Exercise 1

- Preferences: $\log(C_t) + \dfrac{\theta}{1-\eta}(L_t^{1-\eta} - 1)$
- Law of Motion: $K_{t+1} = (1-\delta)K_t + I_t$
- Aggregate Resource Constraint: $C_t + I_t = Y_t$
- Production: $Y_t = Z_t K_t^\alpha (X_t N_t)^{1-\alpha}$
- Trend: $X_t/X_{t-1} = \gamma_X$.
- Technology: $\log(Z_{t+1}) = \rho \log(Z_t) + \epsilon_{t+1}$

Part 1: Change in Hours Worked

A **balanced growth path** means that the levels of certain key variables grow at a constant rate. The fact that hours worked is bounded means that $\gamma_N = 1$. If it was anything more, you'd eventually have N exceeding its upper bound. If it was anything less, you'd eventually have $N = 0$ or less, which isn't valid either. Therefore $N_{t+1}/N_t = 1$, from which it follows that $L_{t+1} = L_t = L$. So let's see if the preferences given in this problem actually possess this feature.

First note that $rk = (r + 1 - \delta)k = k'$ pretty much by definition. It follows from the budget constraint that

$$C_t + K_{t+1} = w(1 - L_t) + (r^k + 1 - \delta)K_t \quad \Longrightarrow \quad C_t = w(1 - L_t).$$

The constrained Bellman equation for this problem is

$$V(\mathbf{K}_t, K_t, Z_t) = \max_{C_t, L_t, K_{t+1}} \log(C_t) + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) - \lambda_t[C_t + K_{t+1} - w(1 - L_t) - (r^k + 1 - \delta)K_t].$$

The first order conditions with respect to c_t and L are

$$\frac{1}{C_t} = \lambda_t,$$

$$L_t^{-\eta} = \lambda_t w.$$

This gives the intratemporal Euler equation

$$\frac{w}{C_t} = L_t^{-\eta}.$$

Now substitute in the fact that $C_t = w(1 - L_t)$.

$$\frac{w}{w(1-L_t)} = L_t^{-\eta} \quad \Longrightarrow \quad 1 = L_t^{-\eta}(1-L_t).$$

So L is not a function of anything; it is just a constant. Ergo, $\gamma_N = 1$, as is required.

Part 2

Consider the law of motion and the growth of capital:

$$\frac{K_{t+1}}{K_t} = \frac{(1-\delta)K_t}{K_t} + \frac{I_t}{K_t}$$
$$\Longrightarrow \quad \gamma_K = (1-\delta) + \frac{I_t}{K_t}$$

In order for γ_K to be constant, we need I_t/K_t to be constant. This implies that

$$\frac{I_t}{K_t} = \frac{I_{t+1}}{K_{t+1}} \quad \Longrightarrow \quad \gamma_K = \frac{K_{t+1}}{K_t} = \frac{I_{t+1}}{I_t} = \gamma_I.$$

Now notice that

$$I_t = Y_t - C_t \quad \Longrightarrow \quad \gamma_I = \frac{Y_t}{I_t} - \frac{C_t}{I_t}.$$

This means we need Y_t/I_t to be constant and C_t/I_t to be constant, which implies

$$\frac{Y_t}{I_t} = \frac{Y_{t+1}}{I_{t+1}} \quad \Longrightarrow \quad \gamma_I = \frac{I_{t+1}}{I_t} = \frac{Y_{t+1}}{Y_t} = \gamma_Y,$$
$$\frac{C_t}{I_t} = \frac{C_{t+1}}{I_{t+1}} \quad \Longrightarrow \quad \gamma_I = \frac{I_{t+1}}{I_t} = \frac{C_{t+1}}{C_t} = \gamma_C.$$

Now let's exploit the production function, noting that $\gamma_Z = 1$.

$$\begin{aligned}
\gamma_Y &= \frac{Z_{t+1}K_{t+1}^{\alpha}(X_{t+1}N_{t+1})^{1-\alpha}}{Y_t} \\
&= \frac{Z_{t+1}K_{t+1}^{\alpha}(X_{t+1}N_{t+1})^{1-\alpha}}{Z_t K_t^{\alpha}(X_t N_t)^{1-\alpha}} \\
&= \gamma_Z \gamma_K^{\alpha} \gamma_X^{1-\alpha} \gamma_N^{1-\alpha} \\
&= \gamma_K^{\alpha} \gamma_X^{1-\alpha} \\
\implies \quad & \gamma_Y \gamma_K^{-a} = \gamma_X^{1-\alpha} \\
\implies \quad & \gamma_Y^{1-\alpha} = \gamma_X^{1-\alpha} \\
\implies \quad & \gamma_Y = \gamma_X.
\end{aligned}$$

And so it has been established that $\gamma_K = \gamma_C = \gamma_I = \gamma_Y = \gamma_X$ in a balanced growth path.

After finding the intratemporal Euler equation, we have

$$\frac{w_t}{C_t} = \theta L_t^{-\eta}.$$

We have just established that $L_t = L$ is a constant, and therefore the ratio w_t/C_t is constant. This implies that

$$\frac{w_t}{C_t} = \frac{w_{t+1}}{C_{t+1}} \quad \implies \quad \gamma_C = \frac{C_{t+1}}{C_t} = \frac{w_{t+1}}{w_t} = \gamma_w.$$

Since $\gamma_C = \gamma_K$, the result is shown.

Part 3: Return on Capital

We want to show that γ_{r^k} is constant along the balanced growth path. This is the marginal product of capital, so

$$\frac{r_{t+1}^k}{r_t^k} = \frac{\alpha Z_{t+1}K_{t+1}^{\alpha-1}X_{t+1}^{1-\alpha}N_{t+1}^{1-\alpha}}{\alpha Z_t K_t^{\alpha-1}X_t^{1-\alpha}N_t^{1-\alpha}} = \gamma_Z \gamma_K^{\alpha-1}\gamma_X^{1-\alpha}\gamma_N^{1-\alpha} = \gamma_K^{\alpha-1}\gamma_X^{1-\alpha} = 1.$$

Exercise 2

- Preferences: $\frac{1}{1-\sigma}\big([C_t v(L_t)]^{1-\sigma} - 1\big)$
- Law of Motion: $K_{t+1} = (1-\delta)K_t + I_t$
- Labor Constraint: $N_t + L_t = 1$.

- Production: $Y_t = Z_t K_t^\alpha N_t^{1-\alpha}$

You can buy a bond B_{t+1} in period t that pays $r_{t+1}B_{t+1}$ in period t_1. Therefore the household's budget constraint is

$$C_t + K_{t+1} + B_{t+1} = w_t N_t + (1 + r_t^k - \delta)K_t + r_t B_t + \pi_t.$$

Real bonds are zero in net supply.

Part 1: Household's Recursive Problem

Households solve

$$V(\mathbf{K}_t, K_t^s, B_t) = \max_{C_t, L_t, K_{t+1}^s, B_{t+1}} \frac{1}{1-\sigma}\big([C_t v(L_t)]^{1-\sigma} - 1\big) + \beta V(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1})$$

subject to

$$C_t + K_{t+1} + B_{t+1} = w_t N_t + (1 + r_t^k - \delta)K_t + r_t B_t + \pi_t.$$

Parts 2-3: First Order Conditions and Returns

Let's ignore labor for now and just focus on the other three state variables.

$$\begin{aligned} V(\mathbf{K}_t, K_t^s, B_t) = \max_{C_t, L_t, K_{t+1}^s, B_{t+1}} & \frac{1}{1-\sigma}\big([C_t v(L_t)]^{1-\sigma} - 1\big) + \beta V(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) \\ & - \lambda_t[C_t + K_{t+1} + B_{t+1} - w_t N_t - (1 + r_t^k - \delta)K_t - r_t B_t + \pi_t]. \end{aligned}$$

Then we have, respectively,

$$\begin{aligned} C_t^{-\sigma} v(L_t)^{1-\sigma} &= \lambda_t, \\ \beta V_K'(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) &= \lambda_t, \\ \beta V_B'(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) &= \lambda_t. \end{aligned}$$

The updated envelope conditions are

$$\begin{aligned} V_K'(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) &= C_{t+1}^{-\sigma} v(L_{t+1})^{1-\sigma}(1 + r_{t+1}^k - \delta), \\ V_B'(\mathbf{K}_{t+1}, K_{t+1}^s, B_{t+1}) &= C_{t+1}^{-\sigma} v(L_{t+1})^{1-\sigma} r_{t+1}. \end{aligned}$$

We get the two Euler equations

$$\beta C_{t+1}^{-\sigma} v(L_{t+1})^{1-\sigma}(1 + r_{t+1}^k - \delta) = C_t^{-\sigma} v(L_t)^{1-\sigma},$$
$$\beta C_{t+1}^{-\sigma} v(L_{t+1})^{1-\sigma} r_{t+1} = C_t^{-\sigma} v(L_t)^{1-\sigma}.$$

So we can see that $r_t = r_t^k + 1 - \delta$.

Part 4: Linearization

Yuck. Linearizing the Euler equations doesn't have any elegant simplification because you'll end up with some $v'(L)$ terms in there screwing things up. In any case, the most you can do is cancel out C terms and exploit the fact that

$$\beta(1 + r^k - \delta) = 1 = \beta r.$$

Linearizing $r_t = r_t^k + 1 - \delta$ gives

$$r\hat{r}_t = r_t^k \hat{r}_t^k = (r - 1 + \delta)\hat{r}_t^k.$$

Exercise 3

With probability p, a person works H hours and consumes c_1. With probability $1 - p$, the person works 0 hours and consumes c_2. This gives rise to expected utility

$$p\big[\log(c_1) + \log\big(v(1-H)\big)\big] + (1-p)\big[\log(c_2) + \log\big(v(1)\big)\big].$$

A feasible allocation must satisfy $pc_1 + (1-p)c_2 = c$, where c is aggregate per capital consumption.

Part 1: Household Optimality and Risk Sharing

The Lagrangian for this problem is

$$\mathscr{L} = p\big[\log(c_1) + \log\big(v(1-H)\big)\big] + (1-p)\big[\log(c_2) + \log\big(v(1)\big)\big] - \lambda_t[pc_1 + (1-p)c_2 - c].$$

The FOC with respect to c_1 and c_2 give

$$\frac{p}{c_1} = \lambda_t p,$$
$$\frac{1-p}{c_2} = \lambda_t (1-p).$$

It follows that $c_1 = c_2$. It then follows from the allocation constraint that $c_1 = c_2 = c$. This is what *complete risk sharing* implies—the same consumption regardless of whether the good or the bad state occurs.

Part 2: Rewriting Expected Utility

Therefore the expected utility is

$$\log(c) + p \log\left(\frac{v(1-H)}{v(1)}\right) + \log\big(v(1)\big).$$

The average number of hours worked will be

$$N = pH + (1-p)0 = pH.$$

From this it follows that $p = (1-L)/H$. Therefore

$$\log(c) + \frac{1-L}{H} \log\left(\frac{v(1-H)}{v(1)}\right) + \log\big(v(1)\big).$$

Part 3: Decentralized Representative Household

The Bellman equation is

$$V(\mathbf{K}_t, K_t) = \max_{c_t, L_t, K_{t+1}} \log(c) + \frac{1-L}{H} \log\left(\frac{v(1-H)}{v(1)}\right) + \log\big(v(1)\big) + \beta E_t[V(\mathbf{K}_{t+1}, K_{t+1})]$$

subject to the plain ol' combo-constraint

$$c_t + K_{t+1} = w_t(1-L_t) + (r^k + 1 - \delta)K_t + \pi.$$

Part 4: First-Order Conditions

To find the labor supply, take the FOC with respect to consumption and leisure. We get

$$\frac{1}{c_t} = \lambda_t,$$
$$-\frac{1}{H}\log\left(\frac{v(1-H)}{v(1)}\right) = \lambda_t w_t.$$

Therefore the labor supply function is

$$-\frac{1}{H}\log\left(\frac{v(1-H)}{v(1)}\right) = \frac{w_t}{c}.$$

But I think it really just wanted the two conditions separately. I'm not sure.

Part 5: Linearization

Linearization leads to

$$-\hat{c}_t = \hat{\lambda}_t,$$
$$0 = \hat{\lambda}_t + \hat{w}_t.$$

Part 6: Elasticity

The second equation implies that the change in wages is independent of the change in labor supply, i.e. aggregate elasticity of labor supply is infinite. Elasticity of individual labor supply is determined by η, which is implicitly embedded in $v(L)$. Choose $\eta = 1$ to be consistent with micro data. The infinite aggregate labor supply elasticity gives stronger amplification in the model and therefore we no longer require large TFP shocks.