

## Population Regression

When we estimate things, our estimation is going to depend on whatever sample we happen to have obtained. That sample is usually not going to be a perfect representation of the population, and hence any given sample will differ from the population in random ways.

To illustrate, suppose you have a population of 100 people and you want to estimate their income. You take 20 random samples, someone else takes 20 random samples. Chances are you won't sample the exact same 20 people and hence your estimates will be a bit different. We need to account for that sampling variability.

In the context of regressions, we'd like a regression that best fits the population data. It will be given by the formula

$$y = \beta_1 + \beta_2 x,$$

which I will explain in detail momentarily. But think of this as being the line of best fit for the entire population, and we want to estimate  $\beta_1$  and  $\beta_2$  using a sample.

**Assumption 1: True Population Model.** Again, a regression is just the line of *best* fit – it is not the line of *perfect* fit. When we talk about a specific data point  $i$ , we will assume that the true population model is

$$y_i = \beta_1 + \beta_2 x_i + u_i.$$

What this says is we use  $\beta_1 + \beta_2 x_i$  to best “predict” what  $y_i$  should be for a given value of  $x_i$ ; but since the regression line doesn't perfectly capture all data points, the prediction will be off by  $u_i$ . Accordingly,  $u_i$  is called the **disturbance term**, sometimes called **error term**.

**Assumption 2: Zero Conditional Mean.** We assume zero conditional mean:  $E[u_i|x_i] = 0$  for all  $i$ . Consider a specific  $x_i = x^*$ , where  $x^*$  is just any old number. This allows us to take the true population model and write

$$\begin{aligned} E[y_i|x_i = x^*] &= E[\beta_1|x_i = x^*] + E[\beta_2 x_i|x_i = x^*] + E[u_i|x_i = x^*] \\ &= \beta_1 + \beta_2 x^*. \end{aligned}$$

This is true because  $\beta_1$  and  $\beta_2$  are just numbers – there is nothing random about them – so we, uh, expect them to be themselves, regardless of what  $x_i$  is. And because of our zero conditional mean assumption, the disturbance term drops out. Thus, the regression line is what we expect  $y_i$  to be, given  $x_i$ .

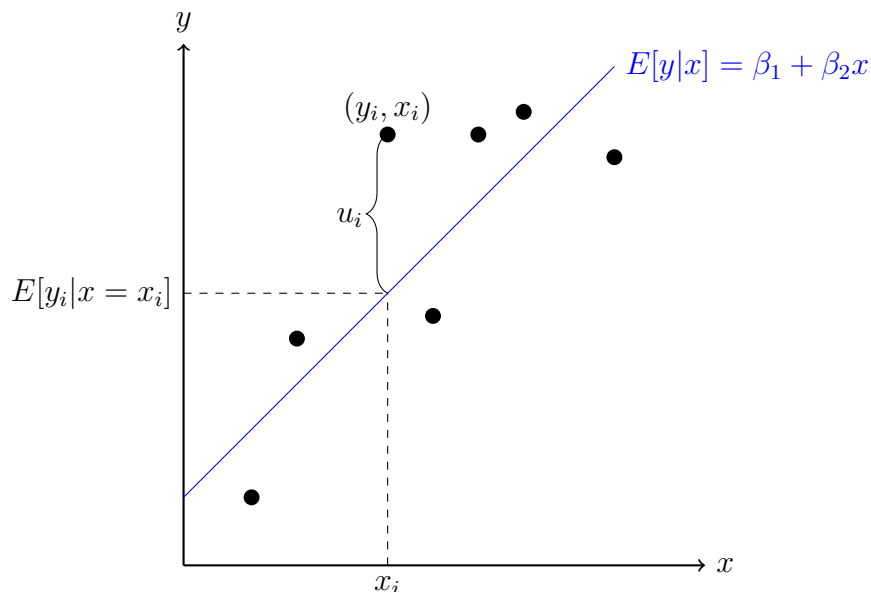


FIGURE 1: Pick some arbitrary data point  $(x_i, y_i)$ . The regression line tells us  $E[y_i|x = x_i]$ , that is, what value we expect  $y_i$  to be for independent variable  $x_i$ . This is the **conditional mean** of  $y_i$  given  $x_i$ . But the regression line is a line of *best fit*, not a line *perfect fit*, so the actual value of  $y_i$  will in general be different than what we expect it to be based on the regression line. The difference between what  $y_i$  actually is and what we expect  $y_i$  to be based on the regression,  $y_i - \beta_1 - \beta_2 x_i$ , is the disturbance term,  $u_i$ .

To summarize the population characteristics:

- The actual value  $y_i$  is given by  $y_i = \beta_1 + \beta_2 x_i + u_i$ .
- The regression line is what we expect  $y_i$  to be, given  $x_i$ . Expressed in the maths,  $E[y_i|x = x_i] = \beta_1 + \beta_2 x_i$ . This is a consequence of assumptions 1 and 2 combined.
- And hence the error term is given by  $u_i = y_i - E[y_i|x = x_i]$ .

We can throw down two more assumptions to make analysis cleaner.

- **Assumption 3: Homoskedasticity.** The variation of  $u_i$  given  $x_i$  is the same number  $\sigma_u^2$  for any  $x_i$ . In math,

$$\text{Var}(u_i|x_i) = \sigma_u^2 \quad \text{for all } i.$$

- **Assumption 4: Independent Errors.** Errors for different observations are statistically independent:  $u_i$  is independent of  $u_j$  whenever  $i \neq j$ .

Adding assumptions 3 and 4 allows us to say that the variation of  $y$  given  $x$  is also constant, and specifically,  $\text{Var}(y|x) = \sigma_u^2$ .

## Estimation Regression

Now we use sample data to estimate  $\beta_1$  and  $\beta_2$  using the ordinary least squares (OLS) technique. Call these estimates  $b_1$  and  $b_2$ , respectively, which are given by equations

$$b_2 = \frac{s_{xy}}{s_x^2} = r_{xy} \times \frac{s_y}{s_x},$$

$$b_1 = \bar{y} - b_2\bar{x},$$

where  $s_{xy}$  is the **sample covariance** defined by

$$s_{xy} \equiv \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}),$$

and  $r_{xy}$  is the **sample correlation coefficient** defined by

$$r_{xy} \equiv \frac{s_{xy}}{s_x s_y}.$$

Under assumptions 1 and 2, the estimates will be unbiased:  $E[b_1] = \beta_1$  and  $E[b_2] = \beta_2$ . That said, they will be different in generality than their population analogues because, well, they're estimates. Hence our estimated regression line will be more or less different than the population regression line, depending on how closely our sample reflects the population.

For our estimated regression, our prediction of  $y_i$  given  $x_i$  is called the **fitted value** and is given by

$$\hat{y}_i = b_1 + b_2 x_i.$$

Much like in the population case, this will in generality be different than the actual value  $y_i$ . We call the difference between the actual value  $y_i$  and our fitted value  $\hat{y}_i$  the **residual**:

$$e_i \equiv y_i - \hat{y}_i.$$

Sometimes you'll also see it as  $\hat{u}_i$ , which I prefer.

Furthermore, assumptions 3 and 4 imply that the variance of the slope estimate  $b_2$  will be

$$\text{Var}(b_2) = \frac{\sigma_u^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \equiv \sigma_{b_2}^2.$$

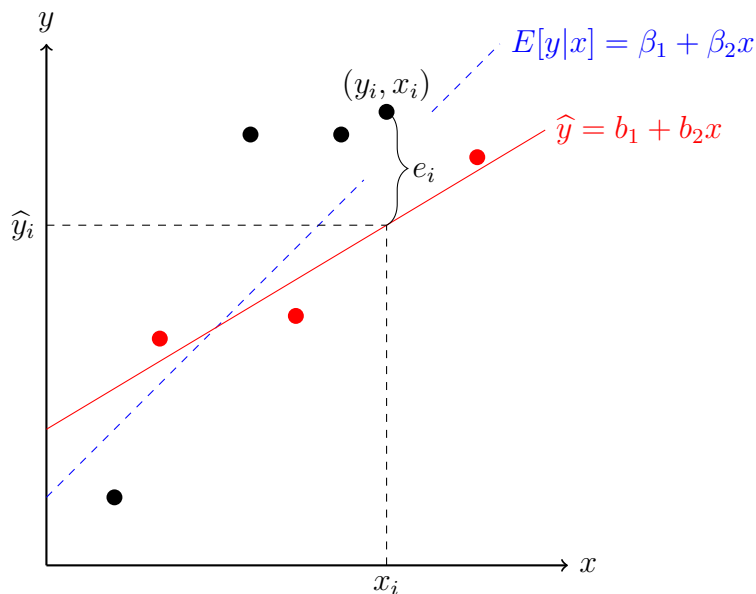


FIGURE 2: Suppose our sample consists of only the red dots. Thus the estimated regression line is in red, which is different than the true population regression line, in blue. For  $x_i$ , it gives us a prediction for  $y_i$ , i.e. the fitted value  $\hat{y}_i$ . The fitted value will not in general be exactly the true value  $y_i$ , and the difference between the true value and the fitted value is the residual,  $e_i = y_i - \hat{y}_i$ .

Assumption 3 is most likely to break down in practice, in which case we will have **heteroskedasticity** – the variance of  $u_i$  will depend on  $x_i$ . In this case we need to use **heteroskedasticity-robust standard errors**.

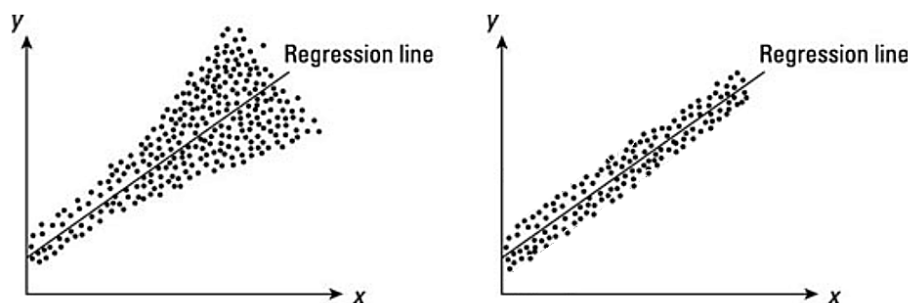


FIGURE 3: The figure on the left is an example of heteroskedasticity; the right an example of homoskedasticity. The left is heteroskedastic because the variation around the regression line gets bigger as  $x$  increases.

## Explained and Unexplained Variation

We define the **residual sum of squares** to be

$$\text{RSS} \equiv \sum_{i=1}^n (y_i - \hat{y}_i)^2.$$

This captures the total error of the estimated regression line, squared so that the errors are positive. Dividing this by  $n - 2$  and taking the square root gives the **standard error of the regression**,

$$s_e \equiv \sqrt{\frac{\text{RSS}}{n-2}} = \sqrt{\frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2},$$

which is the sample analogue of  $\sigma_u$ . You can think of this as being the variation of data around  $\bar{y}$  that cannot be explained by  $x$ .

On the other hand, the variation of data around  $\bar{y}$  that can be explained by  $x$  is the **explained sum of squares**,

$$\text{ESS} \equiv \sum_{i=1}^n (\hat{y}_i - \bar{y})^2.$$

Finally, the total variation of data around  $\bar{y}$  is given by the **total sum of squares**,

$$\text{TSS} \equiv \sum_{i=1}^N (y_i - \bar{y})^2.$$

Based on the intuition it should not be surprising, and it is not difficult to show either, that

$$\text{TSS} = \text{ESS} + \text{RSS}.$$

Total variation is explained variation plus unexplained variation. Great.

The proportion of explained variation around  $\bar{y}$  is called the **R-squared** or **coefficient of determination**, defined as

$$R^2 \equiv \frac{\text{ESS}}{\text{TSS}} = 1 - \frac{\text{RSS}}{\text{TSS}}.$$

If  $R^2$  is high, then  $x$  explains a lot about what's going on with  $y$ ; if  $R^2$  is low, then it doesn't. There is no cutoff for what should be considered "high" or "low," however. Note that  $R^2$  also equals the squared correlation between  $y$  and  $x$ , that is,  $R^2 = r_{xy}^2$ . Also note that  $R^2$  is only valid if the regression includes the intercept.

Note that to test  $R^2$  in a bivariate regression, we use the test statistic

$$F \equiv \frac{R^2}{(1 - R^2)/(n - 2)} \sim F(1, n - 2),$$

or equivalently,

$$F \equiv \frac{ESS}{RSS/(n - 2)} \sim F(1, n - 2),$$

## Estimator Properties

Under assumptions 1-4, our slope estimator  $b_2$  has expected value of  $\beta_2$  because it is unbiased; and it also has variance  $\sigma_{b_2}^2$ . Thus we can write

$$b_2 \sim (\beta_2, \sigma_{b_2}^2).$$

For sufficiently large sample size (greater than 30), the  $z$ -score is approximately standard normal, that is,

$$Z \equiv \frac{b_2 - \beta_2}{\sigma_{b_2}} \sim \mathcal{N}(0, 1).$$

But we don't actually know what  $\sigma_{b_2}$  is because we don't know what  $\sigma_u$  is. Instead we must use the sample estimate of  $\sigma_u$ , given earlier as  $s_e$ . This then allows us to conclude that the sample standard error of  $b_2$  is

$$\text{se}(b_2) = \frac{s_e}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}}$$

So under assumptions 1-4, for sufficiently large sample size (which does *not* have a clear cut rule-of-thumb in this case), we use the  $t$ -statistic

$$t \equiv \frac{b_2 - \beta_2}{\text{se}(b_2)} \sim T(n - 2),$$

where the distribution is approximate. If we add an additional assumption that the disturbance terms are normally distributed, or if  $n \rightarrow \infty$ , then we can say that  $t \sim T(n - 2)$  exactly.