Game Frames and Strategies

Definition 1. *A **game frame** consists of the following objects.*

- *Players $I = \{1, 2, \ldots, n\}$, where $n \geq 2$.*
- *For every $i \in I$ there is a set of **strategies** S_i, e.g. $S_i = \{\text{split}, \text{steal}\}$. The Cartesian product $S = S_1 \times S_2 \times \ldots \times S_n$ is the set of **strategy profiles**.*
- *A set of **outcomes** W.*
- *An outcome function $z : S \to W$.*

In order to have a proper game, we need to include preferences of players, i.e. we know their objectives and know that $\succsim_i$ is defined over W for all i.

Example 1. Consider a game of rock, paper, scissors. The strategies are $S_1 = S_2 = \{R, P, S\}$. Let W_i denote player i winning and D denote a draw. The set of outcomes is $W = \{W_1, W_2, D\}$. We assume that player i's preferences satisfy

$$W_i \succsim D \succsim W_j.$$

The set of strategy profiles is captured by the cells in the table below.

	R	P	S
R	D	W_2	W_1
P	W_1	D	W_2
S	W_2	W_1	D

For now, let's define **rationality** as: a player chooses their most preferred outcome based on their beliefs. We're not making any judgment about whether the beliefs themselves are rational, however—just that they're responding to their beliefs optimally.

We can represent preferences in a few different ways. One is simply by ordering them from best to worst. For instance, top being best,

$$\begin{array}{l} W_2 \\ W_6 \\ W_5 \\ W_4, W_1 \\ W_3. \end{array}$$

In this case, $W_4 \sim W_1$.

We can also order them as a preference relation:

$$W_2 \succ W_6 \succ W_5 \succ W_4 \sim W_1 \succ W_3.$$

Or we can attach ordinal numbers to each outcome, for instance

W_1	W_2	W_3	W_4	W_5	W_6
3	6	2	3	4	5

The larger numbers represent preference, equal numbers indifference. Utilities makes it easy to use a table.

Since $z : S \to W$ and $U_i : W \to \mathbb{R}$, we can compose the two functions into the **payoff function** $\pi_i : S \to \mathbb{R}$. In other words, $\pi_i(s) = u\big(z(s)\big)$.

Example 2. Here are the utilities for player 2 for each outcome.

	c	d	e
a	$\cdot, 3$	$\cdot, 6$	$\cdot, 2$
b	$\cdot, 3$	$\cdot, 4$	$\cdot, 5$

Notice that no matter what player 1 does, player 2 would get a higher payoff if they'd chosen d instead of c. In this case, d strictly dominates c for player 2.

Definition 2. *Strategy $a \in S_i$ is said to **strictly dominate** strategy b if for any $x \in S_j$, $z(a, x) \succ_i z(b, x)$, or in terms of payoffs,*

$$\pi_i(a, x) > \pi_i(b, x).$$

Definition 3. *Strategy $a \in S_i$ is said to **weakly dominate** strategy b if*

- *for any $x \in S_j$, $z(a, x) \succsim_i z(b, x)$*
- *there exists $\hat{x} \in S_j$ such that $z(a, \hat{x}) \succ_i z(b, \hat{x})$.*

Or in terms of payoffs,

- *for any $x \in S_j$, $\pi_i(a, x) \geq \pi_i(b, x)$*
- *there exists $\hat{x} \in S_j$ such that $\pi_i(a, \hat{x}) > \pi_i(b, \hat{x})$.*

Definition 4. *Strategy $a \in S_i$ is **strictly dominant** if it strictly dominates every other $s \in S_i$.*

Definition 5. *Strategy $a \in S_i$ is **weakly dominant** if for any other $x \in S_i$, either*

- *a weakly dominates x, or*
- *a is equivalent to x.*

Consider the table

	d	e
a	1	2
b	1	2
c	0	2

Strategy a weakly dominates c; it is the superior choice if P2 plays d, but is indifferent if P2 plays e. Strategy a is equivalent to b; no matter what P2 plays, P1 receives the same payoff. So a and b are both weakly dominant strategies.

Note if if you read just "dominant," it is assumed to mean *weakly* dominant.

Second Price Auctions

There are two players. S_i is the set of possible bids of player i. The outcomes is who wins and what they pay: (i, p). If there's a tie, P1 is declared the winner. Given bids (b_1, b_2), the outcome function is

$$z(b_1, b_2) = \begin{cases} (1, b_2) & \text{if } b_1 \geq b_2, \\ (2, b_1) & \text{if } b_1 < b_2. \end{cases}$$

So if P1 has the (weakly) highest bid, then she wins and pays P2's bid. If P2 has the highest bid, then she wins and pays P1's bid.

V_1 is the value of the object to P1; it is the maximum price they'd be willing to pay.

- $(1, p) \succ_1 (1, p')$ iff $p < p'$. In words, P1 would prefer paying less when winning.
- $(1, V_1) \sim_1 (2, p)$ for any p. In words, P1 is indifferent between losing and having to pay their full valuation of the object.
- $(2, p) \sim_1 (2, p')$ for any p, p'. So P1 doesn't care what P2 pays if P1 doesn't even win anyway.

We can represent these preferences with the payoff function

$$\pi_1(b_1, b_2) = \begin{cases} V_1 - b_2 & \text{if } b_1 \geq b_2, \\ 0 & \text{if } b_1 < b_2. \end{cases}$$

Theorem 1. *Bidding the true value is a weakly dominant strategy under these preferences.*

"Proof." Consider all possible actions of P2.

- Start with $b_2 = V_1$. In this case, P1 gets payoff 0 no matter what P1 does.
- Now suppose $b_2 > V_1$. Then P1 could get payoff of zero for bidding less than b_2, in particular, by bidding $b_1 = V_1$. However, if voting greater than or equal to b_2, then P1 gets negative payoff.
- Suppose $b_2 < V_1$. Then P1 could get $V_1 - b_2 > 0$ payoff for bidding greater than or equal to b_2. Bidding less than b_2 gives zero payoff.

So bidding truthfully is weakly dominant—it does no worse than other strategies in any situation; in one case it does better than bidding above b_1; in one case it does better than bidding below b_1.

Suppose that the assumption where $(2, p) \sim_1 (2, p')$ does not hold. In particular, suppose P1 wants P2 to pay as much as possible in the case that P1 loses. Then P1's behavior would change.

Nash Equilibria

Two prisoner's have a choice of confessing or not confessing. If they both confess, they both get nine years in prison. If they both remain silent, they both get one year. If one confesses and the other remains silent, the one who confesses goes free and the one who goes silent gets 11 years.

	C	N
C	$9, 9$	$0, 11$
N	$11, 0$	$1, 1$

We can't say what happens because we don't know preferences.

Definition 6. *Suppose $a \in S_1$ and $b \in S_2$. Strategy profile (a, b) is a* ***Nash equilibrium*** *if*

- $\pi_1(a, b) \geq \pi_1(x, b)$ *for all* $x \in S_1$,
- $\pi_2(a, b) \geq \pi_2(a, y)$ *for all* $y \in S_2$.

It's a sort of self-enforcing agreement. Given what P2 is doing, P1 doesn't want to play anything else; and given what P1 is doing, P2 doesn't want to play anything else.

Consider the game below (with payoffs from now on unless stated otherwise):

	D	E	F
A	$1, 0$	$\underline{2}, \underline{3}$	$3, 1$
B	$\underline{3}, 3$	$1, \underline{5}$	$\underline{4}, 4$
C	$\underline{3}, \underline{2}$	$0, 1$	$3, 0$

Strategy profiles (C, D) and (A, E) are Nash equilibria.

We can also talk in terms of a **best response function**. For a strategy of player i, what give player j the highest payoff? Notice that $BR_2(A)$ maps to E,

$$\begin{array}{lll} BR_1(D)=\{B,C\} & BR_1(E)=\{A\} & BR_1(F)=\{B\} \\ BR_2(A)=\{E\} & BR_2(B)=\{E\} & BR_2(C)=\{D\} \end{array}$$

and $BR_1(E)$ maps to A. So (A,E) is a Nash equilibrium. And since $BR_2(C)$ maps to D and $BR_1(D)$ maps to C, (C,D) is another Nash equilibrium.

Iterative Dominance

One way to solve a game is to remove any dominated strategies. The first approach is to delete*strictly dominated* strategies; then check if any of the remaining strategies have become dominated after the previous deletion; and proceed until no more deletions can be made. If there are two weakly dominated strategies that could be deleted, then it doesn't matter which one is deleted first.

Example 3. Consider the game with payoff matrix

	L	C	R
U	$1,1$	$2,0$	$2,2$
M	$0,3$	$1,5$	$4,4$
D	$2,4$	$3,6$	$3,0$

Notice that U is strictly dominated by D. A rational P1 will never play R, so if P2 believes P1 is rational, and P1 is indeed rational, then U is irrelevant in the game. So we can delete it.

	L	C	R
M	$0,3$	$1,5$	$4,4$
D	$2,4$	$3,6$	$3,0$

Now C strictly dominates both L and R. So if P1 is rational, and P2 believes that P1 is rational, in addition to the rationality requirements that lead to the first deletion, then we can delete L and R as well.

	C
M	$1,5$
D	$3,6$

So now D strictly dominates M. Subject to more rationality and beliefs about rationality, we can conclude that the solution is (D,C).

If we carry out the same approach using by deleting weakly dominated strategies, then the order of deletion matters. So make the procedure well-defined, we have to adjust a little bit:

i. Identify all dominated strategies, weak or strict, and delete them all.
ii. Repeat
iii. Stop when you can't delete anymore.

If the remaining profile is a unique, then it is a *unique iterated weak dominant strategy equilibrium.* If there one strategy profile remaining, then it is Nash. If there are multiple remaining strategy profiles, then there is no guarantee that is it Nash.

Theorem 2. *Suppose $S_i \in \mathbb{R}^m$, $\pi_i : S \to \mathbb{R}$, and there are n players. If*

- *S_i is convex, closed, and bounded,*
- *$\pi_i(s_i, s_{-i})$ is concave in s_i,*
- *π is continuous,*

then a Nash Equilibrium exists.

Cournot Game

There are a fixed number n of firms who choose to produce output q_i. There exists an inverse demand function $P = P(Q)$ where Q is the industry output, i.e. $\sum_{i=1}^n q_i = Q$. Production is homogeneous. The cost functions are $C_i(q_i)$. The strategy set is output $S_i = [0,\infty)$ with strategy profile $q = (q_1,\dots,q_n) \in S$. Therefore firms have profit functions $\pi_i(q) = q_iP(Q) - c_i(q_i)$.

Theorem 3. *Suppose q^* is a Cournot-Nash equilibrium where $q_i^* > 0$ for all i and $\frac{dP}{dQ}(Q^*) < 0$. Then there exists some $\hat{q}$ such that*

- *$\hat{q}_i > q_i^*$ for all i,*
- *$\pi_i(\hat{q}) > \pi_i(q^*)$ for all i.*

Does a Cournot-Nash equilibrium always exist? One problem is that S_i is not bounded. Let's impose the existence of some $\bar{Q}$ such that $P(\bar{Q}) = 0$. Then firms will never want to produce beyond $\bar{Q}$ because they would receive negative payoff. Therefore $S_i = [0,\bar{Q}]$, so S_i is now convex, closed, and bounded. Woo. Let's further suppose that $P(Q)$ and $C_i(q_i)$ are both continuous so that π_i is continuous. For concavity, we need to assume that $d^2C_i/dq_i^2 \geq 0$.

Bertrand Game

Bertrand decided that prices are the variable chosen instead of quantity. The product is still homogeneous, and therefore whoever offers the lowest price gets all of the demand. Meaning that the demand function is

$$D_i(p_1, \dots, p_n) = \begin{cases} 0 & \text{if } \exists j \neq i \text{ s.t. } p_j < p_i \\ D(p_i) & \text{if } p_i < p_j \text{ for all } j \neq i \\ \frac{1}{m+1} D(p_i) & \text{if } p_i = p_k \text{ for } m \text{ firms} \\ & \text{and } p_i \leq p_j \text{ for all } j. \end{cases}$$

So if i charges more than anyone else, they get nothing. If they charge the lowest price uniquely, they get the entire demand. If they and m other firms charge the lowest price, then they split the demand by $m+1$.

Theorem 4. *If $(p_1^*, \dots, p_n^*)$ is a Nash Equilibrium, then for at least two firms i and j, $p_i^* = p_j^* = c$; and for any other firm, $p_k^* \geq c$.*

In other words, two or more firms will produce competitive equilibrium output.

The problem with the Bertrand model is the extreme discontinuity of the demand function. Turns out that if we use a continuous approximation of the Bertrand world, then $p > MC$ is restored.

Theorem 5 (Anti-Bertrand Theorem)**.** *Suppose that $n = 2$. Firms pay a cost of $C_i\big(D_i(p_1, p_2)\big)$. Each firm is twice continuously differentiable in their demand functions, and they provide substitute goods, i.e.*

$$\frac{\partial D_i}{\partial p_i} < 0, \quad \frac{\partial D_i}{\partial p_j} > 0.$$

If $(p_1^*, p_2^*) \gg 0$ is a Nash Equilibrium with $D_i(p_1^*, p_2^*) > 0$ for both i, then $p_i^* > MC_i$.

Theorem 6. *Suppose (p_1^*, p_2^*) is a Nash Equilibrium with strictly positive prices and demands. Then there exists some $(\hat{p}_1, \hat{p}_2)$ such that $\pi_i(\hat{p}_1, \hat{p}_2) > \pi_i(p_1^*, p_2^*)$ for both i.*