ECN 200D – Week 3 Lecture Notes

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1 Mortensen-Pissarides Model

1.1 The Beveridge Curve

Continuing where we left off with the Mortensen-Pissarides Model, we'd just derive two arrival times:

- **(a)** $q(\theta)$ is the arrival rate of workers to firms,
- **(b)** $\theta q(\theta)$ is the arrival rate of jobs to workers.

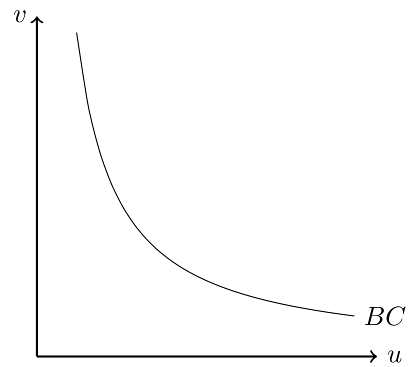
Recall that $\theta = v/u$ is a measure of tightness in the labor market.

We are ultimately dealing with a dynamic model. In a given period, some unemployed people will become employed and some employed people will become unemployed. Since there are u unemployed people and the labor force is normalized to 1, it follows that there are $1 - u$ employed people. Because λ is the probability of losing a job, $\lambda(1 - u)$ people will lose their jobs in a given period. Similarly, $\theta q(\theta)u$ unemployed people will find jobs. Therefore in the steady state, we have

$$\lambda(1 - u) = \theta q(\theta)u \quad \Longrightarrow \quad u = \frac{\lambda}{\lambda + \theta q(\theta)}.$$

This equation actually gives the relationship between v and u, which we call the **Beveridge curve**.

In particular, you can use the implicit function theorem to show that $du/dv < 0$. Note that an increase in matching efficiency will shift the Beveridge curve – for any given number of vacancies, there will be fewer unemployed people (because workers were matched more efficiently to the vacancies). Thus, the Beveridge curve will have shifted to the left.



The Beveridge curve (BC)

1.2 Value Functions

Some intuition was given for this – perhaps more accurately, a pseudo-derivation – but I found it rather cumbersome and unhelpful to go through. So, um, I'm not going to go through it. I'm just going to state it. Suppose r is the fixed real interest rate. Let J denote the value function of a firm that has a worker, and V the value function of a firm that is looking for a worker. Then

$$rV = -pc + q(\theta)[J - V]. \tag{1}$$

The multiplication of V by r is meant to represent discounting. The firm pays the recruitment cost pc, and then we have to account for the arrival rate of the firm finding a worker, $q(\theta)$, at which point the firm would gain the value of having a worker, J, and lose the value of having to continue searching, V.

Recall that firms with a vacancy do not necessarily have to search for a worker. Also recall that we are assuming free entry of firms into the labor market. The implication is that $V = 0$. This is because if $V > 0$, then more firms would enter into the labor market; and if $V < 0$, then firms would leave the labor market.

We can write the value function of a firm who has a worker as

$$rJ = p - w - \lambda J. \tag{2}$$

The firm gains p in production and loses w in wages. Then we have to account for the arrival of job destruction, λ, at which point the firm loses the value of having a worker, J. They do not necessarily gain the value of looking for a new worker V because they don't necessarily start searching right away. And even if they did, $V = 0$ so the equation wouldn't really change.

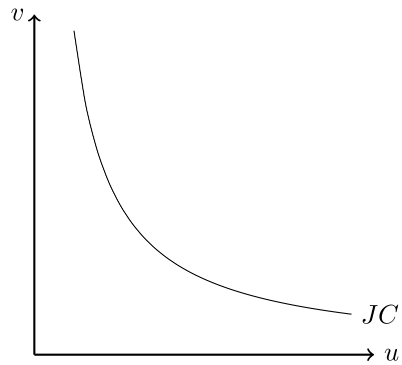
Now look at equation (1). Since $V = 0$ in equilibrium, we can write

$$J = \frac{pc}{q(\theta)} > 0. \tag{3}$$

This allows us to rewrite equation (2) as

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}. \tag{4}$$

We call equation (4) the **job creation curve**. Recall that $q'(\theta) < 0$. An increase in θ thus causes a decrease in $q(\theta)$. Thus means the LHS has a larger negative term. Thus, w will fall. So the JC curve is downward sloping. For this reason, you can kind of think of it as a demand function.



The job creation curve (JC)

So to recap. We have three unknowns: u, v, and w. For many purposes, we can cheat a little a define $\theta = v/u$ and think in terms of only two unknowns, but we ultimately want to solve for u. Thus we will need three equations. So far we only have two: the Beveridge curve, and the job creation curve:

$$w = p - (r+\lambda)\frac{pc}{q(\theta)},$$
$$u = \frac{\lambda}{\lambda + \theta q(\theta)}.$$

We haven't yet looked at the worker side of the market, so that seems like a good place to look for that elusive third equation.

1.3 Workers

The value function for an employed worker is

$$rW = w + \lambda(W - U). \tag{5}$$

They receive the wage w, augmented by the arrival rate of being fired λ, in which case their value function changes from employment W to unemployment U.

The value function for an unemployed worker is

$$rU = z + \theta q(\theta)(W - U). \tag{6}$$

They receive the unemployment compensation of z, augmented by the arrival rate of getting an acceptable job offer $\theta q(\theta)$, in which case they gain the value of being employed W and lose the value of being unemployed U.

We will use Nash bargaining to determine wages. Let $\beta \in [0, 1]$ be the bargaining power of a worker.[1] To that end, we are going to be solving

$$\arg\max_w (W - U)^\beta J^{1-\beta}.$$

As usual, logs are nicer to deal with. So, let's instead consider

$$\arg\max_w \beta \log(W - U) + (1 - \beta)\log(J).$$

Differentiating with respect to w gives

$$\frac{\beta}{W - U}\frac{\partial[W - U]}{\partial w} + \frac{1 - \beta}{J}\frac{\partial J}{\partial w} := 0.$$

First note from a purely economic argument that $\partial U/\partial w = 0$. The value of being unemployed doesn't depend on the wage being *negotiated*. Then from equation (5), we can write

$$W = \frac{w + \lambda U}{r + \lambda} \quad \Longrightarrow \quad \frac{\partial W}{\partial w} = \frac{1}{r + \lambda}. \tag{7}$$

From equation (2), we can write

$$J = \frac{p - w}{r + \lambda} \quad \Longrightarrow \quad \frac{\partial J}{\partial w} = -\frac{1}{r + \lambda}. \tag{8}$$

[1] "Nash bargaining make sense because any meeting is a match." I am not sure what this means, but it was said and seems important.

Therefore the first order condition becomes

$$\frac{\beta}{W-U}\frac{1}{r+\lambda} = \frac{1-\beta}{J}\frac{1}{r+\lambda} \quad \Longrightarrow \quad \beta J = (1-\beta)(W-U). \tag{9}$$

We can substitute using the equations (7) and (8) to get

$$\beta\left(\frac{p-w}{r+\lambda}\right) = (1-\beta)\left(\frac{w-rU}{r+\lambda}\right) \quad \Longrightarrow \quad \beta\,(p-w) = (1-\beta)\,(w-rU)$$

Solving for w, we get

$$w = \beta p + (1-\beta) rU.$$

Plug in equation (6) for rU to get

$$w = \beta p + (1-\beta) z + (1-\beta)\theta q(\theta)(W-U).$$

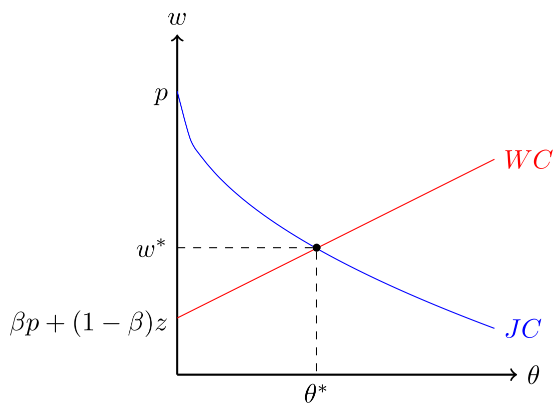
Using the first order condition in equation (9), and subsequently equation (3), we can write

$$\begin{aligned} w &= \beta p + (1-\beta) z + \theta q(\theta)\beta J \\ &= \beta p + (1-\beta) z + \beta\theta p c \qquad (10) \\ &= z + \beta(p - z + \theta p c). \qquad (11) \end{aligned}$$

This result is known as the **wage curve**. Equation (11) gives the intuition behind the answer solution to w. It is similar to what we have seen in other Nash bargaining problems. The wage is the worker's outside offer, namely unemployment benefits, plus their share β of the total surplus, but also with some new term θpc. This new term reflects the fact that the search process, which was costly, is over. In practice, equation (10) is easier to use.

1.4 Equilibrium

By setting the wage curve equal to the job creation curve, we can find an equilibrium with respect to w and θ.



The unique wage and market tightness in equilibrium.

As you can gleam from the plot, existence is guaranteed as long as

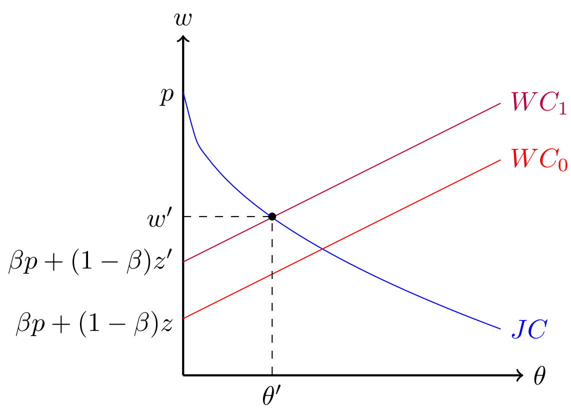
$$p > \beta p + (1-\beta)z \implies p > z.$$

This, as it happens, is an assumption we have been making implicitly anyway. Think about it – if production is less than the alternative of sitting on your ass and doing nothing, then no one is going to do anything ever and this model becomes super boring. So yeah, a solution exists and furthermore is unique. Since we will also know θ^*, we can plug it into the Beveridge curve to get

$$u^* = \frac{\lambda}{\lambda + \theta^* q(\theta^*)}.$$

1.5 Comparative Statics

What happens if z is increased to z'? To answer that, we want to compare the two steady states. Notice that a change in z will really only affect the wage curve, in particular, shifting it upwards.

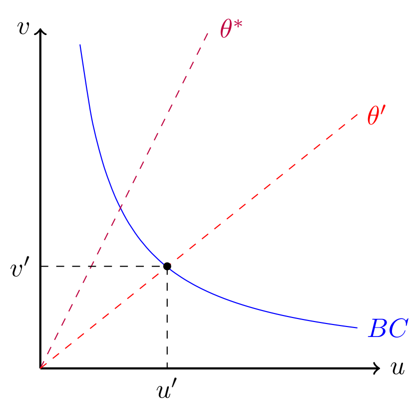


The effect on the equilibrium of increasing z.

As shown above, increasing unemployment benefits increases the wage w and decreases market tightness θ. This is fairly intuitive. People have more incentive to stay at home when z is increased; therefore they need to be tempted with a higher wage in order to actually work. Furthermore, because there is more incentive to not working, we might expect u to increase, giving a decrease in θ. This can also be shown using the Beveridge curve,

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}.$$

The term $\theta q(\theta)$ decreases as θ decreases, making the denominator smaller and thus u larger.



An increase in z decreases θ and thus increases u and decreases v.

1.6 Solution Summary

i. Use the law of motion of unemployment at the steady state to derive the Beveridge curve.

ii. Because the value of searching for a worker $V = 0$ in equilibrium, it follows from the equation for rV that $J = pc/q(\theta)$.

iii. Then plug that result into the equation for rJ to get the job creation curve,

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}.$$

iv. Use Nash bargaining to determine wages. Workers will lose U and gain W; firms will lose $V = 0$ and gain J; so solve

$$\arg\max_{w}(W - U)^{\beta} J^{1-\beta}.$$

v. After some ugliness, you'll get the wage curve

$$w = z + \beta(p - z + \theta pc).$$

vi. The intersection of the wage curve and the job creation curve give the equilibrium θ^* and w^*. Plug θ^* into the Beveridge curve to solve for u^*.

2 Endogenous Job Destruction

Let's complicate shit. We'll have the same model except when λ hits a match, the new productivity of that match is given by px, where p is **general productivity** and x is **idiosyncratic productivity**. We will suppose that $x \sim G$ where $\mathrm{supp}(G) = [0, 1]$. In practice we could have the support of G going to any finite maximum, but working between zero and one is nice so let's just normalize it to that.

The idea is that the worker's idiosyncratic productivity will change every now and then. If the productivity is hit with a negative enough shock, then the match will end and the job is destroyed. In the previous model, the worker was working at full capacity $x = 1$; when λ hit, the idiosyncratic productivity fell to $x = 0$ and the match was ended. Now there is a possibility of a negative shock to idiosyncratic productivity, but the worker might remain productive enough to maintain the match.

We will assume that when a new match is created that the idiosyncratic productivity is $x = 1$. Every hit of λ requires the worker and the firm to decide whether to continue the match and furthermore they must renegotiate the wage. Thus, wage can be written $w(x)$.

As in the previous model, job destruction will be equal to job creation in equilibrium. Job creation is nothing new; simply multiply the job arrival rate by the number of unemployed. The job destruction side will be a bit different, however. We will conjecture that there is a **reservation productivity** R, below which the job will end.[2] The parameter λ is the arrival rate of an idiosyntratic productivity shock, and because G is a cumulative distribution function, there is a $G(R)$ probability that the shock is bad enough to end the relationship. Thus,

$$\theta q(\theta) u = (1 - u)\lambda G(R).$$

Solving for u, we have the Beveridge curve

$$u = \frac{\lambda G(R)}{\lambda G(R) + \theta q(\theta)}. \tag{12}$$

[2]This doesn't mean the worker is fired by the firm. In fact, it means they mutually agree to end the working relationship.

2.1 Firm Value Functions

First, notice that the value of the firm having the worker depends on how productive the worker actually is, so we can write $J(x)$. Again, we will rely on the proposed reservation productivity R, which must be met in order for the match to take place. In other words, the value to a firm of having a worker with productivity R satisfies $J(R) = 0$.

The value function of a firm looking to fill a vacancy is

$$rV = -pc + q(\theta)\big(J(1) - V\big). \tag{13}$$

The cost of search pc is lost, but with arrival rate $q(\theta)$, the firm gets a maximally productive worker with $x = 1$, this switching their state to $J(1)$ from V.

It is still the case that $V = 0$ in equilibrium due to free entry of firms into the labor search market. Therefore we can rewrite equation (13) as

$$J(1) = \frac{pc}{q(\theta)}. \tag{14}$$

This equation tells us the expected cost of opening a vacancy. This is because $q(\theta)$ is the arrival rate of a job, and its inverse give the expected length of time for an arrival.

The value function of a firm having a job is more complicated that what we have seen so far. In a general sense, it is given by

$$rJ(x) = px - w(x) + \lambda \int_0^1 J(s)\, dG(s) - \lambda J(x).$$

The firm receives the production of px, and they lose the wage $w(x)$. Then with the arrival rate of the idiosyncratic production shock, they gain the expected value of $J(s)$ and lose what they had, $J(x)$.

We can do a bit better than this by exploiting our assumption that the reservation productivity is a thing. In particular, if s is below the reservation productivity, then the match is destroyed and the firm goes back into the mass of firms with vacancies for no value. Thus, we can write

$$\begin{aligned} rJ(x) &= px - w(x) + \lambda \int_R^1 J(s)\, dG(s) + \lambda \int_0^R 0\, dG(s) - \lambda J(x) \\ &= px - w(x) + \lambda \int_R^1 J(s)\, dG(s) - \lambda J(x). \end{aligned} \tag{15}$$

2.2 Worker Value Functions

The value of being employed to a worker is going to depend on x since the wage depends on x, so we can write $W(x)$. Keeping in mind that a worker starts a new job with $x = 1$, the value to a worker of being unemployed is

$$rU = z + \theta q(\theta)\big(W(1) - U\big). \tag{16}$$

The value to a worker of having a job with idiosyncratic productivity x, in unsimplified form, is given by

$$rW(x) = w(x) + \lambda \int_R^1 W(s)\, dG(s) + \lambda \int_0^R U\, dG(s) - \lambda W(x).$$

The worker receives a wage of $w(x)$. Then we consider the arrival of an idiosyncratic productivity shock. If the shock is above the reservation productivity, then the worker considers the expected value of that continuation. On the other hand, if the shock is below the reservation productivity, then the worker goes to unemployment. In either case, the worker loses $W(x)$. The minor simplification is to note that U is not a function of productivity shocks, so it can be taken out of the integral. Then the integral simply evaluates to $G(R)$ because G is a CDF. So we can write the function as

$$rW(x) = w(x) + \lambda \int_R^1 W(s)\, dG(s) + \lambda G(R)U - \lambda W(x). \tag{17}$$

2.3 Bargaining

Buckle up, because this is not going to be pretty.

For any $x \geq R$, the Nash bargaining problem is to solve

$$\arg\max_{w(x)} [W(x) - U]^{\beta} J(x)^{1-\beta}.$$

As usual, log it up to get

$$\arg\max_{w(x)} \beta \log[W(x) - U] + (1 - \beta) \log[J(x)].$$

The first order condition gives

$$\begin{aligned} & \frac{\beta}{W(x) - U} \frac{dW(x)}{dw(x)} + \frac{1-\beta}{J(x)} \frac{dJ(x)}{dw(x)} = 0 \\ \implies \quad & \frac{\beta}{W(x) - U} \frac{1}{r + \lambda} - \frac{1-\beta}{J(x)} \frac{1}{r + \lambda} = 0 \end{aligned}$$

From this we can conclude that

$$\beta J(x) = (1 - \beta)\big(W(x) - U\big) \tag{18}$$

Equation (18) will prove important in a few steps. But we're not quite there yet. Solve equation (17) for $W(x)$ and equation (15) for $J(x)$ to get

$$\begin{aligned} & \beta \left[px - w(x) + \lambda \int_R^1 J(s)\ dG(s) \right] \\ = (1 - \beta) & \left[w(x) + \lambda \int_R^1 W(s)\ dG(s) + \lambda G(R)U - rU - \lambda U \right]. \end{aligned}$$

Notice that $\lambda G(R)U - \lambda U$ can be written as

$$\lambda \int_0^R U\ dG(s) - \lambda \int_0^1 U\ dG(s) = -\lambda \int_R^1 U\ dG(s).$$

Using that result, brute force your way through the algebra to get

$$w(x) = \beta p x + (1-\beta) r U + \lambda \int_R^1 \beta J(s) - (1-\beta)\big(W(s) - U\big)\, dG(s).$$

This doesn't look good, but there is one very nice cancellation hiding in plain sight. The integrand, according to equation (18), is equal to zero.

So at this point we have

$$w(x) = \beta p x + (1-\beta) r U.$$

But hey, don't we have an equation for exactly rU? Yes. Yes we do. Equation (16), to be exact. So let's use that bad boy to get

$$w(x) = \beta p x + (1-\beta) z + (1-\beta)\theta q(\theta)[W(1) - U].$$

Let's use equation (18) again. In particular, it follows that

$$\beta J(1) = (1-\beta)\big(W(1) - U\big),$$

which we can use to write

$$w(x) = \beta p x + (1-\beta) z + \theta q(\theta) \beta J(1).$$

Now use equation (14) to write

$$\begin{aligned} w(x) &= \beta p x + (1-\beta) z + \theta q(\theta) \beta \frac{pc}{q(\theta)} \\ &= \beta p x + (1-\beta) z + \theta \beta p c. \end{aligned}$$

Hot damn, now let's just rewrite this a little bit more to get

$$w(x) = (1-\beta)z + \beta p(x + c\theta) \tag{19}$$

$$= z + \beta(px - z + pc\theta). \tag{20}$$

Finally! Woo.

Equation (19) is the **wage curve**. It is usually easiest to work with in this form, but equation (20) might be more intuitive as far as interpretation goes. The wage is the worker's outside offer plus the proportion of the surplus and ended search cost weighted by the worker's bargaining power. The only difference here compared to the exogenous job destruction model is that the surplus depends on x because production px depends on x.

2.4 To Be Continued

We need to solve for u, v, w, and R. We have the wage curve. We also have the Beveridge curve. So we'll need to derive the job creation and job destruction curves. Then we should be able to solve the model. Which is going to happen next week.

3 Solution Summary (So Far)

i. Use the law of motion of unemployment at the steady state to derive the Beveridge curve.

ii. Because the value of searching for a worker $V = 0$ in equilibrium, it follows from the equation for rV that $J(1) = pc/q(\theta)$.

iii. Assume there exists some reservation level of productivity. Rewrite $rJ(x)$ by breaking the integral into two parts: before R and after R.

iv. Break up $rW(x)$ in the same way.

v. Do Nash bargaining. Plus (iii) and (iv) into the bargaining condition.

vi. Rewrite $G(R) - 1$ as $-\int_R^1 dG(s)$ and simplify a bunch.

vii. Plug in rU into (vi).

viii. Evaluate Nash condition at $x = 1$ and plug that into (vii).

ix. Substitute $J(1)$ into (viii) and simplify to get the wage curve,

$$w(x) = (1 - \beta)z + \beta p(x + c\theta).$$