Preferences

Definition 1. The preference relation $\succsim$ on X is **rational** if it possesses the following two properties:

- *Completeness.* For all $x, y \in X$, we have $x \succsim y$ or $y \succsim x$ (or both).
- *Transitivity.* For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, the $x \succsim z$.

Definition 2. The preference relation $\succsim$ on X is **monotone** if $x \in X$ and $y \gg x$ implies $y \succ x$. It is **strongly monotone** if $y \geq x$ and $y \neq x$ implies that $y \succ x$.

In essence, monotonicity is the property that all commodities are "goods," or at least not "bads."

Remark 1. If $\succsim$ is monotone, then we allow the possibility of indifference with respect to an increase in the amount of some, but not all, commodities. In contrast, strong monotonicity says that if y is larger than x for *any* commodity and is no less in any other commodity, then y is strictly preferred to x.

Definition 3. The preference relation $\succsim$ on X is **locally nonsatiated** if for every $x \in X$ and every $\epsilon > 0$, there is a $y \in X$ such that $||y - x|| \leq \epsilon$ and $y \succ x$.

Remark 2. In other words, in any small ϵ ball around x, there exists some y that is strictly preferred to x. A thick indifference curve would be an instance where local nonsatiation fails (since we can fit a ball in that indifference curve and thus nothing in that ball is strictly preferred to x).

Proposition 1. *Let $\succsim$ be a preference relation on X.*

(a) *If $\succsim$ is strongly monotone, then it is monotone.*

(b) *If $\succsim$ is monotone, then it is locally nonsatiated.*

Definition 4. The **indifference set** containing point $x \in X$ is the set of all bundles that are indifferent to x, i.e.

$$\{y \in X : y \sim x\}.$$

Definition 5. The **upper contour set** of a bundle x is the set of all bundles that are at least as good as x, i.e.

$$\{y \in X : y \succsim x\}.$$

Definition 6. The **lower contour set** of a bundle x is the set of all bundles that x is at least as good as, i.e.

$$\{y \in X : x \succsim y\}.$$

Definition 7. The preference relation $\succsim$ on X is **convex** if for every $x \in X$, the upper counter set $\{y \in X : y \succsim x\}$ is convex. That is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1-\alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

Remark 3. Convexity is interpreted as diminishing marginal rates of substitution. Which is to say, for every unit loss of x_1 we require increasingly more x_2 to remain indifferent. (This is easiest seen by drawing an indifference curve.)

We can think of convexity in terms of diversification because it implies that $1/2x + 1/2y$ cannot be worse than either x or y.

Definition 8. The preference relation $\succsim$ on X is **strictly convex** if for every $x \in X$, we have that $y \succsim x$, $z \succsim x$, and $y \neq z$ implies $\alpha y + (1 - \alpha)z \succ x$ for all $\alpha \in (0, 1)$.

Definition 9. A monotone preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is **homothetic** if all indifference sets are related by proportional expansion along rays. That is, if $x \sim y$, then $\alpha x \sim \alpha y$ for any $a \geq 0$.

Definition 10. The preference relation $\succsim$ on $X = (-\infty, \infty) \times R^{L-1}_+$ is **quasilinear** with respect to commodity 1 (the **numeraire** commodity) if

(a) all the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.

(b) Good 1 is desirable; that is, $x + \alpha e_1 \succ x$ for all x and all $\alpha > 0$.

Remark 4. Homothetic and quasilinear preferences make it possible to deduce the consumer's entire preference relation from a single indifference set. Note that quasilinear preferences put no lower bound on the consumption of the numeraire commodity.

Preferences and Utility

Definition 11. The **lexicographic preference relation** in $\mathbb{R}^2$ is defined as: $x \succsim y$ if

- $x_1 \succsim y_1$, or
- $x_1 = y_1$ and $x_2 \succsim y_2$.

Remark 5. The lexicographic preference relation is complete, transitive, strongly monotone, and strictly convex—all "nice" properties. But no utility function exists that can represent this preference ordering. Oooooh, spooky.

Definition 12. The preference relation $\succsim$ on X is **continuous** if it is preserved under limits. Which is to say, suppose we have a sequence of pairs $\{x^n, y^n\}_{n=1}^{\infty}$ such that $x^n \succsim y^n$ for all n, where $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$. Then $\succsim$ is continuous if $x \succsim y$.

Proposition 2. *If $u(\cdot)$ is a continuous utility function representing $\succsim$, then $\succsim$ is continuous.*

Remark 6. An equivalent way of stating continuity is to say that for all $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ and the lower contour set $\{y \in X : x \succsim y\}$ are both closed. (Show equivalence by having $x^n = x$ for all n.)

Proposition 3. *Lexicographic preferences are not continuous.*

Proposition 4. *Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function $u(x)$ that represents $\succsim$.*

From here on out, we will assume that utility functions exist and are twice continuously differentiable.

Definition 13. The utility function $u(\cdot)$ is **quasiconcave** if the set $\{y \in \mathbb{R}^L_+ : u(y) \geq u(x)\}$ is convex for all $x \in X$. It is **strictly quasiconcave** if the set $\{y \in \mathbb{R}^L_+ : u(y) > u(x)\}$ is convex for all $x \neq y$.

Remark 7. So if y and z both give at least as much utility as x, then so does any linear combination of y and z:

$$u\big(\alpha y + [1-\alpha]z\big) \geq u(x).$$

We could alternatively express quasiconcavity as holding if $u\big(\alpha x + (1-\alpha)y\big) \geq \min\{u(x), u(y)\}$ for any $\alpha \in (0,1)$.

Proposition 5. *Preferences are (strictly) convex if and only if $u(\cdot)$ is (strictly) quasiconcave.*

But convexity of $\succsim$ does not imply that $u(\cdot)$ is concave.

Proposition 6. *If every upper and lower contour set in $\mathbb{R}^L_+$ is closed, then $\succsim$ is continuous.*

Proposition 7. *Let $\succsim$ be a continuous preference relation. Then*

- ***(a)*** *$\succsim$ on $X = \mathbb{R}^L_+$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one for all $\alpha \geq 0$.*
- ***(b)*** *$\succsim$ on $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$ is quasilinear with respect to the first commodity if and only if it admits a utility function $u(x)$ of the form $u(x) = x_1 + \theta(x_2, ..., x_L)$.*

Proposition 8. *A continuous $\succsim$ is homothetic if and only if it admits a utility function $u(\cdot)$ that is homogeneous of degree one.*

Remark 8. Monotonicity and convexity of $\succsim$ imply that all utility functions representing $\succsim$ are increasing and quasiconcave—these are ordinal properties of the underlying preferences.

Utility Maximization

Henceforth we assume preferences are rational, continuous, locally nonsatiated, and are represented by $u(\cdot)$.

Definition 14. Given prices $p \gg 0$ and wealth $w > 0$, the **utility maximization problem (UMP)** is

$$\begin{aligned} \max_{x \geq 0} \quad & u(x) \\ \text{s.t.} \quad & p \cdot x \leq w. \end{aligned}$$

Proposition 9. *If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.*

This is true because the budget set is closed and bounded, and therefore compact. A continuous function always has a maximum value on any compact set.

Definition 15. The **Walrasian demand correspondence** is a rule that assigns the set of optimal consumption vectors in the utility maximization problem to each price-wealth situation $(p, w) \gg 0$, and is denoted by $x(p, w) \in \mathbb{R}^L_+$. When $x(p, w)$ is single-valued for all (p, w), then we call it the **Walrasian demand function**.

Proposition 10. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:*

- ***(a)*** Homogeneity of degree zero: *$x(\alpha p, \alpha w) = x(p, w)$ for all p, w and scalar $\alpha > 0$.*
- ***(b)*** Walras law*: $p \cdot x = w$ for all $x \in x(p, w)$.*
- ***(c)*** Convexity/uniqueness*: if $\succsim$ is convex so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ is a singleton.*

Proposition 11. *If $x^* \in x(p, w)$ is a solution to the utility maximization problem, then there exists a Lagrange multiplier $\lambda \geq 0$ such that for all $\ell = 1, ..., L$, we have*

$$\nabla u(x^*) \leq \lambda p,$$

and

$$x^* \cdot [\nabla u(x^*) - \lambda p] = 0.$$

Thus, if we are at an interior optimum (where $x^* \gg 0$), then we must have $\lambda u(x^*) = \lambda p$.

The preceding conditions are known as the (necessary) **Kuhn-Tucker** conditions.

Proposition 12. *If $u(\cdot)$ is quasiconcave, monotone, and $\nabla(x) \neq 0$ for all $x \in \mathbb{R}^L_+$, then the first order conditions are sufficient for maxima.*

Definition 16. Suppose $x^* \gg 0$. Then we have

$$\nabla u(x^*) = \lambda p \quad \Longrightarrow \quad \frac{\partial u(x^*)/\partial x_\ell}{\partial u(x^*)/\partial x_k} = \frac{p_\ell}{p_k}.$$

(Solve for λ and then equate them.) The expression with the partials is called the **marginal rate of substitution** of good ℓ for good k at x^*, denoted $MRS_{\ell k}(x^*)$.

It tells us how much good k the consumer must be given to compensate for a one-unit marginal reduction in her consumption of good ℓ. In the $L = 2$ case, this is the slope of the consumer's indifference set at x^*.

Remark 9. Consider the total derivative of utility,

$$du(x^*) = \frac{\partial u(x^*)}{\partial x_1} dx_1 + \frac{\partial u(x^*)}{\partial x_2} dx_2.$$

For fixed level of utility, we set it equal to zero,

$$du(x^*) = \frac{\partial u(x^*)}{\partial x_1} dx_1 + \frac{\partial u(x^*)}{\partial x_2} dx_2 := 0.$$

Now suppose we vary x_1 by dx_1. Then in order for utility to remain unchanged, we must have

$$dx_2 := -\frac{\left[\dfrac{\partial u(x^*)}{\partial x_1}\right]}{\left[\dfrac{\partial u(x^*)}{\partial x_2}\right]} dx_1 = -MRS_{1,2}(x^*)dx_1.$$

Remark 10. Suppose that we have an interior point $x^* \gg 0$ and that

$$\frac{\partial u(x^*)/\partial x_\ell}{\partial u(x^*)/\partial x_k} > \frac{p_\ell}{p_k}.$$

Then an increase in the consumption of good ℓ of size dx_ℓ, combined with a decrease in consumption of good k of $(p\ell/p_k)dx_\ell$ would be feasible (because we are just moving along the budget line), and it would change utility by

$$\frac{\partial u(x^*)}{\partial x_\ell} dx_\ell - \frac{\partial u(x^*)}{\partial x_k} \frac{p_\ell}{p_k} dx_\ell > 0.$$

In other words, when the MRS is equal to the price ratio, increasing either commodity at the margin will cause utility to fall.

Remark 11. In some cases we might have a boundary point where some $x^*_\ell = 0$. In such a case, $\partial u_\ell(x^*)/\partial x_\ell \leq \lambda p_\ell$, whereas $\partial u_k(x^*)/\partial x_k = \lambda p_k$ for those $x^*_k > 0$. We have an inequality in the price ratio with the x_ℓ terms because the consumer is unable to reduce her consumption of good x_ℓ below zero even though a gain in utility would be achieved by doing so.

Remark 12. The Lagrange multiplier λ gives the marginal or *shadow* value of relaxing the constraint in the utility maximization problem. Which is to say, it is the **marginal utility of wealth** at the optimum. In maths,

$$\frac{\partial u(x^*(p,w))}{\partial w} = \frac{\partial u(x^*(p,w))}{\partial x} \frac{\partial x}{\partial w} = \nabla u(x^*) \cdot D_w x^*(p,w).$$

Since at x^* we have $\nabla u(x^*) = \lambda p$, we get

$$\frac{\partial u(x^*(p,w))}{\partial w} = \lambda p \cdot D_w x^*(p,w) = \lambda,$$

where the last equality follows because of Walras' law. (Solve $px = w$ for x and differentiate with respect to w.)

Continuity of Correspondences

Definition 17. Given a subset $D \subseteq \mathbb{R}^n$ and a closed subset $C \subseteq \mathbb{R}^m$, the function $F : D \to \mathcal{P}(C)$ has a **closed graph** if for any sequences $(x^i, y^i)_{i=1}^\infty$ with $x^i \in D$, $y^i \in F(x^i)$, such that $x^i \to x$ and $y^i \to y$, we have $y \in F(x)$.

In words. F maps every x_i to a set, and every y_i is in that mapped set $F(x_i)$. So in the limit, it must be the case that y is in the mapped set $F(x)$ in order for F to have a closed graph.

Now let's look at a generalization of continuity to correspondences.

Definition 18. Given $D \in \mathbb{R}^n$ and closed $C \in \mathbb{R}^m$, the correspondence $F : D \to \mathcal{P}(C)$ is **upper hemicontinuous** if it has a closed graph and the image of any compact set is bounded.

Recall that in $\mathbb{R}^n$, a set is compact if it is closed and bounded. So taking upper hemicontinuous F of any compact set must be bounded and must have a closed graph.

Proposition 13. *If C is compact, then $F : D \to \mathcal{P}(C)$ is upper hemicontinuous if and only if F has a closed graph.*

Proposition 14. *Given $D \subseteq \mathbb{R}^n$ and the closed set $C \in \mathbb{R}^m$, suppose that $F : D \to \mathcal{P}(C)$ is a single-valued correspondence. (Um, a function.) Then F is upper hemicontinuous if and only if it is continuous.*

Definition 19. The Walrasian demand correspondence $x(p,w)$ is upper hemicontinuous at $(\overline{p}, \overline{w})$ whenever $(p^n, w^n) \to (\overline{p}, \overline{w})$ and $x^n \in x(p^n, w^n)$ for all n implies that $x^n \to x \in x(\overline{p}, \overline{w})$.

Note that $x(p,w)$ is bounded (by the budget set) for all $p \gg 0$.

Proposition 15. *Suppose that $u(\cdot)$ is a continuous function representing locally nonsatiated preferences $\succsim$ on the consumption set X^L_+. Then the demand correspondence $x(p,w)$ is upper hemicontinuous for all $(p,w) \gg 0$. Furthermore, if $x(p,w)$ is a function, then it is continuous at all $(p,w) \gg 0$.*

Indirect Utility

Definition 20. For each $(p,w) \gg 0$, the utility value of the utility maximization problem is denoted $v(p,w)$ and it is equal to $u(x^*)$ for any $x^* \in x(p,w)$. The function $v(p,w)$ is called the **indirect utility function**.

Theorem 1 (Berge's Maximum Theorem). *Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be continuous and have $\gamma : \mathbb{R}^m \to P(\mathbb{R}^m)$ be a continuous, non-empty valued correspondence. Define the* *value function $m : \mathbb{R}^m \to \mathbb{R}$ to be*

$$m(x) = \max_{y \in \gamma(x)} f(x, y),$$

and have

$$\mu(x) = \{y \in \gamma(x) : f(x, y) = m(x)\}.$$

Then we have:

(a) *m is continuous*

(b) *μ is non-empty and compact-valued*

(c) *μ is upper hemicontinuous.*

Remark 13. So have $u(x)$ be $f(x, y)$. Have $\gamma(x)$ be the budget set. $\mu(x)$ is the set of feasible x that maximize $u(x)$, so the arg max set. And $m(x)$ is the indirect utility function. Then we can say that

(a) $v(p, w)$ changes continuously in p and w,

(b) The arg max set is nonempty, compact-valued, and continuous.

Proposition 16. *Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the indirect utility function $v(p, w)$ is*

(a) *homogeneous of degree zero,*

(b) *strictly increasing in w and nonincreasing in p_ℓ for any ℓ,*

(c) *quasiconvex: the set $\{(p, w) : v(p, w) \leq \overline{v}\}$ is convex for any $\overline{v}$. (note: quasi**convex**)*

(d) *continuous in p, w.*

Expenditure Minimization

Definition 21. The following program where $p \gg 0$ and $u > u(0)$ is called the **expenditure minimization problem**:

$$\begin{aligned} \min_{x \geq 0} \quad & p \cdot x \\ \text{s.t.} \quad & u(x) \geq u. \end{aligned}$$

Remark 14. We're moving the budget line inwards as far as we possibly can until hitting $u(x) = u$. If we moved it any further inwards we'd have $u(x) < u$, so we stop at $u(x) = u$. By moving the budget line $p \cdot x$ inwards, we are minimizing expenditure.

Proposition 17. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. Then*

(a) *If x^* is optimal in the utility maximization problem when $w > 0$, then x^* is optimal in the expenditure minimization problem when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this expenditure minimization problem is exactly w.*

(b) *If x^* is optimal in the expenditure minimization problem when the required utility level is $u > u(0)$, then x^* is optimal in the utility maximization problem when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this utility maximization problem is exactly u.*

Remark 15. To have a guaranteed solution to the expenditure minimization problem, the constraint set must be nonempty, which is why we require that $u(x) \geq u > u(0)$.

Definition 22. The value function for the expenditure minimization problem, called the **expenditure function**, is $e : \mathbb{R}^L_{++} \times \mathbb{R} \to \mathbb{R}$ defined by

$$e(p, u) = p \cdot x^*,$$

where x^* is a solution to the expenditure minimization problem.

Proposition 18. *Suppose $u(\cdot)$ is a continuous utility function representing locally nonsatiated preferences on $x = R^L_+$. The expenditure function satisfies:*

(a) *homogeneous of degree one in prices p,*

(b) *strictly increasing in $u(\cdot)$,*

(c) *non-decreasing in p_ℓ,*

(d) *concave in p,*

(e) *continuous in p and $u(\cdot)$.*

Remark 16. For $p \gg 0$, $w > 0$, and $u > u(0)$, we have

$$e\big(p, v(p, w)\big) = w, \quad v\big(p, e(p, u)\big) = u.$$

But $v(p, w) = u$ and $e(p, u) = w$. Therefore

$$e\big(p, v(p, w)\big) = e\big(p, u\big) = w,$$

$$v\big(p, e(p, u)\big) = v\big(p, w\big) = u.$$

So in a sense, they are inverses of each other.

Definition 23. The **Hicksian demand correspondence** is defined by

$$\begin{aligned} h(p, u) := \arg\min_{x \in X} \quad & p \cdot x \\ \text{s.t.} \quad & u(x) \geq u. \end{aligned}$$

If it is single-valued, then it is the **Hicksian demand function**.

Remark 17. So basically, the Hicksian demand correspondence is the set of optimal commodity vectors in the expenditure minimization problem.

Proposition 19. *Suppose that $u(\cdot)$ is a continuous function representing locally nonsatiated preferences on $X \in \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ satisfies*

(a) *Homogeneity of degree zero in p:*

$$h(\lambda p, u) = h(p, u) \quad \forall p, u \text{ and } \lambda > 0$$

(b) *No excess utility. For every $x \in h(p,u)$, we have $u(x) = u$.*

(c) *Convex-valued. If $\succsim$ is convex, then $h(p,u)$ is a convex set. If $\succsim$ is strictly convex, then $h(p,u)$ is a singleton and thus we have a Hicksian demand function.*

Proposition 20. *If preferences are convex, then $h(p,u)$ is convex. If $u(x)$ is strictly convex, then $h(p,u)$ is single-valued.*

Proposition 21. *If $u(\cdot)$ is homogeneous of degree one, then $h(p,u)$ and $e(p,u)$ are homogeneous of degree one in u.*

Proposition 22. *Let $u(\cdot)$ be a continuously differentiable function representing locally nonsatiated preferences on $X \in \mathbb{R}^L$. Then for any $p \gg 0$, if x^* is a solution to the expenditure minimization problem with respect to $\overline{u}$, then for some $\lambda > 0$, we have*

$$p \geq \lambda \nabla u(x^*) \quad \textit{and} \quad x^* \cdot [p - \lambda \nabla(x^*)] = 0.$$

Remark 18. We can relate the Hicksian and Walrasian demand correspondences as follows:

$$h(p,u) = x\big(p, e(p,u)\big) \quad \text{and} \quad x(p,w) = h\big(p, v(p,w)\big).$$

The latter equality explains why Hicksian demand is considered to be *compensated* demand. As prices vary, $h(p,u)$ gives precisely the demand that would arise if the consumer's wealth were simultaneously adjusted to keep her utility at level u. Because it features compensated wealth, it satisfies the compensated law of demand: price and demand move in opposite directions.

Proposition 23. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p,u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function $h(p,u)$ satisfies the compensated law of demand: for all p' and p'', we have*

$$(p'' - p') \cdot [h(p'',u) - h(p',u)] \leq 0.$$

Demands and Value Functions

Proposition 24 (Shephard's Lemma). *Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. For all p and u, the Hicksian demand function $h(p,u)$ is the derivative vector of the expenditure function with respect to prices:*

$$h(p,u) = \nabla_p e(p,u),$$

or in terms of each component,

$$h_\ell(p,u) = \partial e(p,u)/\partial p_\ell \quad \textit{for all } \ell = 1, ..., L.$$

Proposition 25. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatisated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that $h(\cdot,u)$ is continuously differentiable at(p,u) and denote its $L \times L$ derivative matrix by $D_p h(p,u)$. Then*

(a) $D_p h(p,u) = D_p^2 e(p,u)$.

(b) $D_p h(p,u)$ *is negative semidefinite.*

(c) $D_p h(p,u)$ *is symmetric.*

(d) $D_p h(p,u) p = 0$.

Definition 24. Two goods ℓ and k are **substitutes** at (p,u) if $\partial h_\ell(p,u)/\partial p_k \geq 0$, and they are **complements** if $\partial h_\ell(p,u)/\partial p_k \leq 0$. (If we use $x(p,u)$ instead of $h(p,u)$, then we say they are **gross substitutes** or **gross complements**.)

Remark 19. Note that $\partial h_\ell(p,u)/\partial p_\ell \leq 0$. From (d) above, it means that there is some p_k such that $\partial h_\ell(p,u)/\partial p_k \leq 0$. For example with $L = 3$ and with respect to commodity 1, (d) would give

$$\frac{\partial h_1(p,u)}{\partial p_1} + \frac{\partial h_1(p,u)}{\partial p_2} + \frac{\partial h_1(p,u)}{\partial p_3} = 0.$$

Then $\partial h_\ell(p,u)/\partial p_\ell \leq 0$ implies that

$$\frac{\partial h_1(p,u)}{\partial p_2} + \frac{\partial h_1(p,u)}{\partial p_3} \geq 0,$$

which means at least one of the above two terms must satisfy ≥ 0, so there exists a substitute for good ℓ.

Proposition 26 (The Slutsky Equation). *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for all (p,w) and $u = v(p,w)$, we have*

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w) \quad \textit{for all } \ell, k.$$

Or in equivalent matrix notation,

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T.$$

Remark 20. If good ℓ is a normal good, then $h\big(p, v(p,\overline{w})\big)$ is relatively inelastic (i.e. relatively vertical) compared to $x_\ell(p,\overline{w})$. This is intuitive. Suppose p_ℓ increases. Then you would expect x_ℓ to fall. But if you compensate wealth, as you do in Hicksian demand, then you would expect x_ℓ to not fall quite as much since it a normal good. So Hicksian demand is relatively price *inelastic* for a normal good.

However, if it is an inferior good, you would expect the opposite. Which is to say, the compensated wealth would make x_ℓ fall even more. So Hicksian demand is relatively price *elastic* for an inferior good.

Remark 21. Notice that the Slutsky equation above is written exactly as the Slutsky matrix from chapter 2. The difference now is that the Slutsky matrix is guaranteed to be symmetric. (But note that Slutsky compensation and Hicksian compensation are not actually equal in general—they are equal only in a differential sense, which we should interpret as just a mathematical artifact.) In any case, utility maximization gives us the nice symmetric substitution matrix that was not guaranteed in a purely choice-based approach.

Proposition 27 (Roy's Identity)**.** *Suppose that $u(\cdot)$ is a continuous utility function representing a locally non-satisated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\overline{p}, \overline{w}) \gg 0$. Then*

$$x(\overline{p}, \overline{w}) = -\frac{1}{\nabla_w v(\overline{p}, \overline{w})} \nabla_p v(\overline{p}, \overline{w}).$$

That is, for every $\ell = 1, ..., L$, we have

$$x_\ell(\overline{p}, \overline{w}) = -\frac{\partial v(\overline{p}, \overline{w})/\partial p_\ell}{\partial v(\overline{p}, \overline{w})/\partial w}.$$

We can find all demand functions and value functions if we know $u(\cdot)$ by using the following algorithms.

(a) If we have $u(\cdot)$, we can solve for $x(p, w)$ by solving UMP.

(b) Then we can plug $x(p, w)$ into $u(\cdot)$ to find $v(p, w)$.

(c) Because $v\big(p, e(p, u)\big) = u$, we can plug $e(p, u)$ into $v(p, w)$ to solve for $e(p, u)$.

(d) Then we can differentiate $e(p, u)$ with respect to prices to find $h(p, u)$.

If we know $v(p, w)$, then we have two intermediate steps:

(a) We can find $u(\cdot)$ by solving

$$\min_p v(p, w) \;\; s.t. \;\; px = w.$$

(b) We can find $x(p, w)$ by using Roy's identity with $v(p, w)$.

(c) We can then find $e(p, u)$ and $h(p, u)$ in the usual way.

Welfare Evaluations[1]

Remark 22. Utility is nice and everything, but it would be useful if we could express utility in money terms. And it turns out we can! For any price $\overline{p}$, we can convert the value function into an expenditure by looking at $e\big((\overline{p}, v(p, w)\big)$. This tells us how much money needs to be spent, when prices are $\overline{p}$, to achieve utility $v(p, w)$. Since $e(p, u)$ is strictly increasing in u, this money-metric satisfies as an indirect utility function.

Definition 25. Consider two prices, p^0 and p^1, with constant level of wealth w. The price-wealth pairs give utility of

$$v(p^0, w) = u^0, \qquad v(p^1, w) = u^1.$$

Suppose we are initially at (p^0, w). Then our money-metric utility is $e(p^0, u^0) = w$. Suppose that the price does not change from p^0 to p^1, but consequently welfare needs to be adjusted as though the price *did* change. The adjustment in utility would be $u^1 - u^0$, or in current prices,

$$EV = e(p^1, u^0) - e(p^0, u^0) = e(p^0, u^1) - w,$$

where EV is the **equivalent variation**. It tells us how much wealth would have to be adjusted to be indifferent to a price change. *"Instead of changing prices to p^1, we're just going to adjust your wealth by EV so you'll be just as well-off as if we had changed prices."*

Definition 26. Now suppose prices *did* change. Then at new prices, your money-metric utility is $e(p^1, u^1)$. Your old welfare, in current prices, was $e(p^1, u^0)$. So your total change in welfare has been

$$CV = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0),$$

where CV is the **compensating variation**. *"Prices have changed to p^1, so we're going to adjust your wealth by CV so that you'll be exactly as well-off as you were before."*

Remark 23. This is a bit confusing in generality (signs change depending on whether it was a price increase or decrease, normal good or inferior good), but it's very straightforward an intuitive with actual numbers.

Remark 24. We can also calculate the equivalent variation as the area to the left of $h_1(p_1, \overline{p}_{-1}, u^1)$ over the interval of change of p_1. That is,

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \overline{p}_{-1}, u^1)\, dp_1.$$

Similarly, compensating variation is the area to the left of $h_1(p_1, \overline{p}_{-1}, u^0)$ over the same interval,

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \overline{p}_{-1}, u^0)\, dp_1.$$

Definition 27. If there are no wealth effects for good ℓ, for instance if it is quasilinear with respect to good ℓ, then we have

$$h_1(p_1, \overline{p}_{-1}, u^0) = x_1(p_1, \overline{p}_{-1}, w) = h_1(p_1, \overline{p}_{-1}, u^1),$$

which implies that $EV = CV$. In such a case, we can use the **area variation measure**,

$$AV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x_1(p_1, \overline{p}_{-1}, w)\, dp_1.$$

Proposition 28. *Suppose that the consumer has a locally nonsatiated rational preference relation $\succsim$. If*

$$(p^1 - p^0) \cdot x^0 < 0,$$

then the consumer is strictly better off under price-wealth situation (p^1, w) than under (p^0, w).

[1] I fucking *hate* this section.

Remark 25. Easy proof—just distribute the x^0 and apply WARP. If there are wealth effects, then AV will have some error. But if the wealth effects are small, then the error is small. Similarly, if $p_1^1 - p_1^0$ is small, then the error is small.

The Strong Axiom (SARP)

Remark 26. There was an example in chapter 2 where consumer choice satisfied the weak axiom but couldn't be generated by a rational preference relation. (It was not transitive. See example 2.F.1.) So then what do we need in order for consumer choice to guarantee a rational preference relation? You guessed it—SARP.

Proposition 29. *The market demand function $x(p, w)$ satisfies the* ***strong axiom of revealed preference*** *if for any list*

$$(p^1, w^1), ..., (p^N, w^N),$$

with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n \leq N-1$, we have $p^N x(p^1, w^1) > w^N$ whenever $p^n x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \leq N-1$.

Remark 27. Okay, that's a lot to take in. An example with $N = 3$ might help. Suppose $p^2 x(p^3, w^3) \leq w^2$. This means that $x(p^2, w^2)$ has been revealed preferred to $x(p^3, w^3)$. Also suppose that $p^1 x(p^2, w^2) \leq w^1$. This means that that $x(p^1, w^1)$ has been revealed preferred to $x(p^2, w^2)$. So for the sake of consistency, we can't have $x(p^1, w^1)$ being affordable at (p^3, w^3) because it would suggest that $x(p^3, w^3)$ is revealed preferred to $x(p^1, w^1)$ and we would break transitivity. Thus, the strong axiom dictates that $p^3 x(p^1, w^1) > w^3$.

Proposition 30. *If the Walrasian demand function $x(p, w)$ satisfies the strong axiom of revealed preference, then there is a rational preference relation $\succsim$ that rationalizes $x(p, w)$. That is, $x(p, w) \succ y$ for every $y \neq x(p, w)$ with $y \in B_{p,w}$, and this holds for all (p, w).*