_1, c_2, m'} \{u(c_1, c_2) + \beta V(m')\}.$$

The budget constraint is given by

$$pc_1 + pc_2 + m' = pe + m + T,$$

and the CIA constraint is

$$pc_1 \leq m + T.$$

We can combine the two constraints into

$$p(e - c_2) \leq m' \quad \Longrightarrow \quad e - c_2 \leq \frac{m'}{p}.$$

So now the Bellman equation can be written as

$$V(m) = \max_{c_2, m'} \left\{ u\left(e - c_2 + \frac{m + T - m'}{p}, c_2\right) + \lambda\left(\frac{m'}{p} + c_2 - e\right) + \beta V(m') \right\}, \tag{16}$$

where our complementary slackness condition is given by

$$\lambda\left(\frac{m'}{p} + c_2 - e\right).$$

4.3 First-Order Conditions

With respect to c_2, the first-order condition is

$$u_1(c_1, c_2) = u_2(c_1, c_2) + \lambda.$$

With respect to m', we get

$$\frac{u_1(c_1, c_2)}{p} = \frac{\lambda}{p} + \beta V'(m').$$

The envelope condition is

$$V'(m) = \frac{u_1(c_1, c_2)}{p} \quad \Longrightarrow \quad V'(m') = \frac{u_1(c_1', c_2')}{p'}.$$

Plugging the envelope condition into the second first-order condition, we have

$$\frac{u_1(c_1, c_2)}{p} - \frac{\lambda}{p} = \beta \frac{u_1(c_1', c_2')}{p'} \quad \Longrightarrow \quad u_1(c_1, c_2) - \lambda = \beta \frac{p}{p'} u_1(c_1', c_2'). \tag{17}$$

4.4 Equilibrium Conditions

Again, $m_t = M_t$ in equilibrium because supply must equal demand. This allows us to rewrite the budget constraint as

$$pc_1 + pc_2 + M' = pe + M + T \quad \Longrightarrow \quad p(c_1 + c_2) = pe$$

because $M' = M + T$. It follows that $c_1 + c_2 = e$, which isn't all that surprising—a person ultimately consumes their endowment, even though they go through this rigmarole to get it. It follows that $c_1 = e - c_2$.

From the first section of these notes, we found that

$$\frac{p'}{p} = 1 + \mu \quad \Longrightarrow \quad \frac{p}{p'} = \frac{1}{1 + \mu}.$$

We can use this to rewrite equation (17) and the first-order condition with respect to c_2 as

$$u_1(e - c_2, c_2) - \lambda = \frac{\beta}{1+\mu} u_1(e - c_2', c_2')$$
$$u_1(e - c_2, c_2) - \lambda = u_2(e - c_2, c_2).$$

It follows that

$$\frac{\beta}{1+\mu} u_1(e - c_2', c_2') = u_2(e - c_2, c_2). \tag{18}$$

Hey, this is insightful. The social planner would like the marginal utilities to be equalized. But they may or may not be equalized depending on what μ is. In other words, μ is having a *real* effect on the equilibrium.

4.5 Observations

Suppose that $u = \beta - 1$ so that the Friedman rule is observed. It follows that

$$\frac{\beta}{1+u} = 1,$$

which implies that the marginal utilities in equation (18) are equalized. Indeed, the social planner's solution is only achieved when the Friedman rule is observed.

What about existence and uniqueness? We can get everything we need from c_2. Define

$$G(c_2) = u_2(e - c_2, c_2) - \frac{\beta}{1+\mu} u_1(e - c_2, c_2).$$

Any point where $G(c_2) = 0$ defines an equilibrium, so we'll want to know the shape of $G(c_2)$.

Consider what happens when $c_2 \to 0$. From the Inada conditions, u_2 will go to infinity and therefore $G(c_2)$ will go to infinity. Now consider what happens when $c_2 \to e$. Then the first argument approaches zero, so u_1 will go to infinity and $G(c_2)$ will go to negative infinity. We know the asymptotic behavior, and because $G(c_2)$ is continuous, existence of equilibria is established.

For uniqueness, we need the function to be strictly downward sloping. Evaluating the derivative gives

$$\begin{aligned} G'(c_2) = -u_{21} + u_{22} + \frac{\beta}{1+\mu}u_{11} - \frac{\beta}{1+\mu}u_{12} \\ u_{22} + \frac{\beta}{1+\mu}u_{11} - u_{12}\left(1 + \frac{\beta}{1+\mu}\right). \end{aligned}$$

We have assumed that u_{22} and u_{11} are both strictly negative. We have no idea what sign u_{12} is, however, so we cannot guarantee uniqueness. For some functional forms, we'll be able to establish that $u_{12} \geq 0$, for example if we make the utility separable, i.e. $u(c_1, c_2) = v(c_1) + w(c_2)$.

4.6 Comparative Statics

Suppose there's a unique equilibrium, which requires $u_{12} \geq 0$. We want to know what happens if there's a change in the rate of money growth μ. Equilibrium requires that

$$(1+\mu)u_2(e - c_2, c_2) = \beta u_1(e - c_2, c_2).$$

We'll use the implicit function theorem. Write $c_2(\mu)$ as a function of μ. Note that this implies $c_1'(\mu) = -c_2'(\mu)$. So take the derivative of both sides with respect to μ and we have

$$\begin{aligned} u_2 &= [\beta(u_{12} - u_{11}) + (1+\mu)(u_{12} - u_{22})]c_2'(\mu) \\ \implies \quad c_2'(\mu) &= \frac{u_2/(1+\mu)}{\left(1 + \frac{\beta}{1+\mu}\right)u_{12} - \frac{\beta}{1+\mu}u_{11} - u_{22}}. \end{aligned}$$

The mixed partials are both negative and are both being subtracted, and u_{12} is also assumed nonnegative in this case, so the denominator is positive. We also know that $u_2 > 0$, so we can conclude that $c_2'(\mu) > 0$. The implication is that a higher growth rate of money means a higher rate of inflation, increasing the cost of holding money. People will respond by consuming more of good 2 because doing so requires no money.