ECN 200D—Week 10 Lecture Notes
Lagos-Wright Model

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1 Kiyotaki-Wright

Randy Wright gave me extra credit for going to see *The Contractions* perform live once. Right, so the KW model is a monetary search model consisting of a producers, commodity traders, and money traders.

- If an agent has no good, then they go produce one with Poisson rate a. Then they become a commodity trader.
- As a commodity trader, they either try to find a double coincidence of wants—trader A has what trader B wants, trader B has what trader A wants, so they just swap and go back to production. Or they try to find someone who wants their good who has money, possibly trading it for that (if money isn't completely worthless), and become a money trader.
- As a money trader, they try to find someone who has a good they want and will accept money. Then they go back to production.

I'm not going to go through the details, but it turns out that there are three possible equilibria depending on beliefs about the likelihood of money being accepted—a pure exchange equilibrium, a pure money equilibrium, and a mixed equilibrium.

The KW model is nice and everything, but it's intractable in many ways. Such intractability was addressed in the next model.

2 Lagos-Wright

Every period will have two subperiods. The first subperiod—the "day market"—is a decentralized KW market. The second subperiod—the "night market"—is a Walrasian centralized market in which there is no need for money (but money can still be exchanged if desired).

We will assume that preferences are quasilinear because quasilinear preferences have no wealth effects. This means that the optimal choice is unafected by the number of assets on hand while making a decision in any given period.

- The discount rate is β.
- There are two commodities. The *general good* is traded and consumed at night; the *special good* is produced by sellers during the day, which they do not consume.
- There are two types of agents, buyers B and sellers S. They remain the same type for life.
- Buyers work H hours at night and consume x of the general good. They buy q of the special good in the day market and consume it for utility $u(q)$. So in one period, the buyer's payoff of $u(x) - H + u(q)$. This is quasilinear with respect to H.
- Sellers also work H hours at night, and they receive 1 unit of the general good per hour of work; x is the amount of the general good consumed. However, they sell q units of the special good in the day, so the seller's payoff is $u(x) - H - q$. Again, this is quasilinear with respect to H.
- We will assume that $u' > 0$ and $u'' < 0$, and furthermore that u is twice continuously differentiable. Finally, $u'(0) = \infty$.

- We will assume that there exists some optimal x^* such that $u'(x^*) = 1$.
- We will assume that there exists some optimal q^* such that $u'(q^*) = 1$.
- Goods are nonstorable.
- Money is storable and fiat.
- The central bank chooses $M_{t+1} = (1 + \mu)M_t$.
- A lump sum T is transferred to buyers in the night market.
- Buyers meet sellers in the day market with probability $\sigma \in [0, 1]$.
- In the day market, buyers and sellers bargain via Kalai bargaining, where θ is buyer bargaining power.

So the only feasible trades during day are the barter of the special good and the exchange of special good for money. At night, the only feasible trades involve the general good and money.

3 Night Market Value Functions

The Buyer. The buyer's value function is

$$W(m) = \max_{x,H,m'} \{u(x) - H + \beta V(m')\}$$

such that $x + \phi m' = H \cdot 1 + \phi m + T$, where ϕ is the number of the general good the agent has to give up for one unit of money. So the buyer has to choose how much of the general good to consume, how many hours to work, and how much money to carry to the next period.

Let's first solve the budget constraint for H and plug it into the value function. Then we've assumed that there is some optimal x^*, so let's plug that into the value function as well. Doing so allows us to write

$$W(m) = u(x^*) - x^* + \phi m + T + \max_{m'} \{-\phi m' + \beta V(m')\}.$$

Here we can see that there are no wealth effects—current money holdings m do not affect the max operator. Furthermore, $W(m)$ is linear in m, so we can simplify and write

$$W(m) = \ +\phi m, \tag{1}$$

where Λ is all that other junk.

The Seller. This case is less interesting. Sellers do not want to hold onto money, so they'll sell their money to buyers in the night market. In this case, we'll have

$$W^s(m) = \max_{x,H}\{u(x) - H + \beta V^s(0)\}$$

subject to $x = H + \phi m$. The reason sellers come in with money, however, is that they just sold stuff in the day market.

Let's again solve the budget constraint for H, plug it into the Bellman equation, and then impose the optimal x^* so that the Bellman equation can be written as

$$\begin{aligned} W^s(m) &= u(x^*) - x^* + \beta V^s(0) + \phi m \\ &= \Lambda^s + \phi m, \end{aligned}$$

again where Λ^s is all of the other junk.

4 Day Market Bargaining

Now let's move on to the day market. We want to consider the typical meeting between a buyer who carries m units of money and the seller who carries zero. We'll use Kalai bargaining. Let BS be the buyer surplus and SS be the seller surplus. Then we are to solve

$$\max_{q,d} BS \quad \text{s.t.} \quad BS = \frac{\theta}{1-\theta} SS, \quad d \leq m,$$

the first constraint being the *Kalai constriant*, the second being the *cash constraint*.

Consider buyer surplus. q is purchased for d units of money and then consumed; the buyer continues on with $m - d$ money; and leaves behind state $W(m)$. We can write this as

$$\begin{aligned} u(q) + W(m-d) - W(m) &= u(q) + \Lambda + \phi m - \Lambda - \phi d - \phi m \\ &= u(q) - \phi d. \end{aligned}$$

Now you see why the Λ grouping came in handy.

Similarly, the seller surplus is

$$\begin{aligned} -q + W^s(d) - W^s(0) &= -q + \Lambda^s + \phi d - \Lambda^s \phi \cdot 0 \\ &= -q + \phi d. \end{aligned}$$

And therefore the bargaining problem becomes

$$\begin{aligned} &\max_{q,d} u(q) - \phi d \\ \text{such that} \quad & u(q) - \phi d = \frac{\theta}{1-\theta}(\phi d - q), \\ & d \leq m. \end{aligned}$$

From the Kalai constraint, it follows that

$$\begin{aligned} [1-\theta][u(q) - \phi d] &= \theta[\phi d - q] \\ \implies \quad \phi d &= \theta q + (1-\theta)u(q) \\ &= z(q). \end{aligned}$$

We can use this to rewrite the objective function as

$$\max_{q,d} u(q) - \theta q - (1-\theta)u(q) = \max_{q,d} \theta[u(q) - q],$$

which is still subject to $d \leq m$. There are two cases.

Case 1: $m \gg d$. If m is huge, then the cash constraint is irrelevant. In this case, there is nothing stopping us from just choosing the optimal q^*. Then we can find d in the Kalai constraint, namely,

$$d^* = \frac{z(q^*)}{\phi}.$$

So we need m^* in order to pay d^*, which gives the optimal q^*.

Case 2: $m = d$. In this case, cash could be an issue. Since $m = d$, we look to the Kalai constraint to see that $\phi m = z(q)$. If $m \geq m^*$, then $d = d^*$ and all is well. However, if $m < m^*$, then $d = m$ and $q = \tilde{q}(m)$, where q satisfies $\phi m = z(q)$.

5 Day Market Value Functions

The seller is boring and we already know everything we need to know about them. For the buyer, we have

$$V(m) = \sigma[u\big(q(m)\big) + W\big(m - d(m)\big)] + (1 - \sigma)W(m),$$

where $q(m)$ and $d(m)$ arise from bargaining. We can rewrite this as

$$\begin{aligned}
V(m) &= \sigma[u\big(q(m)\big) + \Lambda + \phi m - \phi d(m)] + (1 - \sigma)(\Lambda + \phi m) \\
&= \sigma[u\big(q(m)\big) + \Lambda + \phi m - \phi d(m)] + \Lambda + \phi m - \sigma(\Lambda + \phi m) \\
&= \sigma[u\big(q(m)\big) - \phi d(m)] + W(m).
\end{aligned}$$

Let's consider the Bellman equation

$$W(m) = u(x^*) - x^* + \phi m + T + \max_{m'}\{-\phi m' + \beta V(m')\}$$

a little more, in particular the max operator. Define

$$\begin{aligned}
J(m) &= -\phi m' + \beta V(m') \\
&= -\phi m' + \beta \left(\sigma[u\big(q(m')\big) - \phi d(m')] + W(m')\right) \\
&= -\phi m' + \beta \left(\sigma[u\big(q(m')\big) - \phi d(m')] + \Lambda' + \phi' m'\right) \\
&= -\phi m' + \beta \phi' m' + \beta \Lambda' + \beta\sigma \left[u\big(q(m')\big) - \phi d(m')\right] \\
&= (-\phi + \beta\phi')m' + \beta\sigma \left[u\big(q(m')\big) - \phi d(m')\right].
\end{aligned}$$

The first term captures the net cost of carrying a unit of money from one period to another. The second term is the discounted expected buyer surplus that one additional unit of money will buy. The $\beta\Lambda'$ term has been removed entirely since it is not a function of m' and doesn't affect our subsequent analysis.

6 Money Growth

Let's assume that $\mu > \beta - 1$, which in turn implies that $i > 0$. We will only consider the Friedman rule as a limiting case.

So far we've denoted m^* as the amount of money required to buy the first-best, and you'll never need to bring more than that. So let's further assume that we will always be in the binding branch of the bargaining solution, i.e. you will in fact never bring more than m^*. This allows us to write

$$J(m') = (-\phi + \beta\phi)m' + \beta\sigma \left\{u\big(\tilde{q}(m')\big) - \phi' m'\right\}. \tag{2}$$

Claim. *In any equilibrium, $\phi \geq \beta\phi'$.*

Proof. The cost of carrying money cannot be negative, although it could be zero. If $\phi < \beta\phi'$, then you'd carry $m' = \infty$. This cannot be the case. So it must be the case that $(\beta\phi' - \phi)m' \leq 0$. □

Claim. *A nonmonetary equilibrium where $\psi = \psi' = 0$ always exists.*

Proof. Since money has no value and no price, it's entirely arbitrary what the price turns out to be—it could be anything, in particular, zero. □

In a monetary equilibrium, $m' > 0$. Taking the first order condition of equation (2) with respect to m', we get

$$\phi = \beta\phi' + \beta\sigma\left\{u'\big(\tilde{q}(m')\big)\tilde{q}'(m') - \phi'\right\}.$$

Recall that $\tilde{q}(m)$ solves $\phi m = z\big(\tilde{q}(m)\big)$. Taking the derivative with respect to m, it follows that

$$\phi = z'\big(\tilde{q}(m)\big)\tilde{q}'(m) \quad \Longrightarrow \quad \tilde{q}'(m') = \frac{\phi'}{z'\big(\tilde{q}(m')\big)}.$$

Plug this into the first order condition for

$$\begin{aligned} \phi &= \beta\phi' + \beta\sigma\left\{\frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)}\phi' - \phi'\right\} \\ &= \beta\phi'\left[1 + \sigma\left\{\frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1\right\}\right] \\ \Longrightarrow \quad \frac{\phi}{\beta\phi'} - 1 &= \sigma\left\{\frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1\right\}. \end{aligned} \tag{3}$$

Equation (4) is the demand for money. Let's focus on the steady state where real balances are equal, i.e. where $\phi M = \phi' M'$. Then we can write

$$\frac{\phi}{\beta\phi'} - 1 = \frac{M(1+\mu)\phi}{M'\beta\phi'} - 1 = \frac{1+\mu}{\beta} - 1 = i,$$

where the last equality follows from the Fisher equation and the illiquid real interest rate. So we can write the demand for money as

$$i = \sigma\left\{\frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1\right\}. \tag{4}$$

Definition 1. A **steady state monetary equilibrium** is a list of objects (q, Z), where q is the amount of the special good in a typical trade (we're interested in how close to q^* it is) and $Z = \phi M > 0$ such that

(a) We can pin down q with

$$i = \sigma \left\{ \frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1 \right\}$$

(b) $\phi M = Z = z(q) = \theta q + (1-\theta)u(q)$.

7 Comparative Statics

Let's see what effect changing i has on q. Differentiate the money demand function with respect to q, i.e. find

$$\frac{di}{dq} = \sigma \frac{d}{dq}\left[\frac{u'}{z'}\right].$$

Since $z = \theta q + (1-\theta)u$, it follows that $z' = \theta + (1-\theta)u'$. Therefore

$$\begin{aligned} \frac{u'}{z'} &= \frac{u'}{\theta + (1-\theta)u'} \\ \implies \quad \frac{d}{dq}\left[\frac{u'}{z'}\right] &= \frac{\theta u'' + (1-\theta)u'u'' - u'(1-\theta)u''}{[\theta + (1-\theta)u']^2} \\ &= \frac{\theta u''}{[\theta + (1-\theta)u']^2} < 0 \end{aligned}$$

because we assume $u'' < 0$. It follows that

$$\frac{di}{dq} = \sigma \frac{\theta u''}{[\theta + (1-\theta)u']^2} < 0 \quad \implies \quad \frac{dq}{di} < 0.$$

So as the interest rate rises, people want to carry less cash. Makes sense since i is the opportunity cost of holding cash.

What happens as $i \to 0$, i.e. when we approach the Friedman rule? Then the money demand function can be written as

$$\begin{aligned}
0 &= \sigma \left\{ \frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1 \right\} \\
\implies \quad & u'(q) = z'(q) \\
\implies \quad & u'(q) = \theta + (1-\theta)u'(q) \\
\implies \quad & u'(q) = 1 \\
\implies \quad & q = q^*.
\end{aligned}$$

So the Friedman rule implies that the optimal quantity of money is carried. That's nice.

8 Monetary Equilibrium

We've established that if $i = 0$, then people hold $q^* > 0$. We also know that q decreases as i increases. The big question is this: can i get so large that people no longer wish to hold money, i.e. $q = 0$? Indeed, it can.

Claim. *A monetary equilibrium exists only if*

$$i < \sigma \frac{\theta}{1-\theta}.$$

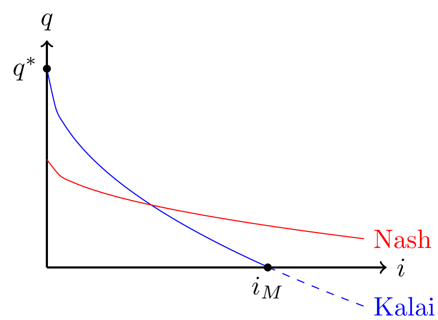
Proof. We'll be considering the money demand function,

$$i = \sigma \left\{ \frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1 \right\}.$$

Recall that $u'(q) = \infty$ as $q \to 0$. Furthermore, $z' = \theta + (1-\theta)u'(q)$. The problem is that we'll have infinity over infinity if we try to take the limit as it. Apply L'Hopital's rule and we can instead take the limit of

$$\sigma\left\{\frac{u''}{(1-\theta)u''}-1\right\}=\frac{\sigma\theta}{1-\theta}=i_M.$$

So $q>0$ if $i\leq i_M$. Otherwise there is no monetary equilibrium. □



Under Kalai bargaining (blue), only interest rates below i_M can sustain a monetary equilibrium. Under Nash bargaining (red), we *are* guaranteed a monetary equilibrium, although we can never have optimal q^*.

9 Two Types of Money

Suppose we have two types of money, m_1 and m_2. Each type of money grows at its own rate μ_i.

Perfect Substitutes. Suppose that the types of money are perfect substitutes—every seller accepts both types of money. For these types of problems, we really only need to pay attention to the J functions. And there is a pattern for writing then.

$$\begin{aligned}J(m_1',m_2')&=(-\phi_1+\beta\phi_1')m_1'+(-\phi_2+\beta\phi_2')m_2'\\&\quad+\beta\sigma\big[u\big(q(m_1',m_2')\big)-\phi_1'd_1(m_1',m_2')-\phi_2'd_2(m_1',m_2')\big].\end{aligned}$$

The first line represents, respectively, the holding cost of m_1 and the holding cost of m_2.

Imperfect Substitutes. Now suppose that with probability $\lambda \in [0,1]$, sellers accept money type 2, whereas everyone accepts money type 1. This means that we must solve two bargaining problems depending on which kind of seller is met. So now the J function is

$$\begin{aligned} J(m_1', m_2') = (-\phi_1 + \beta\phi_1)m_1' + (-\phi_2 + \beta\phi_2)m_2' \\ + \beta\sigma\Big\{\lambda\big[u\big(q^2(m_1', m_2')\big) - \phi_1' d_1^2(m_1', m_2') - \phi_2' d_2^2(m_1', m_2')\big] \\ + (1-\lambda)\big[u\big((q^1(m_1')\big) - \phi_1' d_1^1(m_1')\big]\Big\}. \end{aligned}$$

The first line is the cost carrying each type of money. The second line is the case where you run into a seller who accepts either type of money, the superscript indicating the bargaining outcomes of such an encounter. The third line is the case where you run into a seller who only accepts money type 1, but superscript again indicating the relevant bargaining outcome.