

## Replication and Put-Call Parity

Two portfolios with identical payoff diagrams must have equal prices by arbitrage argument – this is a pricing technique call **replication**.

**Put-call parity** is a specific type of replication, the form of which was given in lecture and discussions, via equation

$$P + S_0 = PV(K) + C.$$

- $K$  is the strike price on both options as well as the bond's face value.
- Portfolio 1:  $P$  is the price of the put option;  $S_0$  is the price at which the share of stock was originally purchased.
- Portfolio 2:  $PV(K)$  is the present value of the bond's face value;  $C$  is the price of the call option.

For stocks with dividends, we augment the equation to

$$P + S_0 - PV(\text{dividends}) = PV(K) + C,$$

where  $PV(\text{dividends})$  is the present value of any dividends.

## Risk

### Variance

The **variance** of random variable  $X$  is

$$\text{Var}(X) = \sum_{i=1}^n p_i (X_i - E[X])^2.$$

### Risk Averse Preferences

An investor is **risk-averse** if she is (negatively) sensitive to variance. As an expositional device, think in terms of utility function

$$u(X) = E[X] - A \text{Var}(X), \quad A \geq 0.$$

$A$  is the *risk aversion parameter*. The point is this: utility is decreasing in variance (i.e risk) whenever  $A > 0$ . If  $A = 0$ , then we have risk-neutral investors.

### Covariance and Correlation

**Covariance** measures *comovement* between random variables, that is, whether they both tend to move in the same direction from their respective means.

$$\text{Cov}(X, Y) = \rho_{X,Y} \sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)},$$

where  $\rho_{X,Y}$  is the **correlation coefficient** of  $X$  and  $Y$ . Note that  $-1 \leq \rho_{X,Y} \leq 1$ .

### Weighted Portfolio

Consider risky assets  $X$  and  $Y$  with respective rates of return  $r_X$  and  $r_Y$ . Your portfolio consists of  $\alpha_X$  proportion of asset  $X$  and  $\alpha_Y$  proportion of asset  $Y$ . Hence the rate of return from your weighted portfolio  $W$  is

$$r_W = \alpha_X r_X + \alpha_Y r_Y, \quad \text{such that } \alpha_X + \alpha_Y = 1.$$

The expected rate of return for  $W$  is

$$E[r_W] = \alpha_X E[r_X] + \alpha_Y E[r_Y].$$

The variance of the rate of return for  $W$  is

$$\text{Var}(r_W) = \alpha_X^2 \text{Var}(r_X) + \alpha_Y^2 \text{Var}(r_Y) + 2\alpha_X \alpha_Y \text{Cov}(r_X, r_Y).$$

### Diversification

When  $X$  and  $Y$  are negatively correlated, it follows that  $\text{Cov}(r_X, r_Y) < 0$ . Hence  $\text{Var}(r_W)$  is smaller than when  $\text{Cov}(r_X, r_Y) \geq 0$ . This is a benefit of diversification – owning negatively correlated assets minimizes portfolio risk.

Intuitively, when one asset is doing poorly, the other is doing well, the effects cancel each other out to some extent.

### Perfect Negative Correlation

If  $\rho_{X,Y} = -1$ , then

$$\text{Var}(r_W) = [\alpha_X SD(r_X) - \alpha_Y SD(r_Y)]^2.$$

Therefore risk is zero when

$$\alpha_X SD(r_X) = \alpha_Y SD(r_Y).$$

If we are given standard deviations, then substituting  $\alpha_Y = 1 - \alpha_X$  will allow us to solve for  $\alpha_X$ .

## Portfolio Choice

### Terminal Value

Let  $\widehat{W}$  denote the **terminal value** of an investment. Example: you buy a one year bond for \$100 and with rate of return of 10%. One year from now you receive that \$100 back in addition to \$10 in interest payments. So  $\widehat{W} = \$110$ .

In general, you have a choice of buying  $k$  different kinds of *risky* assets, and you spend  $a_i$  dollars on risky asset  $i$ .

Let  $\hat{r}_i$  denote interest rate on risky asset  $i$ . Total return on risky assets is

$$\sum_{i=1}^k a_i (1 + \hat{r}_i).$$

Let  $W_0$  denote initial wealth. Whatever is left over will be invested into a safe asset with interest rate  $r_f$ , giving

$$\left( W_0 - \sum_{i=1}^k a_i \right) (1 + r_f).$$

The sum of the two is the terminal wealth of the portfolio:

$$\widehat{W} = \left( W_0 - \sum_{i=1}^k a_i \right) (1 + r_f) + \sum_{i=1}^k a_i (1 + \hat{r}_i).$$

### Portfolio Optimization Problem

You want to choose how much to spend on each asset in such a way that the terminal value of your portfolio maximizes your expected utility of holding such a portfolio:

$$\max_{a_1, \dots, a_k} E \left[ u(\widehat{W}) \right].$$

Take the partial derivative with respect to arbitrary asset purchase  $a_i$  to get the first order condition

$$E \left[ u'(\widehat{W})(\hat{r}_i - r_f) \right] = 0.$$

Note that investing in risky asset  $i$  is optimal if and only if its expected interest rate is greater than that of the safe return, that is,  $E[\hat{r}_i] > r_f$ .