ECN 200D: Week 2 Lecture Notes
Mortensen-Pissarides Model

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These notes will briefly discuss Poisson processes and bargaining. Then it will begin to introduce the Mortensen-Pissarides model of labor markets.

1 Poisson Processes

Let $N_t(\omega)$ denote the number of occurrences of the event ω up to time t. The set $N = \{N_t, t \geq 0\}$ is an **arrival process**. We will be focusing on the **Poisson process**.

(a) Any jump in N will be of size 1.

(b) For any $t, s \geq 0$, the difference $N_{t+s} - N_t$ is independent of u for any $u < t$. (*Poisson is memoryless.*)

(c) For any t, t', the difference $N_{t+s} - N_t = N_{t'+s} - N_{t'}$. (*Poisson is time invariant.*)

Recall that the probability mass function for a Poisson distributed random variable is

$$P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

It turns out that λ is the **arrival rate**, that is, it tells us how many times an event will happen per unit time.

Considering $E[N_t]$, we can calculate it as

$$\begin{aligned} E[N_t] &= \sum_{k=0}^{\infty} kP(N_k = k) \\ &= \sum_{k=0}^{\infty} k\frac{e^{-\lambda t}(\lambda t)^k}{k!}. \end{aligned}$$

Since the first term will evaluate to zero, we can start the sum at $k = 1$ instead, giving

$$e^{-\lambda t}\sum_{k=1}^{\infty} k\frac{(\lambda t)^k}{k!}.$$

We have $k/[k(k-1)(k-2)\ldots]$, which we can simplify to

$$e^{-\lambda t}\sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}.$$

We can shift the index back down to zero, but we have to compensate by shifting the k terms in the summand up by one, giving

$$e^{-\lambda t}\sum_{k=0}^{\infty} \frac{(\lambda t)^{k+1}}{k!} = e^{-\lambda t}(\lambda t)\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}.$$

The sum is the Taylor series expansion for $e^{\lambda t}$, so we can finally simplify to

$$e^{-\lambda t}(\lambda t)e^{\lambda t} = \lambda t.$$

Great, so $E[N_t] = \lambda t$. This should be intuitive. If we something happens, on average,λ times in 1 unit of time, then we should expect it to happen $t\lambda$ times in t units of time.

2 Bargaining

We'll be considering a game between two players—a firm and worker. The idea is that firms have machines, workers have the skill and labor to use the machines. Alone they're unproductive, but together they can produce output. The question is, how is the output divided between the two? In order words, they need to bargain over a wage.

If they make an agreement, then their payoffs will be in some set S. In particular, there will be some function $f(S, d) \to S$ that gives the solution to the bargaining problem. In other words, the function looks at the set of agreements payoffs S and the disagreement payoffs and assigns an "optimal" agreement. The particular agreement assigned will depend on the bargaining power of the players. Let $\theta \in [0, 1]$ be player 1's **bargaining power**. (Then $1 - \theta$ is player 2's bargaining power.)

We will consider two axiomatic approaches to bargaining.

2.1 Nash Bargaining

The axioms John Nash decided would be reasonable for a bargaining equilibrium are the following.

(a) *Irrelevance to equivalent utility representations.* The set S represents preferences, which we'll usually be able to represent as utility functions. Utility functions, of course, can be subjected to positive monotone transformations without altering the underlying preferences. Thus, we want the same solutions to emerge with any utility function that represents the preferences.

(b) *Independence of Irrelevant Alternatives.* Suppose that $f(S, d)$ maps to S', where $S' \subset S$. Then it should be the case that the solution of $f(S, d) = f(S', d)$. In other words, none of the agreements in $S \setminus S'$ were a solution, so we shouldn't expect the solution to change if those agreements become infeasible.

(c) *Pareto Efficiency.*

The collection of these axioms give rise to **Nash bargaining**.

There is a rather long and ugly proof, but it turns out that the solution for the bargaining problem is actually the Stone-Geary function

$$f_\theta(S, d) = \arg\max_{s \in S} (s_1 - d_1)^\theta (s_2 - d_2)^{1-\theta}.$$

The coordinate $s = (s_1, s_2)$ contains two points in the agreement set for player 1 and 2, respectively. Player 1 moving to s_1 generates a surplus of $s_1 - d_1$, and player 2 moving to s_2 generates a surplus of $s_2 - d_2$. Finding the agreement points that maximize these surpluses, weighted by the relative bargaining power, is what solves the bargaining problem.

Example 1. We have a pie! Um, it's a pie of size 1. We want to choose what fraction q of the pie will go to player 1. Then player 2 will get fraction $1 - q$. If no agreement is made, then they get d_1 and d_2 fractions, respectively.

We can write the Nash bargaining problem as

$$\arg\max_{q \in [0,1]} (q - d_1)^\theta (1 - q - d_2)^{1-\theta}.$$

Since we are only interested in the maximizing argument q, we can take a monotonic transformation of the objective function without altering the maximizers. So hey, why not take a logarithm of it? Then we're solving

$$\arg\max_{q \in [0,1]} \theta \log(q - d_1) + (1 - \theta) \log(1 - q - d_2).$$

It's a function only of θ, so we can take the derivative to find the critical point. This first-order condition gives

$$\frac{\theta}{q - d_1} - \frac{1 - \theta}{1 - q - d_2} = 0 \quad \Longrightarrow \quad \frac{\theta}{q - d_1} = \frac{1 - \theta}{1 - q - d_2}.$$

Do some algebra and you'll find that

$$q = d_1 + \theta(1 - d_1 - d_2).$$

This is unsurprising, since this is indeed the solution to the Stone-Geary utility maximization problem. You can think of it like this. Player 1 automatically gets fraction d_1 of the pie, player 2 automatically gets fraction d_2 of the pie. Then, of the remaining pie, player 1 gets θ of it. Similarly, player 2 gets

$$1 - q = d_2 + (1 - \theta)(1 - d_1 - d_2). \qquad \blacksquare$$

Example 2. Now let's suppose that a firm meets a worker. If they agree to work, then they'll produce output of p. The worker can also just sit at home and collect an unemployment payment of z. So they'll need to bargain over some wage w for the worker. Let θ be the bargaining power of the worker.

If the job is accepted, then the worker gets w; if not, then z. So the surplus generated by the worker would be $w - z$. If the job is accepted, then the firm produces p; if not, then the firm still has that w in wage payments. So the surplus generated by the firm would be $p - w$. Thus, the bargaining problem is

$$\arg\max_{w} (w - z)^{\theta}(p - w)^{1-\theta}.$$

Again, let's log it up to instead solve

$$\arg\max_{w} \theta \log(w - z) + (1 - \theta)\log(p - w).$$

It's a function of one variable w, so just take the derivative with respect to w and set it equal to zero to find the critical point. As you might have expected, the solution is

$$w = z + \theta(p - z). \qquad \blacksquare$$

2.2 Kalai Bargaining

Add one axiom to Nash bargaining and we have **Kalai bargaining**.

(d) *Monotonicity.* If $S \subseteq S'$, then $f_j(S', d) \geq f_j(S, d)$, for players $j = 1, 2$. In other words, if we are considering a larger set of possible agreements, then no one can be worse off.

I find this to be rather questionable since choice paralysis is a well-studied phenomenon. But whatever. The nice thing about incorporating this extra axiom is that the solution becomes even simpler, in particular,

$$f_\theta^k(S, d) = \arg\max_{s \in S}(s_1 - d_1) \quad \text{s.t.} \quad s_1 - d_1 = \frac{\theta}{1 - \theta}(s_2 - d_2).$$

The takeaway is that we now have proportionate solutions to the bargaining situation.

Example 3. Let's split that pie! Sounds sexy. The problem we want to solve is

$$\max_q (q - d_1) \quad \text{s.t.} \quad q - d_1 = \frac{\theta}{1 - \theta}(1 - q - d_2).$$

Well ah, the constraint already gives us everything we need. Just solve it for q to find that $q = d_1 + \theta(1 - d_1 - d_2)$. This is the same solution given by the Nash bargaining problem! Indeed, for simple models like this, the two will give the same results. ■

3 Mortensen-Pissarides Model

3.1 The Environment

(a) Time is continuous an infinite in horizon.

(b) The labor force is normalized to 1.

(c) u denotes the proportion of unemployed workers. (Also the number of unemployed workers because of the previous point.)

(d) v is the number of vacancies.

(e) Workers receive z per unit of time when unemployed.

(f) There exist infinitely many firms that could potentially open a vacancy.

(g) Each firm has one vacancy.

(h) While the vacancy is unfilled, firms pay cost $p \cdot c$, where p is the amount of good that would have been produced were the vacancy filled. Think of c as a recruitment cost. We multiply them together to capture the idea that more productive jobs have higher recruitment costs.

(i) The rate of job destruction is given by λ, the arrival rate of a Poisson process.

(j) After λ hits, the worker goes back to unemployment, and the firm goes back to the pool of firms that could potentially open a vacancy.

3.2 The Matching Function

We will be using a **matching function** $m(u, v) \to m$ that describes the amount of **matches** made, i.e. jobs formed, within a period. Once a match is formed, any surplus is split between the firm and the worker according to Nash bargaining, where β denotes the bargaining power of the worker.

(a) $m(u, v)$ is increasing in both arguments. The intuition is that if there are more unemployed workers, then firms will have an easier time finding the right worker; and if there are more vacancies, then workers will have an easier time finding the right job.

(b) $m(u, v)$ is concave. This isn't really important for us.

(c) $m(u, v)$ is homogeneous of degree one, which is to say, $m(\lambda u, \lambda v) = \lambda m(u, v)$ for $\lambda > 0$.[1]

Define $\theta = v/u$. This represents **market tightness**. If the number of vacancies v is relatively large compared to the number of job seekers u, then the labor market is considered *tight*. On the other hand, when the number of vacancies is relatively small compared to the number of job seekers, then the labor market is considered *slack*.[2]

Now let's derive some useful objects from the matching function. Suppose this period begins with 100 vacancies and 10 matches are made. Then the proportion of filled vacancies is 10/100, which is the (Poisson) arrival rate of workers to a firm. More generally, the arrival rate of workers to firms is $m(u, v)/v$. Since $m(u, v)$ is homogeneous of degree one, we can write

$$\frac{1}{v} m(u, v) = m\left(\frac{u}{v}, 1\right) = m\left(\frac{1}{\theta}, 1\right) = q(\theta).$$

We can similarly define the (Poisson) arrival rate of jobs to workers as

$$\frac{1}{u} m(u, v) = m(1, \theta) = \theta m\left(\frac{1}{\theta}, 1\right) = \theta q(\theta).$$

Notice that $q'(\theta) < 0$. This is because increasing θ decreases the first argument of m, and m is increasing in both arguments. Thus, the negative derivative. Intuitively, this means that when there are more vacancies, job seekers are more likely to find a job elsewhere. Alternatively, when u becomes smaller, there are fewer workers from which to match, so $q(\theta)$ would become smaller.

On the other hand, $d[\theta q(\theta)]/d\theta > 0$. We can write $\theta q(\theta) = m(1, \theta)$. Since $m(1, \theta)$ is increasing in both arguments, the positivity of the derivative is

[1] It was said in lecture to exhibit constant returns to scale, but I find that terminology awkward since we are not even dealing with returns.

[2] I find it helpful to think of workers as the "commodity." If it's hard to find a worker, then the market is tight; and vice versa.

clear. This should be intuitive. When v becomes larger, there are more job openings posted, and thus workers will have an easier time landing jobs. Similarly, when u becomes smaller, there are less people looking at the existing pool of vacancies, and thus it becomes easier for them to land those jobs.

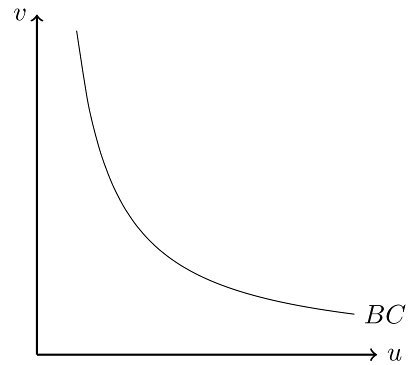
3.3 Beveridge Curve

We are ultimately dealing with a dynamic model. In a given period, some unemployed people will become employed and some employed people will become unemployed. Since there are u unemployed people and the labor force is normalized to 1, it follows that there are $1 - u$ employed people. Because λ is the probability of losing a job, $\lambda(1 - u)$ people will lose their jobs in a given period. Similarly, $\theta q(\theta)u$ unemployed people will find jobs. Therefore in the steady state, we have

$$\lambda(1 - u) = \theta q(\theta)u \quad \Longrightarrow \quad u = \frac{\lambda}{\lambda + \theta q(\theta)}.$$

This equation actually gives the relationship between v and u, which we call the **Beveridge curve**.

In particular, you can use the implicit function theorem to show that $du/dv < 0$. Note that an increase in matching efficiency will shift the Beveridge curve—for any given number of vacancies, there will be fewer unemployed people (because workers were matched more efficiently to the vacancies). Thus, the Beveridge curve will have shifted to the left.



The Beveridge curve (BC)

3.4 Firm Value Functions and Job Creation

Some intuition was given for this—-perhaps more accurately, a pseudo-derivation—but I found it rather cumbersome and unhelpful to go through. So, um, I'm not going to go through it. I'm just going to state it. Suppose r is the fixed real interest rate. Let J denote the value function of a firm that has a worker, and V the value function of a firm that is looking for a worker. Then

$$rV = -pc + q(\theta)[J - V]. \tag{1}$$

The multiplication of V by r is meant to represent discounting. The firm pays the recruitment cost pc, and then we have to account for the arrival rate of the firm finding a worker, $q(\theta)$, at which point the firm would gain the value of having a worker, J, and lose the value of having to continue searching, V.

Recall that firms with a vacancy do not necessarily have to search for a worker. Also recall that we are assuming free entry of firms into the labor market. The implication is that $V = 0$. This is because if $V > 0$, then more firms would enter into the labor market; and if $V < 0$, then firms would leave the labor market.

We can write the value function of a firm who has a worker as

$$rJ = p - w - \lambda J. \tag{2}$$

The firm gains p in production and loses w in wages. Then we have to account for the arrival of job destruction, λ, at which point the firm loses the value of having a worker, J. They do not necessarily gain the value of looking for a new worker V because they don't necessarily start searching right away. And even if they did, $V = 0$ so the equation wouldn't really change.

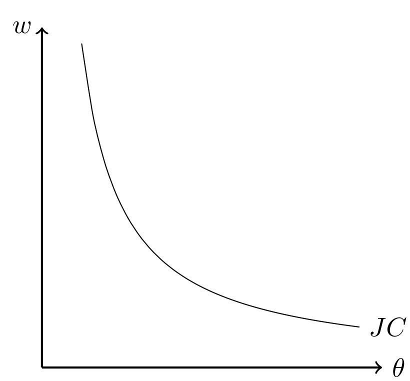
Now look at equation (1). Since $V = 0$ in equilibrium, we can write

$$J = \frac{pc}{q(\theta)} > 0. \tag{3}$$

This allows us to rewrite equation (2) as

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}. \tag{4}$$

We call equation (4) the **job creation curve**. Recall that $q'(\theta) < 0$. An increase in θ thus causes a decrease in $q(\theta)$. Thus means the LHS has a larger negative term. Thus, w will fall. So the JC curve is downward sloping. For this reason, you can kind of think of it as a demand function.



The job creation curve (JC)

So to recap. We have three unknowns: u, v, and w. For many purposes, we can cheat a little a define $\theta = v/u$ and think in terms of only two unknowns, but we ultimately want to solve for u. Thus we will need three equations. So far we only have two: the Beveridge curve, and the job creation curve:

$$w = p - (r+\lambda)\frac{pc}{q(\theta)},$$
$$u = \frac{\lambda}{\lambda + \theta q(\theta)}.$$

We haven't yet looked at the worker side of the market, so that seems like a good place to look for that elusive third equation.

3.5 Worker Value Functions and the Wage Curve

The value function for an employed worker is

$$rW = w + \lambda(U - W). \tag{5}$$

They receive the wage w, augmented by the arrival rate of being fired λ, in which case their value function changes from employment W to unemployment U.

The value function for an unemployed worker is

$$rU = z + \theta q(\theta)(W - U). \tag{6}$$

They receive the unemployment compensation of z, augmented by the arrival rate of getting an acceptable job offer $\theta q(\theta)$, in which case they gain the value of being employed W and lose the value of being unemployed U.

We will use Nash bargaining to determine wages. Let $\beta \in [0, 1]$ be the bargaining power of a worker.[3] To that end, we are going to be solving

$$\arg\max_w (W-U)^\beta J^{1-\beta}.$$

As usual, logs are nicer to deal with. So, let's instead consider

$$\arg\max_w \beta \log(W-U) + (1-\beta)\log(J).$$

Differentiating with respect to w gives

$$\frac{\beta}{W-U}\frac{\partial[W-U]}{\partial w} + \frac{1-\beta}{J}\frac{\partial J}{\partial w} := 0.$$

First note from a purely economic argument that $\partial U/\partial w = 0$. The value of being unemployed doesn't depend on the wage being *negotiated*. Then from equation (5), we can write

$$W = \frac{w+\lambda U}{r+\lambda} \quad \Longrightarrow \quad \frac{\partial W}{\partial w} = \frac{1}{r+\lambda}. \tag{7}$$

From equation (2), we can write

$$J = \frac{p-w}{r+\lambda} \quad \Longrightarrow \quad \frac{\partial J}{\partial w} = -\frac{1}{r+\lambda}. \tag{8}$$

Therefore the first order condition becomes

$$\frac{\beta}{W-U}\frac{1}{r+\lambda} = \frac{1-\beta}{J}\frac{1}{r+\lambda} \quad \Longrightarrow \quad \beta J = (1-\beta)(W-U). \tag{9}$$

We can substitute using the equations (7) and (8) to get

$$\beta\left(\frac{p-w}{r+\lambda}\right) = (1-\beta)\left(\frac{w-rU}{r+\lambda}\right) \quad \Longrightarrow \quad \beta\,(p-w) = (1-\beta)\,(w-rU)$$

[3] "Nash bargaining make sense because any meeting is a match." I am not sure what this means, but it was said and seems important.

Solving for w, we get

$$w = \beta p + (1-\beta)rU.$$

Plug in equation (6) for rU to get

$$w = \beta p + (1-\beta)z + (1-\beta)\theta q(\theta)(W-U).$$

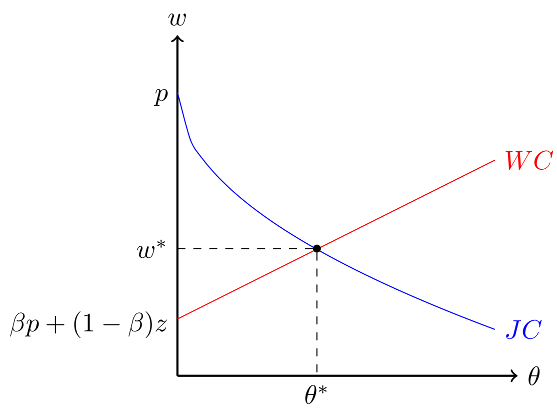
Using the first order condition in equation (9), and subsequently equation (3), we can write

$$\begin{aligned} w &= \beta p + (1-\beta)z + \theta q(\theta)\beta J \\ &= \beta p + (1-\beta)z + \beta\theta pc \qquad (10) \\ &= z + \beta(p - z + \theta pc). \qquad (11) \end{aligned}$$

This result is known as the **wage curve**. Equation (11) gives the intuition behind the answer solution to w. It is similar to what we have seen in other Nash bargaining problems. The wage is the worker's outside offer, namely unemployment benefits, plus their share β of the total surplus, but also with some new term θpc. This new term reflects the fact that the search process, which was costly, is over. In practice, equation (10) is easier to use.

3.6 Equilibrium

By setting the wage curve equal to the job creation curve, we can find an equilibrium with respect to w and θ.



The unique wage and market tightness in equilibrium.

As you can gleam from the plot, existence is guaranteed as long as

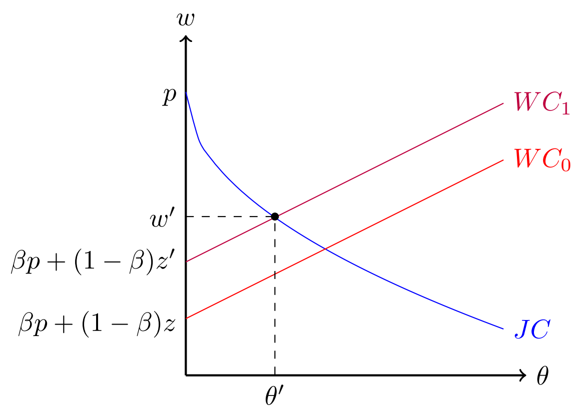
$$p > \beta p + (1-\beta)z \quad \Longrightarrow \quad p > z.$$

This, as it happens, is an assumption we have been making implicitly anyway. Think about it—if production is less than the alternative of sitting on your ass and doing nothing, then no one is going to do anything ever and this model becomes super boring. So yeah, a solution exists and furthermore is unique. Since we will also know θ^*, we can plug it into the Beveridge curve to get

$$u^* = \frac{\lambda}{\lambda + \theta^* q(\theta^*)}.$$

3.7 Comparative Statics

What happens if z is increased to z'? To answer that, we want to compare the two steady states. Notice that a change in z will really only affect the wage curve, in particular, shifting it upwards.

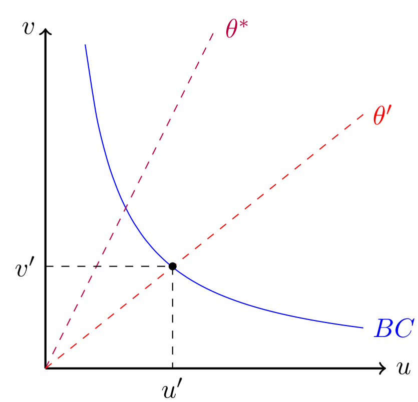


The effect on the equilibrium of increasing z.

As shown above, increasing unemployment benefits increases the wage w and decreases market tightness θ. This is fairly intuitive. People have more incentive to stay at home when z is increased; therefore they need to be tempted with a higher wage in order to actually work. Furthermore, because there is more incentive to not working, we might expect u to increase, giving a decrease in θ. This can also be shown using the Beveridge curve,

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}.$$

The term $\theta q(\theta)$ decreases as θ decreases, making the denominator smaller and thus u larger.



An increase in z decreases θ and thus increases u and decreases v.

4 Solution Summary (Bargaining)

i. For Nash bargaining, solve

$$\arg\max_{s \in S}(s_1 - d_1)^{\theta}(s_2 - d_2)^{1-\theta}$$

where d_1, d_2 are payoffs in absence of an agreement; s_1 and s_2 are potential agreement points; $\theta \in [0, 1]$ is the bargaining power of agent 1; and $1 - \theta$ the bargaining power of agents 2. Taking a logarithmic transformation is usually quite helpful because it doesn't change the argmax yet is easier to work with.

ii. For Kalai bargaining, solve

$$\arg\max_{s \in S}(s_1 - d_1) \quad \text{subject to} \quad \frac{s_1 - d_1}{\theta} = \frac{s_2 - d_2}{1 - \theta}.$$

Usually the constraint alone gives all of the required information.

5 Solution Summary (Mortensen-Pissarides)

i. Use the law of motion of unemployment at the steady state to derive the Beveridge curve.

ii. Because the value of searching for a worker $V = 0$ in equilibrium, it follows from the equation for rV that $J = pc/q(\theta)$.

iii. Then plug that result into the equation for rJ to get the job creation curve,

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}.$$

iv. Use Nash bargaining to determine wages. Workers will lose U and gain

W; firms will lose $V = 0$ and gain J; so solve

$$\arg\max_w (W - U)^\beta J^{1-\beta}.$$

v. After some ugliness, you'll get the wage curve

$$w = z + \beta(p - z + \theta pc).$$

vi. The intersection of the wage curve and the job creation curve give the equilibrium θ^* and w^*. Plug θ^* into the Beveridge curve to solve for u^*.