ECN 200B—Debreu-Scarf Theorem

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Fix an exchange economy $\mathcal{E} = \{I, (u^i, w^i)_{i\in I}\}$. Now suppose we clone each individual i exactly once, where each i has the same preferences and the same endowment. (Visually, the dimension of the Edgeworth box will double as well.) Or suppose clone everyone three times instead. More generally, if we clone everyone $N \in \mathbb{N}$ times, then we define the ***N*-fold replica** of $\mathcal{E}$ to be

$$\mathcal{E}^N = \{I \times \{1, \ldots, N\}, (u^{i,n}, w^{i,n}) = (u^i, w^i)_{i,n}^{I,N}\}.$$

In other words, we have N individuals possessing (u^i, w^i) for each i. If $x = (x^1, \ldots, x^I)$ is an allocation for the original economy, then $(x)^N = (x, x, \ldots, x)$, N times, is the new allocation. So we're just taking the original allocation and repeating it N times so that each $x^{i,n} = x^i$ has the same allocation.

Definition 1. Let x be an allocation for $\mathcal{E}^N$. We say that x has the **equal treatment property** if for every i, n, n', the consumption bundle $x^{i,n} = x^{i,n'}$.

Note that we're not imposing any sort of equilibrium condition on x. So you can think of a social planner giving individual $(1,1)$ a bundle $x^{1,1}$ that is different from individual $(1,2)$'s bundle $x^{1,2}$. But under some equilibrium concepts or other conditions, we might expect that individuals with identical preferences and identical endowments would have the same bundles.

Theorem 1. *If every u^i is strictly quasiconcave and if x is a competitive equilibrium of $\mathcal{E}^N$, then x satisfies the equal treatment property.*

Theorem 2. *Suppose that x is in the core of $\mathcal{E}^N$. If each u^i is strictly quasiconcave, then x has the equal treatment property.*

Proof. Suppose otherwise. Then there exists some i, n, n' such that $x^{i,n} \neq x^{i,n'}$. For simplicity of exposition, assume that $I = \{1, 2\}$ and that $x^{1,1} \neq x^{1,N}$. Pick n_1 and

n_2 such that

$$u^1(x^{1,n_1}) = \min\{u^1(x^{1,1}), \ldots, u^1(x^{1,N})\},$$
$$u^2(x^{2,n_2}) = \min\{u^2(x^{2,1}), \ldots, u^2(x^{2,N})\}.$$

So person $(1, n_1)$ has the lowest utility among the individual 1 clones and person $(2, n_2)$ has the lowest utility among the individual 2 clones. Consider the coalition of these two clones,

$$H = \{(1, n_1), (2, n_2)\}.$$

Also define $\hat{x}^{1,n_1}$ and $\hat{x}^{2,n_2}$ to be the mean consumption bundle for individual 1 clones and individual 2 clones, respectively:

$$\hat{x}^{1,n_1} = \frac{1}{N}\sum_{n=1}^{N} x^{1,n}, \quad \hat{x}^{2,n_2} = \frac{1}{N}\sum_{n=1}^{N} x^{2,n}.$$

Claim 1. $u'(\hat{x}^{1,n_1}) > u^1(x^{1,n_1})$. This follows by strict quasiconcavity—the utility of the convex combination of preferred bundles is strictly greater than the minimum utility. //

Claim 2. $u'(\hat{x}^{2,n_2}) \geq u^2(x^{2,n_2})$. We do not know whether there are any $x^{2,n} \neq x^{2,n'}$, so we cannot have a strict inequality. //

Because we're dealing with clones, it follows that $w^{i,n} = w^i$. And because x is feasible, we can write

$$\begin{aligned}
\hat{x}^{1,n_1} + \hat{x}^{2,n_2} &= \frac{1}{N}\sum_{n=1}^{N} x^{1,n} + x^{2,n} \\
&= \frac{1}{N}\sum_{n=1}^{N} w^{1,n} + w^{2,n} \\
&= \frac{1}{N}\sum_{n=1}^{N} w^{1} + w^{2} \\
&= w^1 + w^2.
\end{aligned}$$

So clones $(1, n_1)$ and $(2, n_2)$ could, as a coalition, respectively consume $\hat{x}^{1,n_1}$ and $\hat{x}^{2,n_2}$, which would make clone $(1, n_1)$ strictly better off and clone $(2, n_2)$ no worse off. This implies that x is not in the core of $\mathcal{E}^N$, which is a contradiction. Thus, the equal treatment property must hold. □

Theorem 3. *If (p, x) is a competitive equilibrium of $\mathcal{E}$, then $\big(p, (x)^N\big)$ is a competitive equilibrium of $\mathcal{E}$.*

Define $W \in \mathbb{R}_+^{LI}$ to be the set of competitive equilibrium allocations for $\mathcal{E}$. For any $N \in \mathbb{N}$, define the **dimension-free core** to be

$$C_N = \{x \in \mathbb{R}_+^{LI} \mid (x)^N \text{ is in the core of } \mathcal{E}^N\}.$$

In other words, we're "collapsing" repeated allocations into one. For example, if $(x)^3 = (x, x, x)$ and $(y)^3 = (y, y, y)$ are in the core of $\mathcal{E}^3$, then $C_3 = \{x, y\}$.

Theorem 4 (Debreu-Scarf)**.** *Suppose that each u^i is continuous, strictly quasiconcave, and strongly monotone. Then*

$$\bigcap_{N=1}^{\infty} C_N = W.$$

"Proof." For expositional simplicity, again assume that $I = \{1, 2\}$.

Step 1. Take an allocation $\hat{x} \in \cap_{N=1}^{\infty} C_N$. This means that $\hat{x}$ is in the core for all E^N, which means market clearing is already satisfied. We need to show that there exists some price such that $(p, \hat{x})$ is a competitive equilibrium.

Define the "better-than" sets

$$\mathcal{U}^1 = \{z^1 \mid u^1(w^1 + z^1) > u^1(\hat{x}^1)\},$$
$$\mathcal{U}^2 = \{z^2 \mid u^2(w^2 + z^2) > u^2(\hat{x}^2)\}.$$

From what I can tell, this is the set of all "divergences" from the endowment, as given by z^i, the give greater utility than the core allocation.

Claim 3. *$\mathcal{U}^1$ and $\mathcal{U}^1$ are convex sets.* This follows from quasiconcavity. //

Claim 4. *$\mathcal{U}^1 + \mathcal{U}^1$ is convex, as is the mean.* The sum of convex sets is convex, and so is its mean. //

Define the set

$$P_0 = \{z \mid \exists n_1, n_2 \in \mathbb{N}, z^1 \in \mathcal{U}^1, z^2 \in \mathcal{U}^2 \text{ where } \frac{n_1}{n_1 + n_2} z^1 + \frac{n_2}{n_1 + n_2} z^2 = z.\}$$

In other words, P_0 is any vector that can be written as a rational-numbered convex combination of "divergences" for each individual.

Claim 5. $0 \notin P_0$. To see why, suppose otherwise. Then for some n_1, n_2, z^1, and z^2,

$$\frac{n_1}{n_1+n_2} z^1 + \frac{n_2}{n_1+n_2} z^2 = 0 \implies \quad n_1 z^1 + n_2 z^2 = 0.$$

Let $\hat{N} = \max\{n_1, n_2\}$. Furthermore, define the coalition

$$H = \begin{Bmatrix} (1,1), (1,2), \ldots, (1,n_1), \\ (2,1), (2,2), \ldots, (2,n_2) \end{Bmatrix}.$$

For any $n \leq n_1$, let's try to have $x^{1,n} = w^1 + z^1$. Similarly, for any $n \leq n_2$, have $x^{2,n} = w^2 + z^2$. We don't yet know if these are actually feasible bundles for the coalition. But since each $z^i \in \mathcal{U}^i$, it follows that

$$u^1(x^{1,n}) > u^1(\hat{x}^1) \quad \text{and} \quad u^2(x^{2,n}) > u^2(\hat{x}^2).$$

By summing up the consumption of all individuals in the coalition, we have

$$\begin{aligned} \sum_{n=1}^{n_1} x^{1,n} + \sum_{n=1}^{n_2} x^{2,n} &= \sum_{n=1}^{n_1} [w^1 + z^1] + \sum_{n=1}^{n_2} [w^2 + z^2] \\ &= n_1(w^1 + z^1) + n_2(w^2 + z^2) \\ &= n_1 w^1 + n_2 w^2 + n_1 z^1 + n_2 z^2 \\ &= n_1 w^1 + n_2 w^2. \end{aligned}$$

Each clone $(1,n)$ has an endowment of w^1, and each clone $(2,n)$ has an endowment of w^2. The coalition H has n_1 clones of individual 1, so together they have aggregate endowment of $n_1 w^1$. Similarly for clones $(2,n)$, they have aggregate endowment of $n_2 w^2$. Which means that the coalition can afford to have $x^{1,n}$ and $x^{2,n}$ for all $(i,n) \in H$, giving greater utility than $\hat{x}$, which they would therefore block. So $\hat{x}$ wouldn't be in the core of $C_{\hat{N}}$, a contradiction. So it must be the case that $0 \notin P_0$. //

$\square$