Geometric Distribution

A geometric distribution is the number of trials until one success in a series of Bernoulli trials. (Or number of failures on the right!)

$$P(X=k)=(1-p)^{k-1}p \qquad P(X=k)=(1-p)^k p$$

$$E(X)=\frac{1}{p} \qquad E[X]=\frac{1-p}{p}$$

$$var(X)=\frac{1-p}{p^2} \qquad var(X)=\frac{1-p}{p^2}$$

Negative Binomial Distribution

Let X denote the the number of trials needed until getting the r^{th} success. A negative binomial distribution, denoted $NB(r,p)$, is the probability of the number of trials undertaken until the rth success.

$$NB(r,p)=P(X=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}$$

$$E(X)=\frac{r}{p} \qquad var(X)=\frac{r(1-p)}{p^2}$$

Poisson Distribution

Let λ be the parameter which indicates the average number of events in the given time interval. The range of a Poisson Distribution is $\{0,1,2,...\}$. The Poisson Distribution is given by

$$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}, \quad E(X)=\lambda, \quad var(X)=\lambda.$$

Suppose $X\sim Poisson(\lambda)$ and $Y\sim Poisson(\mu)$ and that X,Y are independent. Then we have

(a) $E(X+Y)=E(X)+E(Y)=\lambda+\mu$,

(b) $X+Y\sim Poisson(\lambda+\mu)=P(X+Y=z)$

$$=e^{\lambda-\mu}\sum_{y=0}^{z}\frac{\lambda^{z-y}}{(z-y)!}\frac{\mu^y}{y!}.$$

Exponential Distribution

The exponential distribution is commonly used to model waiting times between occurrences of rare events. Given a rate parameter λ, we have for $x\geq 0$,

$$f_X(x)=\lambda e^{-\lambda x} \qquad F_X(x)=1-e^{-\lambda x},$$

$$E(X)=\frac{1}{\lambda}, \qquad var(X)=\frac{1}{\lambda^2}.$$

Gamma Distribution

Let Y denote the sum of r number of $exponential(\lambda)$ random variables. Then Y denotes the number of time until the r^{th} event.

$$f(y)=e^{-\lambda y}\frac{(\lambda y)^{y-1}}{(r-1)!}.$$

Uniform Distribution

$$f(x)=\frac{1}{b-a} \qquad F(x)=\frac{x-a}{b-a}$$

$$E[X]=\frac{1}{2}(a+b) \qquad var(X)=\frac{1}{12}(b-a)^2$$

Normal Distribution

$$\text{CLT: } \frac{(\overline{X}-\mu)}{\sigma/\sqrt{n}}\sim N(0,1)$$

$$X\sim N(\mu,\sigma^2)\Longrightarrow X=\mu+\sigma Z$$

$$X\sim N(\mu,\sigma^2)\Longrightarrow \alpha X\sim N(\mu,\alpha^2\sigma^2)$$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Bivariate Normal Distribution

If $X\sim N(0,1)$, $Y\sim N(0,1)$, $corr(X,Y)=\rho$, then

$$Y|X=x\sim N(\rho x,1-\rho^2)$$

$$X|Y=y\sim N(\rho y,1-\rho^2)$$

$$f(x,y)=\frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} \quad \text{for } X,Y\sim N(0,1)$$

$$Y=\rho X+(1-\rho^2)Z$$

$$x=r\cos(\theta), y=r\sin(\theta), \theta=\arctan\left(\frac{y}{x}\right)$$

Expectation

$$E[X]=\sum_x xP(X=x)$$

$$E[X]=\int_x xP(X=x)\,dx$$

$$E[X+Y]=E[X]+E[Y]$$

$$E[XY]=E[X]E[Y], \quad \text{for } X,Y \text{ independent}$$

$$E[XY]=\sum_x\sum_y xyP(X=x,Y=y)$$

Variance

(a) $var(c)=0$, where c is a constant

(b) $var(X)=E(X^2)-E(X)^2$

(c) $var(aX)=a^2\cdot var(X)$, where a is a constant

(d) $var(X+Y)=var(X)+var(Y)$, independent X,Y

(e) $var(X+Y)=var(X)+var(Y)+2cov(X,Y)$

Covariance

(a) $cov(X,X)=var(X)$

(b) $cov(X,c)=0$, where c is a constant

(c) $cov(X,Y)=E(XY)-E(X)E(Y)$

(d) $cov(X,Y)=0$ for independent X,Y

(e) $cov(X,Y)=cov(Y,X)$

(f) $cov(aX,Y)=a\cdot cov(X,Y)$

(g) $cov(X,Y+Z)=cov(X,Y)+cov(X,Z)$

Correlation

(a) $corr(X,Y)=\dfrac{cov(X,Y)}{\sqrt{var(X)var(Y)}}$

(b) $corr(X,X)=1$

(c) $corr(X,Y)=corr(Y,X)$

(d) X,Y independent $\Rightarrow corr(X,Y)=0$

(e) $-1\leq corr(X,Y)\leq 1$

(f) $corr(aX+b,Y)=corr(X,Y)$ for $a>0$, $-corr(X,Y)$ for $a<0$

Conditional Expectation

(a) $E[X|Y=y] = \sum_{\text{all } x} xP(X=x|Y=y)$

(b) $E[X|Y=y] = \int_x x f_{X|Y}(x,y)\,dx$

(c) $f_Y(y) = \int_{\text{over } x} f(x,y)\,dx$

(d) $E[E(Y|X)] = E[Y]$

(e) $E[X+Y|Z] = E[X|Z] + E[Y|Z]$

(f) $E[Y] = \sum_x E[Y|X=x]P(X=x)$

(g) $E[g(X)Y|X] = g(X)E[Y|X]$

(h) $var(X) = E\Big(var(\ X \mid Y\ )\Big) + var\Big(E[\ X \mid Y\ ]\Big)$

Strategy: separate into independent and dependent parts.

Moment Generating Functions

(a) $M_X(t) = E\left[e^{tX}\right] = \sum_x e^{tx} p_X(x)$

(b) $M_X(t) = E\left[e^{tX}\right] = \int_x e^{tx} f_X(x)\,dx$

(c) $M_{X+Y}(t) = M_X(t)\cdot M_Y(t)$

(d) $\frac{d}{dt}[M_X(t)] = E[X]$

(e) $\frac{d^2}{dt^2}[M_X(t)] = E[X^2]$

(f) $Bernoulli(p) :\ 1-p+pe^t$

(g) $Binomial(n,p) :\ (1-p+pe^t)^n$

(h) $Geometric(p) :\ \dfrac{pe^t}{1-(1-p)e^t}$ for $t < -\ln(1-p)$

(i) $Poisson(\lambda) :\ e^{\lambda(e^t-1)}$

(j) $Exponential(\lambda) :\ \left(1-\dfrac{t}{\lambda}\right)^{-1}$

(k) $Gamma(r,\lambda) :\ \left(1-\dfrac{t}{\lambda}\right)^{-r}$

(l) $NB(r,p) :\ \dfrac{(1-p)^r}{(1-pe^t)^r}$

(m) $N(\mu,\sigma^2) :\ e^{t\mu+\frac{1}{2}\sigma^2t^2}$

Joint, Marginal, Conditional Probabilities

$$P(X=x, Y=y) = P(x,y)$$

$$P(X=x) = \sum_{\text{all } y} P(X=x, Y=y) = \sum_{\text{all } y} P(X=x|Y=y)P(Y=y)$$

$$f(x,y) = \int_x \int_{y(x)} f(x,y)\,dxdy = 1$$

$$f_X(x) = \int_{y(x)} f(x,y)dy$$

$$f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)}$$

Change of Variables

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}$$

Geometric Series

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

Poisson Approximation

$Binomial(n,p)$ with large n and small p is approximated by $Poisson(np)$.

Chebyshev's Inequality

$$P(|X-\mu| \geq \epsilon) \leq \frac{var(X)}{\epsilon^2}$$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Markov's Inequality

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Law of Large Numbers

Let X_i be i.i.d, and let $\overline{X}_n = (X_1 + ... + X_n)/n$ represent the average. Then

$$P(|\bar{X}_n - \mu| > \epsilon) \to 0 \quad \text{as } n \to \infty.$$

Conditional Probability

(a) $P(AB) = P(A|B)P(B) = P(B|A)P(A)$

(b) $P(A|B) = \dfrac{P(AB)}{P(B)}$

(c) $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

Series

$$e^x = \sum_{n=0}^{n} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$