ECN 200D: Week 6 Lecture Notes
Stochastic Economies

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1 The Setup

Suppose there now exists a source of uncertainty in each period t captured by the random variable z_t, which can take on values $\{z^1, \ldots, z^N\}$, where N is finite. As an example, we could have the aggregate endowment per period be randomized. We still have infinite horizon, but every period is going to be different. This will complicate shit.

Let h_t be the **history** up to time t,

$$h_t : \{z_0, z_1, \ldots, z_t\},$$

and let H_t be the set of all possible histories up to time t. We are interested in the probability of a history h_t being realized, which we'll denote $\pi(h_t)$.

There are a number of ways in which the history could matter for a random variable. In the i.i.d case, the random variable z_{t+1} does not depend on the history, i.e. on any previous random variables, that is,

$$P(z_{t+1} = z^j | z_0, \ldots, z_t) = P(z_{t+1} = z^j).$$

In a **first-order Markov process**, only the previous realization matters, so

$$P(z_{t+1} = z^j | z_0, \dots, z_t) = P(z_{t+1} = z^j | z_t).$$

These are the two scenarios we'll mostly be dealing in.

2 Arrow-Debreu Equilibrium

Definition 1. An **Arrow-Debreu equilibrium** is a list of prices $\{\hat{p}_t(h_t)\}_{t=0, h_t \in H_t}^{\infty}$ and allocations $\{\hat{c}_t^i(h_t)\}_{t=0, h_t \in H_t}^{\infty}$ that satisfy the following conditions.

(a) Given prices, the equilibrium allocation solves

$$\max u(c^i) = \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \beta^t \pi(h_t) u\big(c_t^i(h_t)\big)$$

such that $c_t^i(h_t) \geq 0$ for all t, h_t.

(b) Lifetime nominal value of consumption equals nominal wealth, that is,

$$\sum_{t=0}^{\infty} \sum_{h_t \in H_t} \hat{p}_t(h_t) \hat{c}_t^i(h_t) = \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \hat{p}_t(h_t) e_t^i(h_t).$$

(c) For any period t and history h_t, aggregate consumption equals aggregate endowments, that is,

$$\hat{c}_t^1(h_t) + \hat{c}_t^2(h_t) = e_t^1(h_t) + e_t^2(h_t).$$

2.1 Characterizing the ADE

The Lagrangian of the problem is

$$\mathscr{L}^i = \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \beta^t \pi(h_t) u\big(c_t^i(h_t)\big) - \lambda^i \left[\sum_{t=0}^{\infty} \sum_{h_t \in H_t} \hat{p}_t(h_t) [\hat{c}_t^i(h_t) - e_t^i(h_t)] \right].$$

We're going to take the first order conditions with respect to $c_t^i(h_t)$ and $c_0^i(h_0)$, where the latter will give us a nice numeraire. What we end up with is, respectively,

$$\begin{aligned}
\beta^t \pi(h_t) u'\left(c_t^i(h_t)\right) &= \lambda^i p_t(h_t), \\
\pi(h_0) u'\left(c_0^i(h_0)\right) &= \lambda^i p_0(h_0).
\end{aligned}$$

As mentioned, we will normalize $p_0(h_0) = 1$. Then if we divide the two conditions, we get

$$p_t(h_t) = \beta^t \frac{\pi(h_t) u'\big(c_t^i(h_t)\big)}{\pi(h_0) u'\big(c_0^i(h_0)\big)}. \tag{1}$$

To this point we haven't been using any specific functional form for the utility. It turns out that using a **constant relative risk aversion (CRRA)** utility function,

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

will be quite helpful because $u'(c) = c^{-\sigma}$. Using this in equation (1), we get

$$p_t(h_t) = \beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{c_t^i(h_t)}{c_0^i(h_0)}\right]^{-\sigma}, \tag{2}$$

which is true for any i, and therefore

$$\beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{c_t^1(h_t)}{c_0^1(h_0)}\right]^{-\sigma} = \beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{c_t^2(h_t)}{c_0^2(h_0)}\right]^{-\sigma} \quad \Longrightarrow \quad \frac{c_t^1(h_t)}{c_0^1(h_0)} = \frac{c_t^2(h_t)}{c_0^2(h_0)}.$$

Rewriting to have the $t = 0$ terms on the same side of the equation gives

$$\frac{c_t^2(h_t)}{c_t^1(h_t)} = \frac{c_0^2(h_0)}{c_0^1(h_0)} = c. \tag{3}$$

So the ratio of optimal consumption will be the same in any t and after any possible history.

From equation (3), we can write $\hat{c}_t^2 = c\hat{c}_t^1(h_t)$, which when plugged into the market clearing condition gives

$$\hat{c}_t^1(h_t) = \frac{e_t^1(h_t) + e_t^2(h_t)}{1+c}.$$

Okay great, so individual 1 gets share $1/(1+c)$ of the endowment each period, individual 2 gets the remainder. Plug this into equation (2) for individual 1 and we get

$$p_t(h_t) = \beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{c_t^1(h_t)}{c_0^1(h_0)}\right]^{-\sigma} = \beta^t \frac{\pi(h_t)}{\pi(h_0)} \left[\frac{e_t^0(h_t) + e_0^2(h_t)}{e_t^1(h_t) + e_t^2(h_t)}\right]^{\sigma}.$$

And there we have it—a solution for the price in any period.

2.2 ADE Solution Summary

i. Take the first order conditions of the Lagrangian with respect to $c_t^1(h_t)$ and $c_0^1(h_0)$.

ii. Normalize $p_0(h_0) = 1$.

iii. Divide the two conditions for $p_t(h_t)$.

iv. Assume $u(c) = c^{1-\sigma}/(1-\sigma)$. Plug derivatives of $u\big(c_t^1(h_t)\big)$ and $u\big(c_0^1(h_0)\big)$ into (iii).

v. Evaluate (iv) for $i = 1, 2$ and set them equal to each other. Solve for ratio of consumption c.

vi. Plug c into resource constraint and solve for $\hat{c}_t^1(h_t) = e_t(h_t)/(1+c)$.

vii. Plug $\hat{c}_t^1(h_t)$ into (iii) to solve for prices.

3 Sequential Markets Equilibrium

An **Arrow security** is a one-period bond that pays one unit of the consumption good if state $j = \{1, ..., N\}$ occurs. If there exists an Arrow security for every t and every j, then we say that the environment is characterized by **complete** markets.[1] If there are any missing markets, then the equivalence between the SME and the SP problem breaks down.

Let $q_t(h_t, z_{t+1} = z^j)$ denote the price of an Arrow security for the occurrence of state j in period $t+1$ given history h_t. Let $a^i_{t+1}(h_t, z_{t+1} = z^j)$ denote individual i's demand for the corresponding bond. If z^j actually does occur in period $t+1$, then agent i receives one unit of the good in in $t+1$.

Definition 2. A **sequential markets equilibrium** is a list of prices and allocations such that

(a) given prices of assets, the allocation solves the agent's utility maximization problem

$$\max u(c^i) = \max \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \beta^t \pi(h_t) u\big(c^i_t(h_t)\big);$$

(b) such that $c^i_t(h_t) \geq 0$ for any t and any h_t;

(c) $a^i_{t+1}(h_t, z_{t+1} = z^j) \geq -A$ for any t and h_t;

(d) $c^i_t(h_t) + \sum_{j=1}^{N} q_t(h_t, z_{t+1} = z^j) a^i_{t+1}(h_t, z_{t+1} = z^j) = e^i_t(h_t) + a^i_t(h_t)$ for any t and h_t;

(e) $c^1_t(h_t) + c^2_t(h_t) = e^1_t(h_t) + e^2_t(h_t)$ for any t and h_t;

(f) $a^1_{t+1}(h_t, z_{t+1} = z^j) + a^2_{t+1}(h_t, z_{t+1} = z^j) = 0$ for any t, h_t, and $j \in \{1, \ldots, N\}$.

Theorem 1. *If markets are complete, then the SME and the ADE coincide.*

Consequently, we can use the SP problem followed by pricing methods.

[1] More generally, a complete market has perfect information and there is a price for every asset in every possible state of the world.