Problem 1

Suppose the nominal money supply grows from $M_0 = 500$ to $M_1 = 550$, real income grows from $Y_0 = 100$ to $Y_1 = 105$, and the velocity of money is constant. Using the quantity theory of money, solve for the exact rate of change in the price level, as well as the approximate rate of change in the price level using the most common approximation.

Solution 1. The equation of exchange is $MV = PY$. In words, nominal income (PY) is given by how rapidly (V) the money supply (M) is circulates through the economy via transactions. This is merely an accounting identity: there is no economic theory yet.

Suppose that the velocity of money V is constant: this assumption gives us the quantity theory of money, which we can now write as

$$P = \frac{M\overline{V}}{Y}. \tag{1}$$

We want to find the rate at which the price level is changing from $t = 0$ to $t = 1$, i.e. the rate of inflation, defined as

$$\pi_0 \equiv \frac{P_1 - P_0}{P_0} = \frac{P_1}{P_0} - 1.$$

Using equation (1), we can write inflation as

$$\pi_0 = \frac{P_1}{P_0} - 1 = \frac{\left(\frac{M_1\overline{V}}{Y_1}\right)}{\left(\frac{M_0\overline{V}}{Y_0}\right)} - 1 = \frac{M_1}{M_0}\frac{Y_0}{Y_1} - 1.$$

I'm going to add 1 to both sides, and then multiply both sides by Y_1/Y_0, which gives

$$(1 + \pi_0)\frac{Y_1}{Y_0} = \frac{M_1}{M_0}.$$

Now I'm going to do a little trick: I'm going to add $-Y_0 + Y_0$ into the left-hand side numerator and $-M_0 + M_0$ into the right-hand side numerator. (This is kosher because it's really just adding zero, oui?) Doing so and simplifying a bit gives

$$\begin{aligned} & (1+\pi_0)\left(\frac{Y_1 - Y_0 + Y_0}{Y_0}\right) = \left(\frac{M_1 - M_0 + M_0}{M_0}\right) \\ \implies \quad & (1+\pi_0)\left(\frac{Y_1 - Y_0}{Y_0} + 1\right) = \left(\frac{M_1 - M_0}{M_0} + 1\right). \end{aligned}$$

This is useful because the expression $(Y_1 - Y_0)/Y_0 \equiv g_0$ is the rate of real income growth, and the expression $(M_1 - M_0)/M_0 \equiv \mu_0$ is the rate of nominal money supply growth. Therefore we can simplify further into

$$(1 + \pi_0)(1 + g_0) = (1 + \mu_0). \tag{2}$$

Now we can solve for the exact rate of inflation,

$$\begin{aligned}\pi_0 &= \frac{1+\mu_0}{1+g_0} - 1 = \frac{1+0.10}{1+0.05} - 1 \\ &= 1/21 \\ &\approx 0.048.\end{aligned}$$

In practice, we can use a simpler equation. Expand out equation (2) and you get

$$1 + g_0 + \pi_0 + g_0\pi_0 = 1 + \mu_0.$$

Obviously the 1s on each side cancel each other out. Here's the observation that will let us take things a bit further: inflation rates and growth rates are usually pretty small, and it follows that $g_0\pi_0$ is usually *very* small. We can therefore treat $g_0\pi_0$ as though it's approximately zero, which gives the approximate formula

$$\pi_0 = \mu_0 - g_0.$$

In words: the rate of inflation is (approximately) equal to the (nominal) money supply growth rate minus the real income growth rate. A popular saying is that inflation occurs when "more money chases fewer goods," and that's what we see here: if there's more money (larger μ) or fewer goods (smaller g), then inflation π is higher.

Using this approximation, the rate of inflation is roughly $\pi_0 \approx 0.10 - 0.05 = 0.05$.

Problem 2

Suppose you lend \$100 for one year and the expected annual rate of inflation is $\pi^e = 0.04$. What annual nominal interest rate i will give you ex-ante real rate of return $r^e = 0.02$?

Solution 2. If you lent \$100 for one year at annual nominal interest rate i, then your nominal payment will be $100(1+i)$. To find the expected real payment, we must discount via the expected rate of inflation, however, which gives an expected real payment

$$100\left(\frac{1+i}{1+0.04}\right).$$

We want this real payment to be equal to $100(1+0.02)$. Therefore we must solve

$$100\left(\frac{1+i}{1+0.04}\right) = 100(1+0.02) \quad \Longrightarrow \quad (1+i) = (1+0.02)(1+0.04)$$

At this point you could solve for i just by subtracting 1 from both sides. But I'm going to go for another useful approximation.

Expand the whole thing out and you get

$$1 + i = 1 + 0.02 + 0.04 + (0.02 \times 0.04).$$

We can cancel out the 1s, and note that 0.02×0.04 is so close to zero that it is of practical irrelevance. Hence we can instead just use

$$i \approx 0.02 + 0.04 = 0.06.$$

This is the Fisher equation: $i \approx r^e + \pi^e$, often written as $r^e \approx i - \pi^e$, which says that the real interest rate is the nominal interest rate minus the extent to which inflation lessens the purchasing power of that nominal interest. We'll be thinking in terms of expected or ex-ante real interest rate (r^e) since we can only have expectations of what inflation will do (π^e) by the time the loan reaches maturity (i.e. we can't read the future so we don't exactly know what π_t will be for time t).

Problem 3

Suppose the expected rate of inflation is 4 percent and the ex-ante real interest rate is 2 percent. Suddenly the expected rate of inflation increases to 6 percent. According to the Fisher effect, what is the new ex-ante real interest rate?

Solution 3. The Fisher effect says that when there is a change in expected inflation, the nominal interest rate will change one-to-one, and therefore the ex-ante real interest rate doesn't change at all. (This is an example of money neutrality, i.e. the change in a nominal variable like inflation does not affect a real variable like the real interest rate). On the other hand, we can say that the initial nominal interest rate was $4 + 2 = 6$ percent, but now it's $6 + 2 = 8$ percent.

Problem 4

Suppose the annual nominal interest rate is always $i = 0.05$.

(a) What is the future value of 100 of today's dollars?

Solution. You take \$100, you invest it for one year at 5% interest, so one year later you get back $100(1 + 0.05) = \$105$.

(b) What is the present value of 200 dollars you'd receive one year from now?

Solution. We're going in the opposite direction, i.e. we're essentially asking "what would I have to invest today (the principal) in order to get back 200 dollars one year from now?" Mathematically,

$$P(1.05) = 200 \qquad \Longrightarrow \qquad P = \frac{200}{1.05} \approx \$190.48.$$

Note that we are essentially using the opportunity cost of investment (the interest rate) to convert the value of money from now into the future, and vice versa.

(c) What is the present value of 200 dollars you'd receive one year from now, plus 300 dollars you'd receive two years from now?

Solution. When we discount money to be received 1 year in the future, we divide the amount by (1.05). When we discount money to be received 2 years in the future, we divide the amount by $(1.05)^2$. Therefore the present value is

$$\text{PV} = \frac{200}{(1.05)} + \frac{300}{(1.05)^2} = \$462.59.$$

(d) What is the present value of 200 dollars you'd receive annually and perpetually, starting one year from now?

Solution. When we discount money to be received t years in the future, we divide the amount by $(1.05)^t$. Since $t = 1, 2, 3, \ldots$, we can write the present value as

$$\text{PV} = \frac{200}{(1.05)^1} + \frac{200}{(1.05)^2} + \frac{200}{(1.05)^3} + \ldots$$

To make things simpler analytically, let $\beta = 1/(1.05)$. Then the present value is

$$\text{PV} = \beta(200) + \beta^2(200) + \beta^3(200) + \ldots$$

It's an infinite sum, but it actually converges to a constant. Let's find it.

Multiply both sides of the equation by β to get

$$\beta\text{PV} = \beta^2(200) + \beta^3(200) + \beta^4(200) + \ldots$$

Now write out $\text{PV} - \beta\text{PV}$, and note that everything will cancel out except for the $\beta(200)$ term. We're left with

$$\text{PV} - \beta\text{PV} = \beta(200) \quad \Longrightarrow \quad (1-\beta)\text{PV} = \beta(200) \quad \Longrightarrow \quad \text{PV} = \frac{\beta}{1-\beta}(200).$$

We just need to replace the β back with the original i, which requires

$$\frac{\beta}{1-\beta} = \frac{\frac{1}{1.05}}{1 - \frac{1}{1.05}} = \frac{1}{1.05 - 1} = \frac{1}{0.05},$$

or $1/i$ more generally. Great, so the answer is

$$\text{PV} = \frac{200}{0.05} = \$4000.$$