

ECN 200B—First Fundamental Theorem of WE

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Preliminary Results

To begin with, we'll start with a lemma and a theorem.¹

Lemma. *Suppose that $u^i(\cdot)$ is locally nonsatiated and x^* maximizes $u(x)$ subject to $px \leq m$. Then $u(x') \geq u(x^*)$ implies that $px' \geq m$.*

Proof. Suppose there exists some \hat{x} such that $u(\hat{x}) \geq u(x^*)$ and $p\hat{x} < m$. Because preferences are locally nonsatiated, we know that we can construct an ϵ -ball around \hat{x} and inside such a ball exists some x' such that $u(x') > u(\hat{x})$, and furthermore we can choose ϵ so that $px' \leq m$. This means that x' is affordable and is preferred to x^* , meaning that x^* is not actually the maximizer. This is a contradiction and the lemma is established. \square

Theorem. *Suppose that x^* maximizes $u(x)$ subject to $px \leq m$. Then $u(x) > u(x^*)$ implies $px > m$.*

Proof. Note that local nonsatiation has not been assumed. But in any case, this follows directly from the definition of a maximizer. If x is preferred to x^* and is also affordable, then x^* isn't even the maximizer. Thus it has to be the case that x is unaffordable. \square

FFToWE for Exchange Economies

Theorem. *In an exchange economy where all individuals have locally nonsatiated preferences, every competitive equilibrium allocation is Pareto efficient.*

Proof. We will suppose that (p, x) is a competitive equilibrium, but that x is not in the core. The latter implies that there exists some coalition H with allocation $\{\hat{x}_i\}_{i \in H}$ such that

¹No, I do not know why one qualifies as a lemma and the other a theorem.

- (a) $u(\hat{x}^i) \geq u(x^i)$ for all $i \in H$,
- (b) $u(\hat{x}^j) > u(x^j)$ for some $j \in H$,
- (c) $\sum_{i \in H} \hat{x}^i = \sum_{i \in H} w^i$.

Because x is assumed to be a competitive equilibrium allocation, it follows that x^j maximizes the utility of individual j subject to her budget constraint. The implication is that at prices p , it must be the case from a revealed preference argument that \hat{x}^j is unaffordable for the agent, that is,

$$p \cdot \hat{x}^j > p \cdot x^j = w^j.$$

For every other $i \in H \setminus \{j\}$, we can similarly say that

$$p \cdot \hat{x}^i \geq p \cdot x^i = w^i.$$

Sum over H and we get

$$p \sum_{i \in H} \hat{x}^i > p \sum_{i \in H} x^i = p \sum_{i \in H} w^i.$$

This contradicts condition (c) for the coalition. Therefore it must be the case that x is in the core. \square

FFToWE for Production Economies

Theorem. *Fix a production economy,*

$$\{I, J, (Y_j)_{j \in J}, (u^i, w^i, s^{i,j})_{i \in I, j \in J}\}.$$

Suppose that all $u^i(\cdot)$ are locally nonsatiated. If (p, x, y) is a competitive equilibrium, then (x, y) is Pareto efficient.

The Setup. Suppose that (p, x, y) is a competitive equilibrium but (x, y) is not Pareto efficient. Since it is a competitive equilibrium, we know that

- (a) For all i , x^i maximizes $u^i(\tilde{x})$ subject to $p\tilde{x} \leq pw^i + \sum_{j \in J} s^{i,j} p \cdot y^j$.
- (b) For any j , y^j maximizes $p \cdot \tilde{y}$ subject to $\tilde{y} \in Y^j$.
- (c) $\sum_{i \in I} x^i = \sum_{i \in I} w^i + \sum_{j \in J} y^j$.

And because the allocation is not Pareto efficient, there exists some allocation (\hat{x}, \hat{y}) such that

- (i) for all i , $\hat{x}^i \in \mathbb{R}_+^L$,
- (ii) for all j , $\hat{y}^j \in Y^j$,
- (iii) $\sum_{i=1}^I \hat{x}^i = \sum_{i=1}^I w^i + \sum_{j=1}^J \hat{y}^j$,
- (iv) for all i , $u^i(\hat{x}^i) \geq u^i(x^i)$,
- (v) for some i^* , $u^{i^*}(\hat{x}^{i^*}) > u^{i^*}(x^{i^*})$.

The Proof. By the previous theorem, it follows that x^{i^*} must not be affordable, that is,

$$p\hat{x}^{i^*} > pw^{i^*} + \sum_{j \in J} s^{i^*,j} p \cdot y^j.$$

For any of the other individuals, by the previous lemma we must have

$$p\hat{x}^i \geq pw^i + \sum_{j \in J} s^{i,j} p \cdot y^j,$$

and therefore for all i , we have

$$p\hat{x}^i \geq pw^i + \sum_{j \in J} s^{i,j} p \cdot y^j.$$

Summing over all i and taking prices out of the sums, we have

$$p \sum_{i=1}^I \hat{x}^i > p \sum_{i=1}^I w^i + p \sum_{i \in I, j \in J} s^{i,j} \cdot y^j.$$

When we sum over the shares in firm j , each $\sum_{i=1}^I s^{i,j} = 1$, and therefore

$$p \sum_{i=1}^I \hat{x}^i > p \sum_{i=1}^I w^i + p \sum_{j \in J} y^j. \tag{1}$$

Because y^j was assumed to be the (feasible) profit maximizer, it must be the case for any j that $p \cdot \hat{y}^j \leq p \cdot y^j$. So if we sum over all j and take prices out the sums, we get

$$p \cdot \sum_{j=1}^J \hat{y}^j \leq p \cdot \sum_{j=1}^J y^j.$$

Add the aggregate nominal value of endowments to both sides for

$$p \cdot \sum_{i=1}^I w^i + p \cdot \sum_{j=1}^J \hat{y}^j \leq p \cdot \sum_{i=1}^I w^i + p \cdot \sum_{j=1}^J y^j.$$

Combining this with equation (1), it follows that

$$p \cdot \sum_{i=1}^I \hat{x}^i > p \cdot \sum_{i=1}^I w^i + p \cdot \sum_{j=1}^J \hat{y}^j,$$

which contradicts point (iii) above. Thus the competitive equilibrium allocation must be Pareto efficient. \square