Absolute Risk Aversion

Definition 1. *For an increasing and concave utility of wealth $u(\cdot)$, its **index/coefficient of absolute risk aversion**, denoted $R(W)$, is defined as*

$$R(W) \equiv -\frac{u''(W)}{u'(W)}.$$

Intuitively, it reflects the curvature of the utility function (concavity proportional to slope) that reflects risk aversion.

Graphically, one investor is more risk averse than another if at all levels of income, the graph of the utility function of the more risk averse investor is flatter.

Theorem 1. *Let A and B denote two investors with respective utility of wealth function u_A and u_B. Let R_A and R_B denote their respective coefficients of absolute risk aversion. Then $R_A(W) \geq R_B(W)$ if and only if there is a strictly increasing and concave function $G(\cdot)$ such that $u_A(W) = G\big(u_B(W)\big)$.*

The intuition is that $R_A(W) \geq R_B(W)$ implies u_B is less concave (relative to its slope) than u_A. As a purely mathematical result, shoving u_B into a concave function $G(\cdot)$ would make it more concave. What we're saying is that there exists for sure such a function $G(\cdot)$ that would make u_B the same as u_A. The argument also flows in reverse (hence the "if and only if" claim).

tl;dr: mathematically, you can make a less risk averse utility function more risk averse by shoving it into a strictly increasing and concave function $G(\cdot)$.

Arrow-Pratt Theorems

Theorem 2 (Arrow-Pratt 1)**.** *Suppose that $R(W)$ is decreasing in W. Then an investor's optimal choice of investment in the risky asset, denoted $a(W_0)$, is increasing as a function of initial income W_0.*

If $R(W)$ is decreasing in W, it means as a person becomes wealthier they become less averse to risk. Hence they will increase their allocation in the risky asset a. There are two analogous results from the homework for different assumptions about $R(W)$.

Theorem 3 (Arrow-Pratt 2)**.** *Suppose that $R_A(W) \geq R_B(W)$ for all W, and that $a_B(W_0) = W_0$. Then $a_A(W_0) \leq a_B(W_0)$.*

Investors A and B have the same wealth, W_0. Investor B has allocated all of her wealth in the risky asset a. Investor A is (weakly) more risk averse than B. Hence investor A cannot have more of the risky asset than investor B.