Stone-Geary Production Function

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We will consider a setting with $L+1$ commodities. The first L commodities serve as inputs and the $L+1$th commodity is the output. Input commodity ℓ has input price w_ℓ, and the output commodity has price p. We consider the general Stone-Geary production function

$$f(z) = A\prod_{\ell=1}^{L}(z_\ell - \gamma_\ell)^{\alpha_\ell},$$

where A represents the total-factor productivity, an exogenous measure of the level of "technology." Notice how similar this is to Cobb-Douglas technology. The key difference are these γ_ℓ terms, which represent a sort of "subsistence level" below which the function is not defined. In other words, it must be the case that $z_\ell \geq \gamma_\ell$. In the context of production, this means that at least γ_ℓ units of good ℓ must be used in production – the interpretation is not as clear in production theory as it is in consumer theory. Perhaps the firm signed a contract for γ_ℓ units of input ℓ but have recently changed their production process so that they no longer need to use input ℓ, and it would be too much of a hassle to try to sell them to someone else.

1 Factor Demand and Cost Minimization

The cost minimization problem is

$$\min_{z\geq 0} w_1 z_1 + ... + w_L z_L \quad \text{s.t.} \quad A\prod_{\ell=1}^{L}(z_\ell - \gamma_\ell)^{\alpha_\ell} \geq q.$$

We will find the factor demand functions first, and then utilize those to find the cost function.

1.1 Conditional Factor Demand Functions

To find the conditional factor demands, we want to solve

$$\arg\min_{z \geq 0} w_1 z_1 + ... + w_L z_L \quad \text{s.t.} \quad A \prod_{\ell=1}^{L} (z_\ell - \gamma_\ell)^{\alpha_\ell} \geq q.$$

Not only is an interior solution where all $z_\ell > 0$ is necessary, but it also has to be the case that all $z_\ell > \gamma_\ell$. Otherwise we'll have some $z_\ell = \gamma_\ell$ and production will be zero, not satisfying the requirement of production of at least q. Thus, the Lagrangian is

$$L(z, \lambda) = w_1 z_1 + ... + w_L z_L + \lambda \left[q - A \prod_{\ell=1}^{L} (z_\ell - \gamma_\ell)^{\alpha_\ell} \right].$$

The first order conditions are

$$\frac{\partial L(z, \lambda)}{\partial z_k} = w_k - \lambda \frac{\alpha_k}{(z_k - \gamma_k)} A \prod_{\ell=1}^{L} (z_\ell - \gamma_\ell)^{\alpha_\ell} := 0, \tag{1}$$

$$\lambda \left[q - A \prod_{\ell=1}^{L} (z_\ell - \gamma_\ell)^{\alpha_\ell} \right] := 0, \tag{2}$$

$$z_\ell > \gamma_\ell, \tag{3}$$

$$\gamma_\ell \geq 0. \tag{4}$$

We can use equation (1) to show that

$$\frac{w_\ell (z_\ell - \gamma_\ell)}{\alpha_\ell} = \lambda f(z),$$

and therefore we have the string of equalities

$$\frac{w_1 (z_1 - \gamma_1)}{\alpha_1} = \frac{w_2 (z_2 - \gamma_2)}{\alpha_2} = ... = \frac{w_L (z_L - \gamma_L)}{\alpha_L}.$$

Solve for $(z_2 - \gamma_2)^{\alpha_2}$ in terms of z_1 to get

$$(z_2 - \gamma_2)^{\alpha_2} = \left(\frac{\alpha_2 w_1}{\alpha_1 w_2} \right)^{\alpha_2} (z_1 - \gamma_1)^{\alpha_2}.$$

Doing this for all z_ℓ, we have the system

$$\begin{aligned}
(z_1 - \gamma_1)^{\alpha_1} &= \left(\frac{w_1 \alpha_1}{\alpha_1 w_1}\right)^{\alpha_1} (z_1 - \gamma_1)^{\alpha_1}, \\
(z_2 - \gamma_2)^{\alpha_2} &= \left(\frac{w_1 \alpha_2}{\alpha_1 w_2}\right)^{\alpha_2} (z_1 - \gamma_1)^{\alpha_2}, \\
&\vdots \\
(z_L - \gamma_L)^{\alpha_L} &= \left(\frac{w_1 \alpha_L}{\alpha_1 w_L}\right)^{\alpha_L} (z_1 - \gamma_1)^{\alpha_L}.
\end{aligned}$$

Notice that from equation (1), we can't have $\lambda = 0$ because otherwise $w_k = 0$ whereas we assume $w \gg 0$. Therefore from equation (2), we must have

$$A \prod_{\ell=1}^{L} (z_\ell - \gamma_\ell)^{\alpha_\ell} = q.$$

Thus, we can write

$$A \left(\frac{w_1 \alpha_1}{\alpha_1 w_1}\right)^{\alpha_1} (z_1 - \gamma_1)^{\alpha_1} \left(\frac{w_1 \alpha_2}{\alpha_1 w_2}\right)^{\alpha_2} (z_1 - \gamma_1)^{\alpha_2} \ldots \left(\frac{w_1 \alpha_L}{\alpha_1 w_L}\right)^{\alpha_L} (z_1 - \gamma_1)^{\alpha_L} = q.$$

Let $\sum_{\ell=1}^{L} \alpha_\ell = \alpha$. We can rewrite the preceding equation as

$$A(z_1 - \gamma_1)^{\alpha} \left(\frac{w_1}{\alpha_1}\right)^{a} \prod_{\ell=1}^{L} \left(\frac{\alpha_\ell}{w_\ell}\right)^{\alpha_\ell} = q.$$

Finally, we can solve for z_1 to get

$$z_1 = \frac{\alpha_1}{w_1} \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell}\right]^{\alpha_\ell} \right)^{1/\alpha} + \gamma_1.$$

Notice that this is exactly the Cobb-Douglas conditional factor demand but with γ_1 added. More generally,

$$z_k = \frac{\alpha_k}{w_k} \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell}\right]^{\alpha_\ell} \right)^{1/\alpha} + \gamma_k.$$

1.2 Cost Function

To find the cost function, we can plug in the conditional factor demands into the objective function. Let's look more closely at the first term,

$$w_1 z_1 = w_1 \left[\frac{\alpha_1}{w_1} \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} + \gamma_1 \right] \quad \Longrightarrow \quad \alpha_1 \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} + w_1 \gamma_1 .$$

When we sum all of them up, we'll have the cost function

$$c(w, q) = \alpha \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} + \sum_{\ell=1}^{L} w_\ell \gamma_\ell .$$

2 Output Supply Function and Profit Function

2.1 Output Supply Function

To find the output supply function, we want to solve

$$\arg\max_{q \geq 0} pq - \alpha \left(\frac{q}{A} \prod_{\ell=1}^{L} \left[\frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} - \sum_{\ell=1}^{L} w_\ell \gamma_\ell .$$

All we need to do is take the derivative with respect to q and find the critical point. Note that we need $\alpha < 1$ to ensure that the function is concave in q and thus the critical point is a maximum. Also notice that since there is no q in the sum, we get the same result as with Cobb-Douglas:

$$q(p, w) = \left(p^\alpha A \prod_{\ell=1}^{L} \left[\frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} .$$

2.2 Profit Function

Now for the profit function, we plug the output supply function into the objective function. Again, it's going to be practically identical to the Cobb-Douglas case except with the sum tacked on at the end:

$$\pi(p, w) = (1 - \alpha) \left(pA \prod_{\ell=1}^{L} \left[\frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} + \sum_{\ell=1}^{L} w_\ell \gamma_\ell .$$

2.3 Input Demand Functions

We can appeal to Cobb-Douglas again. Using Shepherd's lemma, we find the input demand functions to be

$$z_k(p,w) = -\frac{\partial \pi(p,w)}{\partial w_k} = \frac{\alpha_k}{w_k}\left(pA\prod_{\ell=1}^{L}\left[\frac{\alpha_\ell}{w_\ell}\right]^{\alpha_\ell}\right)^{1/(1-\alpha)} + \gamma_k.$$