ECN 200B—Cournot and Bertrand

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1 Cournot

There are a fixed number n of firms who choose to produce output q_i. We have an inverse demand function $P = P(Q)$ where Q is the industry output, i.e. $\sum_{i=1}^{n} q_i = Q$. Production is homogeneous. The cost functions are $C_i(q_i)$. The strategy set is output $S_i = [0, \infty)$ with strategy profile $q = (q_1, \ldots, q_n) \in S$. Therefore firms have profit functions $\pi_i(q) = q_i P(Q) - C_i(q_i)$.

Theorem 1. *Suppose q^* is a Cournot-Nash equilibrium where $q_i^* > 0$ for all i and $\frac{dP}{dQ}(Q^*) < 0$. Then there exists some $\hat{q}$ such that*

- *$\hat{q}_i > q_i^*$ for all i,*
- *$\pi_i(\hat{q}) > \pi_i(q^*)$ for all i.*

Proof. Since q_i^* is a Nash equilibrium strategy, we know that it must be a best response to q_{-i}^*, so $\pi_i(\cdot, q_{-i}^*)$ is maximized at q_i^*. Since we are assuming the strategy profile is interior, this means that

$$\frac{\partial \pi_i}{\partial q_i}(q_i^*, q_{-i}^*) = 0.$$

Now take $\hat{q}$ close to q^*. We can do a first order approximation of the difference between the two:

$$\begin{aligned}
\pi_i(\hat{q}) &\approx \pi_i(q^*) + \sum_{j=1}^{n} \frac{\partial \pi_i}{\partial q_j}(q^*)(\hat{q}_j - q_j^*) \\
\implies \quad \pi_i(\hat{q}) - \pi_i(q^*) &\approx \sum_{j=1}^{n} \frac{\partial \pi_i}{\partial q_j}(q^*)(\hat{q}_j - q_j^*) \\
&\approx \frac{\partial \pi_i}{\partial q_i}(q^*)(\hat{q}_i - q_i^*) + \sum_{j=1, j\neq i}^{n} \frac{\partial \pi_i}{\partial q_j}(q^*)(\hat{q}_j - q_j^*) \\
&\approx \sum_{j=1, j\neq i}^{n} \frac{\partial \pi_i}{\partial q_j}(q^*)(\hat{q}_j - q_j^*).
\end{aligned}$$

The last step follows from the first order condition above.

Differentiating the profit function $\pi_i(q) = q_i P(Q) - C_i(q_i)$ with respect to $j \neq i$, we have

$$\frac{\partial \pi_i}{\partial q_j}(\tilde{q}) = q_i \frac{dP}{dQ}(\tilde{Q}) \frac{\partial Q}{\partial q_j}(\tilde{q}) \quad \forall \tilde{Q}.$$

Evaluating at q^*, we'll get

$$\frac{\partial \pi_i}{\partial q_j}(q^*) = q_i^* \frac{dP}{dQ}(Q^*) \frac{\partial Q}{\partial q_j}(q^*).$$

We know that $q_i^* > 0$ by supposition. We know that $dP/dQ < 0$ by supposition. And since $Q = \sum_{i=1}^{n} q_j$, we know that $\partial Q/\partial q_j = 1$. This means that for $j \neq i$,

$$\frac{\partial \pi_i}{\partial q_j}(q^*) < 0.$$

So if we make sure that $\hat{q} \ll q^*$, then we can conclude that

$$\pi_i(\hat{q}) - \pi_i(q^*) \approx \sum_{j=1, j\neq i}^{n} \frac{\partial \pi_i}{\partial q_j}(q^*)(\hat{q}_j - q_j^*) > 0. \qquad \square$$

Theorem 2. *Suppose $S_i \in \mathbb{R}^m$, $\pi_i : S \to \mathbb{R}$, and there are n players. If*

- *S_i is convex, closed, and bounded,*
- *$\pi_i(s_i, s_{-i})$ is concave in s_i,*
- *π is continuous,*

then a Nash Equilibrium exists.

Does a Cournot-Nash equilibrium always exist? Well, $S_i = [0, \infty)$ is not bounded. Let's impose the existence of some $\bar{Q}$ where $P(\bar{Q}) = 0$. Then firms will never want to produce beyond $\bar{Q}$ because they would receive negative payoff. Therefore $S_i = [0, \bar{Q}]$, so S_i is now convex, closed, and bounded. Woo. Let's further suppose that $P(Q)$ and $C_i(q_i)$ are both continuous so that π_i is continuous.

Concavity is a bit more involved. What we want is

$$\frac{\partial^2 \pi_i}{\partial q_i^2} < 0.$$

So let's differentiate it twice:

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = 2\frac{dP}{dQ} + q_i \frac{d^2 P}{dQ^2} - \frac{d^2 C_i}{dq_i^2}.$$

Let's impose sufficient conditions for concavity.

- Impose $dP/dQ < 0$ for $Q \in [0, \bar{Q}]$. Demand is downward sloping.
- Since $q_i \geq 0$, we'll impose $d^2P/dQ^2 \leq 0$. Demand is straight or concave.
- Impose $d^2C_i/dq_i^2 \geq 0$. Marginal cost is either constant or is increasing.

Recall that demand is, roughly speaking, the derivative of the indirect utility function. So dP/dQ is, in the same rough sense, the second derivative of the indirect utility function. And therefore d^2P/dQ^2 is the third derivative of the indirect utility function—and we usually don't even think about the third derivative.

Example 1. Suppose $P(Q) = a - bQ$ where $a, b > 0$ and $c < a$. The cost functions will be $C_i(q_i) = cq_i$. Therefore the profit function is

$$\pi_i = q_i[a - b(q_1 + \ldots + q_n)] - cq_i.$$

For q^* to be Cournot-Nash, we'll need for every i

$$\frac{\partial \pi_i}{\partial q_i}(q_i^*, q_{-i}^*) = a - bQ - bq_i^* - c = 0.$$

This is a symmetric game, which means that we can write $Q^* = nq_i^*$. It follows that the Cournot-Nash condition above can be expressed as

$$\frac{\partial \pi_i}{\partial q_i}(q_i^*, q_{-i}^*) = a - b(n+1)q_i^* - c = 0.$$

Therefore Cournot-Nash output for the firm will be

$$q_i^* = \frac{a - c}{b(n+1)}.$$

It follows that industry output will be

$$Q^* = \frac{n}{n+1}\frac{a-c}{b},$$

and the market price will be

$$P^* = \frac{a}{n+1} + \frac{n}{n+1}c.$$

As the number of firms $n \to \infty$, price goes to marginal cost c, and therefore profits approach zero. That's exactly what we'd hope to find. ■

2 Bertrand

Bertrand decided that prices are the variable chosen instead of quantity. The product is still homogeneous, and therefore whoever offers the lowest price gets all of the demand. Meaning that the demand function is

$$D_i(p_1, \ldots, p_n) = \begin{cases} 0 & \text{if } \exists j \neq i \text{ such that } p_j < p_i \\ D(p_i) & \text{if } p_i < p_j \text{ for all } j \neq i \\ \frac{1}{m+1} D(p_i) & \text{if } p_i = p_k \text{ for } m \text{ firms, and } p_i \leq p_j \text{ for all } j. \end{cases}$$

So if i charges more than anyone else, they get nothing. If they charge the lowest price uniquely, they get the entire demand. If they and m other firms charge the lowest price, then they split the demand by $m + 1$.

Theorem 3. *If $(p_1^*, \ldots, p_n^*)$ is a Nash Equilibrium, then for at least two firms i and j, $p_i^* = p_j^* = c$; and for any other firm, $p_k^* \geq c$.*

In other words, two or more firms will produce competitive equilibrium output.

Proof. Let $p_i^* = \min\{p_1^*, \ldots, p_n^*\}$. If there are multiple lowest prices, then just take the first index that qualifies. Let $p_j^* = \min\{p_1^*, \ldots, p_n^*\} \setminus \{p_i^*\}$. Then $p_j^* \geq p_i^*$.

We will now show that it must be the case that $p_i^* = c$. Suppose otherwise, first that $p_i^* < c$. If price is less than marginal cost, then $\pi_i < 0$, which isn't going to happen. So suppose that $p_1^* > c$. Then there are two possibilities:

(a) $p_j^* > p_i^*$. In this case, j is getting no demand at all and thus no profit. They could drop their price to $p_j^* = p_i^*$ and take half of i's profit, which is an improvement. So this can't be sustained.

(b) $p_j^* = p_i^* > c$. Then j could reduce their price by ϵ so that $p_i^* > p_j^* - \epsilon > c$; then j would receive all of the profit.

So in equilibrium it must be the case that $p_i^* = c$. It also has to be the case that $p_j^* = c$. We've already concluded that $p_j^* < 0$ will generate zero profits, so that can't be the case. If $p_j^* > c$, then p_i^* could charge ϵ more to increase profit. Therefore we have at least two firms charging at marginal cost. This flies in the face of the Cournot results. $\square$

The problem with the Bertrand model is the extreme discontinuity of the demand function. Turns out that if we use a continuous approximation of the Bertrand world, then $p > MC$ is restored. Hotelling addressed the issue by questioning the extreme homogeneity assumption.

Theorem 4 (Anti-Bertrand Theorem)**.** *Suppose that $n = 2$. Firms pay a cost of $C_i\big(D_i(p_1, p_2)\big)$. Each firm is twice continuously differentiable in their demand functions, and they provide substitute goods, i.e.*

$$\frac{\partial D_i}{\partial p_i} < 0, \quad \frac{\partial D_i}{\partial p_j} > 0.$$

If $(p_1^*, p_2^*) \gg 0$ is a Nash Equilibrium with $D_i(p_1^*, p_2^*) > 0$ for both i, then $p_i^* > MC_i$.

Proof. Since we have interior prices, we know that

$$\frac{\partial \pi_i}{\partial p_i}(p_1^*, p_2^*) = 0.$$

And therefore

$$\begin{aligned} \frac{\partial \pi_i}{\partial p_i}(p_1^*, p_2^*) &= D_i(p_1^*, p_2^*) + p_i^* \frac{\partial D_i}{\partial p_i}(p_1^*, p_2^*) - \frac{\partial C}{\partial q_i}\big(D_i(p_1^*, p_2^*)\big)\frac{\partial D_i}{\partial p_i}(p_1^*, p_2^*) \\ &= D_i(p_1^*, p_2^*) + \frac{\partial D_i}{\partial p_i}(p_1^*, p_2^*)\left[p_i^* - \frac{\partial C}{\partial q_i}\big(D_i(p_1^*, p_2^*)\big)\right] \\ &= 0. \end{aligned}$$

The first term is positive by supposition. The partial is negative. For this to sum to zero, then, we need the term in the brackets to be positive. $\square$

Theorem 5. *Suppose* (p_1^*, p_2^*) *is a Nash Equilibrium with strictly positive prices and demands. Then there exists some* $(\hat{p}_1, \hat{p}_2)$ *such that* $\pi_i(\hat{p}_1, \hat{p}_2) > \pi_i(p_1^*, p_2^*)$ *for both* i.

Proof. Take $(\hat{p}_1, \hat{p}_2)$ close to (p_1^*, p_2^*). Do a first order approximation to get

$$\pi_1(\hat{p}_1, \hat{p}_2) \approx \pi_1(p_1^*, p_2^*) + \frac{\partial \pi_1}{\partial p_1}(p_1^*, p_2^*)(\hat{p}_1 - p_1^*) + \frac{\partial \pi_1}{\partial p_2}(p_1^*, p_2^*)(\hat{p}_2 - p_2^*).$$

From first order interior conditions, we know $\frac{\partial \pi_1}{\partial p_1}(p_1^*, p_2^*) = 0$. It follows that

$$\pi_1(\hat{p}_1, \hat{p}_2) - \pi_1(p_1^*, p_2^*) \approx \frac{\partial \pi_1}{\partial p_2}(p_1^*, p_2^*)(\hat{p}_2 - p_2^*).$$

So let's actually derive this thing on the RHS. Recall that

$$\pi_1(p_1, p_2) = p_1 D_1(p_1, p_2) - C_1\big(D(p_1, p_2)\big).$$

Right, so then

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_2}(p_1^*, p_2^*) &= p_1 \frac{\partial D_1}{\partial p_2}(p_1^*, p_2^*) - \frac{\partial C_1}{\partial q_1}\big(D_1(p_1^*, p_2^*)\big)\frac{\partial D_1}{\partial p_2}(p_1^*, p_2^*) \\ &= \frac{\partial D_1}{\partial p_2}(p_1^*, p_2^*)\left[p_1 - \frac{\partial C_1}{\partial q_1}\big(D(p_1^*, p_2^*)\big)\right]. \end{aligned}$$

Because goods are substitutes, the first factor is positive. The previous theorem showed that price exceeds marginal cost, so this entire term is positive. Therefore if we have $\hat{p} \gg p^*$, then

$$\pi_1(\hat{p}_1, \hat{p}_2) - \pi_1(p_1^*, p_2^*) > 0. \qquad \square$$