1 Population Regression

When we estimate things, our estimation is going to depend on whatever sample we happen to have obtained. That sample is usually not going to be a perfect representation of the population, and hence any given sample will differ from the population in random ways.

To illustrate, suppose you have a population of 100 people and you want to estimate their income. You take 20 random samples, someone else takes 20 random samples. Chances are you won't sample the exact same 20 people and hence your estimates will be a bit different. We need to account for that sampling variability.

In the context of regressions, we'd like a regression that best fits the population data. It will be given by the formula

$$y_i = \beta_1 + \beta_2 x_i,$$

which I will explain momentarily.

Assumption 1. Again, this is just the line of best fit – it is not the of perfect fit. In generality there will be no line of perfect fit. So when we talk about a specific data point i, the true population model is

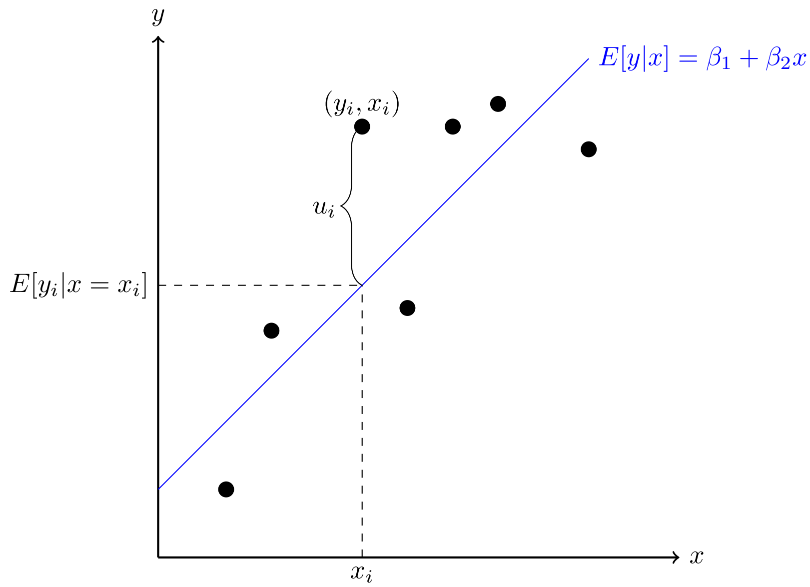
$$y_i = \beta_1 + \beta_2 x_i + u_i,$$

where u_i, called the **error**, is the difference between the actual point y_i given x_i and the regression line's estimate with x_i.

Assumption 2. Now use assumption 2, the zero conditional mean: $E[u_i|x_i] = 0$. This allows us to take the true population model and write

$$\begin{aligned} E[y_i|x = x_i] &= E[\beta_1|x = x_i] + E[\beta_2 x_i|x = x_i] + E[u_i|x = x_i] \\ &= \beta_1 + \beta_2 x_i. \end{aligned}$$

This is true because β_1 and β_2 are just numbers – there is nothing random about them – so we, uh, expect them to be themselves. And because of our zero conditional mean assumption, the error term drops out. Thus, the regression line is what we expect y_i to be, given x_i.



Pick some arbitrary data point (x_i, y_i). The regression line tells us $E[y_i|x = x_i]$, that is, what value we expect y_i to be for independent variable x_i. This is the **conditional mean** of y_i given x_i. But the regression line is a line of *best* fit, not a line *perfect* fit, so the actual value of y_i will in general be different than what we expect it to be based on the regression line. The difference between what we expect y_i to be based on the regression and what y_i actual is is called the **error**, denoted u_i.

To summarize the population characteristics:

- The actual value y_i is given by $y_i = \beta_1 + \beta_2 x_i + u_i$.
- The regression line is what we expect y_i to be, given x_i. Expressed in the maths, $E[y_i|x = x_i] = \beta_1 + \beta_2 x_i$. This is a consequence of assumptions 1 and 2 combined.
- And hence the error term is given by $u_i = y_i - E[y_i|x = x_i]$.

We can throw down two more assumptions to make analysis easier.

- **Assumption 3: homoskedasticity.** The variation of u given x_i is the same for any x_i. In math,
$$\text{Var}(u_i|x_i) = \sigma_u^2 \quad \forall i.$$
- **Assumption 4: independent errors.** Errors for different observations are statistically independent: u_i is independent of u_j whenever $i \neq j$.

Assumptions 3 and 4 allow us to say that the variation of y given x is also constant, and specifically, $\text{Var}(y|x) = \sigma_u^2$.

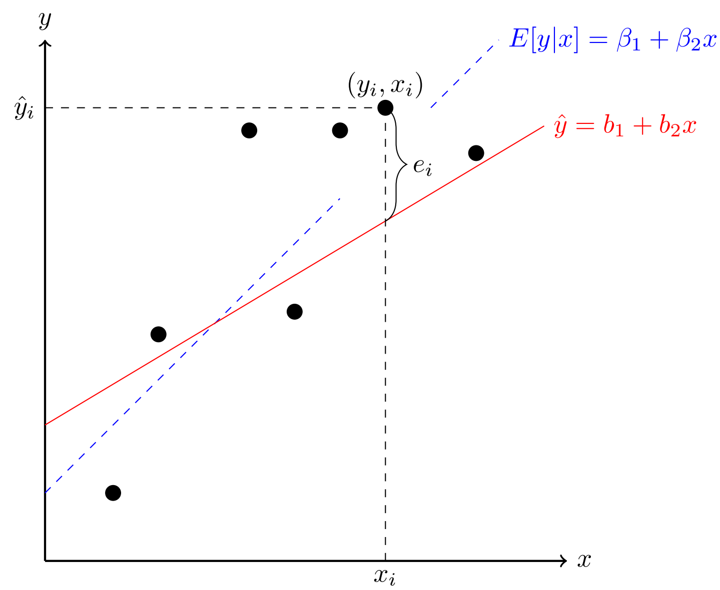
2 Estimation Regression

Now we use sample data to estimate β_1 and β_2 using the ordinary least squares (OLS) technique. Call these estimates b_1 and b_2, respectively. Under **assumptions 1 and 2**, the estimates will be unbiased: $E[b_1] = \beta_1$ and $E[b_2] = \beta_2$. That said, they will be different in generality than their population analogues. Hence our estimated regression line will be more or less different than the population regression line, depending how well we can estimate them.

For our estimated regression, our prediction of y_i given x_i is called the **fitted value** and is given by

$$\hat{y}_i = b_1 + b_2 x_i.$$

Much like in the population case, this will in generality be different than the actual value y_i. We call the difference between the actual value y_i and our fitted value $\hat{y}_i$ the **residual**, denoted e_i. (I find this confusing as hell – why not denote the **e**rror term with e instead? Sigh, economics.)

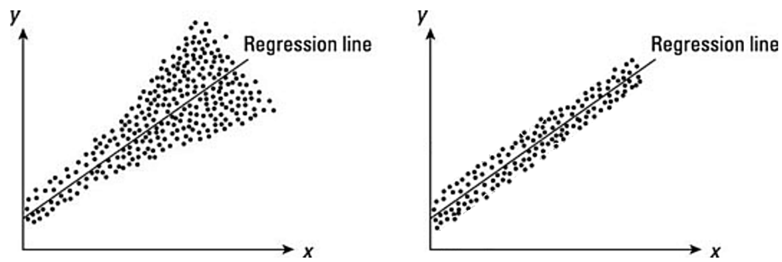


Our estimated regression line is in red. It will almost always be a bit different than the true population regression line, in blue. For x_i, it gives us a prediction for y_i, i.e. the fitted value $\hat{y}_i$. The fitted value will not in general be exactly the true value y_i, and the difference between the true value and the fitted value is the residual $e_i = y_i - \hat{y}_i$.

Assumptions 3 and 4 imply that the variance of the slope estimator b_2 will be

$$\mathrm{Var}(b_2) = \frac{\sigma_u^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \equiv \sigma_{b_2}^2.$$

Assumption 3 is most likely to break down, in which case we will have **heteroskedasticity** – the variance of u_i will depend on x_i. In this case we need to use **heteroskedasticity-robust standard errors**, which are given by the Stata option `vce(robust)`.



The figure on the left is an example of heteroskedasticity; the right an example of homoskedasticity. The left is heteroskedastic because the variation around the regression line gets bigger as x increases.

3 Estimation Properties

Under **assumptions 1-4**, our slope estimator b_2 has expected value of β_2 because it is unbiased; and it also has variance $\sigma^2_{b_2}$. Thus we can write

$$b_2 \sim (\beta_2, \sigma^2_{b_2}).$$

The z-score is standard normal, i.e.

$$\frac{b_2 - \beta_2}{\sigma_{b_2}} \sim \mathcal{N}(0, 1).$$

But we don't actually know σ_{b_2}, so we have to divide by the standard error $se(b_2)$ instead. Under **assumptions 1-4**,

$$T = \frac{b_2 - \beta_2}{se(b_2)} \sim \mathcal{N}(0, 1) \quad \text{as } n \to \infty.$$

If we add an **additional assumption** that the error terms are normally distributed with mean zero, i.e. $u_t \sim \mathcal{N}(0, \sigma^2_u)$, then we can say that $T \sim t(n-1)$ exactly. But we can still use $t(n-2)$ as an approximation even if errors are not normally distributed (which we will when doing hypothesis testing and confidence intervals).