Exercise 2

Utility is given by

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma},$$

subject to resource constraint and law of motion

$$z_t k_t^\alpha = c_t + i_t,$$

$$k_{t+1} = (1-\delta)k_t + i_t.$$

z_t is a shock to productivity. Assume that the shock process for $\ln(z_t)$ follows an AR(1) process with persistence parameter ρ.

Part 1: Recursive Planner's Problem

$$V(k,z) = \max_{c_t, k_{t+1}} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta E[V(k_{t+1}, z_{t+1})] \right\}$$

subject to

$$c_t + k_{t+1} = (1-\delta)k_t + z_t k_t^\alpha,$$

$$\ln(z_{t+1}) = \rho \ln(z_t) + \epsilon_{t+1}.$$

Part 2: First-Order Conditions

I'm going to rewrite the Bellman equation as

$$V(k_t, z_t) = \max_{c_t, k_{t+1}} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta E[V(k_{t+1}, z_{t+1})] - \lambda[c_t + k_{t+1} - (1-\delta)k_t - z_t k_t^\alpha] \right\}.$$

The intertemporal Euler equation is found by differentiating with respect to c_t and k_{t+1} and then combining them after using the envelope theorem. Doing so gives

$$c_t^{-\sigma} = \lambda_t,$$

$$\beta E[V_k'(k_{t+1}, z_{t+1})] = \lambda_t,$$

from which it follows that $c_t^{-\sigma} = \beta E[V_k'(k_{t+1}, z_{t+1})]$. Now envelope it:

$$V_k'(k_t, z_t) = \lambda_t[\alpha z_t k_t^{\alpha-1} + (1-\delta)] = c_t^{-\sigma}[(1-\delta) + \alpha z_t k_t^{\alpha-1}].$$

Update the envelope by a period and you done get

$$V_k'(k_{t+1}, z_{t+1}) = c_{t+1}^{-\sigma}[\alpha z_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)].$$

So plug this into the FOC combo and we get

$$c_t^{-\sigma} = \beta E\left[c_{t+1}^{-\sigma}(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta)\right].$$

This is the intertemporal Euler equation. Woo.

We have four variables in this setup: c, k, i, z. So we need four equilibrium conditions. Indeed, we have

- the Euler equation
- the resource constraint
- the law of motion of capital
- the shock process

Let's simplify the exposition by defining $r_t = \alpha z_t k_t^{\alpha-1} + 1 - \delta$, which is the gross marginal product of capital. Then we'll have one more variable and one more condition, so it's kosher.

Part 3: Linearization

Euler Equation. The Euler equation can now be written as

$$c_t^{-\sigma} = \beta E\left[c_{t+1}^{-\sigma} r_{t+1}\right].$$

First, take the total differential, evaluated at the steady state:

$$-\sigma c^{-\sigma-1} dc_t = \beta E[-\sigma c^{-\sigma-1} r dc_{t+1} + c^{-\alpha} dr_{t+1}].$$

We define $\hat{x} = \dfrac{x_t - x}{x}$, and we define $dx = x_t - x$. Therefore $dx = x\hat{x}$. We can use this to rewrite the above equation as

$$-\sigma c^{-\sigma} \hat{c}_t = \beta E[-\sigma c^{-\sigma} r \hat{c}_{t+1} + c^{-\alpha} r \hat{r}_{t+1}].$$

Because $\beta = 1/r$, as is shown later, it follows that $\beta r = 1$. We can use this to simplify the equation further into

$$-\sigma c^{-\sigma} \hat{c}_t = E[-\sigma c^{-\sigma} \hat{c}_{t+1}] + E[c^{-\alpha} \hat{r}_{t+1}].$$

Since c is today's consumption, we already know what it is, so we can take it outside of the expectations. This allows us to cancel the c terms and what are are left with is

$$-\sigma \hat{c}_t = -\sigma E[\hat{c}_{t+1}] + E[\hat{r}_{t+1}].$$

So this is the *linearized Euler equation.*

Resource Constraint. We have $c_t + i_t = z_t k_t^\alpha$. Totally differentiate, evaluating at the steady state, and we get

$$dc_t + di_t = k^\alpha dz_t + \alpha z k^{\alpha-1} dk_t.$$

Replace dc_t with $c\hat{c}_t$, and so forth for the other variables, to end up with

$$c\hat{c}_t + i\hat{i}_t = k^\alpha z \hat{z}_t + \alpha z k^{\alpha-1} k \hat{k}_t \quad \Longrightarrow \quad c\hat{c}_t + i\hat{i}_t = zk^\alpha[\hat{z}_t + \alpha \hat{k}_t].$$

This is the *linearized resource constraint.*

Law of Motion of Capital. $k\hat{k}_{t+1} = (1-\delta)k\hat{k}_t + i\hat{i}_t$.

Shock Process. $\hat{z}_{t+1} = \rho \hat{z}_t + \epsilon_{t+1}$.

Marginal Product of Crapital. Yep. Craptial. Use the fact that $r = \alpha z k^{\alpha-1} + 1 - \delta$, solving for $\alpha z k^{\alpha-1} = r - (1-\delta)$ to get

$$\hat{r}_t = \frac{r-(1-\delta)}{r}[(\alpha-1)\hat{k}_t + \hat{z}_t].$$

The Linearized System.

$$\begin{aligned}
-\sigma \hat{c}_t &= -\sigma E[\hat{c}_{t+1}] + E[\hat{r}_{t+1}] \\
c\hat{c}_t + i\hat{i}_t &= zk^\alpha[\hat{z}_t + \alpha \hat{k}_t] \\
k\hat{k}_{t+1} &= (1-\delta)k\hat{k}_t + i\hat{i}_t \\
\hat{z}_{t+1} &= \rho \hat{z}_t + \epsilon_{t+1} \\
\hat{r}_t &= \frac{r-(1-\delta)}{r}[(\alpha-1)\hat{k}_t + \hat{z}_t]
\end{aligned}$$

It might be helpful to rearrange the system by putting everything with $t+1$ on one side, everything with t on another.

$$
\begin{aligned}
-\sigma E[\hat{c}_{t+1}] + E[\hat{r}_{t+1}] &= -\sigma \hat{c}_t \\
0 &= z k^{\alpha}[\hat{z}_t + \alpha \hat{k}_t] - c\hat{c}_t - i\hat{i}_t \\
k\hat{k}_{t+1} &= (1-\delta)k\hat{k}_t + i\hat{i}_t \\
\hat{z}_{t+1} &= \rho \hat{z}_t + \epsilon_{t+1} \\
0 &= \frac{r-(1-\delta)}{r}[(\alpha-1)\hat{k}_t + \hat{z}_t] - \hat{r}_t
\end{aligned}
$$

Part 4: Solving

Suppose $\beta = 0.99$, $\sigma = 1$, $\delta = 0.025$, $\alpha = 0.3$, and $\rho = 0.95$. **Find everything else.**

Because it's an AR(1) process, we know that $z = 1$. We can use the original (unlinearized) system to solve. For instance, we know that in the steady state,

$$c^{\sigma} = \beta E[c^{-\sigma} r],$$

from which it follows that $\beta r = 1$. We used this result in the linearization earlier. Therefore $r = 1/0.99$. We can take

$$r = \alpha k^{\alpha-1} + 1 - \delta \quad \Longrightarrow \quad \frac{1}{0.99} = (0.3)k^{-0.7} + 1 - 0.025 \quad \Longrightarrow \quad k \approx 21.4.$$

The law of motion then gives

$$k = (1-\delta)k + i \quad \Longrightarrow \quad i = \delta k \quad \Longrightarrow \quad i \approx 0.5.$$

Finally, the resource constraint gives

$$c + i = k^{\alpha} \quad \Longrightarrow \quad c \approx 2.$$