Dynamic Programming – Infinite Horizon

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December 23, 2016

These notes borrow heavily from Stokey and Lucas, re-written in a way that I find easier to follow. That sometimes means more exposition, more explanation, worked-out examples, and added (occasionally silly) comments. Also probably some added typos and other mistakes.

1 Infinite Horizon

Let's begin by considering the Cobb-Douglas case where $f(k) = k^\alpha$ and $u(c) = \ln(c)$. Recall the law of motion of capital from the finite horizon problem,

$$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha. \tag{1}$$

As T becomes very large, then because $0 < \alpha\beta < 1$, it follows that the coefficient in front of k_t^α will become very close to $\alpha\beta$ for most of the sequence. And at this point, we might wonder why we cannot just take the limit as T goes to infinity as the solution for the infinite horizon problem. Doing so would give an solution of

$$k_{t+1} = \alpha\beta k_t^\alpha.$$

Well, it turns out we *can* just take the limit for the infinite horizon problem, and this is true in general. To prove it, one must first establish that interchanging max and lim operators is kosher, and this is actually rather difficult to show.

So nuts to that. Let's try a different approach. The limit suggests that there is a fixed rule that is used in every period. In this example, the rule seems like it should be to take a fixed proportion of your available capital to the next period, in particular, $\alpha\beta$ of it. More generally, we can conjecture that there is a fixed "savings function" or **policy function** of the form

$$k_{t+1} = g(k_t).$$

This is fairly intuitive. The social planner problem looks the same from any given period; the only meaningful difference is the capital stock in each particular period.

But finding that savings function could be challenging. There isn't any general way we can do it from looking at first order and boundary conditions. The change of variable $z_t = k_t/k_{t-1}^{\alpha}$ is specific to the example. So we need a new approach entirely.

Let's try thinking in terms of only *today* and *tomorrow*, in particular, of choosing consumption today c_0 and tomorrow's beginning-of-period capital k_1. Let $v(\cdot)$ be the **value function**, supposing one exists, that gives the value of the maximized objective function; it's not unlike an indirect utility function, as far as interpretation goes. So $v(k_0)$ is the level of utility we'd get from solving the planner's problem, for instance. Or, given some beginning-of-period capital stock k_1 in period 1, the best that could be done from that point on is $v(k_1)$. Using this, we can rewrite the social planner's problem as

$$\begin{aligned} &\max_{c_0,k_1} \left[U(c_0) + \beta v(k_1)\right] \\ \text{s.t.} \quad & c_0 + k_1 \leq f(k_0), \\ & c_0, k_1 \geq 0, \quad k_0 > 0 \text{ given.} \end{aligned}$$

Of course, we don't actually know what $v(\cdot)$ is. All we really know about it is that is solves the maximization problem. That is,

$$v(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \left\{ U\big(f(k_0) - k_1\big) + \beta v(k_1) \right\}.$$

Now that we are dealing with only two indices, we can adopt the more convenient k for current and k' for next-period. Also, recall our conjecture (which is actually true) that there exists a time-invariant policy rule $k' = g(k)$ that is followed in optimum. Thus, we can actually write the maximization problem as

$$v(k) = U\big(f(k) - g(k)\big) + \beta v\big(g(k)\big) \quad \text{s.t.} \quad 0 \leq g(k) \leq f(k). \tag{2}$$

Equation (2) is called the **Bellman equation**. Notice that we are able to drop the max operator since $g(k)$ is defined to be the optimal policy rule – the fact that we are using it means we will attain the maximum.

So we have one equation with an unknown function $v(\cdot)$, which we call a **functional equation**. Optimizing such problems via functional equations is what constitutes **dynamic programming**. This will be our bread and butter.

Let's suppose that $v(\cdot)$ is differentiable and that there is an interior solution. Then the first order condition (differentiate with respect to $g(k)$ and set equal to zero) is

$$U'_{g(k)}\big(f(k) - g(k)\big) = \beta v'_{g(k)}\big(g(k)\big). \tag{3}$$

To find the envelope condition, differentiate with respect to k to get

$$v_k'(k) = U_{f(k)}'\big(f(k) - g(k)\big) f_k'(k) - U_{g(k)}'\big(f(k) - g(k)\big) g_k'(k) + \beta v_{g(k)}'\big(g(k)\big) g_k'(k).$$

Notice that equation (3) implies that the latter two terms sum to zero. Thus, the envelope condition is

$$v'(k) = U_{f(k)}'\big(f(k) - g(k)\big) f_k'(k). \tag{4}$$

Stokey and Lucas give interpretations for conditions (3) and (4), although I don't find either to particularly intuitive.

> The first of these conditions equates the marginal utility of consuming current output to the marginal utility of allocating it to capital and enjoying augmented consumption next period. The second condition states that the marginal value of current capital, in terms of total discounted utility, is given by the marginal utility of using the capital in current production and allocating its return to current consumption.

2 Example: Cobb-Douglas Revisited

Let's try to calculate $v(\cdot)$ given the conjecture $g(k) = \alpha\beta k^\alpha$. The value function can be written as

$$\begin{aligned}
v(k_0) &= \ln(k_0^\alpha - \alpha\beta k_0^\alpha) + \beta \ln(k_1^\alpha - \alpha\beta k_1^\alpha) + \beta^2 \ln(k_2^\alpha - \alpha\beta k_2^\alpha) + ... \\
&= \alpha \ln(k_0) + \ln(1 - \alpha\beta) + \alpha\beta \ln(k_1) + \beta \ln(1 - \alpha\beta) + ... \\
&= [\alpha \ln(k_0) + \alpha\beta \ln(k_1) + \alpha\beta^2 \ln(k_2) + ...] + \ln(1 - \alpha\beta)(1 + \beta + \beta^2 + ...) \\
&= [\alpha \ln(k_0) + \alpha\beta \ln(k_1) + \alpha\beta^2 \ln(k_2) + ...] + \frac{1}{1-\beta} \ln(1 - \alpha\beta)
\end{aligned}$$

The sum involving the k terms looks like it could be problematic. It would be really nice if we could combine those logarithms, wouldn't it? Well, by using our conjectured policy function $g(k) = \alpha\beta k^\alpha$, we can write each term in the sum in terms of k_0. For example, the first four terms in the sum are

$$\begin{array}{lllll}
\ \ \alpha \ln(k_0) & & & & \\
+\, \alpha^2\beta \ln(k_0) & +\, \alpha\beta \ln(\alpha\beta) & & & \\
+\, \alpha^3\beta^2 \ln(k_0) & +\, \alpha\beta^2 \ln(\alpha\beta) & +\, \alpha^2\beta^2 \ln(\alpha\beta) & & \\
+\, \alpha^4\beta^3 \ln(k_0) & +\, \alpha\beta^3 \ln(\alpha\beta) & +\, \alpha^2\beta^3 \ln(\alpha\beta) & +\, \alpha^3\beta^3 \ln(\alpha\beta) & \\
+\, \dots & +\, \dots & +\, \dots & +\, \dots & +\, \dots
\end{array}$$

What we're going to do is take each column as its own series. The first column, for instance, results in

$$\frac{\alpha}{1-\alpha\beta}\ln(k_0),$$

which takes care of the k_0 terms. The rest of them all have $\ln(\alpha\beta)$ involved. The second, third, and fourth columns give respective sums of

$$\ln(\alpha\beta)\frac{\alpha\beta}{1-\beta}, \qquad \ln(\alpha\beta)\frac{\alpha^2\beta^2}{1-\beta}, \qquad \ln(\alpha\beta)\frac{\alpha^3\beta^3}{1-\beta},$$

and so on. There are an infinite number of columns, so *these* constitute a series, the sum of which evaluates to

$$\frac{\ln(\alpha\beta)}{1-\beta}\frac{\alpha\beta}{1-\alpha\beta}.$$

Great, so let's combine everything into

$$v(k_0) = \frac{\alpha}{1-\alpha\beta}\ln(k_0) + \frac{1}{1-\beta}\left(\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right). \tag{5}$$

Note that $v(k)$ takes the form $A\ln(k) + B$. We will be considering general functional forms like this more later on.

Anyway, consider our newly found value function in terms of the Bellman equation. In particular,

$$v(k) = \ln\left(k^\alpha - \alpha\beta k^\alpha\right) + \beta\left[A\ln(\alpha\beta k^\alpha) + B\right] \tag{6}$$

Let's see if it satisfies the first order condition and the envelope condition. For the first order condition, we want to differentiate with respect to $\alpha\beta k^\alpha$, resulting in

$$\frac{1}{k^\alpha - \alpha\beta k^\alpha} = \frac{\beta A}{\alpha\beta k^\alpha} \quad \Longrightarrow \quad \frac{\alpha k^\alpha}{k^\alpha - \alpha\beta k^\alpha} = \frac{\alpha}{1-\alpha\beta} = A.$$

Okay, so the first-order condition checks out – this is indeed how A is defined in equation (5). Now to test the envelope condition, we want to take the derivative with respect to k, which results in

$$\begin{aligned}
v'(k) &= \frac{1}{k^\alpha - \alpha\beta k^\alpha}(\alpha k^{\alpha-1} - \alpha^2\beta k^{\alpha-1}) + \frac{\beta A}{\alpha\beta k^\alpha}\alpha^2\beta k^{\alpha-1} \\
&= \frac{\alpha}{k(1-\alpha\beta)}(1-\alpha\beta) + \frac{A}{k}\alpha\beta \\
&= \frac{\alpha}{k(1-\alpha\beta)}(1-\alpha\beta) + \frac{\alpha}{1-\alpha\beta}\frac{\alpha\beta}{k} \\
&= \frac{\alpha}{(1-\alpha\beta)k}.
\end{aligned}$$

This must equal the result from equation (4), which gives

$$\begin{aligned} U'_{f(k)}\big(f(k) - g(k)\big) f'_k(k) &= \frac{1}{k^\alpha - \alpha\beta k^\alpha} \alpha k^{\alpha-1} \\ &= \frac{\alpha}{k(1-\alpha\beta)}. \end{aligned}$$

So both the first order condition and the envelope condition are satisfied. Hooray!

The nice thing about this Cobb-Douglas example is that we can solve for $g(\cdot)$ just using a pencil some paper if we're vigilant enough, thus giving us the optimal sequence of capital. Unfortunately, most of the time we will not be able to derive an explicit, closed-form solution. We will often have to resort to approximating $g(\cdot)$.

Sometimes we'll be able to deduce qualitative features of the policy function that are valid under varying assumptions on $f(\cdot)$ and $U(\cdot)$. In such a case we can characterize $g(\cdot)$ and thus establish solutions for the capital sequence. For instance, suppose the policy function says to save a constant fraction of $f(k)$, that is, $g(k) = sf(k)$ where $s > 0$. The policy function will inherit properties of $f(k)$, in particular, i tis strictly increasing and strictly concave, continuously differentiable, comes from the origin, and is strictly positive. If we draw $g(k)$ along with a $45^{\cdot}$ line, we can trace through a trajectory of k_t. Granted, we won't know the precise values of any k_t, but we can still that it will gravitate to a steady state and furthermore, one of the steady-states (the unstable one) will be at the origin.