ECN 200D – Week 5 Lecture Notes
Deterministic Equilibria Concepts

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There are a bunch of different types of equilibria. We'll be applying a sort of Arrow-Debreu equilibrium concept and comparing it to the social planner's outcome.

1 Simple Endowment Economy

There are two agents—you can think of them as islands. The endowments, which are in terms of only one good and normalized to 1, are given as

$$e_t^1 = \begin{cases} 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases} \qquad e_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ 1 & \text{if } t \text{ is odd} \end{cases}.$$

You can think of t as being seasons; during some seasons, island 1 can produce but island 2 can't; and one period later that is reversed. The good is non-storable. The allocation is $\{c_t^1, c_t^2\}_{t=0}^{\infty}$, and preferences are given by

$$u^i(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i).$$

1.1 Arrow-Debreu Equilibrium

1.1.1 Introduction

The market opens once. Agents can trade any amount of the good they can by forming contracts, e.g.

> "I, agent 1, promise to pay x units of the good in period 2016 in exchange for y units of the good in 2017."

Definition 1. An **Arrow-Debreu equilibrium (ADE)** is a list of prices and allocations, $\{\hat{p}_t\}_{t=0}^{\infty}$ and $\{\hat{c}_t^1, \hat{c}_t^2\}$, subject to the following set of constraints. Given equilibrium prices, we must have

$$\{\hat{c}_t^i\}_{t=0}^{\infty} \in \arg\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \qquad \textit{(maximized consumption)}$$

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i = \sum_{t=0}^{\infty} \hat{p}_t e_t^i \qquad \textit{(budget constraint)}$$

$$\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 = 1 \qquad \textit{(market clearing)}$$

$$c_t^i \geq 0 \qquad \textit{(nonnegativity)}$$

Recall that in this model we are allowing trade across time. Thus, it has to be the case that the value of *total* consumption is equal to the value of *total* endowment. This is an important feature of any ADE. However, in any given period, no one can buy or sell more than 1 unit of the good because that's all there is in that period; hence the market clearing condition.

1.1.2 Characterization of ADE

Have $\mathscr{L}^i$ be individual i's Lagrangian,

$$\mathscr{L}^i = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) - \lambda^i \left[\sum_{t=0}^{\infty} p_t (c_t^i - e_t^i) \right].$$

That's a hell of a thing. Let's differentiate it with respect to c_t^i and c_{t+1}^i and we get

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t \quad \text{and} \quad \frac{\beta^{t+1}}{c_{t+1}^i} = \lambda^i p_{t+1} \quad \Longrightarrow \quad \beta^t p_t c_t^i = p_{t+1} c_{t+1}^i. \tag{1}$$

Now sum this over both agents and we have

$$\beta^t p_t (c_t^1 + c_t^2) = p_{t+1}(c_{t+1}^1 + c_{t+1}^2).$$

But we know that the sum of consumption in any given period has to sum to one, so we can simplify to

$$\beta^t \hat{p}_t = \hat{p}_{t+1}. \tag{2}$$

Oh hey, that looks really useful. And hell, why not just normal $p_0 = 1$, from which we can write $\hat{p}_t = \beta^t$. So we have the sequence of equilibrium prices. Woo. Plugging this into equation (1), we get

$$\beta^t p_t c_t^i = p_{t+1} c_{t+1}^i \quad \Longrightarrow \quad \beta^t p_t c_t^i = \beta^t p_t c_{t+1}^i \quad \Longrightarrow \quad c_t^i = c_{t+1}^i = \hat{c}^i. \tag{3}$$

Hey, now we know that consumption is irrespective of t. That makes sense—in an infinite horizon scenario where each period is the same as the last, there's no reason to expect consumption to change. Consumption is smoothed over each period instead of jerking around from 0 to 1 each time—this is a consequence of the concave utility functions.

But do not be fooled into thinking that $\hat{c}^i = 1/2$. In fact, the individual who receives 1 on the first period has an advantage because they begin without any discounting; whereas the second person begins with a discount of β.

Indeed, let's look at the budget constraint with respect to individual 1:

$$\begin{aligned}
\sum_{t=0}^{\infty} \hat{p}_t c_t^1 = \sum_{t=0}^{\infty} \hat{p}_t e_t^1 \quad &\implies \quad \sum_{t=0}^{\infty} \beta^t \hat{c}_t^1 = [1 + 0\beta + 1\beta^1 + 0\beta^3 + 1\beta^4 + \ldots] \\
&\implies \quad \frac{\hat{c}_t^1}{1-\beta} = \frac{1}{1-\beta^2} \\
&\implies \quad \frac{\hat{c}_t^1}{1-\beta} = \frac{1}{(1-\beta)(1+\beta)} \\
&\implies \quad \hat{c}_t^1 = \frac{1}{1+\beta}.
\end{aligned}$$

So there's the allocation sequence for individual 1. It follows that

$$\hat{c}_t^2 = 1 - \frac{1}{1+\beta} = \frac{\beta}{1+\beta}.$$

Because $\beta < 1$, you can see that individual 2 gets some fraction of what individual 1 gets.

Notice that as β approaches 1, they approach equality of allocation. This makes sense because the effect of discounting is what leads to the discrepancy in allocation in the first place. But $\beta \neq 0$ in actuality. And of $\beta = 0$, then individual 1 gets everything.

1.1.3 Solution Summary

1. Take first order conditions of Lagrangian.
2. Sum (1) over both agents and simplify.
3. Normalize $p_0 = 1$ to get prices.
4. Plug prices into (1) to get $\hat{c}_i$.
5. Evaluate budget constraint with $\hat{c}^i$ and $\{\hat{p}\}_{t=0}^{\infty}$ as geometric series to get $\hat{c}^i = 1/(1+\beta)$.

So that's the ADE approach—on to the next approach.

1.2 Social Planner Equilibrium

1.2.1 The Problem

The benevolent social planner wants to maximize with respect to $\{c_t^1, c_t^2\}$ the objective function

$$\sum_{t=0}^{\infty} \beta^t \left[\alpha \ln(c_t^1) + (1-\alpha)\ln(c_t^2)\right],$$

where $\alpha \in [0,1]$ is the **Pareto weight**. A higher Pareto weight means the planner believes agent 1's utility is more important relative to agent 2's.

The social planner doesn't give a damn about money or prices, but only allocations. Thus, the only constraints the social planner must adhere to are the nonegativity of consumption and endowment limitations, namely,

$$c_t^1 + c_t^2 = 1. \tag{4}$$

1.2.2 Solving for Allocations

The social planner's Lagrangian function is

$$\mathscr{L}^{SP} = \sum_{t=0}^{\infty} \beta^t \left[\alpha \ln(c_t^1) + (1-\alpha)\ln(c_t^2)\right] - \sum_{t=0}^{\infty} \mu^t (c_t^1 + c_t^2 - 1).$$

With respect to c_t^1 and c_t^2, take the first order conditions to get

$$\frac{\alpha\beta^t}{c_t^1} = \mu^t \quad \text{and} \quad \frac{(1-\alpha)\beta^t}{c_t^2} = \mu^t \quad \Longrightarrow \quad c_t^2 = \frac{1-\alpha}{\alpha} c_1^t.$$

From equation (4), we can deduce that

$$c_t^1 + \frac{1-\alpha}{\alpha} c_1^t = 1 \quad \Longrightarrow \quad \hat{c}^1 = \alpha.$$

Hooray, more consumption smoothing. Similarly, $\hat{c}^2 = 1 - \alpha$. So given α, the **Pareto frontier set** is given by

$$PO(\alpha) = \{c_t^1 = \alpha, c_t^2 = 1 - \alpha\}_{t=0}^{\infty}.$$

That's great and everything, but we still need prices to have a complete description of the equilibrium. You might think that we'll want to pin down α to a specific value in the process. You'd be right.

1.2.3 The Negishi Method

Suppose the social planner has chosen some arbitrary α. Let the **transfer function**, denoted $t^i(\alpha)$, indicate be the amount that the social planner needs to transfer from agent i, evaluated at prices μ (these are *shadow prices* of the social planner problem), so that the agent can afford the allocation given by α. We want to find the value of α that requires no transfers to take place in order for the agents to afford the social planner's allocations—in other words, we want to find the value of α that satisfies the budget constraints.[1]

So define the transfer function explicitly, and set it equal to zero, with

$$t^i(\alpha) := \sum_{t=0}^{\infty} \mu_t(c_t^i(\alpha) - e_t^i) = 0.$$

It can be shown that $\mu_t = \beta^t$. Focusing on individual 1, we then have

$$t^1(\alpha) := \sum_{t=0}^{\infty} \mu_t(\alpha - e_t^1) := 0 \quad \Longrightarrow \quad t^1(\alpha) := \sum_{t=0}^{\infty} \beta^t(\alpha - e_t^1) := 0.$$

From this it follows that

$$\frac{\alpha}{1-\beta} = \frac{1}{1-\beta^2} \quad \Longrightarrow \quad \hat{\alpha} = \frac{1}{1+\beta}.$$

[1] The social planner ignored the budget constraint. This is how we are re-incorporating it.

Great, so $\hat{\alpha}$ is the one value of α such that $PO(\alpha)$ requires zero transfers and thus satisfies the budget constraints. This also coincides with the result from the Arrow-Debreu equilibrium. More importantly, we have the sequence of prices, namely, $\hat{p}_t = \mu_t$.

1.2.4 Solution Summary

1. Take first order conditions of Lagrangian with respect to c_t^1 and c_t^2, then solve for c_t^2.
2. Use the market clearing constraint with (1) to get $\hat{c}^1 = \alpha$.
3. Use the Negishi transfer function

$$t^1(\alpha) := \sum_{t=0}^{\infty} \mu_t \big(c_t^i(\alpha) - e_t^i\big) = 0,$$

   where μ_t are the Lagrange multipliers from (1). Turns out that $\mu_t = \beta^t$ is the price in period t.
4. Plug $\hat{c}^1 = \alpha$ and $\hat{p}_t = \beta^t$ into $t^1(\alpha) = 0$ to get $\hat{a} = 1/(1+\beta) = \hat{c}_i$.

2 Endowments with Assets

2.1 Sequential Markets Equilibrium

2.1.1 Setup

We're going to introduce bonds into the model—or more accurately, assets, but bonds are easier to conceptualize. Suppose there is only one good in this economy. An agent can buy a bond in period t for a price of q_t. One period later, they will receive one unit of the good. Let a_{t+1}^i denote the demand of agent i for bonds that pay in $t+1$.

Endowments will again be given by

$$e_t^1 = \begin{cases} 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases} \qquad e_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ 1 & \text{if } t \text{ is odd} \end{cases}.$$

Let's also continue to assume that the good is non-storeable, so anything not consumed in period t is wasted.

Definition 2. A **sequential markets equilibrium** is a list of prices $\{\hat{q}_t\}_{t=0}^{\infty}$ and allocations $\{\hat{c}_t^i, \hat{a}_{t+1}^i\}_{t=0}^{\infty}$ such that

(a) given equilibrium prices $\{\hat{q}_t\}_{t=0}^{\infty}$, the allocation solves

$$\max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \quad \text{such that} \quad c_i^t + \hat{q}_t a_{t+1}^i = e_t^i + a_t^i,\ c_t^i \geq 0;$$

(b) for any period t, $\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 = 1$;

(c) for any period t, $\hat{a}_t^1 + \hat{a}_t^2 = 0$;

(d) for any t, $a_{t+1}^i \geq -A$, where $A = (0, \infty)$.

Point (b) means that aggregate consumption in a period cannot exceed the aggregate endowment of that period, which in this case is 1. Point (c) means that the demand agent i has for bonds in period t has to equal agent j's supply of bonds—agent i has to borrow from someone, after all. Point (d) means that agents have a limitation to how much they can actually lend (and borrow) in a period.

2.1.2 Consumption Equilibrium

Let's start by solving the budget constraint with respect to c_t^i and plug that into the objective function. What we end up with is

$$\max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln\left(e_t^i + a_t^i - q_t a_{t+1}^i\right).$$

Take the first order conditions with respect to a^i_{t+1} to get

$$\frac{\beta^t q_t}{e^i_t + a^i_t - q_t a^i_{t+1}} = \frac{\beta^{t+1}}{e^i_{t+1} + a^i_{t+1} - q_t a^i_{t+2}},$$

which results in

$$q_t[e^i_{t+1} + a^i_{t+1} - q_t a^i_{t+2}] = \beta[e^i_t + a^i_t - q_t a^i_{t+1}].$$

Since this is the first order condition, it must be satisfied at the equilibrium, and therefore

$$q_t[e^i_{t+1} + \hat{a}^i_{t+1} - q_t \hat{a}^i_{t+2}] = \beta[e^i_t + \hat{a}^i_t - q_t \hat{a}^i_{t+1}] \implies q_t c^i_{t+1} = \beta c^i_t.$$

Sum this up for both individuals and we have

$$\begin{aligned} &q_t \left[e^1_{t+1} + e^2_{t+1} + \hat{a}^1_{t+1} + \hat{a}^2_{t+1} - q_t(\hat{a}^1_{t+2} + \hat{a}^2_{t+2})\right] \\ = &\beta \left[e^1_t + e^2_t + \hat{a}^1_t + \hat{a}^2_t - q_t(\hat{a}^1_{t+1} + \hat{a}^2_{t+1})\right]. \end{aligned}$$

Okay, we know that endowments in each period add to one; and that demand for bonds sum to zero in each period. So this simplifies quite nicely to

$$\hat{q}_t = \beta \implies \hat{c}^i_{t+1} = \hat{c}^i_t = \hat{c}^i. \tag{5}$$

Consumption smoothing! Woo.

Let's look closer at agent 1's budget constraint, $c^t_i + \hat{q}_t a^i_{t+1} = e^i_t + a^i_t$, over

T periods. Without loss of generality, suppose that T is even. Then we have

$$
\begin{aligned}
&t = 0, && \hat{c}^1 + \beta \hat{a}_1^1 = 1 \\
&t = 1, && \hat{c}^1 + \beta \hat{a}_2^1 = 0 + \hat{a}_1^1 \\
&t = 2, && \hat{c}^1 + \beta \hat{a}_3^1 = 1 + \hat{a}_2^1 \\
&\vdots && \\
&t = T-1, && \hat{c}^1 + \beta \hat{a}_T^1 = 0 + \hat{a}_{T-1}^1 \\
&t = T, && \hat{c}^1 + \beta \hat{a}_{T+1}^1 = 1 + \hat{a}_T^1.
\end{aligned}
$$

Multiply the tth row by β^t and we have

$$
\begin{aligned}
&t = 0, && \hat{c}^1 + \beta \hat{a}_1^1 = 1 \\
&t = 1, && \beta \hat{c}^1 + \beta^2 \hat{a}_2^1 = \beta \hat{a}_1^1 \\
&t = 2, && \beta^2 \hat{c}^1 + \beta^3 \hat{a}_3^1 = \beta^2 + \beta^2 \hat{a}_2^1 \\
&\vdots && \\
&t = T-1, && \beta^{T-1} \hat{c}^1 + \beta^T \hat{a}_T^1 = \beta^{T-1} \hat{a}_{T-1}^1 \\
&t = T, && \beta^T \hat{c}^1 + \beta^{T+1} \hat{a}_{T+1}^1 = \beta^T + \beta^T \hat{a}_T^1.
\end{aligned}
$$

Now let's add everything up. Notice that the column of $\hat{c}^i$ will end up being a geometric series as $T \to \infty$, so that just amounts to $\hat{c}^1/(1-\beta)$. Also notice that we'll have mostly repeats on each side of the equation as far as $\hat{a}$ terms go; indeed, the only remaining term will be $\beta^{T+1}\hat{a}_{T+1}^1$, which goes to zero in the limit anyway since $\beta \in (0,1)$. Then the right-hand side will have $1 + \beta^2 + \beta^4 + \ldots$, which will evaluate to $1/(1-\beta^2)$. So what we have is

$$
\frac{\hat{c}^1}{1-\beta} = \frac{1}{(1+\beta)(1-\beta)} \quad \implies \quad \hat{c}^1 = \frac{1}{1+\beta}.
$$

And because aggregate consumption in any period must sum to 1, it follows that $\hat{c}^2 = \beta/(1+\beta)$. And hot damn, this is exactly the same result as in the

ADE from the previous set of notes.

2.1.3 Bond Demand Equilibrium

Of course, we still need to solve for the demand for bonds to be thorough. In period $t = 0$, the budget constraint $c_i^t + \hat{q}_t a_{t+1}^i = e_t^i + a_t^i$ gives

$$\hat{c}^1 + \beta \hat{a}_1^1 = 1 \quad \Longrightarrow \quad \frac{1}{1+\beta} + \beta \hat{a}_1^1 = 1 \quad \Longrightarrow \quad \hat{a}_1^1 = \frac{1}{1+\beta}.$$

And therefore $\hat{a}_1^2 = -1/(1+\beta)$. In period $t = 1$, we'll have

$$\hat{c}^1 + \beta \hat{a}_2^1 = \hat{a}_1^1 \quad \Longrightarrow \quad \frac{1}{1+\beta} + \beta \hat{a}_2^1 = \frac{1}{1+\beta} \quad \Longrightarrow \quad \hat{a}_2^1 = 0 = \hat{a}_2^2.$$

More generally,

$$\hat{a}_t^1 = \begin{cases} 0 & \text{if } t \text{ is even} \\ \dfrac{1}{1+\beta} & \text{if } t \text{ is odd} \end{cases} \qquad \hat{a}_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ -\dfrac{1}{1+\beta} & \text{if } t \text{ is odd} \end{cases}$$

I like to write this as a little chain of events to make it clearer to me what's happening.

(a) In period $t = 0$, individual 1 has an endowment, but knows they have no endowment tomorrow.

(b) So they buy a bond from individual 2 that will pay out in period $t = 1$.

(c) This means that individual 2 can actually afford stuff in period $t = 0$ from having sold that bond.

(d) And since individual 1 gets paid the bond value in period $t = 1$, they can actually afford stuff in period $t = 1$.

This chain of events will repeat itself on every even numbered period, and consumption will consequently be smoothed.

2.1.4 Solution Summary

i. Solve the budget constraint for c_t^i and replace that in the objective function.

ii. Take first order conditions of the Lagrangian with respect to a_{t+1}^i.

iii. Replace c_i^t back into the first order conditions.

iv. Sum the first order condition for both individuals and simplify for $\hat{c}^i$.

v. With individual 1, write down the budget constraint for T periods. For all $t = 1, \ldots T$, multiply the constraint for period t by β^t.

vi. Sum over the constraints and simplify, then solve for $\hat{c}_1 = 1/(1+\beta)$.

vii. Evaluate the budget constraint in period $t = 0$ to get $\hat{a}_1^1 = 1/(1+\beta)$. This will be the demand for all odd-period bonds.

viii. Evaluate the budget constraint in period $t = 1$ to get $\hat{a}_2^1 = 0$. This will be the demand for all even-period bonds.