Cobb-Douglas Production Function

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January 3, 2017

We will consider a setting with $L+1$ commodities. The first L commodities serve as inputs and the $L+1$th commodity is the output. Input commodity ℓ has input price w_ℓ, and the output commodity has price p. We consider the general Cobb-Douglas production function

$$f(z) = A \prod_{\ell=1}^{L} z_\ell^{\alpha_\ell},$$

where A represents the total-factor productivity, an exogenous measure of the level of "technology."

The parameters α_ℓ acquire an important interpretation in the context of production. Whenever $\sum_{\ell=1}^{L} \alpha_\ell = 1$, this production function has constant returns to scale, meaning that if inputs double then output will double. On the other hand if $\sum_{\ell=1}^{L} \alpha_\ell < 1$, the production function has decreasing returns to scale; and if $\sum_{\ell=1}^{L} \alpha_\ell > 1$ the production function has increasing returns to scale.

1 Factor Demand and Cost Minimization

The cost minimization problem is

$$\min_{z \geq 0} w_1 z_1 + ... + w_L z_L \quad \text{s.t.} \quad A \prod_{\ell=1}^{L} z_\ell^{\alpha_\ell} \geq q.$$

1.1 Conditional Factor Demands

Solving for the conditional factor demands will allow us to derive the cost function. To find the conditional factor demands, we want to find the vector of inputs that solves

$$\arg\min_{z\geq 0} w_1 z_1 + ... + w_L z_L \quad \text{s.t.} \quad A\prod_{\ell=1}^{L} z_\ell^{\alpha_\ell} \geq q.$$

Notice that we cannot have a corner solution for $q > 0$ because if any $z_\ell = 0$, then production is zero. Therefore, writing out the Lagrangian, we have

$$\mathcal{L}(z,\lambda) = w_1 z_1 + ... + w_L z_L + \lambda\left[q - A\prod_{\ell=1}^{L} z_\ell^{\alpha_\ell}\right].$$

This leads to first order conditions

$$\frac{\partial \mathcal{L}(z,\lambda)}{\partial z_\ell} = w_\ell - \lambda\frac{\alpha_\ell}{z_\ell}f(z) = 0 \quad \text{for all } \ell = 1, ..., L, \tag{1}$$

$$\lambda\left[q - A\prod_{\ell=1}^{L} z_\ell^{\alpha_\ell}\right] = 0, \tag{2}$$

$$\lambda \geq 0, \tag{3}$$

$$z_\ell \geq 0 \quad \text{for all } \ell = 1, ..., L. \tag{4}$$

Notice from equation (1) that $\lambda = 0$ implies $w_\ell = 0$, which violates the assumption that $w \gg 0$. Thus, $\lambda > 0$. Also from equation (1), we can write

$$\frac{w_\ell}{\alpha_\ell}z_\ell = \lambda f(z).$$

In other words, we have the series of equalities

$$\frac{w_1}{\alpha_1}z_1 = \frac{w_2}{\alpha_2}z_2 = ... = \frac{w_L}{\alpha_L}z_L.$$

We can express every z_ℓ in terms of z_1 to get the following system of equations:

$$\begin{aligned} z_1 &= z_1, \\ z_2 &= \frac{w_1}{w_2}\frac{\alpha_2}{\alpha_1}z_1, \\ &\vdots \\ z_L &= \frac{w_1}{w_L}\frac{\alpha_L}{\alpha_1}z_1. \end{aligned}$$

Because $\lambda > 0$, we have from equation (2) that $f(z) = q$, that is,

$$\begin{aligned} f(z) &= Az_1^{\alpha_1} \times z_2^{\alpha_2} \times ... \times x_L^{\alpha_L} \\ &= Az_1^{\alpha_1} \times \left[\frac{w_1}{w_2}\frac{\alpha_2}{\alpha_1} z_1\right]^{\alpha_2} \times ... \times \left[\frac{w_1}{w_L}\frac{\alpha_L}{\alpha_1} z_1\right]^{\alpha_L} \\ &= Az_1^{\alpha_1+\alpha_2+...+\alpha_L} \times \left[\frac{w_1}{w_2}\frac{\alpha_2}{\alpha_1}\right]^{\alpha_2} \times ... \times \left[\frac{w_1}{w_L}\frac{\alpha_L}{\alpha_1}\right]^{\alpha_L} \\ &= q. \end{aligned}$$

For clarity, let $\alpha = \sum_{\ell=1}^{L} \alpha_\ell$. Solving for z_1, we get

$$\begin{aligned} z_1 &= \left(\frac{q}{A} \times \left[\frac{w_1}{w_1}\frac{\alpha_1}{\alpha_1}\right]^{\alpha_1} \times \left[\frac{w_2}{w_1}\frac{\alpha_1}{\alpha_2}\right]^{\alpha_2} \times ... \times \left[\frac{w_L}{w_1}\frac{\alpha_1}{\alpha_L}\right]^{\alpha_L}\right)^{1/\alpha} \\ &= \left[\frac{\alpha_1}{w_1}\right] \left(\frac{q}{A}\left[\frac{w_1}{\alpha_1}\right]^{\alpha_1} \left[\frac{w_2}{\alpha_2}\right]^{\alpha_2} \times ... \times \left[\frac{w_L}{\alpha_L}\right]^{\alpha_L}\right)^{1/\alpha}. \end{aligned}$$

This can be generalized to any z_k:

$$\begin{aligned} z_k(w,q) &= \frac{\alpha_k}{w_k}\left(\frac{q}{A}\left[\frac{w_1}{\alpha_1}\right]^{\alpha_1} \left[\frac{w_2}{\alpha_2}\right]^{\alpha_2} \times ... \times \left[\frac{w_L}{\alpha_L}\right]^{\alpha_L}\right)^{1/\alpha} \\ &= \frac{\alpha_k}{w_k}\left(\frac{q}{A}\prod_{\ell=1}^{L}\left[\frac{w_\ell}{\alpha_\ell}\right]^{\alpha_\ell}\right)^{1/a}. \end{aligned}$$

And so we have the conditional factor demand functions.

1.2 Cost Function

To solve the cost function, we can plug each conditional factor demand into the objective function

$$w_1 z_1 + w_2 z_2 + ... + w_L z_L.$$

This will look more intimidating than it actually is. Let's write it all out to see exactly what's happening.

$$\begin{aligned} w_1 z_1 &= w_1 \left[\frac{\alpha_1}{w_1}\right] \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell}\right]^{\alpha_\ell}\right)^{1/a}, \\ w_2 z_2 &= w_2 \left[\frac{\alpha_2}{w_2}\right] \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell}\right]^{\alpha_\ell}\right)^{1/a}, \\ &\vdots \\ w_L z_L &= w_L \left[\frac{\alpha_L}{w_L}\right] \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell}\right]^{\alpha_\ell}\right)^{1/a}. \end{aligned}$$

Right away we can cancel out w_k from each line k. Furthermore there are many common terms in each equation. Indeed, after the aforementioned cancellations, the only terms that won't be common will be each α_ℓ, which will be summed for α after factoring out like-terms. So the cost function is

$$c(w, q) = \alpha \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell}\right]^{\alpha_\ell}\right)^{1/a}.$$

2 Output Supply Function and Profit Function

2.1 Output Supply Function

Now that we have the cost function, we can find the output supply function by solving

$$\arg\max_{q \geq 0} pq - \alpha \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell}\right]^{\alpha_\ell}\right)^{1/a}.$$

Notice that we must have $\sum_{\ell=1}^{L} \alpha_\ell \leq 1$ to guarantee that the objective function is concave in q, so we are not considering the case of increasing returns to scale. If satisfied, then all we really have to do is take the derivative with respect to q, set it equal to zero, and solve. What we get is

$$p - q^{(1-\alpha)/\alpha} \left(\frac{1}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/a} := 0 \quad \Longrightarrow \quad q = \left(pA \prod_{\ell=1}^{L} \left[\frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} .$$

Thus, the the profit maximizing level of output is

$$q(p, w) = \left(p^\alpha A \prod_{\ell=1}^{L} \left[\frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} .$$

Note that this analysis is not applicable if $\sum_{\ell=1}^{L} \alpha_\ell = 1$ since the exponents for $q(p, w)$ would be undefined. Thus, we are not considering the case of constant returns to scale. Therefore this analysis only holds for decreasing returns to scale, that is, when $\sum_{\ell=1}^{L} \alpha_\ell < 1$.

2.2 Profit Function

To find the profit function, we can plug the output supply function into the revenue minus costs equation to get

$$p \left(p^\alpha A \prod_{\ell=1}^{L} \left[\frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} - \alpha \left[\left(p^\alpha A \prod_{\ell=1}^{L} \left[\frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} \frac{1}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell} \right]^{\alpha_i} \right]^{1/a} ,$$

which, after some messy algebra, can be expressed as

$$\pi(p, w) = (1-\alpha) \left(pA \prod_{\ell=1}^{L} \left[\frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} .$$

2.3 Input Demand Functions

To find the input demand function $z_k(p, w)$, apply Hotelling's lemma to the profit function to obtain

$$z_k(p, w) = -\frac{\partial \pi(p, w)}{\partial w_k} = \frac{\alpha_k}{w_k} \left(pA \prod_{\ell=1}^{L} \left[\frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} .$$