

# Stone-Geary Technology

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December 17, 2016

We will consider a setting with  $L + 1$  commodities. The first  $L$  commodities serve as inputs and the  $L + 1$ th commodity is the output. Input commodity  $\ell$  has input price  $w_\ell$ , and the output commodity has price  $p$ . We consider the general Stone-Geary production function

$$f(z) = A \prod_{\ell=1}^L (z_\ell - \gamma_\ell)^{\alpha_\ell},$$

where  $A$  represents the total-factor productivity, an exogenous measure of the level of "technology." Notice how similar this is to Cobb-Douglas technology. The key difference are these  $\gamma_\ell$  terms, which represent a sort of "subsistence level," below which the function is not defined. In other words, it must be the case that  $z_\ell \geq \gamma_\ell$ . In the context of production, this means that at least  $\gamma_\ell$  units of good  $\ell$  must be used in production – the interpretation is not as clear in production theory as it is in consumer theory. Perhaps the mafia has threatened the firm into using  $\gamma_\ell$  inputs of commodity  $\ell$ .

## 1 Factor Demand and Cost Minimization

The cost minimization problem is

$$\min_{z \geq 0} w_1 z_1 + \dots + w_L z_L \quad \text{s.t.} \quad A \prod_{\ell=1}^L (z_\ell - \gamma_\ell)^{\alpha_\ell} \geq q.$$

We will find the factor demand functions first, and then utilize those to find the cost function.

## 1.1 Conditional Factor Demand Functions

To find the conditional factor demands, we want to solve

$$\arg \min_{z \geq 0} w_1 z_1 + \dots + w_L z_L \quad \text{s.t.} \quad A \prod_{\ell=1}^L (z_\ell - \gamma_\ell)^{\alpha_\ell} \geq q,$$

which, supposing an interior solution (i.e.  $z_\ell > \gamma_\ell$  for all  $\ell$ ), gives the Lagrangian

$$L(z, \lambda) = w_1 z_1 + \dots + w_L z_L + \lambda \left[ q - A \prod_{\ell=1}^L (z_\ell - \gamma_\ell)^{\alpha_\ell} \right].$$

The first order conditions are

$$\frac{\partial L(z, \lambda)}{\partial z_k} = w_k - \lambda \frac{\alpha_k}{(z_k - \gamma_k)} A \prod_{\ell=1}^L (z_\ell - \gamma_\ell)^{\alpha_\ell} := 0, \quad (1)$$

$$\lambda \left[ q - A \prod_{\ell=1}^L (z_\ell - \gamma_\ell)^{\alpha_\ell} \right] := 0, \quad (2)$$

$$z_\ell \geq \gamma_\ell. \quad (3)$$

We can use equation 1 to show that

$$\frac{w_\ell (z_\ell - \gamma_\ell)}{\alpha_\ell} = \lambda f(z),$$

and therefore we have the string of equalities

$$\frac{w_1 (z_1 - \gamma_1)}{\alpha_1} = \frac{w_2 (z_2 - \gamma_2)}{\alpha_2} = \dots = \frac{w_L (z_L - \gamma_L)}{\alpha_L}.$$

Solve for  $z_2 - \gamma_2$  in terms of  $z_1$  to get

$$z_2 - \gamma_2 = \frac{\alpha_2 w_1}{\alpha_1 w_2} (z_1 - \gamma_1).$$

Doing this for all  $z_\ell$ , we have the system

$$\begin{aligned}
z_1 - \gamma_1 = \frac{\alpha_1 w_1}{\alpha_1 w_1} (z_1 - \gamma_1) &\implies (z_1 - \gamma_1)^{\alpha_1} = \left( \frac{w_1 \alpha_1}{\alpha_1 w_1} \right)^{\alpha_1} (z_1 - \gamma_1)^{\alpha_1}, \\
z_2 - \gamma_2 = \frac{\alpha_2 w_1}{\alpha_1 w_2} (z_1 - \gamma_1) &\implies (z_2 - \gamma_2)^{\alpha_2} = \left( \frac{w_1 \alpha_2}{\alpha_1 w_2} \right)^{\alpha_2} (z_1 - \gamma_1)^{\alpha_2}, \\
&\vdots \\
z_L - \gamma_L = \frac{\alpha_L w_1}{\alpha_1 w_L} (z_1 - \gamma_1) &\implies (z_L - \gamma_L)^{\alpha_L} = \left( \frac{w_1 \alpha_L}{\alpha_1 w_L} \right)^{\alpha_L} (z_1 - \gamma_1)^{\alpha_L}.
\end{aligned}$$

Notice that from equation 1, we can't have  $\lambda = 0$  because otherwise  $w_k = 0$  whereas we assume  $w \gg 0$ . Therefore from equation 2, we must have

$$A \prod_{\ell=1}^L (z_\ell - \gamma_\ell)^{\alpha_\ell} = q.$$

Thus, we can write

$$A \left( \frac{w_1 \alpha_1}{\alpha_1 w_1} \right)^{\alpha_1} (z_1 - \gamma_1)^{\alpha_1} \left( \frac{w_1 \alpha_2}{\alpha_1 w_2} \right)^{\alpha_2} (z_1 - \gamma_1)^{\alpha_2} \dots \left( \frac{w_1 \alpha_L}{\alpha_1 w_L} \right)^{\alpha_L} (z_1 - \gamma_1)^{\alpha_L} = q.$$

Let  $\sum_{\ell=1}^L \alpha_\ell = \alpha$ . We can rewrite the above equation as

$$A (z_1 - \gamma_1)^\alpha \left( \frac{w_1}{\alpha_1} \right)^a \prod_{\ell=1}^L \left( \frac{\alpha_\ell}{w_\ell} \right)^{\alpha_\ell} = q.$$

Finally, we can solve for  $z_1$  to get

$$z_1 = \frac{\alpha_1}{w_1} \left( \frac{q}{A} \prod_{\ell=1}^L \left[ \frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} + \gamma_1.$$

Notice that this is exactly the Cobb-Douglas conditional factor demand but with  $\gamma_1$  added. More generally,

$$z_k = \frac{\alpha_k}{w_k} \left( \frac{q}{A} \prod_{\ell=1}^L \left[ \frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} + \gamma_k.$$

## 1.2 Cost Function

To find the cost function, we can plug in the conditional factor demands into the objective function. Let's look more closely at the first term,

$$w_1 z_1 = w_1 \left[ \frac{\alpha_1}{w_1} \left( \frac{q}{A} \prod_{\ell=1}^L \left[ \frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} + \gamma_1 \right] \implies \alpha_1 \left( \frac{q}{A} \prod_{\ell=1}^L \left[ \frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} + w_1 \gamma_1.$$

When we sum all of them up, we'll have the cost function

$$c(w, q) = \alpha \left( \frac{q}{A} \prod_{\ell=1}^L \left[ \frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} + \sum_{\ell=1}^L w_\ell \gamma_\ell.$$

## 2 Output Supply Function and Profit Function

### 2.1 Output Supply Function

To find the output supply function, we want to solve

$$\arg \max_{q \geq 0} pq - \alpha \left( \frac{q}{A} \prod_{\ell=1}^L \left[ \frac{w_\ell}{\alpha_\ell} \right]^{\alpha_\ell} \right)^{1/\alpha} - \sum_{\ell=1}^L w_\ell \gamma_\ell.$$

All we need to do is take the derivative with respect to  $q$  and find the critical point. Note that we need  $\alpha < 1$  to ensure that the function is concave in  $q$  and thus the critical point is a maximum. Also notice that there is no  $q$  in the sum, we get the same result as with Cobb-Douglas:

$$q(p, w) = \left( p^\alpha A \prod_{\ell=1}^L \left[ \frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)}.$$

### 2.2 Profit Function

Now for the profit function, we plug the output supply function into the objective function. Again, it's going to be practically identical to the Cobb-Douglas case except with the sum tacked on at the end:

$$\pi(p, w) = (1 - \alpha) \left( p A \prod_{\ell=1}^L \left[ \frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} + \sum_{\ell=1}^L w_\ell \gamma_\ell.$$

## 2.3 Input Demand Functions

We can appeal to Cobb-Douglas again. Using Shepherd's lemma, we find the input demand functions to be

$$z_k(p, w) = -\frac{\partial \pi(p, w)}{\partial w_k} = \frac{\alpha_k}{w_k} \left( pA \prod_{\ell=1}^L \left[ \frac{\alpha_\ell}{w_\ell} \right]^{\alpha_\ell} \right)^{1/(1-\alpha)} + \gamma_k.$$