Solution 1

Part a. You can connect everything, so the common knowledge partition is the partition of every state.

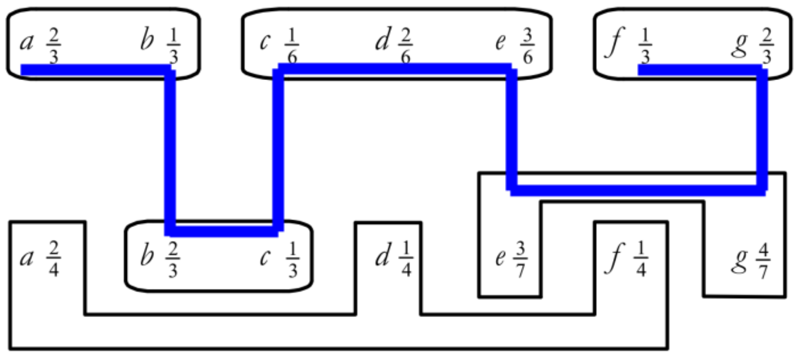


Figure 1: One way of showing that all states are connected. There are, of course, other ways.

Apropos the common prior, one strategy is to focus on a single probability and write all other probabilities in terms of that one probability. Here, I will arbitrarily choose to write all other probabilities in terms of $P(a)$. For example, we can directly relate a to b, d, and f in the following ways (call this Step 1):

- $P(b) = \frac{1}{2}P(a)$ (from Player 1)
- $P(d) = \frac{1}{2}P(a)$ (from Player 2)
- $P(f) = \frac{1}{2}P(a)$ (from Player 2)

Now let's relate a to c, e, and g, using results from Step 1 as "intermediate" steps.

- $P(b) = 2P(c) \implies P(c) = \frac{1}{2}P(b) = \frac{1}{4}P(a)$ (from Player 2)
- $P(e) = 3P(c) \implies P(e) = 3P(c) = \frac{3}{4}P(a)$ (from Player 1 and previous step)
- $P(g) = 2P(f) \implies P(g) = 2P(f) = P(a)$ (from Player 1)

Great. So the reason for doing this is now we can write

$$\begin{aligned}
1 &= P(a) + P(b) + P(c) + P(d) + P(e) + P(f) + P(g) \\
&= P(a) + \frac{1}{2}P(a) + \frac{1}{4}P(a) + \frac{1}{2}P(a) + \frac{3}{4}P(a) + \frac{1}{2}P(a) + P(a) \\
&= \frac{4}{4}P(a) + \frac{2}{4}P(a) + \frac{1}{4}P(a) + \frac{2}{4}P(a) + \frac{3}{4}P(a) + \frac{2}{4}P(a) + \frac{4}{4}P(a) \\
&= \frac{18}{4}P(a).
\end{aligned}$$

From this we can conclude that $P(a) = 4/18$. Because we've expressed every other probability in terms of $P(a)$, we are essentially done:

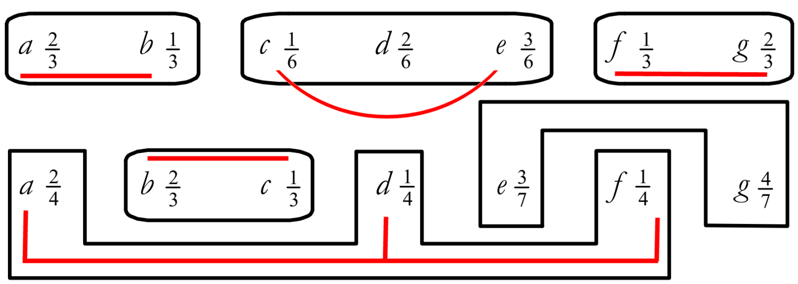
$$P(b) = \frac{1}{2}\frac{4}{18} = \frac{2}{18}, \quad P(c) = \frac{1}{4}\frac{4}{18} = \frac{1}{18}, \quad P(d) = \frac{1}{2}\frac{4}{18} = \frac{2}{18},$$
$$P(e) = \frac{3}{4}\frac{4}{18} = \frac{3}{18}, \quad P(f) = \frac{1}{2}\frac{4}{18} = \frac{2}{18}, \quad P(g) = \frac{1}{1}\frac{4}{18} = \frac{4}{18}.$$

Hence our *candidate* for the common prior is

$$\begin{pmatrix} a & b & c & d & e & f & g \\ \frac{4}{18} & \frac{2}{18} & \frac{1}{18} & \frac{2}{18} & \frac{3}{18} & \frac{2}{18} & \frac{4}{18} \end{pmatrix}$$

I say *candidate* because we still need to make sure that these probabilities are consistent with all partitions. The easy way to do this is to see if the multiples between probabilities are always maintained. For instance, if we look at Player 2's partition for e and g, they think that g is $4/3$ as likely as e. Comparing this to the common prior, it is also the case that g is $4/3$ as likely as e. Any relationship that we didn't use when calculating the candidate needs to be checked for this consistency; the relationships we did use are already satisfied by the way the candidate was constructed.

To that end, it might be worth visually keeping track of which relationships are used. Below I illustrated which relationships I used.



Hence we need to check that e and g are consistent with the candidate (which I just did above), and that d is consistent with c and e (which it clearly is: 1/6, 2/6, 3/6 compared to 1/18, 2/18, 3/18).

Technically what we're doing here is a probability update. Let Q denote Player 1's center partition, that is, $Q = \{c, d, e\}$. Then, given the common prior, the probability of being in Q is $(1 + 2 + 3)/18 = 6/18$. Then, conditional upon being in that partition, the probability of getting d is

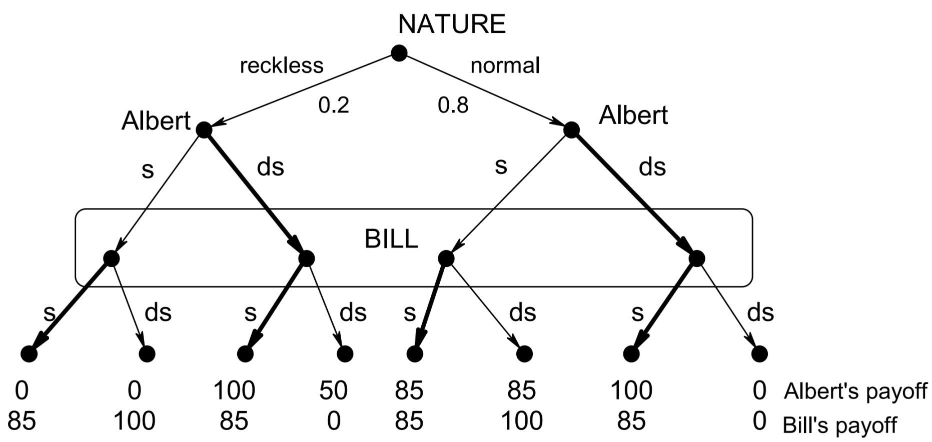
$$P(d|Q) = \frac{2/18}{6/18} = 2/6.$$

Hooray.

If we find some inconsistency while checking the "unused" probabilities, then there is no common prior and it's game over.

Solution 2

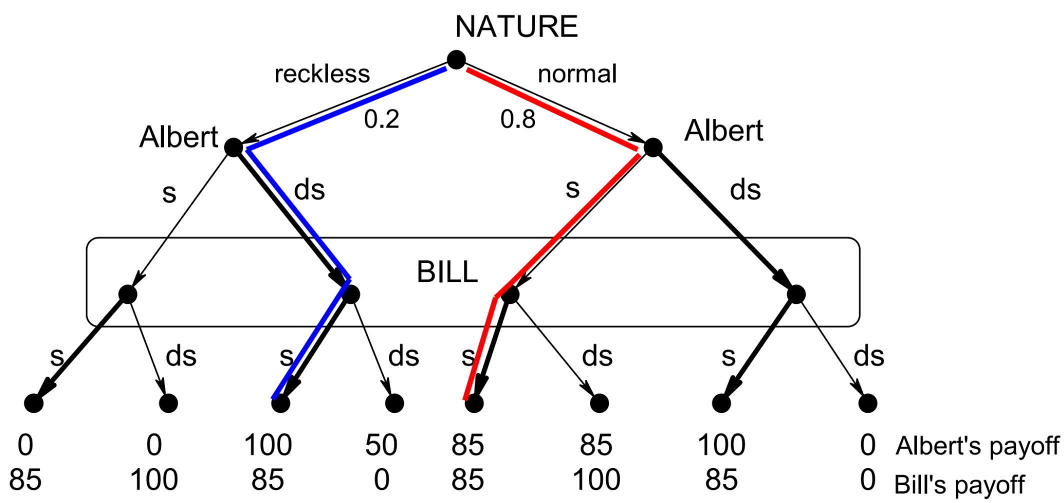
Part b. The Harsanyi transformation captures Bill's uncertainty via Nature. Albert knows his true state, so he faces no imperfect information. Bill, on the other hand, cannot distinguish between either Albert's types or Albert's choice to swerve or not. So the extensive-form Harsanyi representation is



The strategic-form frame is

	s	ds
s, s	?	?
s, ds	?	?
ds, s	?	?
ds, ds	?	?

We have to calculate the expected payoffs of each strategy profile. For example, the way we'd calculate the payoff of $(ds, s), s$ is illustrated by the following figure.



There is a 0.2 probability of the blue path, a 0.8 probability of the red path, and therefore strategy profile $(ds, s), s$ has expected payoff

$$(ds, s), s: \quad 0.2[100, 85] + 0.8[85, 85] = [88, 85].$$

Repeat the procedure for all eight strategy profiles and you get expected payoffs

$$\begin{aligned}
(s, s), s: &\quad 0.2[0, 85] + 0.8[85, 85] &&= [68, 85],\\
(s, s), ds: &\quad 0.2[0, 100] + 0.8[85, 100] &&= [68, 100]\\
(s, ds), s: &\quad 0.2[0, 85] + 0.8[100, 85] &&= [80, 85],\\
(s, ds), ds: &\quad 0.2[0, 100] + 0.8[0, 0] &&= [0, 20],\\
(ds, s), s: &\quad 0.2[100, 85] + 0.8[85, 85] &&= [88, 85],\\
(ds, s), ds: &\quad 0.2[50, 0] + 0.8[85, 100] &&= [78, 80],\\
(ds, ds), s: &\quad 0.2[100, 85] + 0.8[100, 85] &&= [100, 85],\\
(ds, ds), ds: &\quad 0.2[50, 0] + 0.8[0, 0] &&= [10, 0].
\end{aligned}$$

Now plug these into the game frame above and you get

	s	ds
s, s	$68, 85$	$68, \underline{100}$
s, ds	$80, \underline{85}$	$0, 20$
ds, s	$88, \underline{85}$	$\underline{78}, 80$
ds, ds	$\underline{100}, \underline{85}$	$10, 0$

So the unique pure-strategy NE is $(ds, ds), s$.

Note that there are no mixed-strategy Nash equilibria. First, (ds, s) strictly dominates both (s, s) and (s, ds) for Player 1. So we can delete those via IDSDS procedure to get

	s	ds
ds, s	$88, 85$	$78, 80$
ds, ds	$100, 85$	$10, 0$

Then here s strictly dominates ds for Player 2, so delete ds.

	s
ds, s	$88, 85$
ds, ds	$100, 85$

Finally, (ds, ds) strictly dominates (ds, s) and we are left with a single rationalizable strategy profile, $(ds, ds), s$. Since there is nothing left over which to mix, there can be no mixed-strategy NE.

Solution 3

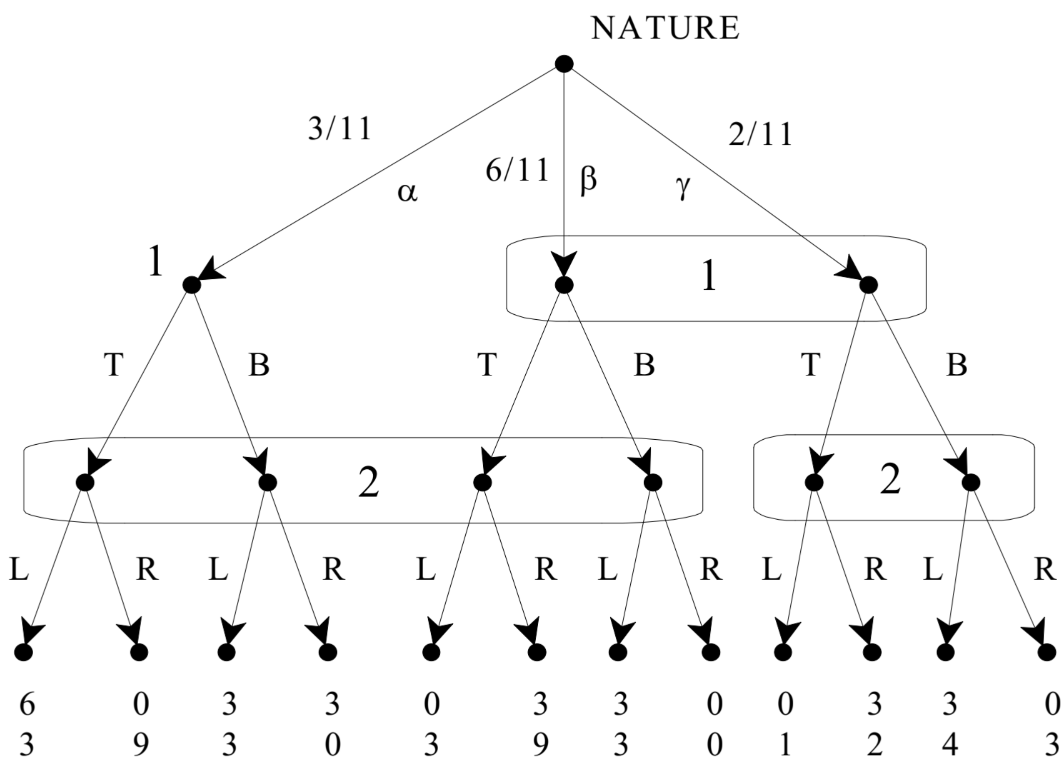
Let's first find the Nature probabilities. We are essentially finding the common prior. From Player 1 we can determine that $P(\gamma) = P(\beta)/3$. From Player 2 we can determine that

$P(\alpha) = P(\beta)/2$. It must be the case that

$$\begin{aligned} 1 &= P(\alpha) + P(\beta) + P(\gamma) \\ &= \frac{1}{2}P(\beta) + P(\beta) + \frac{1}{3}P(\beta) \\ &= \frac{3}{6}P(\beta) + \frac{6}{6}P(\beta) + \frac{2}{6}P(\beta) \\ &= \frac{11}{6}P(\beta), \end{aligned}$$

and therefore $P(\beta) = 6/11$, $P(\alpha) = 3/11$, and $P(\gamma) = 2/11$.

The Harsanyi transformation is



If α is the true state, then Player 1 knows what's up. Player 1 cannot distinguish between β or γ, however, hence the nontrivial information set on the right.

If γ is the true state, then Player 2 knows it for sure. Player 2 is still unable to determine whether Player 1 chooses T or B, however, because we're modeling a simultaneous game here. If either α or β is the true state, then Player 2 doesn't know what Player 1 has chosen, nor do they know which of the two α or β games they're in.