Walras, Budget Sets, Homogeneity

Definition 1. *The **Walrasian budget set***

$$B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$$

is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

*The set $\{x \in \mathbb{R}^L : p \cdot x = w\}$ is called the **budget hyperplane**.*

The price vector p must be orthogonal the budget hyperplane. Which is to say, for any x', x'' on the budget hyperplane, $p \cdot (x' - x'') = 0$.

The Walrasian budget set is a convex set.

Definition 2. *The Walrasian demand correspondence $x(p, w)$ is **homogeneous of degree zero** if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.*

Homogeneity of degree zero implies that we can fix (normalize) the level of one of the $L+1$ independent variables (i.e. $p_1, ..., p_L, w$) with no loss of generality. This usually means setting some $p_k = 1$ or $w = 1$.

Definition 3. *The Walrasian demand correspondence $x(p, w)$ satisfies **Walras' law** if for every $p >> 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p, w)$.*

Wealth and Price Effects

The **wealth effects** are how $x(p, w)$ changes with respect to w. For a single commodity, we have a wealth effect of

$$\frac{\partial x_\ell(p, w)}{\partial w}.$$

For the entire vector x, we have the matrix

$$D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix}.$$

The **price effects** are how $x(p, w)$ changes with respect to p. For a single commodity x_ℓ and the price of good k, we have a price effect of

$$\frac{\partial x_\ell(p, w)}{\partial p_k}.$$

For the entire vector x and the entire price vector p, we have the $L \times L$ matrix

$$D_p x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}.$$

Proposition 1. *If the Walrasian demand function $x(p, w)$ is homogenous of degree zero, then for all p and w,*

$$\sum_{k=1}^{L} \frac{\partial x_\ell(p, w)}{\partial p_k} p_k + \frac{\partial x_\ell(p, w)}{\partial w} w = 0$$

for all $\ell = 1, ..., L$. In matrix notation,

$$D_p x(p, w)p + D_w x(p, w)w = 0.$$

To prove this simply write out the conditions for homogeneity of degree zero and then differentiate with respect to α. Then evaluate the derivative at $\alpha = 1$.

Definition 4. *The **price elasticity** of $x_\ell(p, w)$ with respect to p_k is*

$$\epsilon_{\ell k}(p, w) = \frac{\partial x_\ell(p, w)}{\partial p_k} \frac{p_k}{x_\ell(p, w)},$$

which is expressed in terms of percentages.

Definition 5. *The **wealth elasticity** of $x_\ell(p, w)$ is*

$$\epsilon_{\ell w}(p, w) = \frac{\partial x_\ell(p, w)}{\partial w} \frac{w}{x_\ell(p, w)},$$

which is expressed in terms of percentages.

Proposition 1 can be written using elasticizes (just divide both sides by $x_\ell(p, w)$ as

$$\sum_{k=1}^{L} \epsilon_{\ell k}(p, w) + \epsilon_{\ell w}(p, w) = 0$$

for all $\ell = 1, ..., L$.

Proposition 2 (Cournot Aggregation)**.** *If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w,*

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p, w)}{\partial p_k} + x_k(p, w) = 0$$

for $k = 1, ..., L$. In matrix notation,

$$p \cdot D_p x(p, w) + x(p, w)^T = 0^T.$$

To prove this, simply write out the conditions for Walras' law and then differentiate with respect to prices.

Proposition 3 (Engel Aggregation)**.** *If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w,*

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p, w)}{\partial w} = 1.$$

In matrix notation,

$$p \cdot D_w x(p, w) = 1.$$

Weak Axiom of Revealed Preference

Definition 6. *The Walrasian demand function $x(p,w)$ satisfies the **weak axiom of revealed preference** if the following property holds for any two price-wealth situations (p,w) and (p',w'):*

> *If $p\cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w)$, then $p'\cdot x(p,w) > w'$.*

The idea is that since you prefer bundle $x(p,w)$ at (p,w) even though you could also afford $x(p',w')$, then it must mean that you only choose $x(p',w')$ at (p',w') because $x(p,w)$ is not affordable at (p',w').

Suppose you have (p,w) and thus you choose bundle $x(p,w)$. Now suppose we change p to p'. Since prices have changed, you might not spend all of your wealth on $x(p,w)$ or perhaps you can no longer afford $x(p,w)$. So let's compensate your wealth to w' so that you can just exactly afford $x(p,w)$ still, so $p'x(p,w) = w'$. We can write the adjustment in wealth as

$$\begin{aligned} w' - w &= p'x(p,w) - px(p,w) \\ \implies \quad \Delta w &= \Delta p x(p,w) \end{aligned}$$

This wealth compensation is known as **Slutsky wealth compensation**. A price changes accompanied by such a compensating wealth change is called a **compensated price change**.

Proposition 4. *If $x(p,w)$ is a Walrasian demand function that satisfies the weak axiom, then $x(p,w)$ must be homogeneous of degree zero.*

Proposition 5. *Suppose that the Walrasian demand function $x(p,w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p,w)$ satisfies the weak axiom if and only if the following property holds:*

> *For any compensated price change from an initial situation (p,w) to a new price-wealth pair $(p',w') = \big(p', p'\cdot x(p,w)\big)$, we have*
>
> $$(p'-p)[x(p',w') - x(p,w)] \le 0$$
>
> *with strict inequality whenever $x(p,w) \ne x(p',w')$. Alternatively, $\Delta p \cdot \Delta x \le 0$.*

We call this the **compensated law of demand** because with wealth compensation, demand and price move in opposite directions. The proof uses a lot of Walras' law and the weak axiom. There is another claim the proof uses that is worth mentioning.

Proposition 6. *The weak axiom holds if and only if it holds for all compensated price changes.*

We can rewrite the Slutsky wealth compensation in terms of differentials as $dw = x(p,w)\cdot dp$, and Proposition 4 as $dp\cdot dx \le 0$. The total derivative of $x(p,w)$ is

$$dx = \frac{\partial x(p,w)}{\partial p}dp + \frac{\partial x(p,w)}{\partial w}dw,$$

or in matrix form

$$dx = D_p x(p,w)dp + D_w x(p,w)dw.$$

Plug in the differential form of dw to get

$$dx = D_p x(p,w)dp + D_w x(p,w)[x(p,w)\cdot dp].$$

Now plug this into the differential form of Proposition 4 (and factor out a dp) to get

$$dp\cdot\left[D_p x(p,w) + D_w x(p,w)x(p,w)^T\right]dp \le 0.$$

The expression in the brackets is called the **substitution matrix** $S(p,w)$ with entries (called **substitution effects**) of

$$s_{\ell k}(p,w) = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell}{\partial w}x_k(p,w).$$

Proposition 7. *If a differentiable Walrasian demand function $x(p,w)$ satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any (p,w), the substitution matrix $S(p,w)$ satisfies $v\cdot S(p,w)v^T \le 0$ for any $v\in\mathbb{R}^L$. That is, $S(p,w)$ is negative semidefinite.*

This implies that $s_{\ell\ell}(p,w) \le 0$, which is to say, the substitution effect of a good with respect to its own price is always nonpositive. This in turn implies that a good can be a Giffen good at (p,w) if and only if it is inferior. Likewise, if a commodity is a normal good, then it is an ordinary good (e.g. demand falls as its price rises).

Note that Proposition 6 does not imply that $S(p,w)$ is symmetric, which sucks (see math section). But it is always symmetric for $L=2$.

Proposition 8. *Suppose that the Walrasian demand function $x(p,w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then $p\cdot S(p,w) = 0$ and $S(p,w)p = 0$ for any (p,w).*

This means that $S(p,w)$ is singular. (To see why, suppose that $S(p,w)$ is nonsingular. Then we can have the row-reduced form with non-zero p' equaling zero, which is nonsensical.) So we can't extend this result to negative definiteness.

Math Notes

Proposition 9. *An arbitrary (possibly nonsymmetric) matrix A is negative definite (or semidefinite) if and only if $A + A^T$ is negative definite (or semidefinite).*

Proposition 10. *A symmetric $n\times n$ matrix A is negative definite if and only if $(-1)^k|A_{kk}| > 0$ for all $k\le n$, where A_{kk} is the submatrix obtained by deleting the last $n-k$ rows and columns. For semidefiniteness, we replace the strict inequalities by weak inequalities and require that the weak inequalities hold for all matrices formed by permuting the rows and columns of A.*