Exercise 1: Variable Capital Utilization

The following equations are already de-trended when applicable.

- $E_0\left[\sum_{t=0}^{\infty}\beta^t\log(c_t)+\theta\dfrac{1}{1-\eta}L_t^{1-\eta}\right]$ (lifetime utility)
- $c_t+\gamma k_{t+1}=w_tN_t+\big(1-\delta(u_t)\big)k_t+r_t^ku_tk_t+\pi_t$ (household budget constraint)
- $\gamma k_{t+1}=\big(1-\delta(u_t)\big)k_t+i_t$ (law of motion)
- $y_t=Z_t(u_tk_t)^{\alpha}(N_t)^{1-\alpha}$ (production function)

The variable u_t is the degree of capital utilization. Utilization is costly and causes capital to depreciate more quickly. $X_t/X_{t-1}=\gamma$ is the growth rate of labor augmenting technological change. Keep in mind that this is an extended RBC model, so we have to pay attention to aggregate capital as well as the representative agent's capital.

Part A: First-Order Conditions

Household Conditions. The Bellman equation is

$$\begin{aligned}V(\mathbf{k}_t,k_t,Z_t)=\;&\log(c_t)+\frac{\theta}{1-\eta}L_t^{1-\eta}+\beta E[V(\mathbf{k}_{t+1},k_{t+1},Z_{t+1})]\\&-\lambda_t[c_t+\gamma k_{t+1}-w_tN_t-\big(1-\delta(u_t)\big)k_t-r_t^ku_tk_t-\pi_t].\end{aligned}$$

The first order conditions are

$$\begin{aligned}\text{with respect to } c_t\quad&\Longrightarrow\quad\frac{1}{c_t}=\lambda_t\\ \text{with respect to } k_{t+1}\quad&\Longrightarrow\quad\beta E[V_k'(\mathbf{k}_{t+1},k_{t+1},Z_{t+1})]=\frac{\gamma}{c_t}\\ \text{with respect to } u_t\quad&\Longrightarrow\quad\delta'(u_t)k_t=r_t^kk_t\\ \text{with respect to } L_t\quad&\Longrightarrow\quad\theta L_t^{-\eta}=\frac{w_t}{c_t}\end{aligned}$$

The latter is the intratemporal Euler equation. The envelope condition is

$$V_k'(\mathbf{k}_t,k_t,Z_t)=\frac{1}{c_t}[1-\delta(u_t)+r_t^ku_t]\quad\Longrightarrow\quad V_k'(\mathbf{k}_{t+1}k_{t+1},Z_{t+1})=\frac{1}{c_{t+1}}[1-\delta(u_t)+r_{t+1}^ku_{t+1}]$$

So the intertemporal Euler equation is

$$\beta E\left[\frac{1}{c_{t+1}}\big(1-\delta(u_t)+r_{t+1}^ku_{t+1}\big)\right]=\frac{\gamma}{c_t}.$$

Firm's Conditions. Firms want to maximize profit, i.e.

$$Z_t(u_tk_t)^{\alpha}(N_t)^{1-\alpha} - w_tN_t - r_t^k u_t k_t.$$

The first-order conditions are

$$\begin{aligned}
\text{with respect to } N_t &\implies (1-\alpha)Z_t(u_tk_t)^{\alpha}N_t^{-\alpha} = w_t, \\
\text{with respect to } k_t &\implies \alpha Z_t u_t^{\alpha} k_t^{\alpha-1} N_t^{1-\alpha} = r_t^k u_t, \\
\text{with respect to } u_t &\implies \alpha Z_t u_t^{\alpha-1} k_t^{\alpha} N_t^{1-\alpha} = r_t^k k_t.
\end{aligned}$$

We know that $\delta'(u_t)k_t = r_t^k k_t$, and from the firm's conditions it follows that

$$\alpha Z_t u_t^{\alpha-1} k_t^{\alpha} N_t^{1-\alpha} = \delta'(u_t)k_t.$$

Part B: Linearized Output

Linearizing the production function alone gives

$$\hat{y}_t = \hat{Z}_t + \alpha\hat{u}_t + \alpha\hat{k}_t + (1-\alpha)\hat{N}_t. \tag{1}$$

Linearizing the condition we found in Part A gives

$$\alpha Z u^{\alpha-1}k^{\alpha}N^{1-\alpha}[\hat{Z}_t - (1-\alpha)\hat{u}_t + \alpha\hat{k}_t + (1-\alpha)\hat{N}_t] = \delta''(u)uk\hat{u}_t + \delta'(u)k\hat{k}_t.$$

Also notice from the condition in Part A that in the steady state,

$$\alpha Z u^{\alpha-1}k^{\alpha}N^{1-\alpha} = \delta'(u)k.$$

So we can write the latter linearization as

$$\delta'(u)[\hat{Z}_t - (1-\alpha)\hat{u}_t + \alpha\hat{k} + (1-\alpha)\hat{N}_t] = \delta''(u)u\hat{u}_t + \delta'(u)\hat{k}_t.$$

Subtract $\delta'(u)\hat{k}_t$ from both sides and you'll get

$$\delta'(u)[\hat{Z}_t - (1-\alpha)\hat{u}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{N}_t] = \delta''(u)u\hat{u}_t.$$

Get everything with $\hat{u}_t$ on one side:

$$\delta'(u)[\hat{Z}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{N}_t] = [\delta''(u)u + \delta'(u)(1-\alpha)]\hat{u}_t.$$

Divide both sides by $\delta'(u)$ for

$$[\hat{Z}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{N}_t] = \left[\frac{\delta''(u)u}{\delta'(u)} + 1 - \alpha\right]\hat{u}_t.$$

Now just solve for $\hat{u}_t$ and we get

$$\frac{1}{\left[\frac{\delta''(u)u}{\delta'(u)} + 1 - \alpha\right]}[\hat{Z}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{N}_t] = \hat{u}_t.$$

Define ξ to be

$$\xi = \frac{\delta''(u)u}{\delta'(u)}.$$

Now plug our expression for $\hat{u}_t$ into the linearized production function and we get

$$\hat{y}_t = \hat{Z}_t + \alpha\hat{k}_t + (1-\alpha)\hat{N}_t + \frac{\alpha}{1-\alpha+\xi}[\hat{Z}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{N}_t]. \tag{2}$$

Part C: Interpretation

When $\epsilon \to \infty$, it means the production function converges to the standard form with full capital utilization, i.e. when $u_t = 1$. In this case, there is no variable capital utilization. Makes sense—we're limiting out the pertinent feature of this model, so of course it would converge to the standard. On the other hand, having $\epsilon = 0$ maximizes the impact of $\hat{u}_t$ on the production function. **I'm not sure how specific to be about this sort of question.**

Exercise 2: Optimal Borrowing and Saving

A household chooses a sequence of consumption c_t and future asset holdings b_t to maximize their discounted lifetime utility. The budget constraint is

$$c_t + R^{-1}b_{t+1} = y_t + b_t.$$

In words, today's consumption plus today's purchase of bonds must equal today's output plus today's bond payouts.

There is a transversality condition,

$$\lim_{T\to\infty} R^{-T}b_{t+T} = 0,$$

which essentially says that holding bonds is worthless at the end of time; in other words, you'll want to consume everything right before you die.

We will assume that $b_0 = 0$ and $b_{t+1} \geq 0$.

Part A: Optimal Consumption

Suppose that the endowment process is

$$\{y_t\}_{t=0}^{\infty} = \{y_h, y_\ell, y_h, y_\ell, \ldots\},$$

where $y_h > y_\ell$. The Lagrangian for the problem we are trying to solve is

$$\sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \lambda_t [c_t + R^{-1} b_{t+1} - y_t - b_t].$$

Take the first-order condition with respect to c_t, c_{t+1}, and b_{t+1} and you'll get

$$\begin{aligned} \beta^t u'(c_t) &= \lambda_t, \\ \beta^{t+1} u'(c_{t+1}) &= \lambda_{t+1}, \\ \lambda_t R^{-1} &= \lambda_{t+1}. \end{aligned}$$

The three can be combined into

$$u'(c_t) = \beta R u'(c_{t+1}).$$

Since $\beta R = 1$, perfect consumption smoothing is implied. Let's calculate $c_1 = c_2 = \ldots = \bar{c}$. We know that $b_0 = 0$ and $y_0 = y_h$. Furthermore, we know that $\beta = 1/R$. So we can write each period's budget constraint (assuming without loss of generality that T is even) as

$$\begin{array}{ll} t = 0 & \bar{c} + \beta b_1 = y_h, \\ t = 1 & \bar{c} + \beta b_2 = y_\ell + b_1, \\ t = 2 & \bar{c} + \beta b_3 = y_h + b_2, \\ t = 3 & \bar{c} + \beta b_4 = y_\ell + b_3, \\ \vdots & \vdots \\ t = T - 1 & \bar{c} + \beta b_{T-1} = y_\ell + b_T, \\ t = T & \bar{c} + \beta b_T = y_h + b_T - 1. \end{array}$$

Multiply each period by its respective degree of β to get

$$\begin{array}{ll} t=0 & \bar{c}+\beta b_1 = y_h, \\ t=1 & \beta\bar{c}+\beta^2 b_2 = \beta y_\ell + \beta b_1, \\ t=2 & \beta^2\bar{c}+\beta^3 b_3 = \beta^2 y_h + \beta^2 b_2, \\ t=3 & \beta^3\bar{c}+\beta^4 b_4 = \beta^3 y_\ell + \beta^3 b_3, \\ \vdots & \vdots \\ t=T-1 & \beta^{T-1}\bar{c}+\beta^T b_T = \beta^{T-1} y_\ell + \beta^{T-1} b_{T-1}, \\ t=T & \beta^T\bar{c}+\beta^{T+1} b_{T+1} = \beta^T y_h + \beta^T b_T. \end{array}$$

Notice that if you sum everything up, then the b_t terms will all cancel out since there will be one of each on each side of the equality—with the exception of $\beta^{T+1}b_{t+1}$. So you get

$$\sum_{t=0}^{T} \beta^t \bar{c} + \beta^{T+1} b_{T+1} = (1+\beta^2+\beta^4+\ldots)y_h + (\beta+\beta^3+\beta^5+\ldots)y_\ell.$$

If we take T to infinity, geometric series make everything all nice, in particular,

$$\frac{\bar{c}}{1-\beta} = \frac{y_h}{1-\beta^2} + \beta\frac{y_\ell}{1-\beta^2} \quad \Longrightarrow \quad \bar{c} = \frac{y_h+\beta y_\ell}{1+\beta}.$$

Part B: Savings Plan

Consider period $t=0$. We have from the budget constraint

$$\frac{y_h+\beta y_\ell}{1+\beta} + \beta b_1 = y_h \quad \Longrightarrow \quad b_1 = \frac{y_h - y_\ell}{1+\beta} > 0.$$

Now do period $t=1$ for

$$\frac{y_h+\beta y_\ell}{1+\beta} + \beta b_2 = y_\ell + \frac{y_h - y_\ell}{1+\beta} \quad \Longrightarrow \quad b_2 = 0.$$

Because $b_2 = 0$, period $t=2$ will have the exact same structure as period $t=0$, namely,

$$\frac{y_h+\beta y_\ell}{1+\beta} + \beta b_3 = y_h \quad \Longrightarrow \quad b_3 = \frac{y_h - y_\ell}{1+\beta} > 0.$$

And therefore period $t = 3$ will have the exact same structure as period $t = 1$, so $b_4 = 0$. Since $b_t \geq 0$ for all t, the agent never borrows.

Part C: Shifted Income Process

Now suppose that $\{y_t\}_{t=0}^{\infty} = \{y_\ell, y_h, y_\ell, y_h, \ldots\}$. As far as the consumption plan is goes, we just exchange the y_h and y_ℓ and get

$$\bar{c} = \frac{y_\ell + \beta y_h}{1 + \beta}$$

Then using this in the $t = 0$ budget constraint, we again can just exchange the two, giving

$$b_1 = \frac{y_\ell - y_h}{1 + \beta} < 0.$$

But uh, if we have to assume that $b_{t+1} \geq 0$, then the agent can't actually borrow—she is borrowing constrained. In period $t = 0$, then, we have $c_0 = y_\ell$ because $b_1 = 0$.

Exercise 3: Government Spending Shocks

This economy has no capital, so $y_t = N_t$. Output can either be spent on household consumption or on government—guns or butter, baby. So the resource constraint is $c_t + g_t = y_t$.

Households choose c_t, N_t, and b_{t+1} to maximize

$$\sum_{t=0}^{\infty} \beta^t [\log(c_t) + \gamma \log(g_t) + \theta \log(1 - N_t)]$$

subject to a budget constraint of

$$c_t + \frac{b_{t+1}}{R_t} = b_t + w_t N_t - T_t.$$

Consider g_t to be exogenous but with no uncertainty. There are no government bonds issued in equilibrium, i.e. $g_t = T_t$.

Part A: Setup and First-Order Conditions

Firm's Problem. Since there's only labor, firms want to maximize profit

$$N_t - w_t N_t,$$

which has a first-order condition of $w_t = 1$. Okay then.

Household's Problem. The only state variable is bonds. So the Bellman equation can be written as

$$V(b_t) = \log(c_t) + \gamma \log(g_t) + \theta \log(1 - N_t) + \beta V(b_{t+1}) - \lambda_t \left[c_t + \frac{b_{t+1}}{R_t} - b_t - w_t N_t + T_t \right]$$

The choice variables are c_t, N_t, and b_t. The first order conditions are

$$\begin{aligned}
\text{with respect to } c_t: &\quad \frac{1}{c_t} = \lambda_t, \\
\text{with respect to } N_t: &\quad \frac{\theta}{1 - N_t} = \lambda_t w_t, \\
\text{with respect to } b_{t+1}: &\quad \beta V'(b_{t+1}) = \frac{\lambda_t}{R_t}.
\end{aligned}$$

The intratemporal Euler equation is ready to go, since $w_t = 1$:

$$\frac{\theta}{1 - N_t} = \frac{1}{c_t}.$$

Now envelope it up with b_t to get

$$V'(b_t) = \lambda_t = \frac{1}{c_t} \quad \Longrightarrow \quad V'(b_{t+1}) = \frac{1}{c_{t+1}}.$$

Therefore we have an intertemporal Euler equation of

$$\beta R_t \frac{1}{c_{t+1}} = \frac{1}{c_t}.$$

Part B: Recursive Competitive Equilibrium

A **recursive competitive equilibrium** consists of

- a value function $V(b_t)$,
- policy functions $c_t = \phi^c(b_t)$, $N_t = \phi^N(b_t)$, $b_{t+1} = \phi^b(b_t)$,
- prices $\{R_t\}_{t=0}^{\infty}$ and w_t

such that

- households and firms optimize,

- markets clear: $N_t^s = N_t^d = N_t$, $b_t = B_t = 0$, $y_t = c_t$,
- $g_t = T_t$.

Part C: Government Spending Shocks

The labor supply curve is

$$\frac{\theta}{1 - N_t} = \frac{1}{c_t}.$$

The resource constraint is, in equilibrium,

$$c_t + g_t = N_t.$$

So we can rewrite labor supply and output as

$$c_t = \frac{1 - g_t}{1 + \theta},$$
$$y_t = \frac{1 + \theta g_t}{1 + \theta}.$$

It is clear that an increase in g_t will decrease c_t and increase y_t. However, notice that if $\theta < 1$, then the decrease in c_t will exceed the increase in y_t.

The idea is that higher government consumption induces a negative wealth effect on people; this decreases consumption. People are forced to work more to offset the negative income effect, which in turn increases output.

Part D: Government Consumption and Household Utility

The term g_t is separate from all optimizing conditions, so all choices are made independently of g_t. Which is to say, N_t and c_t are the result of an optimization process that does not depend on g_t.