ECN 200D – Week 7 Lecture Notes

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1 Competitive Growth Model

The word competitive, of course, implies price taking. Anyway, let's begin by describing the economic environment via one huge ass bullet list.

- Time will be discrete, infinite in horizon.
- There will be three commodities: a consumption good, labor services, and capital services.[1]
- Agents will be normalized to one.
- Agents own capital, which they rent to firms.
- Agents are endowed with one unit of time per period.
- The typical agent starts with x_0 units of capital. (Now we *are* talking about stock.)
- Agents own the firms, which makes them claimants of profits.[2]
- Firms have an unspecified measure, but the specifically doesn't matter, as we'll see.
- Firms own only one thing—technology that allows them to use (k_t, n_t) to produce $F(k_t, n_t)$ units of the consumption good.

[1]Services, not stocks!

[2]Although we'll be assuming constant returns to scale so profits will be zero anyway.

- Capital services represents the act of renting machines to firms; labor services is the act of supplying labor to firms.
- Capital deteriorates at a rate of $\delta \in (0, 1)$ per period.
- In every period, $y_t = F(k_t, n_t)$ of the good is produced, which will either be invested or consumed, that is,

$$y_t = c_t + i_t. \tag{1}$$

- The law of motion of capital is given by

$$x_{t+1} = (1 - \delta)x_t + i_t. \tag{2}$$

- The representative agent has utility function

$$u\left(\{c_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{3}$$

- Commodity y_t has price p_t.
- Labor services n_t has a price of w_t, which is denominated in terms of p_t. (Explained in more detail later.)
- Capital services k_t has a price of r_t, which is denominated in terms of p_t. (Also explained in more detail later.)
- The capital stock is x_t.

2 Arrow-Debreu Environment

2.1 Assumptions

(a) We will assume that $u(\cdot)$ is continuously differentiable, strictly increasing, strictly concave, and bounded. (Note that boundedness is sufficient but not necessary for a solution to exist. Log utilities, for instance, are not bounded but will have a solution.) Furthermore we will assume that

$$\lim_{c\to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c\to\infty} u'(c) = 0.$$

These conditions, sans boundedness, constitute the **Inada conditions**.

(b) We will assume that F is continuously differentiable, strictly increasing in both arguments, is concave, and exhibits constant returns to scale. Furthermore, $F(0, n) = F(k, 0) = 0$, and in the limit we assume that

$$\lim_{k=0} F_k(k, 1) = \infty, \quad \text{and} \quad \lim_{k\to\infty} F_k(k, 1) = 0.$$

Finally, define $f(k) = F(k, 1) + (1 - \delta)k$, which represents the available capital—the amount produced plus whatever is left after deprecation takes its toll. Or rather, it *will* represent that once we establish that $k = x$. (Also recall that the final good is a consumption but is also capital.)

2.2 The Firm's Problem

Suppose that $\tilde{r}_t$ and $\tilde{w}_t$ denote the price of capital services and labor services, respectively. Then the firm's profit function in period t would be

$$p_t y_t - \tilde{r}_t k_t - \tilde{w}_t n_t.$$

Now suppose we choose to denominate the latter two prices in terms of p_t, giving $r_t = \tilde{r}_t / p_t$ and $w_t = \tilde{w}_t / p_t$. We can then write the profit function as

$$p_t y_t - p_t r_t k_t - p_t w_t n_t = p_t (y_t - r_t k_t - w_t n_t).$$

So given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, firms want to solve

$$\begin{aligned} \max_{y_t, n_t, k_t} \sum_{t=0}^{\infty} & p_t[y_t - r_t k_t - w_t n_t] \\ \text{subject to} \quad & y_t = F(k_t, n_t), \\ & y_t, k_t, n_t \geq 0. \end{aligned}$$

This is just a bunch of static decisions, and as we have seen in mini-macro, the best decision for one period will be the best decision for any period because the periods all have identical structure and time is infinite.

2.3 The Household's Problem

Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, households solve

$$\begin{aligned} & \max_{c_t, i_t, x_{t+1}, k_t, n_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to} \quad \sum_{t=0}^{\infty} p_t(c_t + i_t) = \sum_{t=0}^{\infty} & p_t(r_t k_t + w_t n_t) + \pi, \\ & c_t, x_{t+1} \geq 0, \\ k_t \in [0, x_t],\ n_t \in [0, 1],\ & x_0 \text{ given}, \\ x_{t+1} = & (1-\delta)x_t + i_t. \end{aligned}$$

Right, so that's a hell of a lot of stuff. Households need to decide the right amount to consume and invest this period, how much capital stock to have next period, and how many capital and labor services to employ this period.

The first constraint means that the amount households spend on the final good—remember that $y_t = c_t + i_t$—must equal their income. Since households ultimately own the firms, their income includes profits.

2.4 The Arrow-Debreu Equilibrium

In a sense, we have a supply and demand for labor and capital services, the firm demanding and the households supplying. Superscript d will denote the demand side of the economy, and superscript s the supply side of the economy.

An **Arrow-Debreu Equilibrium** is a list of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\{y_t, k_t^d, n_t^d\}_{t=0}^{\infty}$ and the households $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ such that

(a) given prices, the firm's allocation solves the firm's problem,

(b) the household's allocation solves the household problem,

(c) $\forall t$, $y_t = c_t + i_t$,

(d) $\forall t$, $n_t^s = n_t^d$,

(e) $\forall t$, $k_t^s = k_t^d$.

In other words, maximization and market clearing are satisfied always and for everyone.

2.5 Observations

Here's a list of observations with no obvious narrative to tie them together.

(i) In equilibrium supply will equal demand, so right off the bat we can just start using $k_t^s = k_t^d = k_t$ and $n_t^s = n_t^d = n_t$.

(ii) Since preferences are strongly monotone, prices must be strictly positive for the utility maximizer to exist.

(iii) Similarly, $F(\cdot, \cdot)$ is strictly increasing in both arguments, so $w_t, r_t > 0$ for any t.

(iv) Because there is a positive price for capital services, and there is no benefit for unused capital, all capital stock will be rented out, i.e. $k_t = x_t$ for all t. (So from now on let's just refer to k_t for both.)

(v) Similarly, because there is a positive price for labor services, and there is no benefit for leisure, $n_t = 1$.

(vi) Recall that $k_{t+1} = (1-\delta)k_t + i_t$. We will use this to remove i_t from the model and instead use $i_t = k_{t+1} - (1-\delta)k_t$.

2.5.1 Euler's Theorem

For the next observation, we'll use the following result.

Theorem 1 (Euler's Theorem)**.** *If $g(\cdot)$ is homogeneous of degree k, then for any $\lambda > 0$,*

$$kg(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial g(x_1, \ldots, x_n)}{\partial x_i} x_i.$$

Proof. Suppose $g(\cdot)$ is homogeneous of degree k. Then it follows that

$$g(\lambda x_1, \ldots, \lambda x_k) = \lambda^k g(x_1, \ldots, x_n).$$

Differentiate both sides with respect to λ for

$$\begin{aligned}
\frac{\partial g(\lambda x)}{\partial \lambda} &= \frac{\partial g(\lambda x)}{\partial \lambda x_1} \frac{\partial \lambda x_1}{\partial \lambda} + \ldots + \frac{\partial g(\lambda x)}{\partial \lambda x_n} \frac{\partial \lambda x_n}{\partial \lambda} \\
&= \frac{\partial g(\lambda x)}{\partial \lambda x_1} x_1 + \ldots + \frac{\partial g(\lambda x)}{\partial \lambda x_n} x_n \\
&= k\lambda^{k-1} g(x_1, \ldots, x_n).
\end{aligned}$$

This holds for any $\lambda > 0$, in particular, for $\lambda = 1$, which gives

$$\frac{\partial g(x)}{\partial x_1} x_1 + \ldots + \frac{\partial g(x)}{\partial x_1} x_n = kg(x_1, \ldots, x_n). \qquad \square$$

2.5.2 The Firm's Profit

If we take the first order conditions of the objective function

$$\max_{y_t, n_t, k_t} \sum_{t=0}^{\infty} p_t[F(k_t, n_t) - r_t k_t - w_t n_t]$$

with respect to k_t and n_t, then respectively we get

$$F_k(k_t, n_t) = r_t, \quad F_n(k_t, n_t) = w_t,$$

which represent the demands of the firm. We are assuming that the firm exhibits constant returns to scale, i.e. is homogeneous of degree one. Therefore from Euler's theorem, we have

$$F(k_t, n_t) = F_k(k_t, n_t)k_t + F_n(k_t, n_t)n_t.$$

Given the first order conditions, in equilibrium it must be the case that

$$F(k_t, n_t) = r_t k_t + w_t n_t.$$

It follows that we have zero profit because

$$\pi = p_t[F(k_t, n_t) - r_t k_t - w_t n_t] = p_t(0) = 0.$$

Because profit equals zero, we can ignore the number of firms in the economy.

2.6 Simplifications

So we've made a lot of observations. Um, let's use them. First, we can rewrite the household's problem as solving

$$
\begin{aligned}
& \max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subject to} \quad \sum_{t=0}^{\infty} p_t[c_t + k_{t+1} - (1-\delta)k_t] &= \sum_{t=0}^{\infty} p_t(r_t k_t + w_t), \\
c_t, k_{t+1} &\geq 0, \\
x_0 & \text{ given.}
\end{aligned}
$$

So we can write the Lagrangian for the problem as

$$
\mathscr{L}^{HH} = \sum_{t=0}^{\infty} \beta^t u(c_t) - \lambda \sum_{t=0}^{\infty} p_t[c_t + k_{t+1} - (1-\delta)k_t - r_t k_t - w_t].
$$

We are interested in the relative consumption over periods, so we'll take first order conditions with respect to c_t and c_{t+1}. We're also interested in the optimal trajectory of capital accumulation, so we'll take the first order condition with respect to k_{t+1} as well. Respectively, we get

- $\beta^t u'(c_t) = \lambda p_t$,
- $\beta^{t+1} u'(c_{t+1}) = \lambda p_{t+1}$,
- $p_t = p_{t+1}[(1-\delta) + r_{t+1}]$.

These three conditions together form the string of equalities

$$
\frac{p_{t+1}}{p_t} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{(1-\delta) + r_{t+1}}. \tag{4}
$$

2.7 More Observations

(i) Recall that $f(k) = F(k,1) + (1-\delta)k$. It follows that $f'(k) = F_k(k,1) + (1-\delta)$. We know that in equilibrium, $F(k_t, 1) = r_t$, and therefore

$$f'(k_t) = r_t + (1-\delta).$$

We will use this in the denominator of the RHS of equalities (4).

(ii) We know that $y_t = F(k_t,1) = c_t + i_t$. So $F(k_t,1) = c_t + k_{t+1} - (1-\delta)k_t$. We can solve for $c_t = F(k_t,1) + (1-\delta)k_t - k_{t+1}$. From the previous point, we can thus write

$$c_t = f(k_t) - k_{t+1}.$$

Now let's consider equalities (4). It tells us that

$$\begin{aligned} & u'(c_t) = \beta u'(c_{t+1})\big[(1-\delta) + r_{t+1}\big] \\ \implies \quad & u'\big(f(k_t) - k_{t+1}\big) = \beta u'\big(f(k_{t+1}) - k_{t+2}\big) f'(k_{t+1}). \end{aligned}$$

If you go back and look at notes from mini-macro, this is precisely the same second order difference equation we had with the social planner's problem.

An important result is that if the second welfare theorem holds, then we can solve for Pareto efficient allocations by solving the social planner's problem. This will ensure that all Pareto efficient allocations are competitive equilibrium allocations. If the first welfare theorem also holds, then we have found *all* competitive equilibrium allocations. The practical importance is that we can instead solve the much easier social planner problem instead of messing around with all of the complicated stuff involved in this set of lecture notes.

The social planner's problem, to be explicit, solves for the allocation of capital only, i.e. $\{k_{t+1}\}_{t=0}^{\infty}$. But if the sequence of capital accumulation has been determined, then we can deduce

- $y_t = F(k_t, 1)$,
- $i_t = k_{t+1} - (1-\delta)k_t$,
- $c_t = F(k_t, 1) - i_t$,
- $n_t = 1$,
- $r_t = F_k(k_t, 1)$,
- $w_t = F_n(k_t, 1)$.

Furthermore, from equalities (4) we can normalize $p_0 = 1$ and solve for the remaining prices. In particular,

$$p_1 = \frac{1}{f'(k_1)}, \quad p_2 = \frac{p_1}{f'(k_2)}, \quad p_3 = \frac{p_2}{f'(k_3)}, \quad \ldots$$

So simply knowing capital accumulation characterizes the entire equilibrium. Woo.