ECN 200E—Week 1 Discussion

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The Setup

The preferences are

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t [\ln(c_t) + \gamma_t \ln(1 - h_t)] \right]$$

where γ_t is a preference shock and h_t is hours worked in a day. Technology is Cobb-Douglas,

$$y_t = k_t^{\alpha} h_t^{1-\alpha}.$$

The resource constraint is

$$c_t + i_t = y_t, .$$

We will suppose $\delta = 1$, so the law of motion of capital is

$$k_{t+1} = i_t.$$

First Order Conditions

Bellman Equation. We can rewrite the resource constraint as

$$c_t + k_{t+1} = k_t^{\alpha} h_t^{1-\alpha}.$$

Then the Bellman equation is

$$V(k_t, \gamma_t) = \ln(c_t) + \gamma_t \ln(1 - h_t) + \beta E\big[V(k_{t+1}, \gamma_{t+1})\big].$$

We can incorporate the resource constraint by writing

$$V(k_t, \gamma_t) = \ln(c_t) + \gamma_t \ln(1 - h_t) + \beta E\big[V(k_{t+1}, \gamma_{t+1})\big] - \lambda[c_t + k_{t+1} - k_t^\alpha h_t^{1-\alpha}].$$

Intratemporal Euler Equation. The choice variables are c_t, h_t, and k_{t+1}, so let's take first order conditions with respect to those three. The first two give

$$\begin{aligned}\frac{1}{c_t} &= \lambda \\ \frac{\gamma_t}{1 - h_t} &= \lambda(1 - \alpha)k_t^\alpha h_t^{-a},\end{aligned}$$

which can be combined into the **intratemporal Euler equation**,

$$\frac{\gamma_t}{1 - h_t} = \frac{1}{c_t}(1 - \alpha)k_t^\alpha h_t^{-a}.$$

It's *intra*temporal because it is confined to one period t.

Intertemporal Euler Equation. Now take the FOC with respect to k_{t+1} to get

$$\beta E[V'(k_{t+1}, \gamma_{y+1})] = \lambda.$$

Envelope it up to get

$$\begin{aligned} & V'(k_t, \gamma_t) = \lambda\alpha k_t^{\alpha-1} h_t^{1-\alpha} \\ \implies \quad & V'(k_t, \gamma_t) = \frac{1}{c_t}\alpha k_t^{\alpha-1} h_t^{1-\alpha} \\ \implies \quad & V'(k_{t+1}, \gamma_{t+1}) = \frac{\alpha k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha}}{c_{t+1}}.\end{aligned}$$

Combine the two for

$$\alpha\beta E\left[\frac{k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}}{c_{t+1}}\right] = \frac{1}{c_t}.$$

This is the **intertemporal Euler equation**.

Guess and Verify

When we see log utility and $\delta = 1$, we'll usually make fractional conjectures. So let's conjecture that $c_t = \theta y_t$ and therefore $k_{t+1} = (1-\theta)y_t$. Substituting the conjectures into the intertemporal Euler equation and simplifying, we get

$$\begin{aligned}
& \alpha\beta E\left[\frac{k_{t+1}^{-1}k_{t+1}^{\alpha}h_{t+1}^{1-\alpha}}{\theta y_{t+1}}\right] = \frac{1}{\theta y_t} \\
\implies \quad & \alpha\beta E\left[\frac{y_{t+1}}{k_{t+1}\theta y_{t+1}}\right] = \frac{1}{\theta y_t} \\
\implies \quad & \alpha\beta E\left[\frac{1}{k_{t+1}}\right] = \frac{1}{y_t} \\
\implies \quad & \alpha\beta E\left[\frac{1}{(1-\theta)y_t}\right] = \frac{1}{y_t} \\
\implies \quad & \alpha\beta = (1-\theta) \\
\implies \quad & \theta = 1-\alpha\beta.
\end{aligned}$$

And therefore

$$c_t = (1-\alpha\beta)y_t, \quad k_{t+1} = \alpha\beta y_t.$$

We are not done, however, because y_t actually contains the control variable h_t. So now let's solve the intratemporal Euler equation h_t by plugging in our conjecture for c_t and substituting $y_t = k_t^{\alpha}h_t^{1-\alpha}$, giving

$$h_t = \frac{1-\alpha}{(1-\alpha\beta)\gamma_t + 1 - \alpha}.$$

Notice that a higher γ implies a lower h_t. So a larger means people will work less and enjoy more leisure. So we can finish off the policy functions by writing

$$
\begin{aligned}
c_t &= (1-\alpha\beta)k_t^\alpha \left[\frac{1-\alpha}{(1-\alpha\beta)\gamma_t + 1-\alpha}\right]^{1-\alpha}, \\
k_{t+1} &= \alpha\beta k_t^\alpha \left[\frac{1-\alpha}{(1-\alpha\beta)\gamma_t + 1-\alpha}\right]^{1-\alpha}.
\end{aligned}
$$