ECN 200B—Second Welfare Theorem Proof

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Preliminary Results

Theorem 1 (Separating Hyperplane Theorem)**.** *Suppose sets $Q, Q' \in \mathbb{R}^A$ are disjoint and convex. Then there exists $p \in \mathbb{R}^A \setminus \{0\}$ and $k \in \mathbb{R}$ such that*

(a) *for all $q \in Q$, $q \cdot p \leq k$,*

(b) *for all $q' \in Q'$, $q \cdot p \geq k$.*

Second Fundamental Theorem of Welfare Economics

Theorem 2. *Suppose that $\{I, J, (u^i, w^i, (s^{i,j}), Y^j)$ is a production economy where all u^i are continuous, locally nonsatiated, and quasiconcave, and each set Y^j is convex and satisfies free disposal. Let $(\hat{x}, \hat{y})$ be a Pareto efficient allocation such that for all i, $x^i \gg 0$. Then there exists prices p and nominal incomes $(m^1, \ldots, m^I)$ such that*

(a) $\sum_{i=1}^I m^i = p \cdot \sum w^i + p \cdot \sum_{j=1}^J \hat{y}^j$,

(b) *for all i, $\hat{x}^i$ maximizes $u^i(x)$ subject to $p \cdot x \leq m^i$,*

(c) *for all j, $\hat{y}^j$ maximizes $p \cdot y$ subject to $y \in Y^j$,*

(d) $\sum_{i=1}^I \hat{x}^i = \sum_{i=1}^I w^i + \sum_{i=1}^I \hat{y}^j$.

Point (d) is actually implied by Pareto efficiency, but whatever.

The Setup. Suppose $(\hat{x}, \hat{y})$ is a Pareto efficient allocation such that for all i, $x^i \gg 0$. We'll be thinking of this allocation as one that a social planner wants to implement. For simplicity, we will assume that $I = 2$ and $J = 1$.

Define the set

$$\mathcal{U}^i = \{x^i \mid u^i(x^i) > u^i(\hat{x}^i)\}.$$

So $\mathcal{U}^i$ consists of all bundles that individual i strictly prefers to the Pareto efficient bundle. Define

$$\mathcal{U} = \mathcal{U}^1 + \mathcal{U}^2 = \{x \mid \exists x^1 \in \mathcal{U}^1 \text{ and } \exists x^2 \in \mathcal{U}^2 \text{ satisfying } x^1 + x^2 = x.\}$$

So $\mathcal{U}$ is the set of points that can be written as a sum of one point from $\mathcal{U}^1$ and one point from $\mathcal{U}^2$.

Also define

$$\mathcal{F} = \sum_{i=1}^{I} w^i + Y = \{x \mid \exists y \in Y \text{ satisfying } y + \sum_{i=1}^{I} w^i = x\}.$$

This is the set of all feasible points for the planner; it represents all possible combinations of commodities the economy could potentially have.

Claim 1. *$\mathcal{U}^i$ are convex.* This follows from quasiconcavity of utility functions. For $x, \tilde{x} \in \mathcal{U}^i$, it follows for any $\lambda > 0$ that

$$u^i(\lambda x + [1-\lambda]\tilde{x}) \geq \min\{u^i(x), u^i(\tilde{x})\} > u^i(\hat{x}^i). \qquad //$$

Claim 2. *$\mathcal{U}$ is convex.* It is the sum of convex sets. //

Claim 3. *$\mathcal{F}$ is convex.* This follows because Y is assumed convex. //

Claim 4. $\mathcal{U} \cap \mathcal{F} = \emptyset$. If $x \in \mathcal{U}$ and $x \in \mathcal{F}$, then we can find some $x^1 \in \mathcal{U}^1$ and x^2 such that $u^1(x^1) > u^1(\hat{x}^1)$ and $u^2(x^2) > u^2(\hat{x}^2)$ where $x^1 + x^2 = x$. Furthermore, we can find some $y \in Y$ such that $y + w^1 + w^2 = x$, i.e. is feasible. Because x is both feasible and superior to $\hat{x}$, it follows that $\hat{x}$ cannot be Pareto efficient. By contradiction, the result follows. //

Claim 5. *There exists some $p \in \mathbb{R}^L \setminus \{0\}$ and some $k \in \mathbb{R}$ such that*

(a) *for any $x \in \mathcal{U}$, $p \cdot x \geq k$,*

(b) *for any $x' \in \mathcal{F}$, $p \cdot x' \leq k$.*

This follows because $\mathcal{U}$ and $\mathcal{F}$ are convex and disjoint, and therefore we can apply the separating hyperplane theorem. //

Claim 6. $p \gg 0$. This follows because Y satisfies free disposal. Suppose, for instance, that $p_1 \leq 0$. We're essentially saying that a firm gets paid to absorb as many inputs as possible with no downside whatsoever. So a firm would choose

$y_1 = -\infty$ and profits would blow up. More specifically, $p(w^1 + w^2 + y) \to \infty$, which is in $\mathcal{F}$. And thus, whatever k happens to be in the above claim, there exists some $x' \in \mathcal{F}$ such that $x' > k$, which is a contradiction. //

Claim 7. Since $(\hat{x}, \hat{y})$ is feasible, we know that $\hat{x}^1 + \hat{y}^1 = w^1 + w^2 + \hat{y} \in \mathcal{F}$ and therefore $p \cdot (w^1 + w^2 + \hat{y}) = p \cdot (\hat{x}^1 + \hat{x}^2) \leq k$. //

Claim 8. Suppose that $u^1(x^1) \geq u^1(\hat{x}^1)$ and $u^2(x^2) \geq u^2(\hat{x}^2)$. By the local nonsatiation of preferences, we can find some bundle $x^i(n) \in \mathcal{U}^i$ such that $\|x^i(n) - x^i\| \leq 1/n$ for any $n \in \mathbb{N}$. It follows that $p \cdot [x^1(n) + x^2(n)] \geq k$ for all n. In the limit, we clearly have $x^i(n) \to x^i$. From continuity it follows that that $p \cdot [x^1 + x^2] \geq k$

The bundles $\hat{x}^1$ and $\hat{x}^2$ satisfy the antecedent, so it follows that $p \cdot (\hat{x}^1 + \hat{x}^2) \geq k$. Combined with the previous claim, it follows that $p \cdot [\hat{x}^1 + \hat{x}^2] = k$. //

Okay, enough with the claims—now we can get to the main points in the theorem itself.

(a) Let $m^i = p \cdot \hat{x}^i$ for each i. Then

$$\sum_{i=1}^{I} m^i = \sum_{i=1}^{I} p \cdot \hat{x}^i = p \cdot \sum_{i=1}^{I} \hat{x}^i = p \cdot \left(\sum_{i=1}^{I} w^i + \hat{y} \right).$$

(b) Suppose x^1 satisfies $u^1(x^1) \geq u^1(\hat{x}^1)$. From claim 8, $p \cdot (x^1 + \hat{x}^2) \geq k$. We also know from claim 8 that $p \cdot (\hat{x}^1 + \hat{x}^2) = k$. It follows that $p \cdot x^1 \geq p \cdot \hat{x}^1 = m^1$. So any bundle that gives at least utility $u(\hat{x}^1)$ is at least as expensive as $\hat{x}$, making $\hat{x}^1$ the expenditure minimizer. Now appeal to duality—because preferences are continuous and locally nonsatiated and the price vector is strictly positive, it must be the case that $\hat{x}^1$ maximizes utility subject to $p \cdot x \leq m^1$. The optimality of bundle $\hat{x}^2$ follows similarly.

(c) Fix $y \in Y$. We know that $p \cdot (w^1 + w^2 + y) \leq k$. We also know that $p \cdot (\hat{x}^1 + \hat{x}^2) = p \cdot (w^1 + w^2 + \hat{y}) = k$. It follows that $y \leq \hat{y}$. So part (c) of the claim has been proved, namely, that $\hat{y}$ is the profit maximizer.

SFToWE for an Exchange Economy

Fix a standard exchange economy and suppose $\hat{x}$ is a Pareto efficient allocation such that $\hat{x}^i \gg 0$ for all i. Then there exists $(\hat{w}^1, \ldots, \hat{w}^I) \in \mathbb{R}^{LI}_{++}$ such that

(i) $\sum_{i=1}^{I} \hat{w}^i = \sum_{i=1}^{I} w^i$,

(ii) $\hat{x}$ is a competitive equilibrium allocation of the economy that has $\left(I, (u^i, \hat{w}^i)_{i=1}^I\right)$.

What's happening is that the social planner is changing the endowments of everyone until the individuals' trading outcome would result in the allocation the social planner deems desirable. (More frankly, the social planner could just choose that allocation itself as the endowment.)

We still need to imply the existence of prices, however. The first fundamental theorem of welfare economics always includes prices. The second theorem needs more assumptions since it doesn't involve prices, which is why convexity of preferences is required.