ECN 200B—Externalities

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March 14, 2017

The Setup

Consider an exchange economy with $I = L = 2$ where $w^i \in \mathbb{R}^2_+$ are the individuals' endowments. The utility function for agent i is separable with respect to the other person's consumption of good 1, that is,

$$\begin{aligned} u^1(x^1, x^2) &= v^1(x^1) + \alpha^1 x_1^2, \\ u^2(x^2, x^1) &= v^2(x^2) + \alpha^2 x_1^1. \end{aligned}$$

So each person cares about the other person's consumption of good 1.

Because each externality term is separable, the marginal utilities and therefore marginal rates of substitution are unaffected by the externality terms—they just drop out when taking the derivatives. The implication is that the externalities will not affect the individuals' behavior, and consequently the competitive equilibrium for this economy is the same as for the economy with $u^i = v^i$, i.e. without the externality terms. But it turns out that efficiency will be affected.

Suppose that $(\bar{p}, \bar{x}^1, \bar{x}^2)$ is a competitive equilibrium for this economy. The first order conditions consist of the usual stuff:

$$\begin{aligned} &Dv^1(\bar{x}^1) = \bar{\lambda}^1 \bar{p}, \quad \bar{p}\bar{x}^1 = \bar{p}w^1, \\ &Dv^2(\bar{x}^2) = \bar{\lambda}^2 \bar{p}, \quad \bar{p}\bar{x}^2 = \bar{p}w^2, \\ &w^1 + w^2 = \bar{x}^1 + \bar{x}^2. \end{aligned}$$

Recall the alternative definition of Pareto efficiency from the problem set: if $(\bar{x}^1, \bar{x}^2)$ is Pareto efficient, then it solves

$$\max_{x_1, x_2} u^1(x^1) \quad \text{s.t.} \quad u^2(x^2) \geq u^2(\bar{x}^2),$$

provided $x_1 + x_2 = w^1 + w^2$.

The Lagrangian for this problem is

$$\begin{aligned} L &= u^1(x^1) - \lambda[u^2(\bar{x}^2) - u^2(x^2)] - \delta_1[x_1^1 + x_1^2 - w_1^1 - w_1^2] - \delta_2[x_2^1 + x_2^2 - w_2^1 - w_2^2] \\ &= v^1(x_1^1, x_2^1) + \alpha^1 x_1^2 - \lambda[v^2(\bar{x}_1^2, \bar{x}_2^2) + \alpha^2 \bar{x}_1^1 - v^2(x_1^2, x_2^2) - \alpha^2 x_1^1] - \dots \end{aligned}$$

So let's take the first order conditions with respect to each x_ℓ^i, evaluating at the maximizer, $\bar{x}$:

$$\begin{aligned} \frac{\partial L}{\partial x_1^1} &= v_{x_1}^1(\bar{x}^1) + \lambda\alpha^2 = \delta_1, \\ \frac{\partial L}{\partial x_2^1} &= v_{x_2^1}^1(\bar{x}^1) = \delta_2, \\ \frac{\partial L}{\partial x_1^2} &= \alpha^1 + \lambda v_{x_1^2}^2(\bar{x}^2) = \delta_1, \\ \frac{\partial L}{\partial x_2^2} &= \lambda v_{x_2^2}^2(\bar{x}^2) = \delta_2. \end{aligned}$$

Equating the multipliers, Pareto efficiency requires

$$v_{x_1^1}^1(\bar{x}^1) + \lambda\alpha^2 = \alpha^1 + \lambda v_{x_1^2}^2(\bar{x}^2), \tag{1}$$

$$\lambda v_{x_2^2}^2(\bar{x}^2) = v_{x_2^1}^1(\bar{x}^1). \tag{2}$$

Let's normalize $\bar{p} = (1, p_2)$. From the competitive equilibrium conditions, we'd have

$$\begin{aligned} Dv^1(\bar{x}^1) = \bar{\lambda}^1 &\implies v_{x_1^1}^1(\bar{x}^1) = \bar{\lambda}^1, \\ Dv^1(\bar{x}^1) = \bar{\lambda}^1 &\implies v_{x_2^1}^1(\bar{x}^1) = \bar{\lambda}^1\bar{p}_2, \\ Dv^2(\bar{x}^2) = \bar{\lambda}^2 &\implies v_{x_1^2}^2(\bar{x}^2) = \bar{\lambda}^2, \\ Dv^2(\bar{x}^2) = \bar{\lambda}^2 &\implies v_{x_2^2}^2(\bar{x}^2) = \bar{\lambda}^2\bar{p}_2. \end{aligned}$$

We can use this to rewrite equation (1) as

$$\bar{\lambda}^1 + \lambda\alpha^2 = \alpha^1 + \lambda\bar{\lambda}^2, \tag{3}$$

and equation (2) as

$$\lambda\bar{\lambda}^2\bar{p}_2 = \bar{\lambda}^1\bar{p}_2 \implies \lambda = \frac{\bar{\lambda}^1}{\bar{\lambda}^2}.$$

Plug this into equation (3) to get

$$\bar{\lambda}^1 + \frac{\bar{\lambda}^1}{\bar{\lambda}^2}\alpha^2 = \alpha^1 + \frac{\bar{\lambda}^1}{\bar{\lambda}^2}\bar{\lambda}^2 \implies \bar{\lambda}^1\alpha^2 = \bar{\lambda}^2\alpha^1.$$

This seems like a pretty damn unlikely situation to actually hold—we need the product of an endogenous variable and an exogenous variable to equal the product of another endogenous variable and exogenous variable.

Return of the Evil Matrix

Recall this monstrosity from one of the homework assignments:

$$F(p, x^1, x^2, \lambda^1, \lambda^2, w^1, w^2) = \begin{bmatrix} Dv^1(x^1) - \lambda^1\overline{p} \\ pw^1 - px^1 \\ Dv^2(x^2) - \lambda^2 p \\ pw^2 - px^2 \\ \tilde{x}^1 + \tilde{x}^2 - \tilde{w}^1 - \tilde{w}^2 \end{bmatrix}. \qquad \begin{pmatrix} \text{2 rows } (L) \\ \text{1 row} \\ \text{2 rows } (L) \\ \text{1 row} \\ \text{1 row} \end{pmatrix}$$

Notice that this is really just a matrix of the first order conditions set equal to zero. This matrix is transverse to zero, i.e. the Jacobian has full rank. The economy is in a competitive equilibrium when $F = 0$.

Now define the function

$$G(p, x^1, x^1, \lambda^1, \lambda^2, w^1, w^2, \alpha^1, \alpha^2) = \begin{bmatrix} F(p, x^1, x^2, \lambda^1, \lambda^2, w^1, w^2) \\ \lambda^1\alpha^2 - \lambda^2\alpha^1 \end{bmatrix}.$$

The economy is in competitive equilibrium and is Pareto efficient when $G = 0$. The Jacobian of G is also transverse to zero. By construction,

$$DG = \begin{bmatrix} DF & 0 & 0 \\ \text{unimportant stuff} & -\lambda^2 & \lambda^1 \end{bmatrix}.$$

Now suppose that $G = 0$. Then it must be the case that $F = 0$, and thus DF has full row rank. Thus, DG has full row rank if λ^2 or λ^1 is nonzero. We can't have $\lambda^1 = 0$ because $Dv^1(x^1) = \lambda^1 p$ and $Dv^1(x^1)$ is assumed to be positive. So DG has full row rank and hence $G \pitchfork 0$.

By the transversality theorem, for any values of $w^1, w^2, \alpha^1, \alpha^2$, we should have

$$G(\,\cdot\,; w^1, w^2, \alpha^1, \alpha^2) \pitchfork 0.$$

So pick some $w^1, w^2, \alpha^1, \alpha^1$ and suppose that

$$G(\bar{p}, \bar{x}^1, \bar{x}^2, \bar{\lambda}^1, \bar{\lambda}^2; w^1, w^2, \alpha^1, \alpha^2) = 0.$$

Then $DG_{\bar{p},\bar{x}^1,\bar{x}^2,\bar{\lambda}^1,\bar{\lambda}^2}$ has full row rank. There is only one price because p_1 has been normalized to 1. There are two arguments in both x^1 and x^2. There are two λ^i. Thus, there are 7 columns in $DG_{\bar{p},\bar{x}^1,\bar{x}^2,\bar{\lambda}^1,\bar{\lambda}^2}$. The big question is how many rows $DG_{\bar{p},\bar{x}^1,\bar{x}^2,\bar{\lambda}^1,\bar{\lambda}^2}$ has. $DF_{\bar{p},\bar{x}^1,\bar{x}^2,\bar{\lambda}^1,\bar{\lambda}^2}$ will have 7 rows and 7 columns, whereas $DG_{\bar{p},\bar{x}^1,\bar{x}^2,\bar{\lambda}^1,\bar{\lambda}^2}$ is 8×7. So $DG_{\bar{p},\bar{x}^1,\bar{x}^2,\bar{\lambda}^1,\bar{\lambda}^2}$ can't have full row rank.

The implication is that if $G = 0$, then its Jacobian must have full rank because it is transverse to zero, but it *can't* have full rank. So for almost all values of $(w^1, w^2, \alpha^1, \alpha^2)$, $G = 0$ is impossible.

So if $F = 0$, then it must be the case that $\bar{\lambda}^1\alpha^2 - \bar{\lambda}^2\alpha^1 \neq 0$. But this equality is a necessary condition for Pareto efficiency. So, finally, for almost all $w^1, w^2, \alpha^1, \alpha^2$, we do not have Pareto efficiency. Thus, almost all competitive equilibria with externalities are Pareto inefficient.