ECN 200E—Week 5 Discussion

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May 20, 2017

The Setup

Consider the social planner problem

$$\max_{c_t, i_t, k_{t+1}} E_t \left[\sum_{t=0}^{\infty} \beta^t \ln(c_t - \sigma c_{t-1}) \right],$$

which is subject to the constraints

$$\begin{aligned}
y_y &= z_t k_t^{\alpha}, \\
c_t + i_t &= y_t, \\
k_{t+1} &= \left[1 - \frac{\gamma}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right] i_t + k_t(1-\delta), \\
z_{t+1} &= z_t^{\rho} e_{t+1}.
\end{aligned}$$

There are two notable things about this model. First, the utility function exhibits habit formation—it actually depends on last period's consumption. Second, the law of motion of capital also has a novel term,

$$\left[1 - \frac{\gamma}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right] i_t,$$

which represents *adjustment costs*. If today's level of investment is significantly different than yesterday's investment, then it will have a negative effect on how much capital accumulates for the next period. We would expect this to have a smoothing effect on investment. After a positive shock, investment would increase only little by little in order to mitigate the adjustment costs, eventually tapering back to the steady state.

This is not the only type of adjustment cost. Another commonly used takes the form

$$k_{t+1} = (1-\delta)k_t + i_t - \frac{\gamma}{2}\left(\frac{i_t}{k_t} - \delta\right)^2.$$

Note that in the steady state of the baseline model, we'll have

$$k_{t+1} = (1-\delta)k_t + i_t \quad \Longrightarrow \quad \frac{i}{k} = \delta.$$

So this cost adjustment will penalize deviations from the steady state. In the baseline model, there is a disproportionately large initial jump in investment relative to the jump in capital—we would expect this cost adjustment to reign in that jump in investment.

The Bellman Equation

The state variables are $z_t, k_t, c_{t-1}, i_{t-1}$, and the control variables are c_t, k_{t+1}, i_t. Normally we like to remove investment from the problem entirely, but investment is no longer a linear element of the constraints so we'll be better off just keeping it. The Bellman equation, constraints included, is

$$\begin{aligned} V(z_t, k_t, c_{t-1}, i_{t-1}) = {} & \ln(c_t - \sigma c_{t-1}) + \beta E_t\left[V(z_{t+1}, k_{t+1}, c_t, i_t)\right] \\ & - \lambda_t[c_t + i_t - z_t k_t^\alpha] \\ & - \lambda_t q_t \left[k_{t+1} - \left(1 - \frac{\gamma}{2}\left[\frac{i_t}{i_{t-1}} - 1\right]^2\right) i_t - k_t(1-\delta)\right]. \end{aligned}$$

If we wanted to we could have a different multiplier on the second constraint. But hell, they're both just going to be numbers so we can always write one as the multiple of the other, hence the q_t. That q_t will have a nice interpretation later on.

The First-Order Conditions

Consumption and Capital. Even though we have this i_t choice floating around now, let's do the usual thing and consider the first order conditions with respect to c_t and k_{t+1}.

Respectively, we get

$$\frac{1}{c_t - \sigma c_{t-1}} + \beta E_t[V_c'(z_{t+1}, k_{t+1}, c_t, i_t)] = \lambda_t,$$
$$\beta E_t[V_k'(z_{t+1}, k_{t+1}, c_t, i_t)] = \lambda_t q_t.$$

Envelope it up with respect to c_{t-1} to get

$$\begin{aligned} & V_c'(z_t, k_t, c_{t-1}, i_{t-1}) = -\frac{\sigma}{c_t - \sigma c_{t-1}} \\ \implies \quad & V_c'(z_{t+1}, k_{t+1}, c_t, i_t) = -\frac{\sigma}{c_{t+1} - \sigma c_t}. \end{aligned}$$

Substituting this into the first-order condition gives

$$E_t\left[\frac{1}{c_t - \sigma c_{t-1}} - \beta\frac{\sigma}{c_{t+1} - \sigma c_t}\right] = \lambda_t.$$

Now envelope it up with respect to k_t to get

$$\begin{aligned} & V_k'(z_t, k_t, c_{t-1}, i_{t-1}) = \lambda_t[\alpha z_t k_t^{\alpha-1} + q_t(1-\delta)] \\ \implies \quad & V_k'(z_t, k_t, c_{t-1}, i_{t-1}) = E_t\left[\frac{1}{c_t - \sigma c_{t-1}} - \beta\frac{\sigma}{c_{t+1} - \sigma c_t}\right][\alpha z_t k_t^{\alpha-1} + q_t(1-\delta)] \\ \implies \quad & V_k'(z_{t+1}, k_{t+1}, c_t, i_t) = E_{t+1}\left[\frac{1}{c_{t+1} - \sigma c_t} - \beta\frac{\sigma}{c_{t+2} - \sigma c_{t+1}}\right][\alpha z_{t+1} k_{t+1}^{\alpha-1} + q_{t+1}(1-\delta)]. \end{aligned}$$

Now substitute this into the first order equation for k_{t+1} and we get

$$\begin{aligned} & \beta E_t[V_k'(z_{t+1}, k_{t+1}, c_t, i_t)] = \lambda_t q_t \\ \implies \quad & \beta E_t\left[E_{t+1}\left[\frac{1}{c_{t+1} - \sigma c_t} - \beta\frac{\sigma}{c_{t+2} - \sigma c_{t+1}}\right][\alpha z_{t+1} k_{t+1}^{\alpha-1} + q_{t+1}(1-\delta)]\right] \\ & = E_t\left[\frac{1}{c_t - \sigma c_{t-1}} - \beta\frac{\sigma}{c_{t+1} - \sigma c_t}\right] q_t. \end{aligned}$$

Finally, just note that we can apply the law of iterated expectations and settle on the Euler equation

$$\begin{aligned}
&\beta E_t\left[\left(\frac{1}{c_{t+1}-\sigma c_t}-\beta\frac{\sigma}{c_{t+2}-\sigma c_{t+1}}\right)\left[\alpha z_{t+1}k_{t+1}^{\alpha-1}+q_{t+1}(1-\delta)\right]\right] \\
&= E_t\left[\frac{1}{c_t-\sigma c_{t-1}}-\beta\frac{\sigma}{c_{t+1}-\sigma c_t}\right]q_t.
\end{aligned}$$

Notice that if $q_t = 1$ and $\sigma = 0$, then we have the same ol' Euler equation. This equation is comparing differences in marginal utilities across periods.

Consumption and Investment. Now FOC this up with respect to investment to get

$$\beta E_t[V_i'(z_{t+1},k_{t+1},c_t,i_t)]-\lambda_t+\lambda_t q_t\left(1-\frac{\gamma}{2}\left[\frac{i_t}{i_{t-1}-1}\right]^2-\gamma\left[\frac{i_t}{i_{t-1}}-1\right]\frac{i_t}{i_{t-1}}\right)=0.$$

Throw down some envelope and we'll get

$$\begin{aligned}
&V_i'(z_t,k_t,c_{t-1},i_{t-1})=\lambda_t q_t\gamma\left[\frac{i_t}{i_{t-1}}-1\right]\frac{i_t}{i_{t-1}^2} \\
\implies\quad &V_i'(z_{t+1},k_{t+1},c_t,i_t)=\lambda_{t+1}q_{t+1}\gamma\left[\frac{i_{t+1}}{i_t}-1\right]\frac{i_{t+1}}{i_t^2}.
\end{aligned}$$

So plug this into the FOC and you get

$$\beta E_t\left[\lambda_{t+1}q_{t+1}\gamma\left(\frac{i_{t+1}}{i_t}-1\right)\frac{i_{t+1}}{i_t^2}\right]-\lambda_t+\lambda_t q_t\left(1-\frac{\gamma}{2}\left[\frac{i_t}{i_{t-1}-1}\right]^2-\gamma\left[\frac{i_t}{i_{t-1}}-1\right]\frac{i_t}{i_{t-1}}\right)=0.$$

The $-\lambda_t$ term reflects the marginal utility of consumption. If you invest one more unit, then you necessarily consume one less unit, so this is a negative immediate utility consequence of investing rather than consuming.

The other term with $\lambda_t q_t$ tells how much more capital next period is had from investing one more unit today, minus adjustment costs. The $\lambda_t q_t$ term translate this into marginal utility terms.

The expectation term is the remaining lifetime benefit from investing one more unit of capital today; it reflects the lessening of adjustment costs over time.

Interpreting q_t

Tobin's q is defined as

$$q = \frac{\text{market value of firm capital}}{\text{replacement cost of capital}}.$$

Tobin reasoned that if the market value of physical capital of a firm exceeded its replacement cost, then capital has more value "in the firm" (the numerator) than outside the firm (the denominator).

Take the investment FOC and divide it out by λ_t, giving

$$\beta E_t \left[\frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \gamma \left(\frac{i_{t+1}}{i_t} - 1 \right) \frac{i_{t+1}}{i_t^2} \right] + q_t \left(1 - \frac{\gamma}{2} \left[\frac{i_t}{i_{t-1} - 1} \right]^2 - \gamma \left[\frac{i_t}{i_{t-1}} - 1 \right] \frac{i_t}{i_{t-1}} \right) = 1.$$

The term λ_{t+1}/λ_t is a *stochastic discount factor.*

The term $\lambda_t q_t$ is the shadow price of one more unit of capital, in utility terms.