

# ECN 200B—Arrow-Debreu Proof

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**Theorem** (Arrow-Debreu Theorem). *Fix an exchange economy. If  $u^i$  is continuous, strictly monotone, and strictly quasiconcave; and if each  $w^i \gg 0$ ; then there exists a competitive equilibrium.*

**The Setup.** Normalize prices to the simplex  $\Delta$ . We will refer to the interior of the simplex as

$$\Delta^o = \Delta \cap \mathbb{R}_{++}^L = \left\{ p \in \mathbb{R}_{++}^L \left| \sum_{\ell=1}^L p_\ell = 1 \right. \right\},$$

and the boundary of the simplex as

$$\Delta^\partial = \Delta \setminus \Delta^o.$$

For any  $i$  and any  $p \in \Delta^o$ , let  $x^i(p)$  maximize individual  $i$ 's utility subject to  $p \cdot x \leq p \cdot w^i$ . Define the excess aggregate demand in the usual way,

$$z(p) = \sum_{i=1}^I x^i(p) - w^i.$$

## Properties of Excess Demand.

**Claim 1.**  $z(p)$  is a continuous function. Since utility is assumed to be strictly quasiconcave, it follows that  $x^i(p)$  is a function, and therefore so is

$z(p)$ . Because utility is also assumed to be continuous, it follows that  $x^i(p)$  is continuous<sup>1</sup>, and therefore  $z(p)$  is continuous as well. //

**Claim 2.** For any  $p \in \Delta^o$ ,  $p \cdot z(p) = 0$ . Since utility is strictly monotone, it is locally nonsatiated, and therefore the budget constraint is an equality, that is,  $x^i(p) = w^i(p)$  for all  $i$ . It follows that  $p \cdot [x^i(p) - w^i(p)] = 0$  for all  $i$ . And therefore

$$p \cdot \sum_{i=1}^I [x^i(p) - w^i(p)] = p \cdot z(p) = 0. //$$

One problem arises when  $p \in \Delta^\partial$ , however. Specifically, if  $p_L = 0$ , then  $x^i(p)_L$  will be undefined—strongly monotone preferences and strictly positive endowments means everyone will want to buy an infinite number of good  $L$ . So we'll need to consider boundary prices and interior prices separately, but not *too* separately. I'll explain in a bit.

**The Gamma Correspondence.** I'll illustrate the idea before defining anything. Suppose  $p \in \Delta^o$ , and  $z(p) = (3, 1, 1, -1)$ . Let's solve

$$\max_{\delta_1, \delta_2, \delta_3, \delta_4} 3\delta_1 + 1\delta_2 + 1\delta_3 - 1\delta_4$$

such that  $\sum_{j=1}^4 \delta_j = 1$ . Clearly we would put all of the weight on  $\delta_1$  because it is the largest positive number; that is, have  $\delta_1 = 1$ , the other  $\delta_{j \neq 1} = 0$ . So the idea is, whichever good has the highest excess demand, we should set its price to 1 and every other price to zero. Let  $\Gamma(p)$  be the maximizing set of  $\delta_j$ , in this case,  $\Gamma(p) = [1, 0, 0, 0]$ .

Now suppose that  $p \in \Delta^\partial$ . In this case, excess demand is not defined. What we'll do is take all possible vectors  $\gamma \in \Delta$  that satisfy  $p \cdot \gamma = 0$ . For instance, if  $p = (.5, .5, 0, 0)$ , then  $\delta_1 = \delta_2 = 0$ , and  $\delta_3 + \delta_4 = 1$ , so  $\delta = (0, 0, .5, .5)$  would work, or  $(0, 0, .1, .9)$ .

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<sup>1</sup>By the Theorem of the Maximum, which states that an optimized function is continuous as its parameter changes, in this case  $p$ , under certain conditions.

Thus the correspondence we'll be working with is

$$\Gamma(p) = \begin{cases} \arg \max_{\gamma \in \Delta} z(p) \cdot \gamma & \text{if } p \in \Delta^o, \\ \{\gamma \in \Delta | p \cdot \gamma = 0\} & \text{if } p \in \Delta^\partial. \end{cases}$$

An important point to note here is that if  $p \in \Delta^o$  and  $z(p) \neq (0, \dots, 0)$ , then  $\Gamma(p) \subseteq \Delta^\partial$ . And thus by the contrapositive, if  $\Delta(p) \not\subseteq \Delta^\partial$  and  $p \in \Delta^*$ , then  $z(p) = (0, \dots, 0)$  and we are done.

Eventually we want to invoke Kakutani's theorem on  $\Gamma(p)$ , but of course  $\Gamma(p)$  must satisfy certain conditions to justify doing so. In the interest of time, we will take for granted a few different properties, although each can be shown.

- (a) We need  $\Gamma$  to map  $\Delta \rightarrow \Delta$ . This condition is rather obvious since each  $\gamma$  is taken from  $\Delta$ .
- (b)  $\Gamma$  is nonempty, compact, and convex-valued.
- (c)  $\Gamma$  is upper-hemicontinuous for  $p \in \Delta^o$ .

Let's show upper-hemicontinuity. Consider a sequence of prices  $(p_n)_{n=1}^\infty \in \Delta^o$  that converges to positive prices in all except one  $\bar{p}_L = 0$ , that is,

$$(p_{1(n)}, \dots, p_{L-1(n)}, p_{L(n)}) \rightarrow (\bar{p}_1, \dots, \bar{p}_{L-1}, 0) = \bar{\delta} \in \Delta^\partial.$$

Let  $\delta_n \in \Gamma(p_n)$  for any  $n$ . Our question is this: if  $p_n$  converges to  $\bar{p}$ , will the correspondence converge to some  $\gamma \in \Gamma(\bar{p})$ ?

We know that  $z_L(p_n) \rightarrow \infty$ . We also know that  $p_{\ell(n)} > 0$  and  $\bar{p}_\ell > 0$  for any  $\ell \neq L$ . Thus, for large enough  $n$ , we'll have

$$z_L(p_n) > z_\ell(p_n).$$

Which means for large enough  $n$ , we'll have  $\Gamma(p_n) = \{0, 0, \dots, 0, 1\}$ . Therefore  $\gamma_n \rightarrow (0, 0, \dots, 0, 1) = \bar{\gamma}$ . Furthermore,  $\bar{p} \cdot \bar{\gamma} = 0$ . So  $\bar{\gamma} \in \Gamma(\bar{p})$ . Hemi-

continuity is established.

This is good—even though we’re treating boundary prices and interior prices with separate correspondences, the case correspondence  $\Gamma(p)$  is still upper-hemicontinuous.

**The Fixed Point.** Okay great, so we can apply Kakutani’s theorem—there exists some  $p^* \in \Delta$  such that  $p^* \in \Gamma(p^*)$ . What can we say about  $p^*$ ?

The most important thing we can say is that it’s not in the boundary. To see why, suppose  $p^* = (1, 1, 0)$ . Then we’ll have  $\delta = (0, 0, 1) \in \Gamma(p^*)$ . But then if we take  $\Gamma(\delta)$ , we’ll get, among other choices,  $\Gamma(\delta) = (.5, .5, 0) \notin \Gamma(p^*)$ .

More generally, suppose  $p^*$  has  $p_L^* = 0$ . Then the correspondence  $\Gamma(p^*)$  will consist of vectors with  $\delta_L > 0$ . So when we plug  $\delta$  back into the correspondence with  $\Gamma(\delta)$ , it will return vectors with  $\tilde{\delta}_L = 0$ , which cannot be in  $\Gamma(p^*)$  because  $\Gamma(p^*)$  consists of vectors with  $\delta_L > 0$ .

Since we know there exists some fixed point  $p^*$ , it follows that it must be in the interior. Furthermore, we know that if  $p^* \in \Delta^\circ$  and  $z(p) \neq (0, \dots, 0)$ , then  $\Gamma(p^*) \subseteq \Delta^\partial$ . By the contrapositive, the fact that  $\Gamma(p^*) \not\subseteq \Delta^\partial$  means that  $z(p^*) = (0, \dots, 0)$ . And thus we know that the equilibrium exists and has strictly positive prices on all commodities.