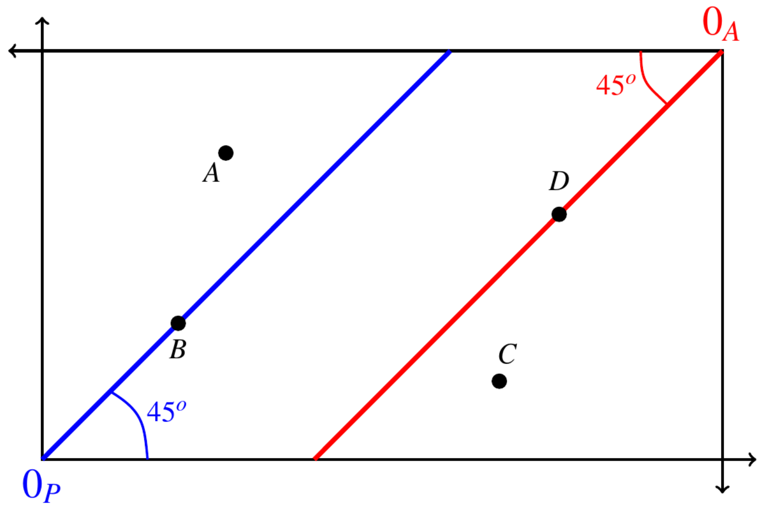
Problem 1 (Exercise 6.6). Using the Edgeworth box below, determine which (if any) contracts A, B, C, or D, are Pareto efficient in the following scenarios.



(a) The Principal is risk averse and the Agent is risk neutral.

Solution. For an interior contract to be Pareto efficient, the two indifference curves must be tangent to each other. In general, the tangency condition can be written as

$$-\frac{p^g}{1-p^g}\frac{U_P'(X^g-w^g)}{U_P'(X^b-w^b)} = -\frac{p^g}{1-p^g}\frac{U_A'(w^g)}{U_A'(w^b)} \quad \Longrightarrow \quad \frac{U_P'(X^g-w^g)}{U_P'(X^b-w^b)} = \frac{U_A'(w^g)}{U_A'(w^b)}.$$

In this case, we can go a bit further. Because the Agent is risk neutral, their indifference curve is a line and therefore slope of their indifference curve is the same everywhere, that is $U_A'(w^g) = U_A'(w^b)$, and therefore $U_A'(w^g)/U_A'(w^b) = 1$. So we can instead write the tangency condition as

$$\frac{U_P'(X^g-w^g)}{U_P'(X^b-w^b)} = 1 \quad \Longrightarrow \quad U_P'(X^g-w^g) = U_P'(X^b-w^b).$$

But the only way this condition can hold is if $X^g - w^g = X^b - w^b$. The quantity $X^g - w^g$ is exactly what's expressed on the horizontal axis for the Principal; the quantity $X^b - w^b$ is exactly what's expressed on the vertical axis for the Principal; therefore we must be on the Principal's 45 degree line.

Which is to say: in this scenario, any Pareto efficient interior contract must have the Principal taking zero risk. (Think about it: the risk neutral person doesn't care about risk at all, the risk averse person hates risk, so it seems like a natural fit if the risk neutral person just takes all of the risk.) That is, we need to be on the Principal's 45 degree line. The only qualifying contract is B.

(b) The Principal is risk neutral and the Agent is risk averse.

Solution. Okay, now the Agent will have zero risk in any Pareto efficient interior contract, i.e. we need to be on the Agent's 45 degree line. Only contract D works.

(c) Both the Principal and the Agent are risk averse.

Solution. Since both parties hate risk, neither are willing to take all of the risk. Therefore they each take some risk as a sort of compromise: the Pareto efficient outcome should be somewhere in between the two 45 degree lines. But uh, there aren't any contracts here between the two 45 degree lines. So none of them are Pareto efficient in this scenario.

(d) Both the Principal and the Agent are risk neutral.

Solution. In this scenario, *any* contract in the Edgeworth box is Pareto efficient.

Problem 2 (Exercise 6.8, Part a). Let the surplus in the good state be $X^g = \$1,206$ and the surplus in the bad state be $X^b = \$676$. The probability of the good state is 25%. The Principal's von Neumann-Morgenstern utility-of-money function is $U_P(m) = m$ while the Agent's von Neumann-Morgenstern utility-of-money function is $U_A(m) = \sqrt{m}$.

Find a Pareto efficient contract in the interior of the Edgeworth box at which the Principal's expected utility is 232.5.

Solution. In the good state, the Principal receives $\$1206 - w^g$; whereas in the bad state, the Principal receives $\$676 - w^b$. If the Principal's expected utility equals 232.5, then it must be the case that

$$E[U_P] = 0.25(1206 - w^g) + 0.75(676 - w^b) = 232.5.$$

This gives us one equation for two unknowns.

The Principal is risk neutral, whereas the Agent is risk averse. Any Pareto efficient interior contract must therefore have the Agent with zero risk. Because the Agent has zero risk in this scenario, their wealth should be the same in both scenarios, that is, $w^g = w^b$. This gives us another equation for the two unknowns.

Substitute and we have

$$0.25(1206 - w^g) + 0.75(676 - w^g) = 232.5 \quad \Longrightarrow \quad w^g = \$576 = w^b.$$

Problem 3 (Exercise 6.16). Mister P wants to hire Miss A to run his firm. If Miss A works for Mister P, then one of two outcomes will occur: the profit of the firm will be \$520 (with probability 45/98) or it will be \$200 (with probability 53/98). Mister P's von Neumann-Morgenstern utility-of-money function is $U(m) = \sqrt{m}$, while Miss A is risk neutral. Consider the following contract, call it C: Miss A will get \$144 if the profit of the firm turns out to be \$520, and she will get \$90 if the profit of the firm turns out to be \$200.

(a) What is Mister P's expected utility from this contract?

Solution. There is a 45/98 probability that the firm makes \$520, in which case Mister P gives \$144 of that to Miss A. There is a 53/98 probability that the firm makes \$200, in which case Mister P gives \$90 of that to Miss A. Mister P has square root utility, so his expected utility is

$$E[U_P(C)] = \frac{45}{98}\sqrt{520 - 144} + \frac{53}{98}\sqrt{200 - 90} \approx 14.576.$$

Before moving on, note two things. First, contract C is strictly within the Edgeworth box: in both cases, the Principal gets more than zero, but less than the full surplus. Second, note that contract C is risky for the Principal, i.e. not on the Principal's 45 degree line. Therefore contract C cannot be Pareto efficient: it is possible to make one party better off without making the other any worse off.

(b) What is Miss A's expected utility from this contract?

Solution. There is a 45/98 probability that the firm makes \$520, in which case Miss A gets \$144. There is a 53/98 probability that the firm makes \$200, in which case Miss A gets \$90. Miss A is risk neutral, so just treat her monetary payoff as utility, yielding

$$E[U_A(C)] = \frac{45}{98}(144) + \frac{53}{98}(90) \approx 114.796.$$

(c) Find a Pareto efficient contract in the Edgeworth box, call it D, that Mister P considers to be just as good as C and Miss A prefers to C.

Solution. First note that when one party is risk neutral and the other is risk averse, a Pareto efficient *interior* outcome will be such that the risk neutral party will take all of the risk and the risk averse party will take none of the risk.

We know that the contract C cannot be Pareto efficient because the risk averse Mister P is taking on some risk and the outcome is not on the boundary. We want to think of a contract that gives Mister P zero risk, the same expected utility 14.576, and makes Miss A better off.

In order for Mister P to have zero risk, it must be the case that $520 - w^g = 200 - w^b$, that is, good state and bad state payoff for Mister P are the same. In order for Mister P to receive the same expected utility as contract C, it must be the case that

$$E[U_P(D)] = \frac{45}{98}\sqrt{520 - w^g} + \frac{53}{98}\sqrt{200 - w^b} := 14.576.$$

But since $520 - w^g = 200 - w^b$, we can write

$$\begin{aligned} E[U_P(D)] &= \frac{45}{98}\sqrt{520 - w^g} + \frac{53}{98}\sqrt{200 - w^b} \\ &= \frac{45}{98}\sqrt{520 - w^g} + \frac{53}{98}\sqrt{520 - w^g} \\ &= \sqrt{520 - w^g}, \end{aligned}$$

which gives us a single equation with a single unknown,

$$\sqrt{520 - w^g} := 14.576,$$

which can be solved for $w^g = 307.540$. Then use $212.460 = 200 - w^b$, which gives $w^b = -12.460$. But uh, we can't have $w^b < 0$, so there is no Pareto efficient *interior* contract that gives Mister P the same expected utility. We have to call an audible and look on the boundary of the Edgeworth box where $w^b = 0$. This gives us the equation

$$E[U_P(D)] = \frac{45}{98}\sqrt{520 - w^g} + \frac{53}{98}\sqrt{200 - 0} := 14.576 \quad \Longrightarrow \quad w^g \approx 292.383.$$

Okay, so we have a contract $D = (w^g, w^b) = (292.383, 0)$ that, by construction, makes Mister P just as well off as he was with contract C. Now we just need to verify that Miss A is better off, specifically,

$$E[U_A(D)] = \frac{45}{98}(292.383) + \frac{53}{98}(0) = 134.258,$$

which is clearly larger than $E[U_A(C)] = 114.796$.

The takeaway: if you try to find an interior Pareto efficient contract but the calculations give you infeasible numbers (in this case: less than zero, greater than 520 in the good state, or greater than 200 in the bad state; would all be infeasible), then set those variables to their respective boundaries and solve from there.

(d) Find a Pareto efficient contract in the Edgeworth box, call it E, that Mister P prefers to C and Miss A considers to be just as good as C.

Solution. Let's start from the same observation: if there is a Pareto efficient interior contract, then Mister P will have no risk at all, and therefore it must be that $520 - w^g = 200 - w^b$. In this question, we need Miss A to have the same expected utility she gets with C, that is, we require

$$E[U_A(E)] = \frac{45}{98}(w^g) + \frac{53}{98}(w^b) := 114.796.$$

We have two equations and two unknowns, so we can solve. Write $w^g = 320 + w^b$ and substitute, which gives

$$\frac{45}{98}(320 + w^b) + \frac{53}{98}(w^b) := 114.796 \quad \Longrightarrow \quad w^b = -32.143.$$

Uh oh, it happened again. Alright, the closest we can get to the negative number is

$w^b = 0$. For Miss A to remain indifferent, it has to be the case that

$$E[U_A(E)] = \frac{45}{98}(w^g) + \frac{53}{98}(0) := 114.796 \quad \Longrightarrow \quad w^g = 250.$$

Okay, so contract $D = (250, 0)$. Let's just confirm that Mister P is in fact better off with this contract, since his expected utility is now

$$E[U_P(E)] = \frac{45}{98}\sqrt{520 - 250} + \frac{53}{98}\sqrt{200} \approx 15.193,$$

which indeed is greater than $E[U_P(C)] = 14.576$.