ECN 200B—Second Welfare Theorem Proof

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February 21, 2017

Preliminary Results

Theorem 1 (Separating Hyperplane Theorem). *Suppose sets $Q, Q' \in \mathbb{R}^A$ are disjoint and convex. Then there exists $p \in \mathbb{R}^A \setminus \{0\}$ and $k \in \mathbb{R}$ such that*

(a) *for all $q \in Q$, $q \cdot p \leq k$,*

(b) *for all $q' \in Q'$, $q \cdot p \geq k$.*

Second Fundamental Theorem of Welfare Economics

Theorem 2. *Suppose that $\{I, J, (u^i, w^i, (s^{i,j}), Y^j)$ is a production economy where all u^i are continuous, locally nonsatiated, and quasiconcave, and each set Y^j is convex and satisfies free disposal. Let $(\hat{x}, \hat{y})$ be a Pareto efficient allocation such that for all i, $x^i \gg 0$. Then there exists prices p and nominal incomes $(m^1, \ldots, m^I)$ such that*

(a) $\sum_{i=1}^I m^i = p \cdot \sum w^i + p \cdot \sum_{j=1}^J \hat{y}^j$,

(b) *for all i, $\hat{x}^i$ maximizes $u^i(x)$ subject to $p \cdot x \leq m^i$,*

(c) *for all j, $\hat{y}^j$ maximizes $p \cdot y$ subject to $y \in Y^j$,*

(d) $\sum_{i=1}^I \hat{x}^i = \sum_{i=1}^I w^i + \sum_{i=1}^I \hat{y}^j$.

Point (d) is actually implied by Pareto efficiency, but whatever.

The Setup. Suppose $(\hat{x}, \hat{y})$ is a Pareto efficient allocation such that for all i, $x^i \gg 0$. We'll be thinking of this allocation as one that a social planner wants to implement. For simplicity, we will assume that $I = 2$ and $J = 1$.

Define the set

$$\mathcal{U}^i = \{x^i \mid u^i(x^i) > u^i(\hat{x}^i)\}.$$

So $\mathcal{U}^i$ consists of all bundles that individual i strictly prefers to the Pareto efficient bundle. Define

$$\mathcal{U} = \mathcal{U}^1 + \mathcal{U}^2 = \{x \mid \exists x^1 \in \mathcal{U}^1 \text{ and } \exists x^2 \in \mathcal{U}^2 \text{ satisfying } x^1 + x^2 = x.\}$$

So $\mathcal{U}$ is the set of points that can be written as a sum of one point from $\mathcal{U}^1$ and one point from $\mathcal{U}^2$.

Also define

$$\mathcal{F} = \sum_{i=1}^{I} w^i + Y = \{x \mid \exists y \in Y \text{ satisfying } y + \sum_{i=1}^{I} w^i = x\}.$$

This is the set of all feasible points for the planner; it represents all possible combinations of commodities the economy could potentially have.

Claim 1. *$\mathcal{U}^i$ are convex.* This follows from quasiconcavity of utility functions. For $x, \tilde{x} \in \mathcal{U}^i$, it follows for any $\lambda > 0$ that

$$u^i(\lambda x + [1-\lambda]\tilde{x}) \geq \min\{u^i(x), u^i(\tilde{x})\} > u^i(\hat{x}^i). \qquad //$$

Claim 2. *$\mathcal{U}$ is convex.* It is the sum of convex sets. //

Claim 3. *$\mathcal{F}$ is convex.* This follows because Y is assumed convex. //

Claim 4. $\mathcal{U} \cap \mathcal{F} = \emptyset$. If $x \in \mathcal{U}$ and $x \in \mathcal{F}$, then we can find some $x^1 \in \mathcal{U}^1$ and x^2 such that $u^1(x^1) > u^1(\hat{x}^1)$ and $u^2(x^2) > u^2(\hat{x}^2)$ where $x^1 + x^2 = x$. Furthermore, we can find some $y \in Y$ such that $y + w^1 + w^2 = x$, i.e. is feasible. Because x is both feasible and superior to $\hat{x}$, it follows that $\hat{x}$ cannot be Pareto efficient. By contradiction, the result follows. //

Claim 5. *There exists some $p \in \mathbb{R}^L \setminus \{0\}$ and some $k \in \mathbb{R}$ such that*

(a) *for any $x \in \mathcal{U}$, $p \cdot x \geq k$,*

(b) *for any $x' \in \mathcal{F}$, $p \cdot x' \leq k$.*

This follows because $\mathcal{U}$ and $\mathcal{F}$ are convex and disjoint, and therefore we can apply the separating hyperplane theorem. //

Claim 6. $p \gg 0$. This follows because Y satisfies free disposal. Suppose, for instance, that $p_1 \leq 0$. We're essentially saying that a firm gets paid to absorb as many inputs as possible with no downside whatsoever. So a firm would choose

$y_1 = -\infty$ and profits would blow up. More specifically, $p(w^1 + w^2 + y) \to \infty$, which is in $\mathcal{F}$. And thus, whatever k happens to be in the above claim, there exists some $x' \in \mathcal{F}$ such that $x' > k$, which is a contradiction. //

Claim 7. Since $(\hat{x}, \hat{y})$ is feasible, we know that $\hat{x}^1 + \hat{y}^1 = w^1 + w^2 + \hat{y} \in \mathcal{F}$ and therefore $p \cdot (w^1 + w^2 + \hat{y}) = p \cdot (\hat{x}^1 + \hat{x}^2) \leq k$. //

Claim 8. Suppose that $u^1(x^1) \geq u^1(\hat{x}^1)$ and $u^2(x^2) \geq u^2(\hat{x}^2)$. By the local nonsatiation of preferences, we can find some bundle $x^i(n) \in \mathcal{U}^i$ such that $\|x^i(n) - x^i\| \leq 1/n$ for any $n \in \mathbb{N}$. It follows that $p \cdot [x^1(n) + x^2(n)] \geq k$ for all n. In the limit, we clearly have $x^i(n) \to x^i$. From continuity it follows that that $p \cdot [x^1 + x^2] \geq k$

The bundles $\hat{x}^1$ and $\hat{x}^2$ satisfy the antecedent, so it follows that $p \cdot (\hat{x}^1 + \hat{x}^2) \geq k$. Combined with the previous claim, it follows that $p \cdot [\hat{x}^1 + \hat{x}^2] = k$. //

Okay, enough with the claims—now we can get to the main points in the theorem itself.

(a) Let $m^i = p \cdot \hat{x}^i$ for each i. Then

$$\sum_{i=1}^{I} m^i = \sum_{i=1}^{I} p \cdot \hat{x}^i = p \cdot \sum_{i=1}^{I} \hat{x}^i = p \cdot \left(\sum_{i=1}^{I} w^i + \hat{y} \right).$$

(b) Suppose x^1 satisfies $u^1(x^1) \geq u^1(\hat{x}^1)$. From claim 8, $p \cdot (x^1 + \hat{x}^2) \geq k$. We also know from claim 8 that $p \cdot (\hat{x}^1 + \hat{x}^2) = k$. It follows that $p \cdot x^1 \geq p \cdot \hat{x}^1 = m^1$. So any bundle that gives at least utility $u(\hat{x}^1)$ is at least as expensive as $\hat{x}$, making $\hat{x}^1$ the expenditure minimizer. Now appeal to duality—because preferences are continuous and locally nonsatiated and the price vector is strictly positive, it must be the case that $\hat{x}^1$ maximizes utility subject to $p \cdot x \leq m^1$. The optimality of bundle $\hat{x}^2$ follows similarly.

(c) Fix $y \in Y$. We know that $p \cdot (w^1 + w^2 + y) \leq k$. We also know that $p \cdot (\hat{x}^1 + \hat{x}^2) = p \cdot (w^1 + w^2 + \hat{y}) = k$. It follows that $y \leq \hat{y}$. So part (c) of the claim has been proved, namely, that $\hat{y}$ is the profit maximizer.