ECN 200D—Week 9 Lecture Notes

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1 Asset Pricing

1.1 The Lucas Trees Model (CAPM)

The **Lucas Trees model** is a capital asset pricing model (CAPM). In it, there exist trees that produce fruit of quantity d_t in period t, where $d_t \in \{d_1, \ldots, d_N\}$ possible states of production. Typically we will be assuming a Markov process, i.e.

$$P(d_{t+1} = d_j | d_t = d_i) = \pi_{ij}.$$

The supply of trees T is fixed and exogenous, so we may as well just normalize the supply of trees to $T = 1$. Fruit from the tree is nonstorable. An agent has a share s of the fruit that the tree produces.

The Bellman equation for the representative agent is given by

$$V_i(s) = \max_{c_i, s'} \left\{ u(c_i) + \beta \sum_{j=1}^{N} \pi_{ij} V_j(s') \right\}.$$

Given that the current state of the world is i, the agent wants to choose how much to consume and how much of a share of the tree to have tomorrow based on the expected value of tomorrow's level of production.

Let ψ_i denote the price of a unit of stock when the state of the world is

i. Then the agent's budget constraint is can be written as

$$c_i + \psi_i s' = (\psi_i + d_i)s.$$

In other words, the amount consumed today and the amount of tomorrow's shares purchased today must equal the value of today's shares plus the value of today's fruit, i.e. today's nominal wealth.

Plugging the budget constraint into the Bellman equation gives

$$V_i(s) = \max_{s'} \left\{ u\big([\psi_i + d_i]s - \psi s'\big) + \beta \sum_{j=1}^{N} \pi_{ij} V_j(s') \right\}.$$

Then the first order condition is

$$u'(c_i)\psi_i = \beta \sum_{j=1}^{N} \pi_{ij} V_j'(s'). \tag{1}$$

Plug in the policy function $s' = g(s)$ and then take the first order condition of

$$V_i(s) = u\big([\psi_i + d_i]s - \psi_i g(s)\big) + \beta \sum_{j=1}^{N} \pi_{ij} V_j\big(g(s)\big)$$

with respect to k for the envelope condition

$$V_i'(s) = u'(c_i)(\psi_i + d_i). \tag{2}$$

Updating the envelope condition by a period, change the state of the world to j, and in combination with the first order condition we get

$$u'(c_i)\psi_i = \beta \sum_{j=1}^{N} \pi_{ij} u'(c_j')(\psi_j' + d_j). \tag{3}$$

In equilibrium we will have supply equaling demand. There's a fixed supply

$1 = s = s' = s'' = \ldots$. It follows from the budget constraint that $c_i = d_i$ $c'_j = d_j$. Therefore we can write condition (3) as

$$u'(d_i)\psi_i = \beta \sum_{j=1}^{N} \pi_{ij} u'(d_j)(\psi'_j + d_j). \tag{4}$$

This condition hold for all $i = 1, \ldots N$, so we have N conditions.

1.2 Special Case: $N = 1$

When $N = 1$, the model is not stochastic—we have $d_t = d$ in all cases. So we can rewrite the asset pricing formula in equation (3) as

$$u'(d)\psi = \beta u'(d)(\psi' + d) \quad \Longrightarrow \quad \psi' = \frac{\psi}{\beta} - d.$$

This sequence follows an "explosive path," so the only admissible solution is the steady state solution where

$$\psi = \frac{\psi}{\beta} - d \quad \Longrightarrow \quad \psi^* = \frac{\beta d}{1 - \beta}.$$

So the price of the asset is the value of the discount stream of dividends, i.e. its "fundamental value."

1.3 General Case

Okay, now let's consider the steady state of the general case,

$$u'(d_i)\psi_i = \beta \sum_{j=1}^{N} \pi_{ij} u'(d_j)(\psi_j + d_j),$$

where $i = 1, \ldots, N$. We can write the system as the matrix

$$U\psi = \beta\pi U(\psi + d),$$

where π is the **Markov matrix** in which the (ij)th entry is π_{ij}. Doing some matrix algebra, we can solve for

$$\psi^* = (U - \beta\pi U)^{-1}\beta\pi U'd.$$

This is, without question, something for a computer to solve.

2 Monetary Theory

2.1 The Basics

We need some preliminaries before diving into an actual model. There are four of particular importance.

2.1.1 The Fisher equation

In general, asset price is equal to its discounted payment, i.e.

$$\text{asset price} = \frac{\text{payment}}{(1+r)^n}. \tag{5}$$

We'll be dealing in one period intervals, so we'll have $n = 1$ from now on.

Let p_t be the price of an asset that has a nominal interest rate of i. The real price of the asset is the goods you're giving up, e.g. the real price $1/p_t$. The payment in real terms is $(1 + i)/p_{t+1}$. So from equation (5),

$$\frac{1}{p_t} = \frac{(1+i)/p_{t+1}}{1+r} \quad \Longrightarrow \quad \frac{p_{t+1}}{p_t} = \frac{1+i}{1+r}. \tag{6}$$

The rate of inflation is the rate of growth in the price level, so

$$\pi_t = \frac{p_{t+1} - p_t}{p_t} \quad \Longrightarrow \quad \frac{p_{t+1}}{p_t} = 1 + \pi_t. \tag{7}$$

Plug in equation (6) and we get

$$\frac{1+i}{1+r} = 1 + \pi_t.$$

Solving for i, we get the Fisher equation,

$$i = r + \pi_t + r\pi_t. \tag{8}$$

2.1.2 Inflation and Money Growth

Claim. *In a steady state monetary model where supply of money grows at a constant rate μ, $\pi = \mu$.*

Proof. The money growth rate implies that $M_{t+1} = (1+\mu)M_t$. In a steady state, real variables do not change, and therefore the level of real balances M_t/p_t is constant. It follows that

$$\frac{M_t}{p_t} = \frac{M_{t+1}}{p_{t+1}} \implies \frac{p_{t+1}}{p_t} = \frac{M_{t+1}}{M_t} = 1 + \mu.$$

From equation (refinflation), it follows that

$$1 + \mu = 1 + \pi \implies \mu = \pi. \qquad \square$$

2.1.3 Illiquid Real Interest Rate

Claim. *The real interest rate of a fully illiquid bond in a standard (monetary) model is given by $r = 1/\beta - 1$.*

Think of a world where agents can buy a one period real discount bond which gives you one unit of the numeraire good tomorrow. What price ψ are you willing to pay? Well, given your discount rate of β, you'll be willing to pay $\beta \cdot 1$. It follows from equation (5) that

$$\beta = \frac{1}{1+r} \implies r = \frac{1}{\beta} - 1.$$

2.1.4 The Friedman Rule

Claim. *The rate of money growth is* $\mu = \beta - 1 < 0$ *when the nominal interest rate* $i = 0$.

Proof. From the Fisher equation, we have $1 + i = (1 + r)(1 + \pi)$. From the previous results, we know that $1 + r = 1/\beta$ and $1 + \pi = 1 + \mu$. It follows that

$$1 + i = \frac{1}{\beta}(1 + \mu) \quad \Longrightarrow \quad \mu = \beta(1 + i) - 1.$$

Since the assumption is $i = 0$, the result follows. $\square$