ECN 200D: Week 10 Lecture Notes
Lagos-Wright Model

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1 Kiyotaki-Wright

Randy Wright gave me extra credit for going to see *The Contractions* perform live once. Right, so the KW model is a monetary search model consisting of producers, commodity traders, and money traders.

- **Producers.** If an agent has no good, then they go produce one with Poisson rate a. Then they become a commodity trader.
- **Commodity Traders.** As a commodity trader, they either try to find a double coincidence of wants—trader A has what trader B wants, trader B has what trader A wants, so they just swap and go back to production. Or they try to find someone who wants their good who has money, possibly trading it for that (if money isn't completely worthless), after which they become a money trader.
- **Money Traders.** As a money trader, they try to find someone who has a good they want and will accept money. Then they go back to production.

I'm not going to go through the details (because it was said to be optional), but it turns out that there are three possible equilibria depending on beliefs about the likelihood of money being accepted—a pure exchange equilibrium, a pure money equilibrium, and a mixed equilibrium.

The KW model is nice and everything, but it's largely intractable. The **Lagos-Wright** model addresses this intractability.

2 Lagos-Wright

Every period will have two subperiods. The first subperiod, which we'll call the *day market*, is a decentralized KW market. The second subperiod, which we'll call the *night market*, is a Walrasian centralized market in which there is no need for money (but money can still be exchanged if desired).

We will assume that preferences are quasilinear because wealth effects are jerks and quasilinear preferences have no wealth effects. This means that the optimal choice is unaffected by the number of assets on hand while making a decision in any given period.

The Lagos-Wright model is built around the following environment.

- The discount rate is β.
- There are two commodities. The *general good* is traded and consumed at night. Everyone likes the general good.

  The *special good* is produced by sellers during the day, which they do not consume (so they try to trade it). Not everyone likes the special good, which necessitates the double coincidence of wants or money trading.
- There are two types of agents, buyers B and sellers S. They remain the same type for life.
- Buyers work H hours at night and consume x of the general good. They buy q of the special good in the day market and consume it for utility $u(q)$. So in one period, the buyer's payoff is $u(x) - H + u(q)$. This is quasilinear with respect to H.

- Sellers work H hours at night and they receive 1 unit of the general good per hour of work. They will spend all of their resources on consuming x of the general good. During the day, they sell q units of the special good, so the seller's payoff is $u(x) - H - q$. Again, this is quasilinear with respect to H.
- We will assume that u is twice continuously differentiable and furthermore that $u' > 0$, $u'' < 0$, and $u'(0) = \infty$.
- We will assume that there exists some optimal x^* such that $u'(x^*) = 1$.
- We will assume that there exists some optimal q^* such that $u'(q^*) = 1$.
- Goods are nonstorable.
- Money is storable and fiat.
- The central bank chooses a constant rate of money growth μ such that $M_{t+1} = (1 + \mu)M_t$.
- A lump sum T is transferred to buyers in the night market.
- Buyers meet sellers in the day market with probability $\sigma \in [0, 1]$.
- In the day market, buyers and sellers bargain via Kalai bargaining, where θ is buyer bargaining power.

The only feasible trades during day are the barter of the special good and the exchange of special good for money. At night, the only feasible trades involve the general good and money.

3 Night Market Value Functions

3.1 The Buyer

Let $V(\cdot)$ be the buyer's value function for the day market. Then the buyer's value function for the night market is

$$W(m) = \max_{x,H,m'} \left\{ u(x) - H + \beta V(m') \right\}.$$

They begin in the night market with m dollars. They want to choose how much of the general good x to buy and consume, how many hours to work, and how much money to carry over into the upcoming day market.

The buyer's budget constraint is given by

$$x + \phi m' = H \cdot 1 + \phi m + T,$$

where ϕ is the amount of the general good the agent has to give up for one unit of money. So the amount of consumption plus the real cost of carrying money m' into the next period equals the earnings from hours worked plus the real value of money on hand plus whatever payment is transferred.

Let's first solve the budget constraint for H and plug it into the value function. Then we've assumed that there is some optimal x^*, so let's plug that into the value function as well. Doing so allows us to write

$$W(m) = u(x^*) - x^* + \phi m + T + \max_{m'} \big\{ -\phi m' + \beta V(m') \big\}.$$

Here we can see that there are no wealth effects—current money holdings m do not affect the max operator. Furthermore, $W(m)$ is linear in m, so we can simplify and write

$$W(m) = \Lambda + \phi m, \tag{1}$$

where Λ is all that other junk.

3.2 The Seller

This case is less interesting. Sellers do not want to take money into the day market because it does them no good—as per their namesake, they're only selling stuff in the day market—so they'll sell their money to buyers in the night market in exchange for the general good. In this case, we'll have

$$W^s(m) = \max_{x,H} \{u(x) - H + \beta V^s(0)\}$$

subject to $x = H + \phi m$. (The reason sellers come into the night market with money is that they just sold stuff in the day market.)

Let's again solve the budget constraint for H, plug it into the Bellman equation, and then impose the optimal x^* so that the Bellman equation can be written as

$$\begin{aligned} W^s(m) &= u(x^*) - x^* + \beta V^s(0) + \phi m \\ &= \Lambda^s + \phi m, \end{aligned}$$

where Λ^s is all of the other junk.

4 Day Market Bargaining

We want to consider the typical meeting between a buyer who carries m dollars into the day market and the seller who carries zero. We'll use Kalai bargaining. Let q be the quantity of the special good purchased and d be the number of dollars spent. Let BS be the buyer surplus and SS be the seller surplus. Then we are to solve

$$\max_{q,d} BS \quad \text{s.t.} \quad BS = \frac{\theta}{1-\theta} SS \quad \text{and} \quad d \leq m,$$

the first constraint being the *Kalai constriant*, the second being the *cash constraint*.

4.1 Buyer Surplus

The buyer purchases q units of the special good for d dollars, after which q is consumed. The buyer continues on in the day market with $m - d$ dollars and leaves state $W(m)$. We can write this as

$$\begin{aligned} BS &= u(q) + W(m-d) - W(m) \\ &= u(q) + \Lambda + \phi m - \Lambda - \phi d - \phi m \\ &= u(q) - \phi d. \end{aligned}$$

Now you see why the Λ grouping came in handy. This expression is intuitive: the buyer surplus is the utility gained minus the real value of the dollars spent.

4.2 Seller Surplus

Recalling that the seller begins in the day market with no money, the seller surplus from a trade is

$$\begin{aligned} SS &= -q + W^s(d) - W^s(0) \\ &= -q + \Lambda^s + \phi d - \Lambda^s \phi \cdot 0 \\ &= -q + \phi d. \end{aligned}$$

They gain the real value of the dollars they receive, but lose q of their special good. Again, pretty intuitive.

4.3 Bargaining Solution

And therefore the bargaining problem becomes

$$\begin{aligned} \max_{q,d} \quad & u(q) - \phi d \\ \text{s.t.} \quad & u(q) - \phi d = \frac{\theta}{1-\theta}(\phi d - q), \\ & d \leq m. \end{aligned}$$

From the Kalai constraint, it follows that

$$\begin{aligned} [1-\theta][u(q)-\phi d] &= \theta[\phi d - q] \\ \implies \quad \phi d &= \theta q + (1-\theta)u(q) \\ &= z(q). \end{aligned}$$

We can use this to rewrite the objective function as

$$\max_q u(q) - \theta q - (1-\theta)u(q) = \max_q \theta[u(q)-q],$$

which is still subject to $d \le m$. There are two cases with respect to money holdings that we must consider.

Case 1: Optimality. If m is large enough, then the cash constraint is irrelevant. In this case, there is nothing preventing the bargaining solution from selecting the optimal q^*, paid for with d^* as expressed via the Kalai constraint,

$$d^* = \frac{z(q^*)}{\phi}.$$

In other words, let m^* be the amount of cash needed to be able to afford the optimal q^*. If the buyer has $m \ge m^*$ in cash, then they can pay $d^* = m^*$ and the solution is optimal.

Case 2: Cash Constrained. If $m < m^*$, then the consumer will spend as much as they can, i.e. $d = m < d^*$, and quantity consumed will be $q = \tilde{q}(m) < q^*$, where q satisfies $\phi m = z(q)$. This is a suboptimal outcome since the optimal trade cannot occur.

Thus we can define the solution of the Kalai bargaining to be

$$\big(q(m), d(m)\big) = \begin{cases} (q^*, d^*) & \text{if } m \ge m^*, \\ (\tilde{q}, m) & \text{if } m < m^*. \end{cases}$$

5 Day Market Value Functions

The seller is boring and we already know everything we need to know about them. For the buyer, we have

$$V(m) = \sigma\left[u\big(q(m)\big) + W\big(m - d(m)\big)\right] + (1-\sigma)W(m),$$

where $q(m)$ and $d(m)$ arise from bargaining. The first term represents the probability that a buyer meets a seller, gaining utility for consuming $q(m)$ for $d(m)$ dollars and then continuing to the night market with $m - d(m)$ dollars. The second term represents not being able to find a seller and thus continuing on to the night market with all of their money. Recalling that $W(m) = \Lambda + \phi m$, we can rewrite this as

$$\begin{aligned}
V(m) &= \sigma\left[u\big(q(m)\big) + \Lambda + \phi m - \phi d(m)\right] + (1-\sigma)(\Lambda + \phi m) \\
&= \sigma\left[u\big(q(m)\big) + \Lambda + \phi m - \phi d(m)\right] + \Lambda + \phi m - \sigma(\Lambda + \phi m) \\
&= \sigma\left[u\big(q(m)\big) - \phi d(m)\right] + W(m).
\end{aligned}$$

Now let's consider the original Bellman equation a little more,

$$W(m) = u(x^*) - x^* + \phi m + T + \max_{m'}\{-\phi m' + \beta V(m')\},$$

in particular the max operator. Define

$$\begin{aligned}
J(m) &= -\phi m' + \beta V(m') \\
&= -\phi m' + \beta\left(\sigma[u\big(q(m')\big) - \phi d(m')] + W(m')\right) \\
&= -\phi m' + \beta\left(\sigma[u\big(q(m')\big) - \phi d(m')] + \Lambda' + \phi' m'\right) \\
&= -\phi m' + \beta\phi' m' + \beta\Lambda' + \beta\sigma\left[u\big(q(m')\big) - \phi d(m')\right] \\
&= (-\phi + \beta\phi')m' + \beta\sigma\left[u\big(q(m')\big) - \phi d(m')\right].
\end{aligned}$$

The first term captures the net cost of carrying a unit of money from one period to another. The second term is the discounted expected buyer surplus that one additional unit of money will buy. The $\beta\Lambda'$ term has been omitted entirely since it is not a function of m' and therefore doesn't affect our subsequent analysis.

6 Money Growth

Let's assume that $\mu > \beta - 1$, which in turn implies that $i > 0$. We will only consider the Friedman rule as a limiting case.

So far we've denoted m^* as the amount of money required to buy the first-best, and you'll never need to bring more than that. So let's further assume that we will always be in the binding branch of the bargaining solution, i.e. you will in fact never bring more than m^*. This allows us to write

$$J(m') = (-\phi + \beta\phi)m' + \beta\sigma\left\{u\big(\tilde{q}(m')\big) - \phi' m'\right\}. \tag{2}$$

Claim 1. *In any equilibrium, $\phi \geq \beta\phi'$.*

Proof. The cost of carrying money cannot be negative, although it could be zero. If $\phi < \beta\phi'$, then you'd carry $m' = \infty$. This cannot be the case. So it must be the case that $(\beta\phi' - \phi)m' \leq 0$. □

Claim 2. *A nonmonetary equilibrium where $\psi = \psi' = 0$ always exists.*

Proof. Since money has no value and no price, it's entirely arbitrary what the price turns out to be—it could be anything, in particular, zero. □

In a monetary equilibrium, $m' > 0$. Taking the first order condition of equation (2) with respect to m', we get

$$\phi = \beta\phi' + \beta\sigma\left\{u'\big(\tilde{q}(m')\big)\tilde{q}'(m') - \phi'\right\}.$$

Recall that $\tilde{q}(m)$ solves $\phi m = z\big(\tilde{q}(m)\big)$. Taking the derivative with respect to m, it follows that

$$\phi = z'\big(\tilde{q}(m)\big)\tilde{q}'(m) \quad \Longrightarrow \quad \tilde{q}'(m') = \frac{\phi'}{z'\big(\tilde{q}(m')\big)}.$$

Plug this into the first order condition for

$$\begin{aligned} \phi &= \beta\phi' + \beta\sigma\left\{\frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)}\phi' - \phi'\right\} \\ &= \beta\phi'\left[1 + \sigma\left\{\frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1\right\}\right] \\ \Longrightarrow \quad \frac{\phi}{\beta\phi'} - 1 &= \sigma\left\{\frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1\right\}. \end{aligned} \tag{3}$$

Equation (4) is the demand for money. Let's focus on the steady state where real balances are equal, i.e. where $\phi M = \phi' M'$. Then we can write

$$\frac{\phi}{\beta\phi'} - 1 = \frac{M(1+\mu)\phi}{M'\beta\phi'} - 1 = \frac{1+\mu}{\beta} - 1 = i,$$

where the last equality follows from the Fisher equation and the illiquid real interest rate. So we can write the demand for money as

$$i = \sigma\left\{\frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1\right\}. \tag{4}$$

Definition 1. A **steady state monetary equilibrium** is a list of objects (q, Z), where q is the amount of the special good in a typical trade (we're interested in how close to q^* it is) and $Z = \phi M > 0$ such that

(a) We can pin down q with

$$i = \sigma \left\{ \frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1 \right\},$$

(b) $\phi M = Z = z(q) = \theta q + (1-\theta)u(q)$.

7 Comparative Statics

Let's see what effect changing i has on q. Differentiate the money demand function with respect to q, i.e. find

$$\frac{di}{dq} = \sigma \frac{d}{dq}\left[\frac{u'}{z'}\right].$$

Since $z = \theta q + (1-\theta)u$, it follows that $z' = \theta + (1-\theta)u'$. Therefore

$$\begin{aligned} && \frac{u'}{z'} &= \frac{u'}{\theta + (1-\theta)u'} \\ \implies && \frac{d}{dq}\left[\frac{u'}{z'}\right] &= \frac{\theta u'' + (1-\theta)u'u'' - u'(1-\theta)u''}{[\theta + (1-\theta)u']^2} \\ && &= \frac{\theta u''}{[\theta + (1-\theta)u']^2} < 0 \end{aligned}$$

because we assume $u'' < 0$. It follows that

$$\frac{di}{dq} = \sigma \frac{\theta u''}{[\theta + (1-\theta)u']^2} < 0 \quad \implies \quad \frac{dq}{di} < 0.$$

So as the interest rate rises, people want to carry less cash. Makes sense since i is the opportunity cost of holding cash.

What happens as $i \to 0$, i.e. when we approach the Friedman rule? Then the money demand function can be written as

$$
\begin{aligned}
0 &= \sigma \left\{ \frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1 \right\} \\
\implies \quad & u'(q) = z'(q) \\
\implies \quad & u'(q) = \theta + (1-\theta)u'(q) \\
\implies \quad & u'(q) = 1 \\
\implies \quad & q = q^*.
\end{aligned}
$$

So the Friedman rule implies that the optimal quantity of money is carried. That's nice.

8 Monetary Equilibrium

We've established that if $i = 0$, then people hold $q^* > 0$. We also know that q decreases as i increases. The big question is this: can i get so large that people no longer wish to hold money, i.e. $q = 0$? Indeed, it can.

Claim 3. *A monetary equilibrium exists only if*

$$i < \sigma \frac{\theta}{1-\theta}.$$

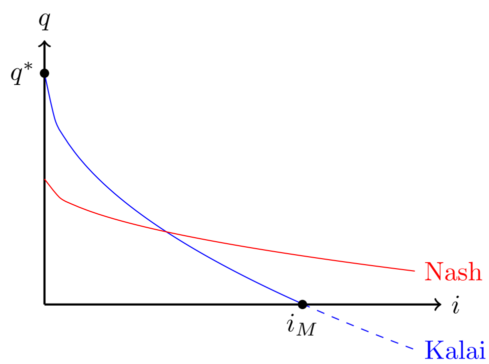
Proof. We'll be considering the money demand function,

$$i = \sigma \left\{ \frac{u'\big(\tilde{q}(m')\big)}{z'\big(\tilde{q}(m')\big)} - 1 \right\}.$$

Recall that $u'(q) = \infty$ as $q \to 0$. Furthermore, $z' = \theta + (1-\theta)u'(q)$. The problem is that we'll have infinity over infinity if we try to take the limit as it. Apply L'Hopital's rule and we can instead take the limit of

$$\sigma \left\{ \frac{u''}{(1-\theta)u''} - 1 \right\} = \frac{\sigma\theta}{1-\theta} = i_M.$$

So $q > 0$ if $i \leq i_M$. Otherwise there is no monetary equilibrium. $\square$



Under Kalai bargaining (blue), only interest rates below i_M can sustain a monetary equilibrium. Under Nash bargaining (red), we *are* guaranteed a monetary equilibrium, although we can never have optimal q^*.

9 Two Types of Money

Suppose we have two types of money, m_1 and m_2. Each type of money grows at its own rate μ_i.

Perfect Substitutes. Suppose that the types of money are perfect substitutes—every seller accepts both types of money. For these types of problems, we really only need to pay attention to the J functions. And there is a pattern for writing then.

$$\begin{aligned} J(m_1', m_2') &= (-\phi_1 + \beta\phi_1')m_1' + (-\phi_2 + \beta\phi_2')m_2' \\ &\quad + \beta\sigma\big[u\big(q(m_1', m_2')\big) - \phi_1' d_1(m_1', m_2') - \phi_2' d_2(m_1', m_2')\big]. \end{aligned}$$

The first line represents, respectively, the holding cost of m_1 and the holding cost of m_2.

Imperfect Substitutes. Now suppose that with probability $\lambda \in [0, 1]$, sellers accept money type 2, whereas everyone accepts money type 1. This means that we must solve two bargaining problems depending on which kind of seller is met. So now the J function is

$$
\begin{aligned}
J(m_1', m_2') = {} & (-\phi_1 + \beta\phi_1)m_1' + (-\phi_2 + \beta\phi_2)m_2' \\
& + \beta\sigma\Big\{\lambda\big[u\big(q^2(m_1', m_2')\big) - \phi_1' d_1^2(m_1', m_2') - \phi_2' d_2^2(m_1', m_2')\big] \\
& + (1-\lambda)\big[u\big((q^1(m_1')\big) - \phi_1' d_1^1(m_1')\big]\Big\}.
\end{aligned}
$$

The first line is the cost carrying each type of money. The second line is the case where you run into a seller who accepts either type of money, the superscript indicating the bargaining outcomes of such an encounter. The third line is the case where you run into a seller who only accepts money type 1, but superscript again indicating the relevant bargaining outcome.