Solution 1

Move through each information set and see if the assessment implies a rational choice.

Start with information set $\{x, y\}$ of Player 2. Their strategy is to always play f. They believe there is a 1/3 chance that node x is the one that is reached, in which case they would get a payoff of 3. They believe there is a 2/3 chance that node y is the one that is reached, in which case they would also get a payoff of 3. Thus the their expected payoff for f is

$$\frac{1}{3}[3] + \frac{2}{3}[3] = 3.$$

If they decided to play g instead, their payoff would be

$$\frac{1}{3}[6] + \frac{2}{3}[0] = 2.$$

So, given Player 2's beliefs, their choice to play f over g is sequentially rational.

Now consider information set $\{w, z\}$ of Player 1. Their strategy is to always play h. They believe there is a 1/2 probability that node w is the one that is reached, in which case h gives them a payoff of 3. They believe there is a 1/2 probability that node z is the one that is reached, in which case h gives them a payoff of 0. So the expected payoff is h is

$$\frac{1}{2}[3] + \frac{1}{2}[0] = 1.5.$$

If they decided to play k instead, then their payoff would be

$$\frac{1}{2}[1] + \frac{1}{2}[1] = 1.$$

So, given Player 1's beliefs, their choice to play h over k is sequentially rational.

Now consider Player 2's choice of d or e. They choose to play e, after which Player 1 plays h, giving a payoff of 2. If they chose instead to play d, they'd get 1. So Player 2's choice of playing e over h is sequentially rational.

Okay, finally, let's look at Player 1's choice at the root between a, b, and c.

- Strategy a is played with probability 1/8. In this case, the node $x \in \{x, y\}$ of Player 2 is reached, and their strategy in that information set is to always play f. So we end up at the outcome $[0, 3]$.
- Strategy b is played with probability 3/8. In this case, the node $y \in \{x, y\}$ of Player 2 is reached, and their strategy in that information set is to always play f. So we end up at the outcome $[0, 3]$.

- Strategy c is played with probability 4/8. In this case, Player 2 then plays e. Then the node $z \in \{w, z\}$ of Player 1 is reached. They choose to play h for outcome $[0, 2]$.

So Player 1's expected payoff is

$$\frac{1}{8}[0] + \frac{3}{8}[0] + \frac{4}{8}[0] = 0.$$

But Player 1 could do better than this. Notice that if Player 1 chooses to play k instead of h, they'd get payoff of $1 > 0$ in the case of c, which is an improvement. (This also implies that Player 1 should always play c instead of mixing over a and b, which both give zero.) Therefore this assessment is not sequentially rational for Player 1.

Solution 2

The first step is to find the probabilities $P_{\text{root},\sigma}(x)$ for all nodes x contained within nontrivial information sets, translated to English as "the probability that x is reached from the root of the tree if σ is implemented." There are four nodes found within nontrivial information sets: x, y, w, and z.

- The probability of node x being reached from the root is simply 1/8, that is, the probability that Player 1 plays a.
- The probability of node y being reached from the root is simply 3/8, that is, the probability that Player 1 plays a.
- The probability of node w being reached is the 4/8 probability Player 1 plays c, times the 3/4 probability that Player 2 plays d, so 3/8.
- The probability of node z being reached is the 4/8 probability Player 1 plays c, times the 1/4 probability that Player 2 plays d, so 1/8.

Now we do Bayesian updating to specify μ. This is very easy to do if everything is in terms of common denominators. Since reaching y is three times as likely as reaching x, Player 2's beliefs should reflect that. Those are the only two nodes within the information set, so it must be the case that $\mu(x) = 1/4$ and $\mu(y) = 3/4$. Similarly, it must be the case that $\mu(w) = 3/4$ and $\mu(z) = 1/4$.

But I'll be more rigorous about it. First, find the probabilities of reaching the nontrivial information sets.

- The probability of reaching $\{x, y\}$ is the probability that Player 1 plays a plus the probability that Player 1 plays b, so 4/8.

- The probability of reaching $\{w, z\}$ is the probability that Player 1 plays c (because all of Player 2's subsequent choices will lead to $\{w, z\}$), that is, 4/8.

Now apply the typical updating mechanics to get

$$\mu(x) = \frac{1/8}{4/8} = \frac{1}{4} \qquad \mu(y) = \frac{3/8}{4/8} = \frac{3}{4} \qquad \mu(w) = \frac{3/8}{4/8} = \frac{3}{4} \qquad \mu(z) = \frac{1/8}{4/8} = \frac{1}{4}.$$

Oh hey, that's what I said it would be earlier. Good. Great. Grand. Wonderful.

So the system of beliefs satisfying Bayesian updating at reached information sets is

$$\mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{array}\right).$$

Solution 3

Let us start by putting full probability on a and working our way through. Player 2 will then have beliefs

$$\begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix},$$

since node y is not reached. Since Player 2 is certain that x is reached, they will choose to play g. Player 1 now has a payoff of 4. If Player 1 chooses to play b instead, they'd get payoff $1 < 4$, so (a, g) checks out.

We still need to specify what's going to happen in the subgame starting with Player 2, however. Suppose Player 2 plays d. Then it's rational for Player 1 to choose h. But if Player 1 plays h, then it is no longer rational for Player 2 to play d because e would give payoff of $2 > 1$. So d cannot be part of a WSE.

Maybe Player 2 plays e then, in which case Player 1 rationally chooses to play k. But if k is played by Player 1, then Player 2 would rather play d for payoff $3 > 1$. So e cannot be part of a WSE either. But uh, Player 2 can't specify anything else for that node. Therefore this game has no pure-strategy WSE.

Solution 4

If the game is simple enough, it might be easiest to find all of the Nash equlibria and then see which ones are WSE. I will do this approach first.

Nash Equilibrium Approach. To that end, the strategic-form of the game is

	E	F
AC	$\underline{1.5}, \underline{0.5}$	$\underline{1}, \underline{0.5}$
AD	$1, \underline{1}$	$\underline{1}, \underline{1}$
BC	$\underline{1.5}, \underline{1}$	$0.5, 0$
BD	$1, \underline{1.5}$	$0.5, 0.5$

So we have four NE to check: (AC, E), (AC, F), (AD, F), and (BC, E). We can represent these as behavioral strategies

$$(AC, E) = \left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array} \right),$$

$$(AC, F) = \left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right),$$

$$(AD, F) = \left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array} \right),$$

$$(BC, E) = \left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array} \right).$$

(AC, E). I'll start from the bottom.

- Node x is not reached.
- Node y is reached. Therefore Bayesian updating requires that

$$\mu = \left(\begin{array}{cc} x & y \\ 0 & 1 \end{array} \right).$$

- E and F are both rational for Player 2 when Player 1 plays C.
- Player 1 can choose C over D regardless of what Player 2 does.
- Player 1 playing A gives payoff 1; Playing B gives payoff 1 as well. So A is rational.

Sequential rationality checks out. This qualifies as a WSE.

(AC, F). I'll start from the bottom.

- Node x is not reached.

- Node y is reached. Therefore Bayesian updating requires that

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

- E and F are both rational for Player 2 when Player 1 plays C.
- Player 1 can choose C over D no matter what Player 2 does at node y.
- Player 1 playing A gives payoff 1; Playing B gives payoff 1 as well. So A is rational.

Sequential rationality checks out. This qualifies as a WSE, too. In fact, I just copied and pasted this from the previous one.

(AD, F). I'll start from the bottom.

- Node x is not reached.
- Node y is not reached either. This means there are no Bayesian restrictions upon the beliefs of Player 2. So we can write

$$\mu = \begin{pmatrix} x & y \\ p & 1-p \end{pmatrix}.$$

  But it still has to be the case for Player 2 that playing F is rational. This will be the case when

$$F: \quad p(0) + (1-p)(0) \geq p(2) + (1-p)(0) \quad : E,$$

  which requires that $p = 0$. Hence we require

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

- Playing A is rational for Player 1 because $1 > 0$.
- Playing D is rational for Player 1 because $1 = 1$.

And so another WSE has been found.

(BC, E). I'll start from the bottom.

- Node x is reached.

- Node y is also reached. This means we must place Bayesian restrictions upon the beliefs of Player 2, in this case,

$$\mu = \begin{pmatrix} x & y \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

  which are determined entirely by Nature.

- Given the beliefs in the last step, Playing E is rational because

$$F: \quad \frac{1}{2}(0) + \frac{1}{2}(0) \leq \frac{1}{2}(2) + \frac{1}{2}(0) \quad : E,$$

- Player 1 choosing B is rational because $1 = 1$.
- Player 1 choosing C is rational because $2 > 1$.

And so another WSE has been found.

So in this question, all NE are WSE. With the previous game, no NE were WSE. In some games, only some of the NE will be WSE. Point is, you gotta work for these, no real shortcuts.

From Scratch Approach. The other way is to just work your way through and pick up the WSE as you go.

- Let's start by trying AC. Then x is not reached but y is, so

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

  Since Player 2 is thus convinced they are in node y, they will choose to play either E or F because they both give payoff zero.

  Suppose they pick E. Then A is rational because it gives payoff 1 compared to $BE \to 1$. C is rational because $CE \to 2$ compared to $D \to 1$. Therefore (AC, E) is a WSE.

  Now suppose they pick F. Then A is rational because it gives payoff 1 compared to $BF \to 0$. C is rational because $CF \to 1$ compared to $D \to 1$. Therefore (AC, F) is a WSE.

- Now let's try AD. Neither x nor y are reached, so Bayes places no restrictions on belief. Thus

$$\mu = \begin{pmatrix} x & y \\ p & 1-p \end{pmatrix}.$$

Is A rational? Yes, it is always rational because it weakly dominates B. Is D rational? Well, D is only rational is F is played. So we need to ask when F is rational. This will depend on the beliefs of Player 2. Specifically, given beliefs p and $1-p$, the expected payoffs must be such that

$$E: \quad p(2) + (1-p)0 \leq p(0) + (1-p)0 \quad : F$$

This implies that F is rational only when $p = 0$. So

$$\mu = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

Therefore (AD, F) is a WSE.

- Let's now try BC. Since x and y are both reached,

$$\mu = \begin{pmatrix} x & y \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

For B to be rational, it must be that E is chosen. C is always rational because C weakly dominates D. So when can Player 2 choose E? Always, because E weakly dominates F.

We have (BC, E) as another WSE.

- Let's try BD. Because x is reached but y isn't,

$$\mu = \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}.$$

This means Player 2 should choose E. This makes B a rational choice compared to A because the payoffs are both 1. But this makes C the rational choice because $CE \to 2$ compared to $D \to 1$. So BD cannot be part of a WSE.

We get the same WSE with both approaches (as we damn well should). Whichever is easiest depends on the problem, especially on how simple the strategic-form of the game is.