ECN 200D—Week 8 Lecture Notes
Competitive Growth Model, Part 2

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1 Recursive Competitive Equilibrium

What if the first welfare theorem does not hold? The link between Arrow-Debreu and the social planner breaks. In this situation, one approach is to solve the *recursive competitive equilibrium*. The word "recursive" should get you thinking about dynamic programming—in particular, about Bellman equations, value functions, state and control variables.

There is a little bit of a conceptual hurdle we must overcome first, however. Specifically, we need to recognize that k refers to the representative agent's available capital, and not aggregate available capital. So let K refer to the aggregate level of capital. We need to differentiate between the two because the agent's choice of capital should not have any effect on market prices—it is a price taking economy—whereas the aggregate capital K most certainly should. The usage will be elucidated upon in the material that follows.

1.1 The Firm's Problem

A representative firm wants to solve

$$\max_{n_t^d, k_t^d} F(k^d, n^d) - wn^d - rk^d.$$

But we'll need to be more specific about what w and r are. It might be temping to write $w = F_n(k,1)$ since we are still assuming constant returns to scale, but this is problematic—it suggests that the wage is a function of this single representative firm's capital k. In actuality, the wage is going to be determined by the aggregate capital in the economy, K. Same thing with the rental price of capital. So we have

$$w = F_n(K,1), \quad r = F_k(K,1).$$

1.2 The Household's Problem

The representative household has the Bellman equation

$$V(k,K) = \max_{c,k'} u(c) + \beta V(k', K').$$

In a typical Bellman equation, everything should either be a state variable, control variable, or a parameter. The aggregate capital K', however, is something else entirely—it is the aggregate capital next period as "chosen" by the entire economy. This makes it a novel object for our analysis.

The constraints of the Bellman equation are

(a) $c + k' = w + (r + 1 - \delta)k$, *(budget constraint)*

(b) $w = w(K) = F_n(K,1)$,

(c) $r = r(K) = F_k(K,1)$,

(d) $K' = H(K)$.

The function $H(K)$ is the agent's **rational expectation** about K', which means that unsettling term in the Bellman equation is now in terms of the state variable K.

1.3 Defining the Equilibrium

A **recursive competitive equilibrium** consists of

(a) the household value function $V : \mathbb{R}^2_+ \to \mathbb{R}$,

(b) policy functions $c, g : \mathbb{R}^2_+ \to \mathbb{R}_+$,

(c) pricing functions $w, r : \mathbb{R}_+ \to R_+$,

(d) an aggregate law of motion for capital, $H : \mathbb{R}_+ \to R_+$ such that

(i) given w, r, and H, V solves the household's Bellman equation with associated policy functions c and g,

(ii) $w = w(K) = F_n(K, 1)$,

(iii) $r = r(K) = F_k(K, 1)$,

(iv) $H(K) = K' = g(K, K)$,

(v) $c + K' = F(K, 1) + (1 - \delta)K$.

Note that because the measure of households is 1, c is both aggregate consumption and individual consumption. Also, the policy function that says how a representative agent should accumulate capital should apply to the economy as a whole, and thus to K as well.

1.4 Example

This example will illustrate what happens if we goof up and use k where we should have used K. The model will be the same basic environment as in the neoclassical growth model, but there will now be a government that taxes a fraction $\tau \in [0, 1]$ of the household's income and then returns the tax as a lump sum.

The household value function is still

$$V(k, K) = \max_{c, k'} u(c) + \beta V(k', K'),$$

now subject to the constraints

(a) $c + k' = (1 - \delta)k + [w + rk](1 - \tau) + T$,

(b) $K' = H(K)$,

(c) $w = F_n(K, 1)$,

(d) $r = F_k(K, 1)$,

(e) $T = \tau[F_n(K, 1) + F_k(K, 1)k] = \tau[F(K, 1)]$.

The last equality follows from Euler's theorem. Note that the lump sum payment T is both the aggregate and individual lump sum payment because the measure of the households is 1. Pedantically, each agent receives $T/1$. Since T is not a parameter nor is it a state or control variable, we'll use the expression with τ.

1.4.1 Ignoring the Distinction Between k and K

If we do ignore the distinction between k and K, then the budget constraint can be written as

$$\begin{aligned}
c + k' &= (1-\delta)k + [w + rk](1-\tau) + T \\
&= (1-\delta)k + [F_n(k, 1) + F_k(k, 1)k](1-\tau) + \tau[F_n(k, 1) + F_k(k, 1)k] \\
&= (1-\delta)k + F_n(k, 1) + F_k(k, 1)k \\
&= (1-\delta)k + F(k, 1) \\
&= f(k).
\end{aligned}$$

You'll find a first order condition of

$$u'\big(f(k) - k'\big) = \beta u'\big(f(k') - k''\big) f(k'),$$

which might make you all warm and fuzzy inside since it's the same first order condition as in the other problems. One might expect that result—I mean, all we're doing differently is taking money away from the households and giving it right back. But it turns out that ***this is wrong!***

2 Growth with Lump-Sum Taxes

We've been using k to denote the individual's level of capital and K to denote the aggregate level of capital. Let's do a notation switcharoo. Have a be the individual's level of capital and k be the aggregate capital.

Now let's actually do it the right way by not mixing up a and k. We had the Bellman equation

$$V(a,k) = \max_{c,a'} u(c) + \beta V(a',k')$$

subject to

- $c + a' = (1-\delta)a + (w + ra)(1-\tau) + T$,
- $w = F_n(k,1)$,
- $r = F_k(k,1)$,
- $k' = H(k)$,
- $T = \tau F(k,1)$.

By solving the budget constraint for c and substituting in the policy function $g(a,k) = a'$, we can write the Bellman equation as

$$V(a,k) = u\big([1-\delta]a + [w + ra][1-\tau] + T - g(a,k)\big) + \beta V\big(g(a,k),k'\big).$$

The first order condition is found by taking the derivative with respect to $g(a,k)$, which gives

$$u'(c) = \beta V'_{g(a,k)}\big(g(a,k),k'\big). \tag{1}$$

Now find the envelope condition by differentiating with respect to a to get

$$\begin{aligned} V'_a(a,k) &= u'(c)\big([1-\delta] + [1-\tau]r - g'_a(a,k)\big) + \beta V'_{g(a,k)}\big(g(a,k),k'\big)g'_a(a,k) \\ &= g'_a(a,k)\left[-u'(c) + \beta V'_{g(a,k)}\big(g(a,k),k'\big)\right] + u'(c)[1-\delta+(1-\tau)r]. \end{aligned}$$

From the first order condition in equation (5), the first bracket term is zero. It follows that

$$V_a'(a,k) = u'(c)[1-\delta+(1-\tau)r]. \tag{2}$$

Notice that the envelope condition contains τ. This gives us a fundamentally different result than when we'd mixed up a and k.

I'll take this opportunity to note that when finding envelope conditions, the terms with $g_a(a,k)$ will almost always disappear. In fact, for our purposes, we can just draw that conclusion henceforth without having to show it each time.

Anyway, update the envelope condition by one period to get

$$V'_{g(a,k)}\big(g(a,k),k'\big) = u'(c')[1-\delta+(1-\tau)r'].$$

This can be substituted into the first order condition for

$$u'(c) = \beta u'(c')[1-\delta+(1-\tau)r']. \tag{3}$$

This equation fully describes the optimal behavior of an individual agent in optimum as a price taker.

Only now that we know the individual agent's behavior can we consider the aggregate economy. In aggregate, the budget constraint will be, unsurprisingly,

$$c+k' = (1-\delta)k + \big[F_n(k,1)+F_k(k,1)k\big](1-\tau)+\tau F(k,1),$$

where $T=\tau F(k,1)$. By Euler's theorem, we can also write this as

$$c+k' = (1-\delta)k + F(k,1)(1-\tau)+\tau F(k,1),$$

from which it follows that

$$c + k' = (1 - \delta)k + F(k, 1) = f(k).$$

And so the first order condition for the aggregate economy can be written as

$$u'\big(f(k) - k'\big) = \beta u'\big(f(k') - k''\big)[1 - \delta + (1 - \tau)F_{k'}(k', 1)].$$

If you'd like, you can rewrite the $F_k(k, 1)$ term in terms of $f(k')$, but it doesn't really matter a whole lot.

Note that if $\tau = 0$, then this equilibrium should coincide with the tax-free model. And they do because

$$1 - \delta + F_{k'}(k', 1) = f'(k').$$

It's worth asking how a change in τ might affect the steady state equilibrium value of k. For simplicity, assume that $F(k, n) = k^\alpha n^{1-\alpha}$. In the steady state, we'll have $k = k' = k'' = \ldots$ and $c = c' = c'' = \ldots$, and therefore $u'(c) = u'(c')$. Then from the first order condition, it follows that

$$1 = \beta[1 - \delta + (1 - \tau)\alpha k^{\alpha - 1}] \quad \Longrightarrow \quad k^* = \left[\frac{\alpha\beta(1 - \tau)}{1 - \beta(1 - \delta)}\right]^{1/(1-\alpha)}. \tag{4}$$

Notice that a higher τ means a smaller steady state k^*. Also notice that when $\tau = 1$, we have $k^* = 0$. There is a free-rider problem present—all agents have their income taken away but receive T. So people will just not work and receive T instead. But then everyone does that so $T = 0$.

Turns out there's a tax rate that optimizes T, illustrated by the **Laffer curve**. We can write the tax revenue as

$$T = \tau F(k^*, 1) = \tau \left[\frac{\alpha\beta(1 - \tau)}{1 - \beta(1 - \delta)}\right]^{\alpha/(1-\alpha)}.$$

Taking the first order condition with respect to τ gives the revenue-maximizing tax rate $\tau^* = 1 - \alpha$.

3 Growth with Technological Progress

We will be considering **labor-augmenting technological progress**. What this means is that workers become more efficient. Let n be the growth rate of labor. Then $n_t = (1+n)^t n_0$ is the labor force in period t. Hell, we may as well normalize $n_0 = 1$ so that $n_t = (1+n)^t$. Similarly, let g be the growth rate of worker efficiency. Then output in period t will be

$$y_t = F\big(K_t, [1+g]^t n_t\big) = F\big(K_t, [1+g]^t [1+n]^t\big),$$

where K_t is the aggregate level of capital in period t. We will be assuming that F exhibits constant returns to scale.

Capital letter variables will indicate aggregates, and lower case will indicate per-capita. So K_t is the aggregate level of capital and $k_t = K_t/n_t$ is the per-capita level of capital. Finally, tilde variable indicates per effective capita, so

$$\tilde{x}_t = \frac{X_t}{[(1+g)(1+n)]^t}.$$

This gives capital per *effective* worker.

3.1 Social Planner's Problem

3.1.1 The Setup

The social planner wants to maximize per-capita utility

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the aggregate resource/budget constraint

$$C_t + K_{t+1} = (1-\delta)K_t + F\big(K_t, [1+g]^t[1+n]^t\big).$$

Let's divide everything by $(1+g)^t(1+n)^t$, which gives

$$\begin{aligned} \tilde{c}_t + \frac{K_{t+1}}{[(1+g)(1+n)]^t} &= (1-\delta)\tilde{k}_t + F\big(\tilde{k}_t, 1\big) \\ \implies \quad \tilde{c} + (1+g)^t(1+n)^t\tilde{k}_{t+1} &= (1-\delta)\tilde{k}_t + F\big(\tilde{k}_t, 1\big) = f(\tilde{k}_t). \end{aligned}$$

This is one of those models that's only analytically tractable for certain functional forms. In this case, we'll adopt a CRRA utility function,

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

where σ is the coefficient of relative risk aversion. Then the utility function can be written as

$$\begin{aligned} \max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} &= \max \sum_{t=0}^{\infty} \beta^t \frac{[(1+g)^t\tilde{c}]^{1-\sigma}}{1-\sigma} \\ &= \max \sum_{t=0}^{\infty} \beta^t [(1+g)^t]^{1-\sigma} \frac{\tilde{c}^{1-\sigma}}{1-\sigma} \\ &= \max \sum_{t=0}^{\infty} \tilde{\beta}^t u(\tilde{c}). \end{aligned}$$

This is nice because now it looks similar to other scenarios we have observed. We do, however, need to make the additional assumption that $\hat{\beta} < 1$.

3.1.2 Solving for the Equilibrium

Protip: when you're short on time and you've given a choice, go with the recursive approach. In this case, we have the Bellman equation

$$V(\tilde{k}) = \max_{\tilde{k}'} u\big(f(\tilde{k}) - (1+n)(1+g)\tilde{k}'\big) + \tilde{\beta}V(\tilde{k}').$$

The first order condition is

$$u'(\tilde{c})(1+n)(1+g) = \tilde{\beta}V'(\tilde{k}'). \tag{5}$$

Now substitute in the policy function,

$$V(\tilde{k}) = u\big(f(\tilde{k}) - (1+n)(1+g)g(\tilde{k})\big) + \tilde{\beta}V\big(g(\tilde{k})\big).$$

The derivative with respect to $\tilde{k}$ is

$$V'(\tilde{k}) = u'(\tilde{c})\big[f'(\tilde{k}) - (1+n)(1+g)g'(\tilde{k})\big] + \tilde{\beta}V'\big(g(k)\big)g'(k).$$

Shortcut alert—everything with $g'(k)$ will end up being zero. Therefore we can write the envelope condition as

$$V'(\tilde{k}) = u'(\tilde{c})f'(\tilde{k}). \tag{6}$$

Update the envelope condition by one period so we can plug it into the first order equation. Then we have

$$u'(\tilde{c})(1+n)(1+g) = \tilde{\beta}u'(\tilde{c}')f'(\tilde{k}'). \tag{7}$$

If we are given a nice functional form, then we can use this to find the closed form solution.

3.1.3 Steady State

In the steady state, we will have $\tilde{k} = \tilde{k}' = \ldots$, and therefore $\tilde{c} = \tilde{c}' = \ldots$. We can use this fact with equation (7) to write

$$u'(\tilde{c})(1+n)(1+g) = \tilde{\beta} u'(\tilde{c}) f'(\tilde{k}'). \quad \Longrightarrow \quad (1+n)(1+g) = \tilde{\beta} f'(\tilde{k}^*).$$

This gives us an implicit description of $\tilde{k}^*$ in the steady state.

The growth rate of per-capita capital in the steady state will be

$$\frac{k_{t+1} - k_t}{k_t} = \frac{(1+g)^{t+1}\tilde{k}_{t+1} - (1+g)^t \tilde{k}_t}{(1+g)^t \tilde{k}_t} = \frac{(1+g)\tilde{k}_t - \tilde{k}_t}{\tilde{k}_t} = g.$$

So the growth rate of capital per capital is also equal to the rate of growth of worker efficiency.

In aggregate, the growth rate of the level of capital is

$$\begin{aligned} \frac{K_{t+1} - K_t}{K_t} &= \frac{[(1+n)(1+g)]^{t+1}\tilde{k}_{t+1} - [(1+n)(1+g)]^t \tilde{k}_t}{[(1+n)(1+g)]^t \tilde{k}_t} \\ &= (1+n)(1+g) - 1 \\ &= n + g + ng. \end{aligned}$$

The ng term is going to be rather small so often times this result will just be simplified to $n + g$.