Solution

The candidates are $\{pass, d, A\}$, $\{pass, e, A\}$, $\{pass, d, B\}$, $\{pass, e, B\}$. Let's just try them in order because why not.

$\{pass, d, A\}$. Let's see if this is rational for Player 1 by comparing *pass* to *play*. We get expected payoffs of

$$\begin{aligned} pass: &\quad \frac{1}{5}[2] + \frac{1}{5}[2] + \frac{3}{5}[2] = 2, \\ play: &\quad \frac{1}{5}[4] + \frac{1}{5}[4] + \frac{3}{5}[1] = 2.2. \end{aligned}$$

Oh no! This can't be a WSE; sequential rationality fails for Player 1.

$\{pass, e, A\}$. Let's see if this is rational for Player 1 by comparing *pass* to *play*. We get expected payoffs of

$$\begin{aligned} pass: &\quad \frac{1}{5}[2] + \frac{1}{5}[2] + \frac{3}{5}[2] = 2, \\ play: &\quad \frac{1}{5}[4] + \frac{1}{5}[2] + \frac{3}{5}[0] = 1.2. \end{aligned}$$

So at least Player 1's strategy is sequentially rational. What about the other two meatballs?

Because Player 1 passes always, information set $\{u, v\}$ is never reached. Therefore it faces no Bayesian restrictions – the beliefs can be anything. Accordingly, let's specify

$$\mu = \begin{pmatrix} u & v \\ p & 1-p \end{pmatrix}.$$

We can set the beliefs ourselves in such a way that will make A preferred to B, if such beliefs are feasible (i.e. are legit probabilities). Compare the expected values of A and B, given by

$$\begin{aligned} A: &\quad p[2] + (1-p)[4] = 4 - 2p, \\ B: &\quad p[0] + (1-p)[5] = 5 - 5p. \end{aligned}$$

Then A is rational when $4 - 2p \geq 5 - 5p$, that is, when $p \geq 1/3$.

Again, because Player 1 always passes, information set $\{x, y\}$ is not reached. Let's write

$$\mu = \begin{pmatrix} x & y \\ q & 1-q \end{pmatrix}.$$

We need e to be rational. Compare the expected values of d and e, given by

$$d: \quad q[10] + (1-q)[2] = 8q + 2,$$
$$e: \quad q[3] + (1-q)[4] = 4 - q.$$

Then e is rational when $4 - q \geq 8q + 2$, that is, when $q \leq 2/9$.

So we have found a WSE; all players are acting rationally, given what the other players are doing and what their beliefs are, given by

$$\sigma = \left(\begin{array}{cc|cc|cc} pass & play & d & e & A & B \\ 1 & 0 & 0 & 1 & 1 & 0 \end{array} \right), \qquad \mu = \left(\begin{array}{ccc|cc|cc} r & s & t & x & y & u & v \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & q & 1-q & p & 1-p \end{array} \right).$$

such that $p \geq 1/3$ and $q \leq 2/9$. So there are an infinite number of beliefs that constitute a WSE. A particularly elegant solution is to just have $p = 1$ and $q = 0$.

$\boldsymbol{\{pass, d, B\}}$**.** Same routine. Let's see if this is rational for Player 1 by comparing *pass* to *play*. We get expected payoffs of

$$pass: \quad \frac{1}{5}[2] + \frac{1}{5}[2] + \frac{3}{5}[2] = 2,$$
$$play: \quad \frac{1}{5}[0] + \frac{1}{5}[5] + \frac{3}{5}[1] = 1.6.$$

Okay, so at least Player 1's strategy is sequentially rational. But what about the other meatballs?

Because Player 1 passes always, information set $\{u, v\}$ is never reached. Therefore it faces no Bayesian restrictions – the beliefs can be anything. Accordingly, let's specify

$$\mu = \left(\begin{array}{cc} u & v \\ p & 1-p \end{array} \right).$$

We can set the beliefs ourselves in such a way that will make B preferred to A, if such beliefs are feasible (i.e. are legit probabilities). Compare the expected values of A and B, given by

$$A: \quad p[2] + (1-p)[4] = 4 - 2p,$$
$$B: \quad p[0] + (1-p)[5] = 5 - 5p.$$

Then B is rational when $4 - 2p \leq 5 - 5p$, that is, when $p \leq 1/3$.

Similarly, because Player 1 always passes, information set $\{x, y\}$ is never reached, so let's write

$$\mu = \begin{pmatrix} x & y \\ q & 1-q \end{pmatrix}.$$

We need d to be rational. Compare the expected values of d and e, given by

$$\begin{aligned} d: &\quad q[0] + (1-q)[2] = 2 - 2q, \\ e: &\quad q[3] + (1-q)[4] = 4 - q. \end{aligned}$$

Then d is rational when $2 - 2q \geq 4 - q$, that is, when $q \leq -2$. So, never. (Notice that when B is played, e strictly dominates d; had we noticed this initially, we wouldn't have had to bother looking at the other two players.) This cannot be sequentially rational for Player 2.

$\boldsymbol{\{pass, e, B\}}$**.** Same routine. Let's see if this is rational for Player 1 by comparing *pass* to *play*. We get expected payoffs of

$$\begin{aligned} pass: &\quad \frac{1}{5}[2] + \frac{1}{5}[2] + \frac{3}{5}[2] = 2, \\ play: &\quad \frac{1}{5}[0] + \frac{1}{5}[2] + \frac{3}{5}[0] = 0.4. \end{aligned}$$

Okay, so at least Player 1's strategy is sequentially rational. But what about the other meatballs?

Because Player 1 passes always, information set $\{u, v\}$ is never reached. Therefore it faces no Bayesian restrictions – the beliefs can be anything. Accordingly, let's specify

$$\mu = \begin{pmatrix} u & v \\ p & 1-p \end{pmatrix}.$$

We can set the beliefs ourselves in such a way that will make B preferred to A, if such beliefs are feasible (i.e. are legit probabilities). Compare the expected values of A and B, given by

$$\begin{aligned} A: &\quad p[2] + (1-p)[4] = 4 - 2p, \\ B: &\quad p[0] + (1-p)[5] = 5 - 5p. \end{aligned}$$

Then B is rational when $4 - 2p \leq 5 - 5p$, that is, when $p \leq 1/3$.

Similarly, because Player 1 always passes, information set $\{x, y\}$ is never reached, so let's write

$$\mu = \begin{pmatrix} x & y \\ q & 1-q \end{pmatrix}.$$

We need e to be rational. Well, we just found above that if B is played, then e strictly dominates d. Since B is played in this strategy profile, we can conclude that e is rational for any beliefs Player 2 might have. Although if you want to get specific about it again, compare the expected values of d and e, given by

$$\begin{aligned} d: &\quad q[0] + (1-q)[2] = 2 - 2q, \\ e: &\quad q[3] + (1-q)[4] = 4 - q. \end{aligned}$$

So e is rational when $4 - q \geq 2 - 2q$, that is, when $q \geq -2$. Since q is a probability and can't be negative, this is always satisfied.

So we have found another WSE; all players are acting rationally, given what the other players are doing and what their beliefs are.

$$\sigma = \left(\begin{array}{cc|cc|cc} pass & play & d & e & A & B \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array} \right), \qquad \mu = \left(\begin{array}{ccc|cc|cc} r & s & t & x & y & u & v \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & q & 1-q & p & 1-p \end{array} \right),$$

such that $q \in [0, 1]$ and $p \leq 1/3$. Having $p = 0$ and $q = 0$ or $q = 1$ are particularly elegant cases.

***play*?** The problem doesn't tell us to find a WSE in which Player 1 plays, but I'll look for one here just for good measure.

Notice that of all the *pass* cases we did, the only one for which *play* was superior was $\{pass, d, A\}$. So that is our only candidate; any other combination of Player 2 and Player 3 choices makes *pass* strictly superior to *play*. Specifically,

$$\begin{aligned} pass: &\quad \frac{1}{5}[2] + \frac{1}{5}[2] + \frac{3}{5}[2] = 2, \\ play: &\quad \frac{1}{5}[4] + \frac{1}{5}[4] + \frac{3}{5}[1] = 2.2. \end{aligned}$$

Now notice that with *play* and d, all information sets information sets are reached with positive probability and therefore necessitate Bayesian updating. The probability from the root to x is $1/5$ and the probability from the root to y is also $3/5$. Therefore the probability to information set $\{x, y\}$ is 4/5. Bayesian updating then requires

$$\mu(x) = \frac{1/5}{4/5} = \frac{1}{4},$$

$$\mu(y) = \frac{3/5}{4/5} = \frac{3}{4}.$$

Same idea for Player 3 in $\{u, v\}$ gives

$$\mu(u) = \frac{1/5}{2/5} = \frac{1}{2},$$

$$\mu(v) = \frac{1/5}{2/5} = \frac{1}{2}.$$

Given these probabilities, we need to ask whether A is rational. The expected payoffs are

$$\begin{aligned} A: &\quad \frac{1}{2}[2] + \frac{1}{2}[4] = 3, \\ B: &\quad \frac{1}{2}[0] + \frac{1}{2}[5] = 2.5. \end{aligned}$$

Great, A is rational for Player 3 given their beliefs.

Now do the same thing for the choice of d in $\{x, y\}$. The expected payoffs are

$$\begin{aligned} d: &\quad \frac{1}{4}[10] + \frac{3}{4}[2] = 4, \\ e: &\quad \frac{1}{4}[3] + \frac{3}{4}[4] = 3.75. \end{aligned}$$

Okay, so d is rational for Player 2 given their beliefs.

So that settles it; we have another WSE of

$$\sigma = \left(\begin{array}{cc|cc|cc} pass & play & d & e & A & B \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array} \right), \qquad \mu = \left(\begin{array}{ccc|cc|cc} r & s & t & x & y & u & v \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & \frac{1}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} \end{array} \right).$$