Problem 5

Part i. First, the safe return on a \$50 bond is $\$50 \times 1.20 = \$60$, and therefore this project must give the same return – the usual start to this kind of problem.

The issue we now need to address is which states result in a default and which do not. In other words, in which states can the promised face value actually be paid off in full?

Note that because this is a risky project, its promised face value must be more than the face value of the safe asset. (The bond owner must be compensated for the possibility of a bad outcome by giving them even more in the event of a good outcome.) Thus we can conclude that the promised face value satisfies $x_{sND} > 60$.

Now let's proceed in parts.

- If the bad state occurs and \$50 is generated, this is not enough to pay back $x_{sND} > 60$. Hence it will be a default state and all \$50 will be paid back to the bond owner.
- Now consider the case where \$200 is generated. We have already concluded that $x_{sND} > 60$, but we haven't established anything about how much greater x_{sND} is than 60. In particular, it might be the case that $x_{sND} > 200$ as well – at this point, we just don't know. If it is the case that $x_{sND} > 200$, it would imply that the full promised face value cannot be paid back when only \$200 is generated, and hence it would also be a default state.

  To rule out this possibility, let's assume that it is indeed the case that $x_{sND} > 200$. Under this assumption, the full promised face value cannot be paid back in either the \$200 or \$50 case. In the \$300 state, the bond owner will either be paid back $x_{sND} > 200$ (if it's less than 300) or 300 (if it's greater than 300 and thus there's another default). But in either case, the payoff in the \$300 state will exceed 200. Hence the expected payoff of the bond is

$$\underbrace{\frac{2}{3} \times 200}_{\text{at least}} + \frac{1}{6} \times 200 + \frac{1}{6} \times 50 > 175 > 60. \qquad \lightning$$

  This payoff exceeds the safe return of \$60 and hence cannot occur – it *must* have a payoff of 60 via perfect capital markets and risk-neutral investors. So we have derived an absurdity from the assumption that $x_{sND} > 200$, from which we conclude that the assumption was made in error and in fact $x_{sND} \leq 200$. Thus, the \$200 state is not a default state and x_{sND} can be paid back in full.
- Since $x_{sND} \leq 200$, we conclude that x_{sND} is paid off in full in the \$300 state as well.

Hence we can write the expected payoff of the bond as

$$\frac{2}{3} \times x_{sND} + \frac{1}{6} \times x_{sND} + \frac{1}{6} \times 50 = 60 \quad \Longrightarrow \quad x_{sND} = \$62.$$

And therefore the promised rate of return is $r_p = (62 - 50)/50 = 0.24$.