Production Sets

A **production vector** is a vector $y = (y_1, ..., y_L)$ that describes the net outputs of the L commodities from a production process. Positive numbers are outputs and negative numbers are inputs. So if we have $y = (-2, 5)$, then we used two units of y_1 to create five units of y_2. If $y_\ell = 0$, then none of commodity ℓ was used.

The set of all production vectors that constitute feasible plans for the firm is known as the **production set** and is denoted $Y \subset \mathbb{R}^L$. Any $y \in Y$ is possible, any $y \notin Y$ is not possible. The set of feasible plans is limited primarily by technological constraints.

Sometimes it is convenient to describe the production set Y using a function $F(\cdot)$ called the **transformation function**. It satisfies

$$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$$

and $F(y) = 0$ if and only if y is an element of the boundary of Y. The set of boundary points of Y, $\{y \in \mathbb{R}^L : F(y) = 0\}$ is known as the **transformation frontier**.

If $F(\cdot)$ is differentiable and if the production vector $\overline{y}$ satisfies $F(\overline{y}) = 0$, then for any commodities ℓ and k, the ratio

$$MRT_{\ell k}(\overline{y}) = \frac{\partial F(\overline{y})/\partial y_\ell}{\partial F(\overline{y})/\partial y_k}$$

is the **marginal rate of transformation** of good ℓ for good k at $\overline{y}$. It is a measure of how much the output of good k can increase given a one marginal unit loss of good ℓ. To see why, we can use the implicit function theorem (because $F(\overline{y}) = 0$) to get

$$\frac{\partial F(\overline{y})}{\partial y_k} dy_k + \frac{\partial F(\overline{y})}{\partial y_\ell} dy_\ell = 0,$$

which results in $MRT_{\ell k} = -\partial k/\partial \ell$. I think it's easier to see the meaning in this form.

Sometimes it's helpful to partition the goods into inputs and outputs. Let $q = (q_1, ..., q_m) \geq 0$ be the production level of the firms M outputs. Let $z = (z_1, ..., z_{L-M}) \geq 0$ denote the firms $L - M$ inputs, which are now nonnegative. (Let's also put goods not used, i.e. goods with a zero entry, as inputs.)

It's sometimes useful to describe a single-output technology by way of a **production function**, denoted $f(z)$. It gives the maximum amount q of output that can be produced using inputs $z = (z_1, ..., z_{L-1})$. The production function then gives rise to the production set

$$Y = \{(-z_1, ..., -z_{L-1}, q) : q - f(z_1, ..., z_{L-1}) \leq 0\},$$

where $(z_1, ..., z_{L-1}) \geq 0$. Notice that $q - f(z_1, ..., z_{L-1})$ is, in a sense, the transformation function. The term q is the maximum amount of output that can be produced given the inputs, so the production function is at most q. Hence the nonpositive difference.

Holding the level of output fixed, define the **marginal rate of technical substitution** of input ℓ input k at $\overline{z}$ to be

$$MRTS_{\ell k}(\overline{z}) = \frac{\partial f(\overline{z}) \partial z_\ell}{\partial f(\overline{z}) \partial z_k}.$$

Supposing $f(\overline{z}) = \overline{q}$ is fixed, we can again use the implicit function theorem to have

$$\frac{\partial f(\overline{z})}{\partial z_k} dk + \frac{\partial f(\overline{z})}{\partial z_y} dy = 0,$$

which implies that $MRTS = -dz_k/dz_\ell$. It is a measure of how much we must increase input k by given a unit loss of input ℓ to remain at output $\overline{q}$. It's also just a special case of the marginal rate of transformation for a single-output, many-input technology.

Properties of Production Sets

Here, have a shit ton of properties that are commonly assumed of production sets.

- **(a)** *Y is nonempty.* This just means the firm can actually plan to do something. Otherwise why the fuck are we even here?
- **(b)** *Y is closed.* So the set Y includes its boundary. This is mostly a technical condition.
- **(c)** *No free lunch.* It is not possible to produce something from nothing. Which is to say, if there are any positive elements in y (outputs), then there must be some negative elements (inputs) in y as well. In other words, you need to actually use inputs to generate outputs.
- **(d)** *Possibility of inaction.* This just says that $0 \in Y$. This is a stronger assumption than it might seem at first glance. If contracts have been signed, for instance, then inaction is not possible — the contracts already constitute an input.
- **(e)** *Free disposal.* The firm can absorb additional amounts of inputs without any reduction in output. Consider some $y \in Y$ and some other $y' \leq y$.

Then $y' \in Y$. The idea is that y' produces at most the same amount of outputs as y using at least the same amount of inputs. The interpretation is that all of those extra inputs can be disposed of with no cost.

(f) *Irreversibility.* If $y \in Y$ and $y \neq 0$, then $-y \notin Y$. Which is to say, you can't swap inputs and outputs and expect everything to work out all nice and fancy. If it takes 2 of good 1 to create 5 of good 2, i.e. the vector $y = (-2, 5)$, then you can't say it takes 5 of good 2 to create 2 of good 1, i.e. the vector $-y = (2, -5)$.

This is somewhat intuitive. If it takes two eggs and a cup of flour to bake a cake, you can't say it takes one cake to produce two eggs and a cup of flour. In fact, it's nonsensical. (I interpret this as an application of the second law of thermodynamics.) It is, in a sense, saying you can't go backwards and un-build something, giving you back the exact same parts you used to build it in the first place.

(g) *Nonincreasing returns to scale.* If we take some fraction $\alpha \in [0, 1]$ of every input and output of the vector $y \in Y$, then $\alpha y \in Y$ as well. Thus, production can be scaled down. Note that this also implies that inaction is possible since we can have $\alpha = 0$. Graphically, this means that the set Y is not bowed inwards.

(h) *Nondecreasing returns to scale.* Now suppose that $\alpha \geq 1$. Then if $y \in Y$, we have $\alpha y \in Y$. Thus, production can be scaled up. Graphically, this means that the set Y is not bowed outwards.

(i) *Constant returns to scale.* Bet you didn't see this coming. Okay, so this just says if $y \in Y$, then $\alpha y \in Y$ for any $\alpha \geq 0$. Geometrically, Y is a cone – the frontier is straight.

(j) *Additivity.* Suppose $y \in Y$ and $y' \in Y$. Then additivity requires that $y + y' \in Y$. In other words, $Y + Y \subset Y$. The economic interpretation is that if y and y' are both possible, then one can set up two plans that do not interfere with each other and produce y and y' independently. The result is the overall production vector $y + y'$.

We can also relate this to free entry. If $y \in Y$ is being produced, then another firm can enter and produce y' with the net result being the vector $y + y'$. So the *aggregate production set* must satisfy additivity whenever free entry is possible.

(k) *Convexity.* This is a biggie. If $y, y' \in Y$ and $\alpha \in [0, 1]$, then $\alpha y + (1 - \alpha) y' \in Y$.

This captures two economics ideas. First, it captures nonincreasing returns as long as we assume $0 \in Y$. This is because $\alpha y = \alpha y + (1 - \alpha) 0 \in Y$.

Second, it captures the idea that "unbalanced" input combinations are not more productive than balanced ones. Or, equivalently, that unbalanced output combinations are not least costly to produce than balanced ones. In particular, if y' and y' produce the same output but with different input combinations, then the production vector that uses the average of the two input combinations will produce at least as much output as either y or y'.

(l) *Y is a convex cone.* This is just a mixture of convexity and constant returns to scale. Formally, this means that for any vectors $y, y' \in Y$ and constant $\alpha \geq 0$ and $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.

Proposition 1. *Suppose that $f(\cdot)$ is the production function associated with a single-output technology, and let Y be the production set of this technology. Then Y satisfies constant returns to scale if and only if $f(\cdot)$ is homogeneous of degree one.*

Proposition 2. *For a single-output technology, Y is convex if and only if the production function $f(z)$ is concave.*

Proposition 3. *The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.*

Profit Maximizing

Have $p \gg 0$, and assume that p is independent of the firm's production plans, i.e. the firm is a price taker. We will also assume that the firm's objective is to maximize its profit. In the process, we will assume that Y is nonempty, closed, and disposal is free.

Alright, so the **profit maximization problem (PIMP)** is given by

$$\arg\max_y \ \ p \cdot y \ \ s.t. \ \ y \in Y.$$

Alternatively we can use a transformation function to describe Y, which gives the PIMP as

$$\arg\max_y \ \ p \cdot y \ \ s.t. \ \ F(y) \leq 0.$$

Note that $p \cdot y$ already includes both revenues and costs since inputs are negative values.

Okay then, the firm's **profit function**, denoted $\pi(p)$, associates with every p the amount

$$\pi(p) = \max\{p \cdot y : y \in Y\}.$$

The firm's **supply correspondence** at p, denoted $y(p)$, is the set of profit maximizing vectors,

$$y(p) = \{y \in Y : p \cdot y = \pi(p)\}.$$

In general, $y(p)$ could be a set. If Y is strictly convex, then we will have a unique maximizer. It is also possible that there is no bound on profit so that $\pi(p) = \infty$.

If $F(\cdot)$ is differentiable, then we can use some first order conditions to characterize the PIMP solution. In particular, if $y^* \in y(p)$, then for some $\lambda \geq 0$, it must be the case that y^* satisfies

$$p_\ell = \lambda \frac{\partial F(y^*)}{\partial y_\ell} \text{ for } \ell = 1, ..., L,$$

or in matrix notation,

$$p = \lambda \nabla F(y^*).$$

Typical Lagrange stuff, n'est pas? It leads to the result where, for all ℓ and k,

$$MRT_{\ell k}(y^*) = \frac{p_\ell}{p_k}.$$

So the slope of the transformation frontier at the profit-maximizing production plan must be equal to the price ratio.

When we have a single-output technology with differentiable $f(z)$, we can view the firm's decision as a choice over its input levels z. Let $p > 0$ be the price of the firms output and $w \gg 0$ be the input prices. Then the input vector z^* maximizes profit given (p, w) if it solves

$$\arg\max_{z \geq 0} pf(z) - w \cdot z.$$

If z^* is optimal, then for all $\ell = 1, ..., L-1$, the first order conditions

$$p \frac{\partial f(z^*)}{\partial z_\ell} \leq w_l, \text{ with equality if } z^*_\ell > 0,$$

or written in matrix notation,

$$p\nabla f(z^*) \leq w \quad \text{and} \quad [p\nabla f(z^*) - w] \cdot z^* = 0.$$

In other words, if z_ℓ is actually used, then the marginal product of z_ℓ must equal w_ℓ / p_ℓ. Also, for $(z^*\ell, z^*_k) \gg 0$, we also have

$$MRTS_{\ell k} = \frac{w_l}{w_k}.$$

Note that if Y is convex, then the first order conditions are both necessary and sufficient for the determination of the PIMP solution.

Now here are a shit ton of properties about the PIMP.

(a) $\pi(\cdot)$ is homogeneous of degree one.

(b) $\pi(\cdot)$ is convex.

(c) If Y is convex, then

$$Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}.$$

(d) $y(\cdot)$ is homogeneous of degree zero.

(e) If Y is convex, then $y(p)$ is a convex-valued set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if nonempty).

(f) (Hotelling's Lemma) If $y(\overline{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\overline{p}$ and $\nabla\pi(\overline{p}) = y(\overline{p})$.

(g) If $y(\cdot)$ is a function differentiable at $\overline{p}$, then $Dy(\overline{p}) = D^2\pi(\overline{p})$ is a symmetric and positive semidefinite matrix with $Dy(\overline{p})\overline{p} = 0$.

The positive semidefiniteness of $Dy(p)$ is the **law of supply**. Which is to say, quantities respond in the same direction as price changes. We can write the law of supply as

$$(p - p') \cdot (y - y') \geq 0,$$

which follows from a revealed-preference type argument by taking $y \in y(p)$ and $y' \in y(p')$.

Cost Minimizing

Cost minimization is a necessary condition for profit maximization. Which is to say, if profit is maximized, then costs are minimized. Here we consider a single-output case where z is a nonnegative vector of inputs and $f(z)$ is the production functions. The number q is a production threshold above which we must produce, and $w \gg 0$ is the vector of input prices. We'll also assume free disposal of output. The **cost minimization problem (CMP)** can be stated as

$$\arg\min_{z \geq 0} w \cdot z \;\; s.t. \;\; f(z) \geq q.$$

The optimized value of the CMP is the **cost function**, denoted $c(w, q)$. The optimizing set of input choices, denoted $z(w, q)$, is the **conditional factor demand correspondence**. The term condition appears because the factor demands are conditional on the requirement that output level q at least be produced.

If z^* is optimal, and if $f(\cdot)$ is differentiable, then for some $\lambda \geq 0$, and for every input $\ell = 1, ..., L-1$, we must have

$$w_\ell \geq \lambda \frac{\partial f(z^*)}{\partial z_\ell}, \text{ with equality if } z^*_\ell > 0,$$

or in matrix notation,

$$w \geq \lambda \nabla f(z^*) \quad \text{and} \quad [w - \lambda \nabla f(z^*)] \cdot z^* = 0.$$

If Y is convex (i.e. if $f(\cdot)$ is concave), then this is necessary and sufficient for z^* to be an optimum.

As in the PIMP, the CMP conditions also imply that if $(z_\ell, z_k) \gg 0$, then $MRTS_{\ell k} = w_\ell / w_k$. Shouldn't be surprising since PIMP implies CMP. The Lagrange multiplier λ can be interpreted as the marginal value of relaxing the constraint $f(z^*) \geq q$. That is, $\lambda = \partial c(w,q)/\partial q$, the **marginal cost of production**.

Now have another shit ton of properties.

Proposition 4. *Suppose that $c(w,q)$ is the cost function of single-output technology Y with production function $f(\cdot)$ and that $z(w,q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then*

- ***(a)*** *$c(\cdot)$ is homogeneous of degree one in w and nondecreasing in q.*
- ***(b)*** *$c(\cdot)$ is a concave function of w.*
- ***(c)*** *If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then*

$$Y = \{(-z, q) : w \cdot z \geq c(w,q) \text{ for all } w \gg 0.\}$$

- ***(d)*** *$z(\cdot)$ is homogeneous of degree zero in w.*
- ***(e)*** *If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w,q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is a strictly convex set, then $z(w,q)$ is single-valued.*
- ***(f)*** *(Shephard's Lemma) If $z(\overline{w}, q)$ consists of a single point, then $c(\cdot)$ is differentiable with respect to w at $\overline{w}$ and $\nabla_w c(\overline{w}, q) = z(\overline{w}, q)$.*
- ***(g)*** *If $z(\cdot)$ is differentiable at $\overline{w}$, then $D_w z(\overline{w}, q) = D_w^2 c(\overline{w}, q)$ is a symmetric and negative semidefinite matrix with $D_w z(\overline{w}, q)\overline{q} = 0$.*
- ***(h)*** *If $f(\cdot)$ is homogeneous of degree one (i.e. exhibits constant returns to scale), then $c(\cdot)$ and $z(\cdot)$ are homogeneous of degree one in q.*
- ***(i)*** *If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q. In particular, marginal costs are nondecreasing in q.*

Single-Output Geometry

Hold prices constant at some $\overline{w} \gg 0$. Let the cost function be $C(q) = c(\overline{w}, q)$. When $q > 0$, we denote the firm's **average cost** as $AC(q) = C(q)/q$. Assuming the derivative exists, the **marginal cost** is denoted $C'(q) = dC(q)/dq$.

For a given output price p, profit maximizing levels of output $q \in q(p)$ must satisfy

$$p \leq C'(q) \quad \text{with equality if } q > 0.$$

If Y is convex, then $C(\cdot)$ is a convex function and thus marginal cost is nondecreasing. Thus, the first order condition is sufficient to establish that q is a profit-maximizing output level at price p. Without convex technology, then satisfying the first order condition does not imply that q is profit maximizing.

The level(s) of production corresponding to the minimum average cost is called the **efficient scale**, denoted by $\overline{q}$ if unique. We will be at $\overline{q}$ when $AC(\overline{q}) = C'(\overline{q})$.

Aggregation

The nice thing about supply is that there are no wealth effects – we only have substitution effects along the production frontier. This makes supply aggregation a lot easier and more powerful.

Suppose there are J production units in the economy, each has its own production set $Y_1, ..., Y_j$, each nonempty, closed, and with free disposal. Each firm has profit $\pi_j(p)$ and supply $y_j(p)$. The **aggregate supply correspondence** is the sum of the individual supply correspondences,

$$\begin{aligned} y(p) &= \sum_{j=1}^{J} y_j(p) \\ &= \{y \in \mathbb{R}^L : y = \sum_j y_j \text{ for some } y_j \in y_j(p), \forall j.\}. \end{aligned}$$

Suppose that every $y_j(\cdot)$ is single-valued and differentiable at p. We already know that $Dy_j(p)$ is symmetric and negative semidefinite. These properties are preserved under addition, so $Dy(p)$ is also symmetric and positive semidefinite. Yay!

The positive semidefiniteness of $Dy(p)$ implies the law of supply in aggregate, and this holds for all price changes:

$$(p - p') \cdot [y(p) - y(p')] \geq 0.$$

Given $Y_1, ..., Y_J$, define the **aggregate production**

set as

$$\begin{aligned} Y &= Y_1 + ... + Y_k \\ &= \{y \in \mathbb{R}^L : y = \sum_j y_j \text{ for some } y_j \in Y_j, j = 1, ..., J\}. \end{aligned}$$

In other words, the aggregate production set Y describes the production vectors that are feasible in aggregate if all the production sets are used together. Let $\pi^*(p)$ and $y^*(p)$ be the profit function and the supply correspondence of Y, respectively. The are the profit function and supply correspondence that would arise if a single price-taking firm were to operate, under the same management, all of the individual production sets.

Proposition 5. *Let $\pi^*(p)$ be the aggregate profit function, $y^*(p)$ be the aggregate supply correspondence, and $\pi_j(p)$, $y_i(p)$ be the individual profit function and supply correspondence. Then we have*

(a) $\pi^*(p) = \sum_j \pi_j(p)$

(b) $y^*(p) = \sum_j y_j(p)$

In words: the aggregate profit obtained by each production unit maximizing profit separately, taking prices as given, is the same as that which would be obtained if they were to coordinate their actions in a joint profit maximization decision. Pretty cool.

Consider the single-output case. The aggregate output produced by the firms is $q = \sum_j q_j$. So the total cost of production is $c(w, q)$, the value of the aggregate cost function. Thus, the allocation of the production of output level q among the firms is cost minimizing. Aggregate output $q(p)$ has a similarly convenient aggregation.

Efficient Production

Definition 1. *A production vector $y \in Y$ is **efficient** if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.*

So no point in the interior of Y can be efficient – we'd always be able to just move in some positive direction in the direction of the boundary. On the other hand, not every boundary vector is efficient – consider a horizontal or flat boundary where we can increase in some direction without decreasing in another.

Here, have a version of the **first fundamental theorem of welfare economics**.

Proposition 6. *If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.*

This proposition holds even if the production set is nonconvex.

If a collection of firms each independently maximizes profits with respect to the same fixed price vector $p \gg 0$, then the aggregate production function is **socially efficient** – no other production plan for the entire economy could more using no additional inputs.

The converse does not necessarily hold. That is, we cannot say that any efficient production vector is profit maximizing for some price system. To get this result, we have to assume that Y is convex, which gives a version of the **second fundamental theorem of welfare economics**.

Proposition 7. *Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero price vector $p \geq 0$.*