Exercise 1

- Preferences: $\log(C_t) + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1)$
- Law of Motion: $K_{t+1} = (1-\delta)K_t + I_t$
- Aggregate Resource Constraint: $C_t + I_t = Y_t$
- Production: $Y_t = Z_t K_t^{\alpha}(X_t N_t)^{1-\alpha}$
- Trend: $X_t / X_{t-1} = \gamma_X$.
- Technology: $\log(Z_{t+1}) = \rho \log(Z_t) + \epsilon_{t+1}$

Part 1: Change in Hours Worked

A **balanced growth path** means that the levels of certain key variables grow at a constant rate. The fact that hours worked is bounded means that $\gamma_N = 1$. If it was anything more, you'd eventually have N exceeding its upper bound. If it was anything less, you'd eventually have $N = 0$ or less, which isn't valid either. Therefore $N_{t+1}/N_t = 1$, from which it follows that $L_{t+1} = L_t = L$. So let's see if the preferences given in this problem actually possess this feature.

First, let's derive the intratemporal Euler equation. Take the first order conditions of the Bellman equation

$$V(\mathbf{K}_t, K_t, Z_t) = \log(c_t) + \frac{\theta}{1-\eta}(L^{1-\eta} - 1)$$
$$- \lambda_t[c_t + k_{t+1} - w_t(1 - L_t) - (r_t^k + 1 - \delta)K_t)]$$

and you'll get

$$\theta L_t^{-\eta} = \frac{w_t}{C_t}.$$

Now we're already deduced that L will really just be some constant. Therefore w_t/C_t must also be constant, so we can conclude that $\gamma_w = \gamma_C$ and $\gamma_N = \gamma_L = 1$.

Part 2: Balanced Growth Path

Consider the law of motion and the growth of capital:

$$\frac{K_{t+1}}{K_t} = \frac{(1-\delta)K_t}{K_t} + \frac{I_t}{K_t}$$
$$\implies \quad \gamma_K = (1-\delta) + \frac{I_t}{K_t}$$

In order for γ_K to be constant, we need I_t/K_t to be constant. This implies that

$$\frac{I_t}{K_t} = \frac{I_{t+1}}{K_{t+1}} \quad \implies \quad \gamma_K = \frac{K_{t+1}}{K_t} = \frac{I_{t+1}}{I_t} = \gamma_I.$$

Now notice that

$$I_{t+1} = Y_{t+1} - C_{t+1} \quad \implies \quad \gamma_I = \frac{Y_t}{I_{t+1}} - \frac{C_t}{I_{t+1}}.$$

This means we need Y_t/I_{t+1} to be constant and C_t/I_{t+1} to be constant, which implies

$$\frac{Y_t}{I_{t+1}} = \frac{Y_{t+1}}{I_{t+2}} \quad \implies \quad \gamma_I = \frac{I_{t+2}}{I_{t+1}} = \frac{Y_{t+1}}{Y_t} = \gamma_Y,$$
$$\frac{C_t}{I_{t+1}} = \frac{C_{t+1}}{I_{t+2}} \quad \implies \quad \gamma_I = \frac{I_{t+2}}{I_{t+1}} = \frac{C_{t+1}}{C_t} = \gamma_C.$$

The shock Z_t doesn't grow over time because it's just a set of shocks, so $\gamma_Z = 1$. Now let's exploit the production function, noting that $\gamma_Z = 1$, giving

$$\begin{aligned}
\gamma_Y &= \frac{Z_{t+1}K_{t+1}^{\alpha}(X_{t+1}N_{t+1})^{1-\alpha}}{Y_t} \\
&= \frac{Z_{t+1}K_{t+1}^{\alpha}(X_{t+1}N_{t+1})^{1-\alpha}}{Z_tK_t^{\alpha}(X_tN_t)^{1-\alpha}} \\
&= \gamma_Z\gamma_K^{\alpha}\gamma_X^{1-\alpha}\gamma_N^{1-\alpha} \\
&= \gamma_K^{\alpha}\gamma_X^{1-\alpha} \\
\implies \quad & \gamma_Y\gamma_K^{-a} = \gamma_X^{1-\alpha} \\
\implies \quad & \gamma_Y^{1-\alpha} = \gamma_X^{1-\alpha} \\
\implies \quad & \gamma_Y = \gamma_X.
\end{aligned}$$

It has been established that $\gamma_w = \gamma_C = \gamma_Y = \gamma_I = \gamma_K = \gamma_X$ in a balanced growth path.

Part 3: Return on Capital

We want to show that γ_{r^k} is constant along the balanced growth path. This is the marginal product of capital, so

$$\frac{r_{t+1}^k}{r_t^k} = \frac{\alpha Z_{t+1}K_{t+1}^{\alpha-1}X_{t+1}^{1-\alpha}N_{t+1}^{1-\alpha}}{\alpha Z_tK_t^{\alpha-1}X_t^{1-\alpha}N_t^{1-\alpha}} = \gamma_Z\gamma_K^{\alpha-1}\gamma_X^{1-\alpha}\gamma_N^{1-\alpha} = \gamma_K^{\alpha-1}\gamma_X^{1-\alpha} = 1.$$

Exercise 2

- Preferences: $\frac{1}{1-\sigma}\big([C_t v(L_t)]^{1-\sigma} - 1\big)$
- Law of Motion: $K_{t+1} = (1-\delta)K_t + I_t$
- Labor Constraint: $N_t + L_t = 1$.
- Production: $Y_t = Z_t K_t^{\alpha} N_t^{1-\alpha}$

You can buy a bond B_{t+1} in period t that pays $r_{t+1}B_{t+1}$ in period t_1. Therefore the household's budget constraint is

$$C_t + K_{t+1} + B_{t+1} = w_t N_t + (1 + r_t^k - \delta)K_t + r_t B_t + \pi_t.$$

Real bonds are zero in net supply.

Part 1: Household's Recursive Problem

Households solve

$$V(\mathbf{K}_t, K_t^s, B_t, Z_t) = \max_{C_t, L_t, K_{t+1}^s, B_{t+1}} \frac{1}{1-\sigma}\big([C_t v(L_t)]^{1-\sigma} - 1\big) + \beta V(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}, Z_{t+1})$$

subject to

$$C_t + K_{t+1} + B_{t+1} = w_t N_t + (1 + r_t^k - \delta)K_t + r_t B_t + \pi_t.$$

Parts 2-3: First Order Conditions and Returns

We are told that Z_t is a constant, so we don't really need to consider it recursively, and furthermore the problem because nonstochastic. Let's ignore labor for now and just focus on the other three state variables.

$$\begin{aligned} V(\mathbf{K}_t, K_t^s, B_t) = \max_{C_t, L_t, K_{t+1}^s, B_{t+1}} & \frac{1}{1-\sigma}\big([C_t v(L_t)]^{1-\sigma} - 1\big) + \beta V(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) \\ & - \lambda_t[C_t + K_{t+1} + B_{t+1} - w_t N_t - (1 + r_t^k - \delta)K_t - r_t B_t - \pi_t]. \end{aligned}$$

Then we have, respectively,

$$
\begin{aligned}
C_t^{-\sigma} v(L_t)^{1-\sigma} &= \lambda_t, \\
\beta V_K'(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) &= \lambda_t, \\
\beta V_B'(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) &= \lambda_t.
\end{aligned}
$$

The updated envelope conditions are

$$
\begin{aligned}
V_K'(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) &= C_{t+1}^{-\sigma} v(L_{t+1})^{1-\sigma}(1 + r_{t+1}^k - \delta), \\
V_B'(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) &= C_{t+1}^{-\sigma} v(L_{t+1})^{1-\sigma} r_{t+1}.
\end{aligned}
$$

We get the two Euler equations

$$
\begin{aligned}
\beta C_{t+1}^{-\sigma} v(L_{t+1})^{1-\sigma}(1 + r_{t+1}^k - \delta) &= C_t^{-\sigma} v(L_t)^{1-\sigma}, \\
\beta C_{t+1}^{-\sigma} v(L_{t+1})^{1-\sigma} r_{t+1} &= C_t^{-\sigma} v(L_t)^{1-\sigma}.
\end{aligned}
$$

So we can see that $r_t = r_t^k + 1 - \delta$.

Part 4: Linearization

Yuck. Linearizing the Euler equations doesn't have any elegant simplification because you'll end up with some $v'(L)$ terms in there screwing things up. In any case, the most you can do is cancel out C terms and exploit the fact that

$$\beta(1 + r^k - \delta) = 1 = \beta r.$$

Linearizing $r_t = r_t^k + 1 - \delta$ gives

$$r \hat{r}_t = r_t^k \hat{r}_t^k = (r - 1 + \delta)\hat{r}_t^k.$$

Exercise 3

With probability p, a person works H hours and consumes c_1. With probability $1 - p$, the person works 0 hours and consumes c_2. This gives rise to expected utility

$$p\big[\log(c_1) + \log\big(v(1 - H)\big)\big] + (1 - p)\big[\log(c_2) + \log\big(v(1)\big)\big].$$

A feasible allocation must satisfy $pc_1 + (1-p)c_2 = c$, where c is aggregate per capital consumption.

Part 1: Household Optimality and Risk Sharing

The Lagrangian for this problem is

$$\mathscr{L} = p\big[\log(c_1) + \log\big(v(1-H)\big)\big] + (1-p)\big[\log(c_2) + \log\big(v(1)\big)\big] - \lambda_t[pc_1 + (1-p)c_2 - c].$$

The FOC with respect to c_1 and c_2 give

$$\frac{p}{c_1} = \lambda_t p,$$
$$\frac{1-p}{c_2} = \lambda_t(1-p).$$

It follows that $c_1 = c_2$. It then follows from the allocation constraint that $c_1 = c_2 = c$. This is what *complete risk sharing* means—the same consumption regardless of whether the good or the bad state occurs.

Part 2: Rewriting Expected Utility

Therefore the expected utility is

$$\log(c) + p\log\left(\frac{v(1-H)}{v(1)}\right) + \log\big(v(1)\big).$$

The average number of hours worked will be

$$N = pH + (1-p)0 = pH.$$

From this it follows that $p = (1-L)/H$. Therefore

$$\log(c) + \frac{1-L}{H}\log\left(\frac{v(1-H)}{v(1)}\right) + \log\big(v(1)\big).$$

Part 3: Decentralized Representative Household

The Bellman equation is

$$V(\mathbf{K}_t, K_t) = \max_{c_t, L_t, K_{t+1}} \log(c_t) + \frac{1-L_t}{H}\log\left(\frac{v(1-H)}{v(1)}\right) + \log\big(v(1)\big) + \beta E_t[V(\mathbf{K}_{t+1}, K_{t+1})]$$

subject to the plain ol' combo-constraint

$$c_t + K_{t+1} = w_t(1 - L_t) + (r^k + 1 - \delta)K_t + \pi.$$

Part 4: First-Order Conditions

To find the labor supply, take the FOC with respect to consumption and leisure. We get

$$\frac{1}{c_t} = \lambda_t,$$

$$-\frac{1}{H}\log\left(\frac{v(1-H)}{v(1)}\right) = \lambda_t w_t.$$

Therefore the labor supply function is

$$-\frac{1}{H}\log\left(\frac{v(1-H)}{v(1)}\right) = \frac{w_t}{c_t}.$$

But I think it really just wanted the two conditions separately. I'm not sure.

Part 5: Linearization

Linearization leads to

$$0 = \hat{\lambda}_t + \hat{c}_t,$$

$$0 = \hat{\lambda}_t + \hat{w}_t.$$

Part 6: Elasticity

The linearized labor supply says that aggregate labor supplied is not a function of wage—if the wage changes, then nothing happens to aggregate labor supplied. So aggregate labor supply is infinitely elastic. (Draw it!) On the other hand, the individual labor supply is totally inelastic—people either work H hours or 0 hours with no flexibility and no choice.

In the baseline RBC model, fluctuation is hours worked is solely captured at the *intensive margin*, that is, fluctuations in hours per worker. However, the fluctuation of hours worked in reality is mainly due to the *extensive margin*, i.e. employment. So the divisible labor model fails to match the data. Furthermore, for the divisible-labor model, the calibration implied labor supply elasticity is roughly 4, which is far higher than the value from the empirical finding, which is less than 1. Adding divisible labor helps rectify the issue.

Also, adding indivisible labor increases consumption volatility, better matching the data.