ECN 200E: Week 6 Discussion

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Money in Utility

The social planner maximizes

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{\left(\frac{M_t}{P_t}\right)^{1-v}}{1-v} \right]$$

subject to the constraints

$$\begin{aligned} y_t &= k_t^\alpha, \\ c_t + i_t &= y_t, \\ k_{t+1} &= (1-\delta)k_t + i_t. \end{aligned}$$

There's also a bond market. You can buy a bond today for Q_{t+1} and tomorrow you'll get back a dollar. Woo. Also note that money is added to the economy each period via lump sum transfers as a constant growth rate of g.

Part A: Variables and the Social Planner's Problem

- The control variables are: c_t, k_{t+1}, b_t, M_t. You control what to consume, how much to invest, how many bonds to buy today, and how much money to hold today.
- The state variables are: $k_t, b_{t-1}, M_{t-1}, p_t$. You start with some capital. Whatever bonds you bought last period pay out today; whatever money you held onto last period can be spent today; and there's going to be some price in each period.

So the resource constraint is

$$p_t c_t + Q_t b_t + p_t[k_{t+1} - (1-\delta)k_t] + M_t = p_t k_t^\alpha + b_{t-1} + M_{t-1} + T_t.$$

Technically, $T = gM_t$. That is, you enter the period with M_t and the government increases the money supply by gM_t. But for now, just consider it T; we don't want to consider that term when doing first-order conditions.

The social planner's problem can thus be written as

$$\begin{aligned} V(k_t, b_{t-1}, M_{t-1}, p_t) =& \frac{c_t^{1-\sigma}}{1-\sigma} + \frac{\left(\frac{M_t}{P_t}\right)^{1-v}}{1-v} + \beta V(k_{t+1}, b_t, M_t, p_{t+1}) \\ & - \lambda_t \left(p_t c_t + Q_t b_t + p_t[k_{t+1} - (1-\delta)k_t] + M_t - p_t k_t^\alpha - b_{t-1} - M_{t-1} - T_t\right). \end{aligned}$$

Oy.

Part B: First-Order Conditions and Money Neutrality

There are four first-order conditions:

$$\begin{aligned} \text{with respect to } c_t &\implies c_t^{-\sigma} = \lambda_t p_t, \\ \text{with respect to } k_{t+1} &\implies \beta V_k'(k_{t+1}, b_t, M_t, p_{t+1}) = \lambda_t p_t, \\ \text{with respect to } M_t &\implies \left(\frac{M_t}{p_t}\right)^{1-v} \frac{1}{p_t} + \beta V_M'(k_{t+1}, b_t, M_t, p_{t+1}) = \lambda_t, \\ \text{with respect to } b_t &\implies \beta V_b'(k_{t+1}, b_t, M_t, p_{t+1}) = \lambda_t Q_t. \end{aligned}$$

There are three envelope conditions:

$$\begin{aligned} V_k'(k_{t+1}, b_t, M_t, p_{t+1}) &= c_{t+1}^{-\sigma}[(1-\delta) + \alpha k_{t+1}^{\alpha-1}], \\ V_M'(k_{t+1}, b_t, M_t, p_{t+1}) &= \frac{c_{t+1}^{-\sigma}}{p_{t+1}}, \\ V_b'(k_{t+1}, b_t, M_t, p_{t+1}) &= \frac{c_{t+1}^{-\sigma}}{p_{t+1}}. \end{aligned}$$

So stuff the envelopes into the first-order conditions and we have

$$\beta c_{t+1}^{-\sigma}\left[(1-\delta)+\alpha k_{t+1}^{\alpha-1}\right] = c_t^{-\sigma}, \tag{1}$$

$$\left(\frac{M_t}{p_t}\right)^{1-v}\frac{1}{p_t}+\beta\frac{c_{t+1}^{-\sigma}}{p_{t+1}} = \frac{c_t^{-\sigma}}{p_t}, \tag{2}$$

$$\beta\frac{c_{t+1}^{-\sigma}}{p_{t+1}} = \frac{c_t^{-\sigma}}{p_t}Q_t. \tag{3}$$

Let's try to interpret each Euler equation.

- The first equation implies money neutrality. Notice that the investment decision (in the brackets) is not affected by any nominal terms. Therefore neither is consumption, and therefore neither is output.
- Now the second equation, which is the money demand equation. The left-hand side gives benefit from holding one more dollar today in consumption terms, plus what that that dollar can buy you in the next period. The right hand side is the benefit today that spending the dollar could give. The two should be equal in equilibrium.
- Yeah, I still can't interpret this one. Um, the cost of buying a bond today, in consumption terms, should equal the discounted benefit that dollar gives in the next period?

Part C: Money Demand Equivalence

In lecture, the money demand equation was

$$\left(\frac{M_t}{p_t}\right)^{-v} = c_t^{-\sigma}(1-R_t^{-1}).$$

We can plug in the third Euler equation into the second Euler equation, which gives

$$\left(\frac{M_t}{p_t}\right)^{1-v}\frac{1}{p_t}+\frac{c_t^{-\sigma}}{p_t}Q_t = \frac{c_t^{-\sigma}}{p_t}.$$

Multiply both sides by p_t, subtract the term with Q_t from both sides, and then define $R_t = 1/Q_t$. Then we have the function as given in class:

$$\left(\frac{M_t}{p_t}\right)^{1-v} = c_t^{-\sigma}(1 - R^{-1}).$$

Part D: Real Price of a Dollar

The *real price of a dollar* is given by $1/p_t$. Suppose $p_t = \$0.50$ per unit of the consumption good. Then $1/p_t = 2$. This means one dollar can be used to purchase two units of the consumption good.

So let's find the real price of a dollar in this economy using the money demand equation. Using two periods, we know that

$$\begin{aligned}
\frac{c_t^{-\sigma}}{p_t} &= \left(\frac{M_t}{p_t}\right)^{1-v}\frac{1}{p_t} + \beta\frac{c_{t+1}^{-\sigma}}{p_{t+1}},\\
\frac{c_{t+1}^{-\sigma}}{p_{t+1}} &= \left(\frac{M_{t+1}}{p_{t+1}}\right)^{1-v}\frac{1}{p_{t+1}} + \beta\frac{c_{t+2}^{-\sigma}}{p_{t+2}}.
\end{aligned}$$

Discount the second equation by β and it can be plugged into the first equation, giving

$$\frac{c_t^{-\sigma}}{p_t} = \left(\frac{M_t}{p_t}\right)^{1-v}\frac{1}{p_t} + \beta\left[\left(\frac{M_{t+1}}{p_{t+1}}\right)^{1-v}\frac{1}{p_{t+1}} + \beta\frac{c_{t+2}^{-\sigma}}{p_{t+2}}\right]$$

Then update the money demand function again,

$$\frac{c_{t+2}^{-\sigma}}{p_{t+2}} = \left(\frac{M_{t+2}}{p_{t+2}}\right)^{1-v}\frac{1}{p_{t+2}} + \beta\frac{c_{t+3}^{-\sigma}}{p_{t+3}},$$

which we can plug into the previous equation for

$$\frac{c_t^{-\sigma}}{p_t} = \left(\frac{M_t}{p_t}\right)^{1-v}\frac{1}{p_t} + \beta\left[\left(\frac{M_{t+1}}{p_{t+1}}\right)^{1-v}\frac{1}{p_{t+1}} + \beta\left(\left(\frac{M_{t+2}}{p_{t+2}}\right)^{1-v}\frac{1}{p_{t+2}} + \beta\frac{c_{t+3}^{-\sigma}}{p_{t+3}}\right)\right].$$

Rearranging a little bit, we can rewrite this as

$$\frac{c_t^{-\sigma}}{p_t} = \left(\frac{M_t}{p_t}\right)^{1-v}\frac{1}{p_t} + \beta\left(\frac{M_{t+1}}{p_{t+1}}\right)^{1-v}\frac{1}{p_{t+1}} + \beta^2\left(\frac{M_{t+2}}{p_{t+2}}\right)^{1-v}\frac{1}{p_{t+2}}.$$

See the pattern? If we keep iterating like that, we'll end up with

$$\frac{c_t^{-\sigma}}{p_t} = \sum_{k=0}^{\infty} \beta^k \left(\frac{M_{t+k}}{p_{t+k}} \right)^{1-v} \frac{1}{p_{t+k}}.$$

Now we have an expression for the real price of money,

$$\frac{1}{p_t} = \frac{\sum_{k=0}^{\infty} \beta^k \left(\frac{M_{t+k}}{p_{t+k}} \right)^{1-v} \frac{1}{p_{t+k}}}{c_t^{-\sigma}}.$$

We only need to make sure that the transversality condition holds, namely,

$$\lim_{T\to\infty} \beta^T c_T^{-\sigma} = 0^1.$$

The numerator of the real price of money is the lifetime utility from holding a dollar forever, discounted. Hold a dollar in period t and you'll get marginal benefit today, and some tomorrow, and some the day after, and so forth.

Part E: Bond Holdings and the Interest Rate

To find bond holdings and the interest rate, we'll use the bond Euler equation,

$$\frac{c_t^{-\sigma}}{p_t} Q_t = \beta \frac{c_{t+1}^{-\sigma}}{p_{t+1}} \implies c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \frac{p_t}{Q_t p_{t+1}}.$$

We have previously defined $R_t = 1/Q_t$ to be the nominal interest rate, so $1/Q_{t+1} = 1 + i_t$. Furthermore, p_{t+1}/p_t is $1 + \pi_t$. Therefore

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \frac{1+i}{1+\pi_t} = \beta c_{t+1}^{-\sigma}(1+r_t).$$

Now recall the consumption-investment Euler equation,

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left[(1-\delta) + \alpha k_{t+1}^{\alpha-1} \right].$$

So $1 + r_t = (1-\delta) + \alpha k_{t+1}^{\alpha-1}$. In other words, the real interest rate is the marginal product of capital.

[1] I am convinced that this should actually be the marginal utility of money instead.

Part F: Steady State Prices and Inflation

In a steady state, we want real money balances to be equal. That is,

$$\frac{M_{t+1}}{p_{t+1}} = \frac{M_t}{p_t}.$$

We are told that money grows at a constant rate, so $M_{t+1} = (1+g)M_t$. Therefore

$$\frac{(1+g)M_t}{p_{t+1}} = \frac{M_t}{p_t} \quad \Longrightarrow \quad 1 + g = 1 + \pi_t.$$

So in the steady state, the rate of inflation will equal the rate of money growth.