Present and Future Value

- The **future value** tells you what the value of a present variable will be in the future given the growth rate g_x:

  $$(\text{present value of } x) \times (1 + g_x) = \text{future value of } x.$$

  Suppose you can invest \$100 today at annual interest rate 10%. Then the future value (one year from now) of your present \$100 is $\$100 \times (1.10) = \$110$.

- The **present value** goes in the opposite direction: it tells you what the value of a future variable is in today's terms, given the growth rate:

  $$\text{present value of } x = \frac{\text{future value of } x}{1 + g_x}.$$

  This process is called **discounting** and $1/(1 + g_x)$ is the **discount factor**.

  Suppose you will receive \$110 in one year, and the interest rate over that period is 10%. Then the present value of that future \$110 is $\$110/1.10 = \$100$. It's like asking, "how much do I have to invest today, given that the interest rate is 10%, so that I get back \$110 in the future?"

Arbitrage ensures that price equals present value.

Problem 1. A bond has a future value of \$140,000 and an interest rate of 12%. What is the price of this bond?

Answer 1. Use the formula:

$$P(1.12) = \$140,000 \implies P = \frac{\$140,000}{1.12} = \$125,000.$$

Problem 2. A bond has a future value of \$136,800 and a price today of \$120,000. What is the interest rate on this bond?

Answer 2. Use the formula, again:

$$\$120,000(1 + R) = \$136,800 \implies R = \frac{\$136,800}{\$120,000} - 1 \implies R = 14\%.$$

Problem 3. A one-year corporate bond pays out $10,000 next year and is selling for $8,000 today in the bond market. A one-year US treasury discount bond pays $1,325 next year and is selling for $1,250 today. Find the risk premium on the corporate bond.

Answer 3. Remember that risk is an undesirable quality, and accordingly those who issue risky bonds have to offer compensation in the form of a higher rate of return. We call this compensation the risk premium, RP. We can then write $R_{\text{risky}} = R_{\text{safe}} + RP$.

A corporate bond is risky, whereas a US Treasury bond is about as safe as it gets (the US has never defaulted). So we conclude that the safe interest rate

$$\$1,250(1 + R_{\text{safe}}) = \$1,325 \quad \Longrightarrow \quad R_{\text{safe}} = \frac{\$1,325}{\$1,250} - 1 = 6\%.$$

The risky corporate bond has an interest rate of

$$\$8,000(1 + R_{\text{risky}}) = \$10,000 \quad \Longrightarrow \quad R_{\text{risky}} = \frac{\$10,000}{\$8,000} - 1 = 25\%.$$

Therefore the extra compensation offered by the corporate bond for its riskiness is

$$RP = 25\% - 6\% = 19\%.$$

Problem 4. A low-risk bond has a future value of \$140,000 and a price today of \$125,000. What is the future value of a high-risk bond with a risk premium of 5% and a price of \$100,000?

Answer 4. First let's find the risk-free interest rate using

$$\$125,000(1 + R_{\text{safe}}) = \$140,000 \quad \Longrightarrow \quad R_{\text{safe}} = \frac{\$140,000}{\$125,000} - 1 = 12\%.$$

We are told that the risk premium is 5%, so it must be the case that the high-risk bond has an interest rate of $R_{\text{risky}} = 12\% + 5\% = 17\%$. We conclude that the future value of the high-risk bond is

$$\$100,000 \times (1.17) = \$117,000.$$

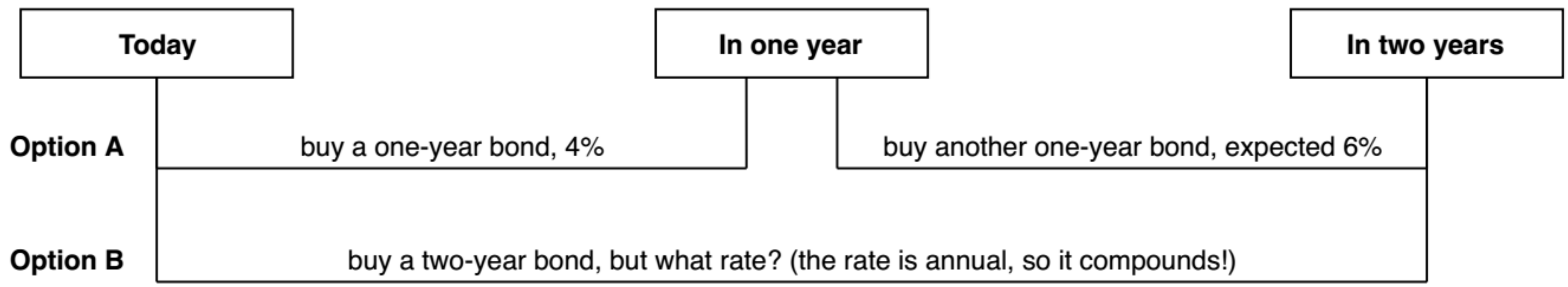
Problem 5. You buy a three-year coupon bond. Each coupon payment is $100, and in the third year you also receive your final payment of $842.38. The interest rate on this coupon bond is 5%. Find the price of the coupon bond.

Answer 5. The coupon bond means in one year you receive $100; in two years you receive $100; in the third and final year you receive another $100 and $842.38 on top of that. To find the price, calculate the present value of each payment and the sum it all up. To find the present value of something two years in the future, you square the discount factor; to find the present value of something t years in the future, you take the discount factor to a power of t.

Years from now	Future Value Payout	Present Value
1	100	$\frac{100}{1.05}$
2	100	$\frac{100}{1.05^2}$
3	942.38	$\frac{942.38}{1.05^3}$

The present values sum to $1,000.

Problem 6. Suppose you are looking at one-year bonds and two-year bonds. You can buy a one-year bond today that will give rate of return 4%. If you buy another one-year bond next year, then you expect it to give rate of return 6%. What is the rate of return on today's two-year bond?



Answer 6. The **expectations hypothesis (EH)** says that the yield on a long-maturity bond is the average of the expected yields on shorter-maturity bonds. In other words, you should expect to do no better by purchasing a sequence of one-year bonds than from just buying the two-year bond.

To illustrate. If you purchased the sequence of one-year bonds, you would expect to get back

$$(1 + 0.04)(1 + 0.06) = 1 + 0.04 + 0.06 + (0.04 \times 0.06)$$

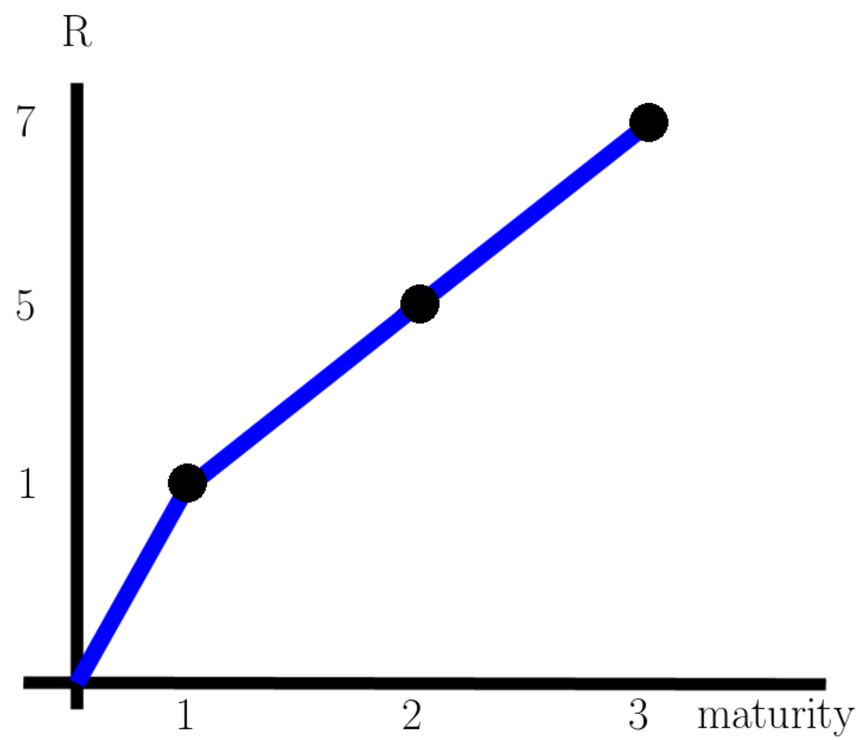
times your initial investment. The product 0.04×0.06 is really small, so let's just get rid of it. So you'll expect to get back approximately 1.10 times your initial investment, a rate of return equaling 10%.

If instead you just buy the two-year bond, then its annual return R_2 must satisfy

$$(1 + R_2)^2 = 1.10 \implies R_2 \approx 5\% = \frac{4\% + 6\%}{2}.$$

So yeah, now you can forget about all of that approximation stuff in your calculations: today's two year rate is the average of today's one-year rate (4%) and next year's expected one year-rate (6%). An analogous result holds for today's three-year rate up to the 30-year rate, the longest offered.

Problem 7. Consider the extremely amateurish-looking yield curve below that I drew in MS Paint, which shows today's annual rates of return on one, two, and three-year bonds.



Find the expected one-year rate of return to be offered a year from now; and the expected one-year rate of return to be offered two years from now.

Answer 7. The EH says the interest rate for a two-year bond purchased today (5%) must be the average of today's one-year rate (1%) and the expected one-year rate of a bond purchased next year (x). That is,

$$5\% = \frac{1\% + x}{2} \implies x = 9\%.$$

Using the same logic, the interest rate for a three-year bond purchased today (7%) must be the average of today's one-year rate (1%), the expected one-year rate of a bond purchased next year (9%), and the expected one-year rate of a bond purchased in two years (x). That is,

$$7\% = \frac{1\% + 9\% + x}{3} \implies x = 11\%.$$

Alternatively, the interest rate for a three-year bond purchased today (7%) must be the average of today's two-year rate (5% twice since it's a two-year bond) and the expected one-year rate of a bond purchased in two years (x). That is,

$$7\% = \frac{5\% + 5\% + x}{3} \implies x = 11\%.$$