ECN 200D – Week 6 Lecture Notes

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1 Asset Market

1.1 Setup

We're going to introduce bonds into the model—or more accurately, assets, but bonds are easier to conceptualize. Suppose there is only one good in this economy. An agent can buy a bond in period t for a price of q_t. One period later, they will receive one unit of the good. Let a^i_{t+1} denote the demand of agent i for bonds that pay in $t+1$.

Endowments will again be given by

$$e^1_t = \begin{cases} 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases} \qquad e^2_t = \begin{cases} 0 & \text{if } t \text{ is even} \\ 1 & \text{if } t \text{ is odd} \end{cases}.$$

Let's also continue to assume that the good is non-storeable, so anything not consumed in period t is wasted.

Definition 1. A **sequential markets equilibrium** is a list of prices $\{\hat{q}_t\}^\infty_{t=0}$ and allocations $\{\hat{c}^i_t, \hat{a}^i_{t+1}\}^\infty_{t=0}$ such that

(a) given equilibrium prices $\{\hat{q}_t\}^\infty_{t=0}$, the allocation solves

$$\max_{\{c^i_t, a^i_{t+1}\}^\infty_{t=0}} \sum_{t=0}^\infty \beta^t \ln(c^i_t) \quad \text{such that} \quad c^t_i + \hat{q}_t a^i_{t+1} = e^i_t + a^i_t,\ c^i_t \geq 0;$$

(b) for any period t, $\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 = 1$;

(c) for any period t, $\hat{a}_t^1 + \hat{a}_t^2 = 0$;

(d) for any t, $a_{t+1}^i \geq -A$, where $A = (0, \infty)$.

Point (b) means that aggregate consumption in a period cannot exceed the aggregate endowment of that period, which in this case is 1. Point (c) means that the demand agent i has for bonds in period t has to equal agent j's supply of bonds—agent i has to borrow from someone, after all. Point (d) means that agents have a limitation to how much they can actually lend (and borrow) in a period.

1.2 Consumption Equilibrium

Let's start by solving the budget constraint with respect to c_t^i and plug that into the objective function. What we end up with is

$$\max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln\left(e_t^i + a_t^i - q_t a_{t+1}^i\right).$$

Take the first order conditions with respect to a_{t+1}^i to get

$$\frac{\beta^t q_t}{e_t^i + a_t^i - q_t a_{t+1}^i} = \frac{\beta^{t+1}}{e_{t+1}^i + a_{t+1}^i - q_t a_{t+2}^i},$$

which results in

$$q_t[e_{t+1}^i + a_{t+1}^i - q_t a_{t+2}^i] = \beta[e_t^i + a_t^i - q_t a_{t+1}^i].$$

Since this is the first order condition, it must be satisfied at the equilibrium, and therefore

$$q_t[e_{t+1}^i + \hat{a}_{t+1}^i - q_t \hat{a}_{t+2}^i] = \beta[e_t^i + \hat{a}_t^i - q_t \hat{a}_{t+1}^i] \quad \Longrightarrow \quad q_t c_{t+1}^i = \beta c_t^i.$$

Sum this up for both individuals and we have

$$
\begin{aligned}
&q_t\left[e_{t+1}^1 + e_{t+1}^2 + \hat{a}_{t+1}^1 + \hat{a}_{t+1}^2 - q_t(\hat{a}_{t+2}^1 + \hat{a}_{t+2}^2)\right] \\
&= \beta\left[e_t^1 + e_t^2 + \hat{a}_t^1 + \hat{a}_t^2 - q_t(\hat{a}_{t+1}^1 + \hat{a}_{t+1}^2)\right].
\end{aligned}
$$

Okay, we know that endowments in each period add to one; and that demand for bonds sum to zero in each period. So this simplifies quite nicely to

$$\hat{q}_t = \beta \quad \Longrightarrow \quad \hat{c}_{t+1}^i = \hat{c}_t^i = \hat{c}^i. \tag{1}$$

Consumption smoothing! Woo.

Let's look closer at agent 1's budget constraint, $c_i^t + \hat{q}_t a_{t+1}^i = e_t^i + a_t^i$, over T periods. Without loss of generality, suppose that T is even. Then we have

$$
\begin{aligned}
&t = 0, && \hat{c}^1 + \beta\hat{a}_1^1 = 1 \\
&t = 1, && \hat{c}^1 + \beta\hat{a}_2^1 = 0 + \hat{a}_1^1 \\
&t = 2, && \hat{c}^1 + \beta\hat{a}_3^1 = 1 + \hat{a}_2^1 \\
&\vdots && \\
&t = T-1, && \hat{c}^1 + \beta\hat{a}_T^1 = 0 + \hat{a}_{T-1}^1 \\
&t = T, && \hat{c}^1 + \beta\hat{a}_{T+1}^1 = 1 + \hat{a}_T^1.
\end{aligned}
$$

Multiply the tth row by β^t and we have

$$
\begin{aligned}
&t = 0, && \hat{c}^1 + \beta\hat{a}_1^1 = 1 \\
&t = 1, && \beta\hat{c}^1 + \beta^2\hat{a}_2^1 = \beta\hat{a}_1^1 \\
&t = 2, && \beta^2\hat{c}^1 + \beta^3\hat{a}_3^1 = \beta^2 + \beta^2\hat{a}_2^1 \\
&\vdots && \\
&t = T-1, && \beta^{T-1}\hat{c}^1 + \beta^T\hat{a}_T^1 = \beta^{T-1}\hat{a}_{T-1}^1 \\
&t = T, && \beta^T\hat{c}^1 + \beta^{T+1}\hat{a}_{T+1}^1 = \beta^T + \beta^T\hat{a}_T^1.
\end{aligned}
$$

Now let's add everything up. Notice that the column of $\hat{c}^i$ will end up being a geometric series as $T \to \infty$, so that just amounts to $\hat{c}^1/(1-\beta)$. Also notice that we'll have mostly repeats on each side of the equation as far as $\hat{a}$ terms go; indeed, the only remaining term will be $\beta^{T+1}\hat{a}^1_{T+1}$, which goes to zero in the limit anyway since $\beta \in (0,1)$. Then the right-hand side will have $1 + \beta^2 + \beta^4 + \ldots$, which will evaluate to $1/(1-\beta^2)$. So what we have is

$$\frac{\hat{c}^1}{1-\beta} = \frac{1}{(1+\beta)(1-\beta)} \quad \implies \quad \hat{c}^1 = \frac{1}{1+\beta}.$$

And because aggregate consumption in any period must sum to 1, it follows that $\hat{c}^2 = \beta/(1+\beta)$. And hot damn, this is exactly the same result as in the ADE from the previous set of notes.

1.3 Bond Demand Equilibrium

Of course, we still need to solve for the demand for bonds to be thorough. In period $t = 0$, the budget constraint gives

$$\hat{c}^1 + \beta\hat{a}^1_1 = 1 \quad \implies \quad \frac{1}{1+\beta} + \beta\hat{a}^1_1 = 1 \quad \implies \quad \hat{a}^1_1 = \frac{1}{1+\beta}.$$

And therefore $\hat{a}^2_1 = -1/(1+\beta)$. In period $t = 1$, we'll have

$$\hat{c}^1 + \beta\hat{a}^1_2 = \hat{a}^1_1 \quad \implies \quad \frac{1}{1+\beta} + \beta\hat{a}^1_2 = \frac{1}{1+\beta} \quad \implies \quad \hat{a}^1_2 = 0 = \hat{a}^2_2.$$

More generally,

$$\hat{a}^1_t = \begin{cases} 0 & \text{if } t \text{ is even} \\ \dfrac{1}{1+\beta} & \text{if } t \text{ is odd} \end{cases} \qquad \hat{a}^2_t = \begin{cases} 0 & \text{if } t \text{ is even} \\ -\dfrac{1}{1+\beta} & \text{if } t \text{ is odd} \end{cases}$$

I like to write this as a little chain of events to make it clearer to me what's happening.

(a) In period $t = 0$, individual 1 has an endowment, but knows they have no endowment tomorrow.

(b) So they buy a bond from individual 2 that will pay out in period $t = 1$.

(c) This means that individual 2 can actually afford stuff in period $t = 0$ from having sold that bond.

(d) And since individual 1 gets paid the bond value in period $t = 1$, they can actually afford stuff in period $t = 1$.

This chain of events will repeat itself on every even numbered period, and consumption will consequently be smoothed.

2 Stochastic Environment

2.1 Setup

Suppose there now exists a source of uncertainty in each period t captured by the random variable z_t, which can take on values $\{z^1, \ldots, z^N\}$, where N is finite. As an example, we could have the aggregate endowment per period be randomized. We still have infinite horizon, but every period is going to be different. This will complicate shit.

Let h_t be the **history** up to time t,

$$h_t : \{z_0, z_1, \ldots, z_t\},$$

and let H_t be the set of all possible histories up to time t. We are interested in the probability of a history h_t being realized, which we'll denote $\pi(h_t)$.

There are a number of ways in which the history could matter for a random variable. In the i.i.d case, the random variable z_{t+1} does not depend on the history, i.e. on any previous random variables, that is,

$$P(z_{t+1} = z^j | z_0, \ldots, z_t) = P(z_{t+1} = z^j).$$

In a **first-order Markov process**, only the previous realization matters, so

$$P(z_{t+1} = z^j | z_0, \ldots, z_t) = P(z_{t+1} = z^j | z_t).$$

These are the two scenarios we'll mostly be dealing in.

Definition 2. An **Arrow-Debreu equilibrium** is a list of prices $\{\hat{p}_t(h_t)\}_{t=0,h_t\in H_t}^{\infty}$ and allocations $\{\hat{c}_t^i(h_t)\}_{t=0,h_t\in H_t}^{\infty}$, such that

(a) given prices, the equilibrium allocation solves

$$\max u(c^i) = \sum_{t=0}^{\infty} \beta^t \sum_{h_t\in H_t} \pi(h_t) u\big(c_t^i(h_t)\big) \quad \text{such that} \quad c_t^i(h_t) \geq 0 \ \text{ for all } t, h_t;$$

(b) lifetime nominal value of consumption equals nominal wealth, that is,

$$\sum_{t=0}^{\infty} \sum_{h_t\in H_t} \hat{p}_t(h_t)\hat{c}_t^i(h_t) = \sum_{t=0}^{\infty} \sum_{h_t\in H_t} \hat{p}_t(h_t) e_t^i(h_t);$$

(c) for any period t and history h_t, aggregate consumption equals aggregate endowments, that is,

$$\hat{c}_t^1(h_t) + \hat{c}_t^2(h_t) = e_t^1(h_t) + e_t^2(h_t).$$

2.2 Characterizing the ADE

The Lagrangian of the problem is

$$L^i = \sum_{t=0}^{\infty} \sum_{h_t\in H_t} \beta^t \pi(h_t) u\big(c_t^i(h_t)\big) - \lambda^i \left[\sum_{t=0}^{\infty} \sum_{h_t\in H_t} \hat{p}_t(h_t)[\hat{c}_t^i(h_t) - e_t^i(h_t)] \right].$$

We're going to take the first order conditions with respect to $c_t^i(h_t)$ and $c_0^i(h_t)$, where the latter will give us a nice numeraire. What we end up with is, respectively,

$$\beta^t \pi(h_t) u'\left(c_t^i(h_t)\right) = \lambda^i p_t(h_t),$$
$$\pi(h_0) u'\left(c_0^i(h_0)\right) = \lambda^i p_0(h_0).$$

As mentioned, we will normalize $p_0(h_0) = 1$. Then if we divide the two conditions, we get

$$p_t(h_t) = \beta^t \frac{\pi(h_t) u'\big(c_t^i(h_t)\big)}{\pi(h_0) u'\big(c_0^i(h_0)\big)}. \tag{2}$$

To this point we haven't been using any specific functional form for the utility. It turns out that using a **constant relative risk aversion (CRRA)** utility function,

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

will be quite helpful because $u'(c) = c^{-\sigma}$. Using this in equation (2), we get

$$p_t(h_t) = \beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{c_t^i(h_t)}{c_0^i(h_0)}\right]^{-\sigma}, \tag{3}$$

which is true for any i, and therefore

$$\beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{c_t^1(h_t)}{c_0^1(h_0)}\right]^{-\sigma} = \beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{c_t^2(h_t)}{c_0^2(h_0)}\right]^{-\sigma} \quad \Longrightarrow \quad \frac{c_t^1(h_t)}{c_0^1(h_0)} = \frac{c_t^2(h_t)}{c_0^2(h_0)}.$$

Rewriting to have the $t = 0$ terms on the same side of the equation gives

$$\frac{c_t^2(h_t)}{c_t^1(h_t)} = \frac{c_0^2(h_0)}{c_0^1(h_0)} = c. \tag{4}$$

So the ratio of optimal consumption will be the same in any t and after any possible history.

From equation (4), we can write $\hat{c}_t^2 = c\hat{c}_t^1(h_t)$, which when plugged into the market clearing condition gives

$$\hat{c}_t^1(h_t) = \frac{e_t^1(h_t) + e_t^2(h_t)}{1+c}.$$

Okay great, so individual 1 gets share $1/(1+c)$ if the endowment each period, individual 2 gets the remainder. Plug this into equation (3) for individual 1 and we get

$$p_t(h_t) = \beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{c_t^1(h_t)}{c_0^1(h_0)}\right]^{-\sigma} = \beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{e_t^0(h_t) + e_0^2(h_t)}{e_t^1(h_t) + e_t^2(h_t)}\right]^{\sigma}.$$

And there we have it—a solution for the price in any period.

2.3 Sequential Markets Equilibrium

An **Arrow security** is a one-period bond that pays one unit of the consumption good if state $j = \{1, ..., N\}$ occurs. If there exists an Arrow security for any t and any j, then we say that the environment is characterized by **complete** markets.[1] If there are any missing markets, then the equivalence between the SME and the SP problem breaks down.

Let $q_t(h_t, z_{t+1} = z^j)$ denote the price of an Arrow security for the occurrence of state j in period $t+1$ given history h_t. Let $a_{t+1}^i(h_t, z_{t+1} = z^j)$ denote individual i's demand for the corresponding bond. If z^j actually does occur in period $t+1$, then agent i receives one unit of the good in in $t+1$.

Definition 3. A **sequential markets equilibrium** is a list of prices and allocations such that

(a) given prices of assets, the allocation solves the agent's utility maximiza-

[1]More generally, a complete market exhibits perfect information and there is a price for every asset in every possible state of the world.

tion problem

$$\max u(c^i) = \max \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \pi(h_t) u\big(c_t^i(h_t)\big);$$

(b) such that $c_t^i(h_t) \geq 0$ for any t and any h_t;

(c) $a_{t+1}^i(h_t, z_{t+1} = z^j) \geq -A$ for any t and h_t;

(d) $c_t^i(h_t) + \sum_{j=1}^N q_t(h_t, z_{t+1} = z^j) a_{t+1}^i(h_t, z_{t+1} = z^j) = e_t^i(h_t) + a_t^i(h_t)$ for any t and h_t;

(e) $c_t^1(h_t) + c_t^2(h_t) = e_t^1(h_t) + e_t^2(h_t)$ for any t and h_t;

(f) $a_{t+1}^1(h_t, z_{t+1} = z^j) + a_{t+1}^2(h_t, z_{t+1} = z^j) = 0$ for any t, h_t, and $j \in \{1, \ldots, N\}$.

Theorem 1. *If markets are complete, then the SME and the ADE coincide.*

Consequently, we can use the SP problem followed by pricing methods.

3 Competitive Growth Model

The word competitive, of course, implies price taking. Anyway, let's begin by describing the economic environment via one huge ass bullet list.

- Time will be discrete and infinite in horizon.
- There will be three commodities: a consumption good, labor services, and capital services.[2]
- Agents will be normalized to one.
- Agents own capital, which they rent to firms.
- Agents are endowed with one unit of time per period.

[2]Services, not stocks!

- The typical agents starts with x_0 units of capital. (Now we *are* talking about stock.)
- Agents own the firms, which makes them claimants of profits.[3]
- Firms have an unspecified measure, but the specifically doesn't matter, as we'll see.
- Firms own only one thing—technology that allows them to use (k_t, n_t) to produce $F(k_t, n_t)$ units of the consumption good.
- Capital services represents the act of renting machines to firms; capital services is the act of renting machines to firms.
- Capital deteriorates at a rate of $\delta \in (0, 1)$ per period.
- In every period, $y_t = F(k_t, n_t)$ of the good is produced, which will either be invested or consumed, that is,

$$y_t = c_t + i_t. \tag{5}$$

- The law of motion of capital is given by

$$x_{t+1} = (1 - \delta)x_t + i_t. \tag{6}$$

- The representative agent has utility function

$$u\left(\{c_t\}_{t=0}^{\infty}\right) \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{7}$$

- Commodity y_t has price p_t.
- Labor services n_t has a price of w_t.
- Capital services k_t has a price of r_t.

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[3] Although we'll be assuming constant returns to scale so profits will be zero anyway.