ECN 200D: Week 1 Lecture Notes
McCall Search Model

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These notes introduce the **McCall search model**, which is a nice introduction into the world of **search theory**. In this case, it is an alternative approach to studying the phenomenon of unemployment, which the typical Walrasian analysis does not account for. Our goal is to maximize the function

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t y_t, \right]$$

where y_t represents the income received in period t and, in this case, the utility received. There will be uncertainty about what precisely y_t will be in each period t, hence the expectation. The subscript on the expectation operator reflects the fact that we are maximizing the expectation from time $t = 0$; we know what y_0 is while maximizing, but not what any other subsequent y_t will be. We will assume that the economy consists of people identical to this person, so any solution we find will reflect the economy's equilibrium.

1 Setup

For simplicity, we will define the income function to be

$$y_t = \begin{cases} w & \text{if employed} \\ z & \text{if unemployed, } 0 < z < \overline{w}. \end{cases}$$

In each period t, unemployed workers receive a job offer distributed according to F, that is, $w \sim F$, where the support of F is the interval $[\underline{w}, \overline{w}]$. If the unemployed worker accepts the job offer, then the job is kept forever. If the worker does not accept the job offer, then they continue to the next period to receive a new offer.

It turns out that there is a unique equilibrium to this problem. In particular, there exists a unique number R such that $\underline{w} < R < \overline{w}$ and people will accept their job offer if it exceeds R. This number R is called the **reservation wage**.

2 Behavior

Let $V(w)$ be the value function of being employed at wage w. Since the worker is employed forever, she therefore receives w in each period, and hence

$$V(w) = w + \beta w + \beta^2 w + \ldots = \frac{w}{1-\beta}.$$

To an unemployed worker, this is seen as the value for accepting the job offer with wage w.

Let U be the value function for being employed. Notice that U has not been made a function of w in the way that $V(w)$ has. This is because the *value* of remaining unemployed does not depend on whatever offer you received today—it is a function of whatever offers you are expected to receive in the future. We will write

$$U = z + \beta E[J(w')], \tag{1}$$

where $J(w')$ is the value of *receiving* the offer w', not the value of actually accepting it. In other words,

$$J(w) = \max\{V(w), U\} = \max\left\{\frac{w}{1-\beta}, z + \beta E[J(w')]\right\}. \tag{2}$$

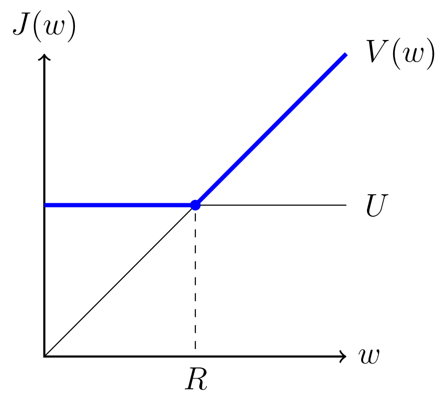
Let's try to digest this a little bit more. Suppose you receive a job offer today with wage w. Then you have two choices.

(a) Accept the job offer and receive $V(w)$ as your payoff, or

(b) Remain unemployed, receive z today, and try your luck again next period—that is, face $J(w')$ with a new random wage w', discounted by a period.

Of those two, you will choose the one that gives the higher payoff, i.e. the maximum. Thus, the value of choosing to be unemployed is to receive z today and face a new choice tomorrow—this is the value function U as written above. This means we have a functional equation and we want to solve for $J(\cdot)$.

As a first line of attack, let's just try to graph $J(w)$. Graphing $V(w)$ is rather easy since it is simply $w/(1-\beta)$. Graphing U is also easy since, as we just said, it is not even a function of w; thus, is it a horizontal line. So the graph is



The thick blue line represents the maximum of $V(w)$ and U. The two functions intersect at the reservation wage $w = R$. This is intuitive. If the value of being unemployed is greater than the value of accepting the job at wage w, then reject the job offer. If the value of the job at wage w is greater than the value of remaining unemployed, then accept the job offer. The wage R is where the worker is indifferent between remaining unemployed or accepting the job at wage R.

We can use this information to write $U = R/(1-\beta)$. Thus, we can rewrite function (2) as

$$J(w) = \max\left\{\frac{w}{1-\beta}, \frac{R}{1-\beta}\right\} = \frac{\max\{w, R\}}{1-\beta}.$$

This is nice because instead of having to solve for a function, we only have to solve for the scalar R, which should be a considerably easier task. Because $U = R/(1-\beta)$, we can use equation (1) to write

$$U = \frac{R}{1-\beta} = z + \beta \int_{\underline{w}}^{\overline{w}} \frac{\max\{w', R\}}{1-\beta} f(w')\, dw',$$

where the integral is the expectation.[1] Multiply both sides by $1-\beta$. Then

[1]He used w instead of w' in his analysis. Nothing changes, but usage of w' seems more consistent with what came prior.

R^* solves

$$R = (1-\beta)z + \beta \int_{\underline{w}}^{\overline{w}} \max\{w', R\} f(w')\, dw'. \tag{3}$$

Great. So how we do actually solve it? We will explore two different approaches.

3 Solution Method 1

Define the function

$$G(R) = (1-\beta)z + \beta \int_{\underline{w}}^{\overline{w}} \max\{w', R\} f(w')\, dw' - R.$$

From equation (3), we know that any R satisfying $G(R) = 0$ is a solution.

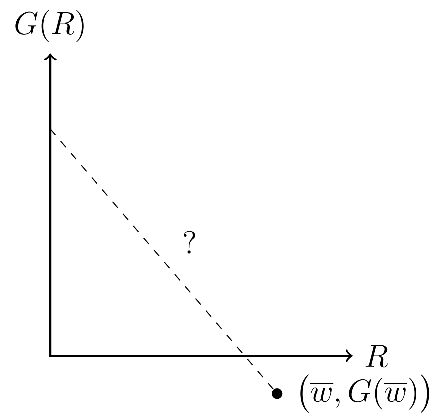
Evaluating $G(\overline{w})$, and noting that $\max\{w, \overline{w}\} = \overline{w}$, we get

$$\begin{aligned} G(\overline{w}) &= (1-\beta)z + \beta \int_{\underline{w}}^{\overline{w}} \max\{w', \overline{w}\} f(w')\, dw' - \overline{w} \\ &= (1-\beta)z + \beta\overline{w} \int_{\underline{w}}^{\overline{w}} f(w')\, dw' - \overline{w}. \end{aligned}$$

Notice that the integral is evaluating a density function over its entire support. Thus, the integral simply evaluates to 1. So we have

$$G(\overline{w}) = (1-\beta)z + \beta\overline{w} - \overline{w} = (1-\beta)(z - \overline{w}).$$

Since $\beta \in (0, 1)$, the first term is positive. And since $z < \overline{w}$, the second term is negative. Thus, $G(\overline{w}) < 0$.



The graph shows $G(\overline{w}) < 0$. But it is not clear what the rest of the function $G(R)$ looks like. Ideally it would look something like that shown because it crosses the R-axis exactly once and thus there exists a unique solution. For this to be satisfied, we require two things. First, we'll want $G'(R) < 0$ so that it slopes down (and never back up). Second, we'll want $G(\underline{w}) > 0$ so that we're guaranteed to cross the R-axis. Let's analyze.

3.1 Derivative Check (Uniqueness)

Let us first (re)familiarize ourselves with the **Leibniz rule**. For our purpose, it can be stated as

$$\frac{d}{da}\left[\int_a^b f(x)\ dx\right] = -f(a).$$

Intuitively, this says that when we increase the lower limit of integration by just a tiny little bit, we'll remove that tiny little slice under the function from the area being integrated. Similarly,

$$\frac{d}{db}\left[\int_a^b f(x)\ dx\right] = f(b).$$

So if we increase the upper limit of integration by just a tiny little bit, we'll add that tiny little slice under the function to the area being integrated.

Now back to the economics. To compute $G'(R)$, let's rewrite $G(R)$ as

$$G(R) = (1-\beta)z + \beta R \int_{\underline{w}}^{R} f(w')\, dw' + \beta \int_{R}^{\overline{w}} w' f(w')\, dw' - R.$$

We can apply Leibniz's rule to this. The result will be

$$\begin{aligned} G'(R) &= \beta \left[R f(R) + \int_{\underline{w}}^{R} f(w')\, dw' - R f(R) \right] - 1 \\ &= \beta \int_{\underline{w}}^{R} f(w')\, dw' - 1. \end{aligned}$$

We're integrating a density function over less than its entire support, and then multiplying it by $\beta \in (0,1)$, so $G'(R) < 0$. This is good.

3.2 Positivity Check (Existence)

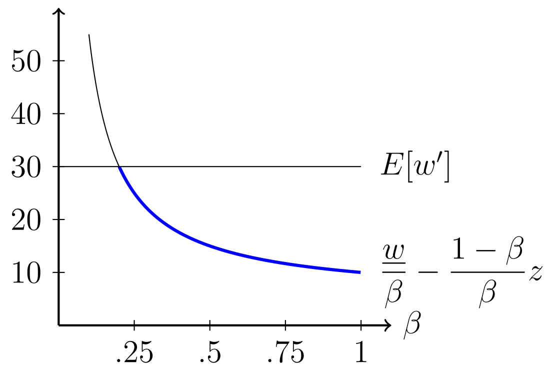
To guarantee a solution, we need $G(R) = 0$ somewhere, which means we need $G(\underline{w}) > 0$ so that the function crosses the R-axis. Evaluating, we have

$$\begin{aligned} G(\underline{w}) &= (1-\beta)z + \beta \int_{\underline{w}}^{\overline{w}} \max\{w', \underline{w}\} f(w')\, dw' - \underline{w} \\ &= (1-\beta)z + \beta \int_{\underline{w}}^{\overline{w}} w' f(w')\, dw' - \underline{w} \\ &= (1-\beta)z + \beta - \underline{w} \\ &= (1-\beta)z + \beta E[w'] - \underline{w}. \end{aligned}$$

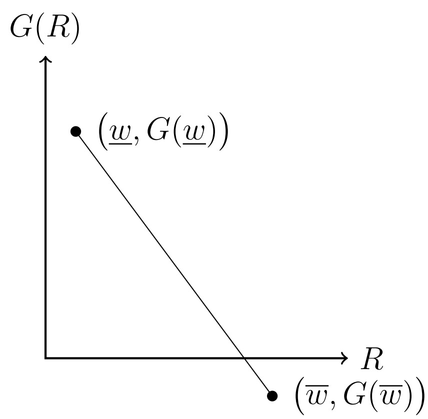
It's not really obvious that this is positive. We would need

$$E[w'] > \frac{\underline{w}}{\beta} - \frac{1-\beta}{\beta} z.$$

To illustrate, suppose that $\underline{w} = 10$, $z = 5$, and $E[w'] = 30$. Then the plot each side of the inequality is



The thick, blue segment of the function indicates the values of β that satisfy positivity of $G(\underline{w})$. Thus, if β is too small, then there is no reservation wage. Recall that a low value for β indicates that people are impatient and thus heavily discount whatever improvement a high wage might provide in the future. If people care about the future little enough, they will never work.



$G(\underline{w})$ is positive, $G(\overline{w})$ is negative, the slope is strictly negative, and the function is continuous. Thus, the function must take on this shape, indicating the existence of a unique reservation wage.

So in order to have a unique solution, we will simply assume that

$$(1-\beta)z + \beta E[w'] - \underline{w} > 0.$$

In certain cases, e.g. where $w \sim unif[0,1]$, the assumption is satisfied.

4 Solution Method 2

Let's go ahead and assume that $w \sim unif[0,1]$. This really amounts to a normalization of the distribution of wages. We're going to try to apply the contraction mapping theorem to solve. Recall that a mapping $T : [0,1] \to [0,1]$ is a contraction if for any $x, y \in [0,1]$, we have

$$d\big(T(x), T(y)\big) \leq kd(x,y), \quad k \in [0,1).$$

We are dealing with real numbers, so we can use the absolute value metric.

Define the function

$$T(R) = (1-\beta)z + \beta \int_0^1 \max\{w, R\} 1 \, dw,$$

where $f(x) = 1$ is the density of $unif[0,1]$. We will show that this is a contraction. For any $x, y \in [0,1]$, the distance $d\big(T(x), T(y)\big)$

$$\begin{aligned}
&= \left| (1-\beta)z + \beta \int_0^1 \max\{w, x\} \, dw - (1-\beta)z - \beta \int_0^1 \max\{w, y\} \, dw \right| \\
&= \beta \left| \int_0^1 \max\{w, x\} \, dw - \int_0^1 \max\{w, y\} \, dw \right| \\
&= \beta \left| x \int_0^x \, dw + \int_x^1 w \, dw - y \int_0^y \, dw - \int_y^1 w \, dw \right| \\
&= \beta \left| \frac{x^2}{2} - \frac{y^2}{2} \right| \\
&= \beta \left| \frac{(x-y)(x+y)}{2} \right| \\
&= \beta \left| x - y \right| \left| \frac{x+y}{2} \right|.
\end{aligned}$$

Since $x, y \in [0,1]$, it follows that $x + y \leq 2$. Therefore, $(x+y)/2 \leq 1$. And since $\beta < 1$, it follows that

$$d\big(T(x), T(y)\big) \leq kd(x, y), \quad k \in [0, 1).$$

So $T(R)$ is in fact a contraction.

Furthermore, because $([0, 1], |\cdot|)$ is a complete space, we know that the contraction mapping theorem holds. Thus we can draw two important conclusions:

(a) There exists a unique R^* such that $T(R^*) = R^*$.

(b) For any $R_0 \in [0, 1]$, define the sequence $R_{n+1} = T(R_n)$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} R_n = R^*$.

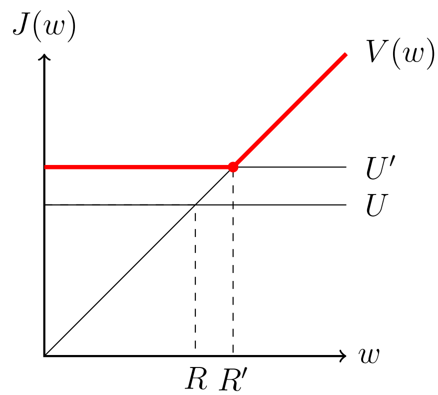
5 Comparative Statics

We might want to ask what effect an increase in z will have on either unemployment or R^*. To find the effect a change in z would have on R^*, we want to use the implicit function theorem and totally differentiate $G(R) = 0$. In particular, we have

$$\frac{d}{dz} G\big(R(z), z\big) = \frac{\partial G}{\partial R}\frac{dR}{dz} + \frac{\partial G}{\partial z} = 0 \quad \Longrightarrow \quad \frac{dR}{dz} = -\frac{\partial G/\partial z}{\partial G/\partial R}.$$

The numerator is $1 - \beta$. What about the denominator? Well, we already showed that $G'(R) < 0$ in subsection 3.1 when we were showing the existence of the reservation wage. Thus, overall, $dR/dz > 0$.

So when there is a higher unemployment benefit, there will be a higher reservation wage. Since wages are random draws, this makes it less likely that a draw will be above the reservation wage and thus less likely that unemployed workers will accept a job offer. Thus, higher unemployment. This can be seen as an upward shift of the U function to U'.



6 Solution Summary

i. Solve the value function of being employed $V(w) = w/(1-\beta)$.

ii. By definition, $U = V(R) = R/(1-\beta)$. Insert this into $J(w)$ to get

$$J(w) = \frac{\max\{w, R\}}{1-\beta}.$$

iii. Now evaluate U with (ii). Solve for R.

iv. Take the equation from (iii) and solve it for zero. Call this $G(R)$. Any R satisfying $G(R) = 0$ is a solution.

v. Evaluate $G(\overline{w})$ to establish negativity.

vi. Use Leibniz's rule to calculate $G'(R) < 0$.

vii. Establish that $G(\underline{w}) > 0$ so that a solution is guaranteed to exist by continuity (and is unique by strictly negative slope). We will often just assume whatever is necessary to satisfy this condition.

Alternatively, you could use the contraction mapping theorem, but that will be rather tedious to execute.