Notes for PHYS 231A: General Relativity

Bill Wolf

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1 Introduction

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1.1 What We'll Cover

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1.2 Misconceptions about GR

1.2.1 GR is Hard

This is a product of people being unable to do calculus on curved manifolds. The underlying theory is very simple and elegant. The details, when they get hairy, can be relegated to a computer. The equations of motion are easy to recover, but exact solutions can be difficult.

1.2.2 SR only describes inertial motion

This is simply not true. Generalizations of Newtonian Mechanics can easily account for accelerated motion. This probably comes from the twin paradox, where an accelerated observer experiences a different proper time than his twin, but this does not mean that SR is unable to handle accelerated motion. It cannot handle motion on a curved metric, though.

1.2.3 GR is based on a principle of general covariance

General covariance is more general than general relativity. In fact, all theoretical physics should be done in a generally covariant way so that a change of coordinates should not affect the theory.

1.3 GR is based on the Equivalence Principle

The **equivalence principle** states essentially that an observer undergoing acceleration is completely equivalent to a different observer that is "at rest" in a gravitational field. For example, a scientist conducting experiments in an accelerating rocket (with acceleration g) will measure the same results as another scientist on Earth at rest, but experiencing a gravitational acceleration of g.

An interesting conclusion of this principle is that of the curved paths of light. For instance, in the rocket scenario mentioned above, if a ray of light enters the rocket perpendicular to the rocket's acceleration, the light would appear to curve in a parabolic shape solely due to the relative motion of the rocket. Thus, we expect light traveling through a gravitational field to likewise be curved in the direction of the field. This can be observed by the shifting of apparent positions of stars near the sun during a solar eclipse.

Additionally, if light were traveling along the direction of motion of the rocket, observers outside ("at rest") and inside would observe different wavelengths of the light. We would then expect a similar result in a gravitational field. In the rocket scenario, this is called the Doppler Shift, but in a gravitational field, we call it the **gravitational redshift**.

This bending of light rays leads us to conclude that "spacetime is curved", in the sense that two parallel light rays may intersect (e.g. two parallel light rays travel on either side of a massive body and are curved into each other). This leads to an entirely geometric understanding of spacetime. Note that it is space*time* that is curved. GR must be consistent with SR, so both space *and time* must be curved under the influence of mass. Indeed, GR wraps space and time together into one entity, whereas Newtonian physics kept the two completely separate.

1.4 Inertial frames

We need not invoke SR of GR to talk of inertial frames. In fact, the idea was originally due to Galileo, using the space-time of Newton. In this paradigm, time is absolute, and is experienced the same everywhere, regardless of their motion. There was such a thing as synchronized clocks, and we could conceive of a variable t, which tells the time for all observers. Additionally, we are dealing with three dimensions in completely flat space. Considering two positions, $\mathbf{x}$ and $\mathbf{x}'$, the distance between the two points would be

$$\delta s^2 = \left[x^1 - (x^1)'\right]^2 + \left[x^2 - (x^2)'\right]^2 + \left[x^3 - (x^3)'\right]^2 = (\mathbf{x} - \mathbf{x}')^2 \tag{1}$$

In inertial frames, where relative motions differ only by constant velocities, each of these coordinates necessarily has a vanishing second time derivative:

$$\ddot{x}^i = 0 \tag{2}$$

We'll now change coordinates in such a way to preserve the structure of (1), which is called an **isometry** (since it preserves distance).

$$\tilde{x}^i(t) = \sum_j R^i{}_j(t)x^j(t) + g^i(t) \tag{3}$$

This transformation gives a time dependent rotation and translation of each coordinate. We need only require that (2) be true in this new coordinate system as well to come up with some reasonable constraints.

$$\ddot{\tilde{x}}^i(t) = \ddot{g}^i(t) + \sum_j \left[\ddot{R}^i{}_j x^j(t) + 2\dot{R}^i{}_j \dot{x}^j + R^i{}_j(t)\ddot{x}^j(t)\right] = 0 \tag{4}$$

(2) mandates that the last term in (4) must vanish. Moreover, this equation must be true for all particles at all locations at all velocities. A particle not moving ($\dot{x}^i = 0$) at the origin ($x^i = 0$)

requires that

$$\ddot{g}^i(t) = 0 \tag{5}$$

So the coordinate system as a whole is only moving at constant velocity. Similarly, only requiring the particle to be at rest now mandates that

$$\ddot{R}^i{}_j(t) = 0 \tag{6}$$

which necessitates that the last term must also vanish:

$$\dot{R}^i{}_j(t) = 0 \tag{7}$$

So, the set of all **Galilean transformations** are those with a constant rotation and a constant relative velocity.