Problem 1: Monetary Policy and Technology Shocks

Consider a NK model without preference shocks or monetary policy shocks. We'll get the equations

- $E_t[\hat{y}_{t+1}] = \hat{y}_t + \dfrac{1}{\sigma}\left(\hat{i}_t - E_t[\hat{\pi}_{t+1}]\right)$
- $\hat{\pi}_t = \beta E_t[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{y}_t^n)$
- $\hat{i}_t = \phi_\pi \hat{\pi}_t$

We'll assume that $\phi_\pi > 1$ so that price indeterminacy is not an issue. Labor productivity is driven by TFP, so

- $\hat{y}_t - \hat{n}_t = \hat{a}_t$,
- $\hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_t, \quad$ i.i.d. ϵ_t

The natural level of output is assumed to be proportional to technology, that is,

$$\hat{y}_t^n = \psi_y \hat{a}_t,$$

where $\psi_y > 1$.

Part A: Equilibrium

Let's use the method of undetermined coefficients to show that each endogenous variable (output, inflation, and hours worked) is proportional to the contemporaneous value of technology, $\hat{a}_t$. In other words, let's assume that

$$\hat{y}_t = \theta_y \hat{a}_t,$$

$$\hat{\pi}_t = \theta_\pi \hat{a}_t,$$

$$\hat{n}_t = \theta_n \hat{a}_t.$$

Plug these and $\hat{i}_t = \phi_\pi \hat{\pi}_t$ and $\hat{y}_t^n = \psi_y \hat{a}_t$ into our system and simplify a little to get

$$E_t[\theta_y \hat{a}_{t+1}] = \theta_y \hat{a}_t + \frac{1}{\sigma}\left(\phi_\pi \theta_\pi \hat{a}_t - E_t[\theta_\pi \hat{a}_{t+1}]\right)$$

$$\theta_\pi \hat{a}_t = \beta E_t[\theta_\pi \hat{a}_{t+1}] + \kappa(\theta_y \hat{a}_t - \psi_y \hat{a}_t)$$

$$\theta_y \hat{a}_t = \hat{a}_t + \theta_n \hat{a}_t.$$

Now exploit the fact that $\hat{a}_t$ is an AR(1) process and you can simplify further to

$$
\begin{aligned}
\theta_y \rho_a \hat{a}_t &= \theta_y \hat{a}_t + \frac{1}{\sigma}\left(\phi_\pi \theta_\pi \hat{a}_t - \theta_\pi \rho_a \hat{a}_t\right) \\
\theta_\pi \hat{a}_t &= \beta \theta_\pi \rho_a \hat{a}_t + \kappa(\theta_y \hat{a}_t - \psi_y \hat{a}_t) \\
\theta_y \hat{a}_t &= \hat{a}_t + \theta_n \hat{a}_t.
\end{aligned}
$$

Notice that all of the $\hat{a}_t$ terms are going to cancel out, giving

$$
\begin{aligned}
\theta_y \rho_a &= \theta_y + \frac{1}{\sigma}\theta_\pi \left(\phi_\pi - \rho_a\right) \\
\theta_\pi &= \beta \theta_\pi \rho_a + \kappa(\theta_y - \psi_y) \\
\theta_y &= 1 + \theta_n.
\end{aligned}
$$

We have three equations and three unknowns, hooray. I think the easiest approach is to solve the second equation for θ_π and then plug that into the first equation to solve for θ_y. What you get is

$$
\theta_y = \frac{(\phi_\pi - \rho_a)\kappa\psi_y}{(1-\rho_a)(1-\beta\rho_a)\sigma + (\phi_\pi - \rho_a)\kappa}.
$$

Then plug this into the second equation to solve for θ_π and you get

$$
\theta_\pi = -\kappa\psi_y \left[\frac{(1-\rho_a)\sigma}{(1-\rho_a)(1-\beta\rho_a)\sigma + (\phi_\pi - \rho_a)\kappa}\right]
$$

Solving for θ_n is then straightforward,

$$
\theta_n = \frac{(\phi_\pi - \rho_a)(\psi_y - 1)\kappa - (1-\rho_a)(1-\beta\rho_a)\sigma}{(1-\rho_a)(1-\beta\rho_a)\sigma + (\phi_\pi - \rho_a)\kappa}.
$$

Part B: Interpretation

Interpreting $\boldsymbol{\theta_y}$. The parameter ϕ_π reflects how strongly monetary policy reacts to inflation. If ϕ_π blows up to infinity, then using L'Hopital's rule it follows that $\theta_y \to \psi_y$. Therefore $\hat{y}_t = \theta_y \hat{a}_t \to \psi_y \hat{a}_t = \hat{y}_t^n$. The output gap closes. The idea is that because $\hat{i}_t = \phi_\pi \hat{\pi}_t$, any inflation whatsoever would cause $\hat{i}_t$ to jump to infinity as well. This means there's practically an infinite return on bonds, so no one would want to consume anything. Firms expect that result and therefore would not raise their prices and thus there isn't any inflation. We have the divine coincidence.

It can be shown that θ_y is increasing in ψ_π. So on the other hand, if ϕ_π approaches 1, then θ_y approaches its lower limit. The idea is that the interest rate doesn't rise much in response to inflation; so firms won't really hesitate to raise prices. The output gap therefore is negative.

The parameter κ captures the degree of price flexibility. A high κ means prices are flexible, a low κ means prices are sticky. As κ explodes, $\theta_y \to \psi_y$ and there is no output gap. The idea is that if prices are completely flexible, then the typical self-correcting mechanism will carry out instantaneously.

On the other hand, if $\kappa \to 0$, then $\theta_y \to 0$. If prices are completely fixed, then firms cannot change their prices. They have to resort to reducing output instead, which creates the output gap.

Interpreting $\boldsymbol{\theta_\pi}$. As $\phi_\pi \to \infty$, it's clear that $\theta_\pi = 0$. The idea is that if the reaction to any inflation at all is infinitely strong, then, well, there will be no inflation. If $\phi_\pi \to 1$, then θ_π approaches its upper limit. The idea is that if the response to inflation is weak, then inflation can actually manifest.

As $\kappa \to \infty$, θ_π goes to zero. As $\kappa \to 0$, θ_π approaches zero as well. And when κ is a positive finite number, θ_π will be negative. The idea is that absolute price stickiness means inflation is zero, which should be obvious. With fully flexible prices,

Interpreting $\boldsymbol{\theta_n}$. It can be shown that θ_n is increasing in ϕ_π. So if monetary policy reacts less strongly to inflation, then labor drops. As $\phi_\pi \to \infty$, $\theta_n \to \psi_y - 1 > 0$.

this is fucking boring, finish it later

Part C: Joint Response of Hours Worked and Output

Recall that $\hat{y}_t - \hat{n}_t = \hat{a}_t$. If there is a positive TFP shock,

Problem 2: Optimal Markovian Policy

Consider an economy where inflation is described by the NK Phillips curve

$$\hat{\pi}_t = \beta E_t[\hat{\pi}_{t+1}] + \kappa \hat{x}_t + u_t,$$

where u_t is a cost-push shock

$$u_t = \rho_u u_{t-1} + \epsilon_t^u.$$

In period 0, the central bank chooses once and for all its policy among a class of Markovian policies in order to minimize the loss function

$$E_0\left[\frac{1}{2}\sum_{t=0}^{\infty}\beta^t(\hat{\pi}_t^2+v\hat{x}_t^2)\right],$$

subject to the sequence of constraints describing the evolution of inflation.

Markovian policy means that the central bank sets the output gap and inflation according to rules of the form

$$\hat{x}_t = \psi_x u_t,$$

$$\hat{\pi}_t = \psi_\pi u_t.$$

Part A: Deriving Optimal Values

We want to find the values of ψ_x and ψ_π that minimize the loss function. Let's first plug in our policy rules into the loss function to get

$$E_0\left[\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left([\psi_\pi u_t]^2+v[\psi_x u_t]^2\right)\right].$$

Let's do the same with the NK Phillips curve, which gives

$$\psi_\pi u_t = \beta E_t[\psi_\pi u_{t+1}] + \kappa\psi_x u_t + u_t.$$

Use the Markov process to rewrite u_{t+1} and we get

$$\psi_\pi u_t = \beta\psi_\pi\rho_u u_t + \kappa\psi_x u_t + u_t.$$

Great, now we can write up the Lagrangian for the problem[1], namely,

$$\mathcal{L} = E_0\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{1}{2}\left([\psi_\pi u_t]^2+v[\psi_x u_t]^2\right)+\lambda_t[\psi_\pi u_t-\beta\psi_\pi\rho_u u_t-\kappa\psi_x u_t-u_t]\right)\right].$$

[1] I have no idea why the constraint is being discounted as well. Because it's a sequence?

Take the first order conditions with respect to ψ_x and ψ_π to get

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\vartheta\psi_x u_t^2\right] = E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t\kappa u_t\right],$$

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\psi_\pi u_t^2\right] = E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t(\beta\rho_u u_t - u_t)\right].$$

Suck the constants out of the expectations and divide them out of the LHS and we'll get

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u_t^2\right] = \frac{\kappa}{\vartheta\psi_x}E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t u_t\right],$$

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u_t^2\right] = \frac{1}{\psi_\pi}E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t(\beta\rho_u u_t - u_t)\right].$$

The LHS are now equal to each other. Woo. So we have the single equality

$$\frac{\kappa}{\vartheta\psi_x}E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t u_t\right] = \frac{1}{\psi_\pi}E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t(\beta\rho_u u_t - u_t)\right].$$

Get the ψ_π on the LHS and then expand out the LHS expectation for

$$\frac{\kappa\psi_\pi}{\vartheta\psi_x}E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t u_t\right] = \beta\rho_u E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t u_t\right] - E_0\left[\sum_{t=0}^{\infty}\beta^t\lambda_t u_t\right].$$

Oh hey, all of the expectations cancel out. Hooray! Now solve to get

$$\psi_x = \frac{\psi_\pi}{\beta\rho_u - 1}\frac{\kappa}{\vartheta}.$$

Seems we need another equation. Johannes took the FOC with respect to λ_t, which is something I've never seen done before. What we'll get is

$$\psi_x = -\frac{\kappa}{(1-\beta\rho_u)^2\vartheta + \kappa^2},$$

$$\psi_\pi = \frac{\vartheta}{(1-\beta\rho_u)\vartheta + \frac{\kappa^2}{1-\beta\rho_u}}.$$

Part B: Markov vs. Discretion

In the discretionary case, we had

$$\hat{x}_t = -\frac{\kappa}{(1-\beta\rho_u)\vartheta + \kappa^2} u_t,$$
$$\hat{\pi}_t = \frac{\vartheta}{(1-\beta\rho_u)\vartheta + \kappa^2} u_t.$$

The big difference with respect to $\hat{x}_t$ is that the Markovian case squares the $(1-\beta\rho_u)$ term in the denominator. Since $0 < (1-\beta\rho_u) < 1$ and we are squaring it, it means we have a smaller denominator in the commitment case, and thus $\hat{x}_t$ will be larger.

With inflation we are comparing the κ^2 term. In particular, we are dividing it by a number between 0 and 1, so that term is larger. And thus we have a larger denominator and a smaller $\hat{\pi}_t$.

So I guess the takeaway is this: Markovian policy places a greater emphasis on fighting inflation but allows for a larger output gap.

Part C: Markov vs. Commitment