STELLAR ATMOSPHERES

Virtually all informations about stars come from the observation of their light. Photons reaching the observer transport informations about the conditions of the gas they had the last interaction with, before leaving the star.
This region of "last interaction" is called Photosphere. The informations about the physical conditions of the photosphere (and hence properties of the star like gravity and surface temperature) can be retrieved by studying the stellar spectra.

To understand and describe the formation of spectral lines one requires to link the degree of ionization and the occupation of the energy levels in the gas to the thermodynamic variables.

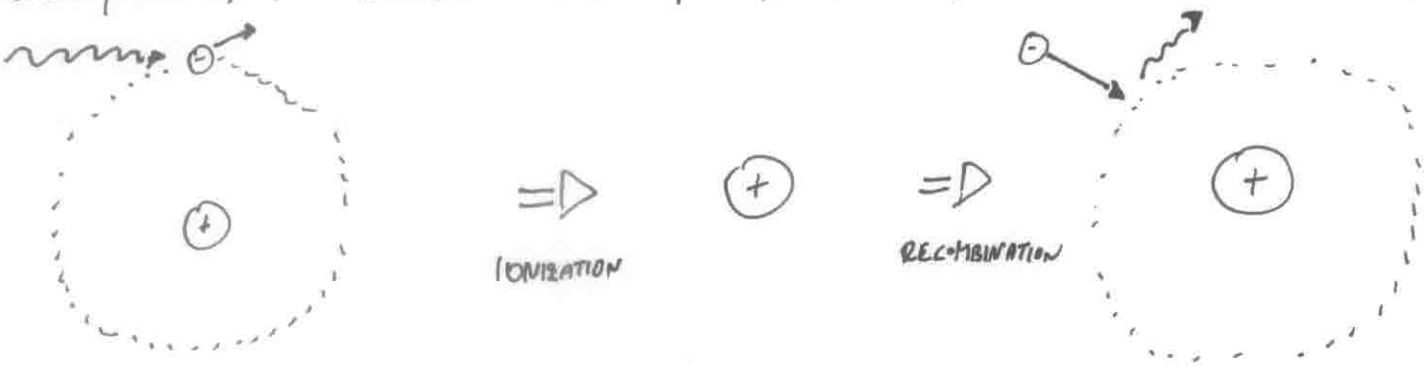
We will assume to be in LOCAL THERMODYNAMIC EQUILIBRIUM (LTE) which basically means the interactions between gas particles are so frequent that an equilibrium situation is reached in which

1) All particles have Maxwellian distribution in velocities (with $T_{MAXWELL}$)
2) Atom populations follow BOLTZMANN equation (with $T_{BOLTZMANN}$)
3) Ionization is described by SAHA equation (with T_{SAHA})

in LTE $\boxed{T_{MAXWELL} = T_{BOLTZMANN} = T_{SAHA}} = T$

If the interactions with the radiation field are also very frequent, then the radiation intensity is given by the PLANCK distribution and $T_{BB} = T$
In this case one talks of THERMODYNAMIC EQUILIBRIUM

We are interested in deriving the SAHA equations, that should allow us to know, for example, what fraction of H is ionized in the solar photosphere. This equation should describe the following process



LET'S START WITH THE MOST GENERAL STATISTICS FOR e^- (F.D. STATISTICS)

$$\boxed{n(p) = \frac{g}{h^3} \frac{1}{\exp[(E_p-\mu)/kT]+1}}$$

where $E_p = p^2/2m_p + m_p c^2$

m_p: particle rest mass

μ = chemical potential

To get the total number density we integrate over $d^3p = 2\,(4\pi p^2 dp)$

$$n = \frac{\overset{=2}{g}}{h^3}\int_0^\infty \frac{4\pi(p^2dp)}{\exp[(E_p-\mu)/kT+1}$$

for electron $g=2$ (2 spin orientation)

volume in phase space occupied by particles with $p \in [p, p+dp]$

if $\bar{\lambda} \gg \lambda$ de Broglie the particles do not know their exclusion statistics.

Below we are in the MAXWELL-BOLTZMANN LIMIT $\exp\left(\frac{E_p-\mu}{kT}\right) \gg \pm 1$

$$n = \frac{4\pi}{h^3}\int_0^\infty p^2dp\,\exp\left(\frac{\mu-\overset{p^2/2m_p+m_pc^2}{E_p}}{kT}\right)$$

using $\int_0^\infty x^2 e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{4a^3}$ for $a>0$

$$\Rightarrow \exp\left(\frac{\mu-m_pc^2}{kT}\right) = \frac{nh^3}{2(2m_pkT\pi)^{3/2}}$$

$$m_Q \equiv \left[\frac{2\pi m kT}{h^2}\right]^{3/2}$$

$$= \frac{n}{2m_Q}$$

So we obtain an expression for the chemical potential $\boxed{\mu_e = kT\ln\left(\frac{n_e}{2m_Q}\right) + m_ec^2}$

Let's consider H ionization / recombination: $e^- + p \rightleftharpoons H + \gamma$

The chemical balance + equilibrium impose $\mu_e + \mu_p = \mu_H$

$$\mu_p = m_pc^2 - kT\ln\left(\frac{g_p m_{Q,p}}{n_p}\right)$$

$$\mu_e = m_ec^2 - kT\ln\left(\frac{g_e m_{Q,e}}{n_e}\right)$$

$$\mu_H = m_Hc^2 - kT\ln\left(\frac{g_H m_{Q,H}}{n_H}\right)$$

and we know that $\boxed{m_Hc^2 = (m_p+m_e)c^2 - 13.6\,\text{eV}}$

$$KT\left[\ln\left(\frac{g_e n_{Q,e}}{n_e}\right)+\ln\left(\frac{g_p n_{Q,p}}{n_p}\right)-\ln\left(\frac{g_H n_{Q,H}}{n_H}\right)\right] = 13.6\ eV$$

$$\ln\left[\frac{g_e n_{Q,e}}{n_e}\ \frac{g_p \cancel{n_{Q,p}}}{n_p}\ \frac{n_H}{g_H \cancel{n_{Q,H}}}\right] = \frac{13.6\ eV}{KT}$$

$$\left[n_Q \equiv \left[\frac{2\pi m KT}{h^2}\right]^{3/2}\right.$$

and $m_p \simeq m_H$

$\rightarrow n_{Q,p} \simeq n_{Q,H}$

$$\frac{n_e n_p}{n_H} = \frac{g_p g_e}{g_H} n_{Q,e} \exp\left(-\frac{13.6}{KT}\right)$$

$$\boxed{\frac{n^- n^+}{n} = \underbrace{\frac{g_e g_p}{g_H}}_{=2/2=1} \left(\frac{2\pi m_e KT}{h^2}\right)^{3/2} \exp\left(-\frac{13.6}{KT}\right)}$$

SAHA equation

One can see that the 1/2 ionisation phase ($n_p = n_e$, $[n_H = n_p] = n_e$)

$$\frac{n^2}{n} = n_p = 2.4 \times 10^{21}\ cm^{-3}\ T_4^{3/2}\ e^{-\frac{15.76}{T_4}}$$

and setting typical values for stellar photosphere ($n_p \sim 10^{11}$) one gets, solving for T, $\boxed{T = 12.400\ K}$. So the 1/2 ionisation point happens at $\sim 10^4$ K for typical photosphere density

One can generalize the result and write the Degree of Ionisation $y \equiv \frac{n_p}{n} = \frac{n_e}{n}$

$$\boxed{\frac{y^2}{1-y} = \frac{4 \times 10^{-9}}{\rho}\ T^{3/2}\ e^{-\frac{1.6}{T_5}}}$$

where one can see that

$y \uparrow$	$\rho \downarrow$
$y \uparrow$	$T \uparrow$

ASK Why?

* Thermal Ionisation: ionisation rate x volume $\propto n$ $\neq$ collisional ionisation / pressure ionisation
* Recombination rate: involves 2 particles $\propto n^2$

What about the center of the SUN ?

$\rho_c \sim 100\ g\ cm^{-3}$

$T_c \sim 1.5 \cdot 10^7 K$

using equation for $y \Rightarrow \frac{y^2}{1-y} \simeq 2.3 \Rightarrow y = 0.75$

which states the center of the Sun is 25% neutral

Are you ok with this ?
If not why ?
What we did wrong ?

$\lambda_{DE\ BROGLIE} = h/p = h/\sqrt{mKT}$

$\lambda \simeq 1/n^{1/3}$

One requires $\lambda \gg \lambda_{DB}$ to apply the SAHA equation.

If $\lambda \sim \lambda_{DB}$ (as in the sun's center) then one has to account for pressure ionization.

<u>ATOMS DO OVERLAP</u> (separation comparable to their (electron cloud) size)

STELLAR SPECTRA

Let's assume a plane parallel atmosphere in LTE and Hydrostatic Eq.

Plane parallel is justified because, for most stars, $\boxed{H_p \equiv \frac{KT}{m_p g} \ll R_*}$

(H_p is the pressure scale height, i.e. the vertical distance over which the pressure decreases by a factor 'e')

$$\left[\text{Derivation: } \frac{dP}{dz} = -g\rho \,,\ \rho KT = m_p P \Rightarrow \frac{dP}{P} = -\frac{g m_p}{KT} dz = -\frac{dz}{H_p}\right]$$

$$g = \frac{GM}{R^2} = 2.7 \cdot 10^4 \frac{M}{R_\odot} \quad \text{(scaling)} + \left[g = \frac{GM}{R^2}\right] \Rightarrow \boxed{g = \frac{GM}{R^{2.5}} = 2.7 \cdot 10^4 \left(\frac{M_\odot}{R}\right)^{1/2}} \ (*)$$

useful scaling

Photosphere: region from which we observe photons emerging from the star. These photons do one last scattering, then reach the observer. Given the flux, one can define the effective temperature of the star as $\boxed{F \equiv \sigma_{SB} T^4}$

CALCULATE PARTICLE DENSITY AT PHOTOSPHERE

If there was no lines, then one would observe a BB with temperature $T_{BB} = T_{eff}$.

Given the def. of OPTICAL DEPTH $\tau \equiv \int K\rho dr$, one defines the stellar photosphere as that region where $\tau_{ph} = 2/3$

$$\Rightarrow \tau_{ph} = 2/3 = K \int \rho dr = K y \Rightarrow y_{ph} = \frac{2}{3K}$$

H.E. $dP = -\rho g dr$

$$\Rightarrow P_{ph} = g\, y_{ph} = g \frac{2}{3K} \overset{\text{g. state}}{=} n K_B T$$

$$\Rightarrow n_{ph} = \frac{2 g m_p}{3 \sigma_{Th} K_B T} \quad \text{(Thomson scattering dominates)}$$

$\left(\cancel{g = \frac{GM}{R^2}}, \cancel{\text{for upper MS } R = M^{0.75}}\right)$

$$\boxed{n_p \approx (10^{16} m^{-3}) \left(\frac{K_B S}{K}\right) \left(\frac{10^4 K}{T}\right) \left(\frac{g}{2.7 \cdot 10^4}\right)} (*) \left(\frac{M_\odot}{M}\right)^{1/2}$$

Let's divide SAHA eq. for Na and H :

$$\frac{n_{Na^+}}{n_{Na}} = \frac{n_p}{n_H} \exp\left[-\frac{(\overset{5.14}{E_{Na}} - \overset{13.6}{E_H})}{KT}\right] \approx 10^7 \frac{n_p}{n_H} \quad @\ 6000K$$

$$\frac{n(He^+)}{n(He)} = \frac{n_p}{n_H} \exp\left[-\frac{(\overset{24.6-13.6}{E_{He} - E_H})}{KT}\right] \approx 6\times10^{-10} \frac{n_p}{n_H} \quad ''$$

So, compared to H, Na is almost completely Ionized. He is NOT
=> Alkali metals are rare, but they account for nearly all free e^-'s.

e.g.: SUN $\left(n_e = 10^{13}\,cm^{-3},\ T \sim 6000\,K\right)$ (as $n_p \sim 10^{16}\,cm^{-3}$)

$$\underbrace{\frac{n_p}{n_H} = 9.3\cdot10^{-4}, \quad \frac{n_{He^+}}{n_{He}} = 2.5\cdot10^{-13}}_{\text{NEUTRAL}}$$

$$\frac{n_{Na^+}}{n_{Na}} = 5.4\cdot10^3 \leftarrow \text{ALL BASICALLY singly Ionized.}$$

TRANSITION T (½-ionization) for a number of n_e's

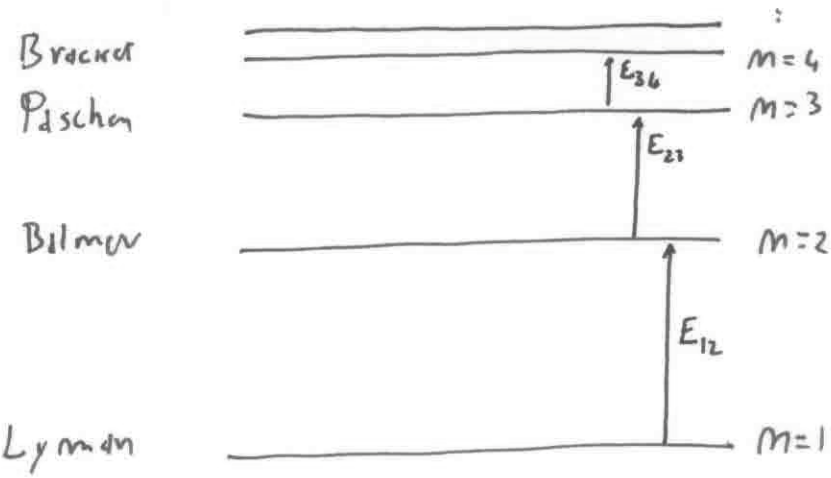
n_e	T_{Na} (5.14)	T_H (13.6)	T_{He} (24.6)
10^{13}	3000	8000	12500
10^{14}	3508	9082	16500
10^{15}	4057	10500	19000
10^{16}	4610	12450	22510

H - lines

$$E_{nm} = \left[+\frac{13.6}{n^2} - \frac{13.6}{m^2} \right] eV$$

(RYDBERG FORMULA)

$$= 13.6 \left[\frac{1}{n^2} - \frac{1}{m^2} \right] eV$$



Lyman Line (UV):	1→2	1215 Å	Ly α
	1→3	1025 Å	Ly β
	1→4	972 Å	Ly α
	⋮		
	1→∞	911 Å	Ly LIMIT
Balmer Line (VISIBLE):	2→3	6563 Å	H-α
	2→4	4861 Å	H-β
	2→5	⋮	⋮
	⋮		
	2→∞		
Paschen Line (IR)	. . ~ - - -		

To see any of the Balmer lines we need an appreciable amount of H in the $n=2$ state

$$\frac{n_2}{n_1} = \exp\left(-\frac{10.2\,eV}{KT}\right) = 10^{-5} \quad @\ T=10^4 K$$

On top of that one requires

$$\sigma_{line\text{-}center} \gg \sigma_{Thomson} = 6.65 \cdot 10^{-25}\,cm^{-2}$$

$$\int \sigma_{line} = \pi \lambda^2 \sim \pi (1 Å)^2 \sim 10^{-16}\,cm^2$$

$$\frac{\sigma_{line}}{\sigma_{Th}} \sim 10^{+8}$$

so as long as $\boxed{\exp\left(-\frac{10.2\,eV}{KT}\right) \cdot \frac{\sigma_{line}}{\sigma_{Th}} \frac{m_H}{m_p} > 1}$ we get BALMER LINES

for $T < 10^4$, we require

$$\exp\left(-\frac{10.2\,eV}{KT}\right) > \frac{\sigma_{Th}}{\sigma_{line}} \sim 6 \cdot 10^{-9} \Rightarrow \boxed{T > 6000\,K}$$

SPECTRAL TYPES

O3	52000 K	120
O5	44500	60
O8	35800	22
B0	30000	17
B2	22000	9
B5	15400	6
A0	9520	3
F0	7200	1.6
G0	6030	1.05
K0	5250	0.79
M0	3850	0.51

Metallic lines (low FIP) : Li, Na, Mg, Al $\lesssim 5\,eV$

Middle species : H, C, N, O $10 < E_{FIP} < 20\,eV$

Noble Gases : He, Ne $> 20\,eV$

H (13.6 eV)
He (24.6 eV) → why He FIP is not 2 · H FIP ? screening from other electrons
Na (5.14 eV)

Let's have a look at Na $\quad Na^+ + e^- \rightleftharpoons Na + \gamma$

$$\frac{n^+ n^-}{n} = \left(\frac{2\pi m_e k T}{h^2}\right)^{3/2} \exp\left[-\frac{E_i}{k_B T}\right]$$

or

$$\boxed{\frac{n^+}{n} = \frac{2.4 \times 10^{15}}{n_e} T^{3/2} \exp\left[-\frac{E_i}{kT}\right]}$$

O STARS : HeII lines (absorption / emission)

B STARS : HeI lines (absorption)

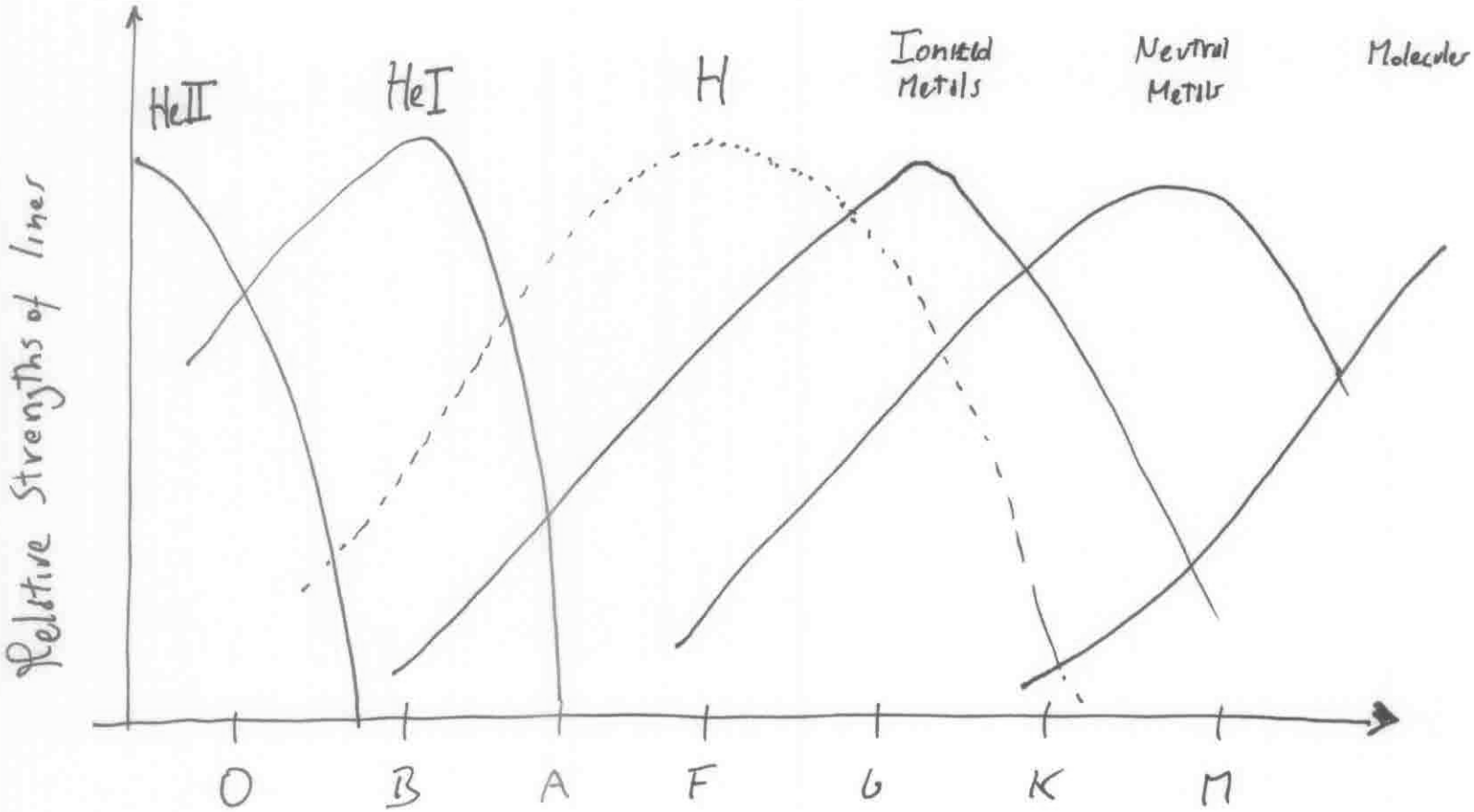
A – F STARS : Balmer (HI) lines

F STARS : Metallic lines become visible

G STARS (SUN-like) : Absorption lines of neutral metallic ~~lines~~ atoms and ions (e.g. CaII)

K STARS : Dominated by Metallic lines

M STARS : Molecular bands (e.g. TiO, titanium oxyde)



H^- ion : bound state of $p + 2e^-$

Important contributor to opacity for cool stars. The second e^- is barely bound (0.75 eV)

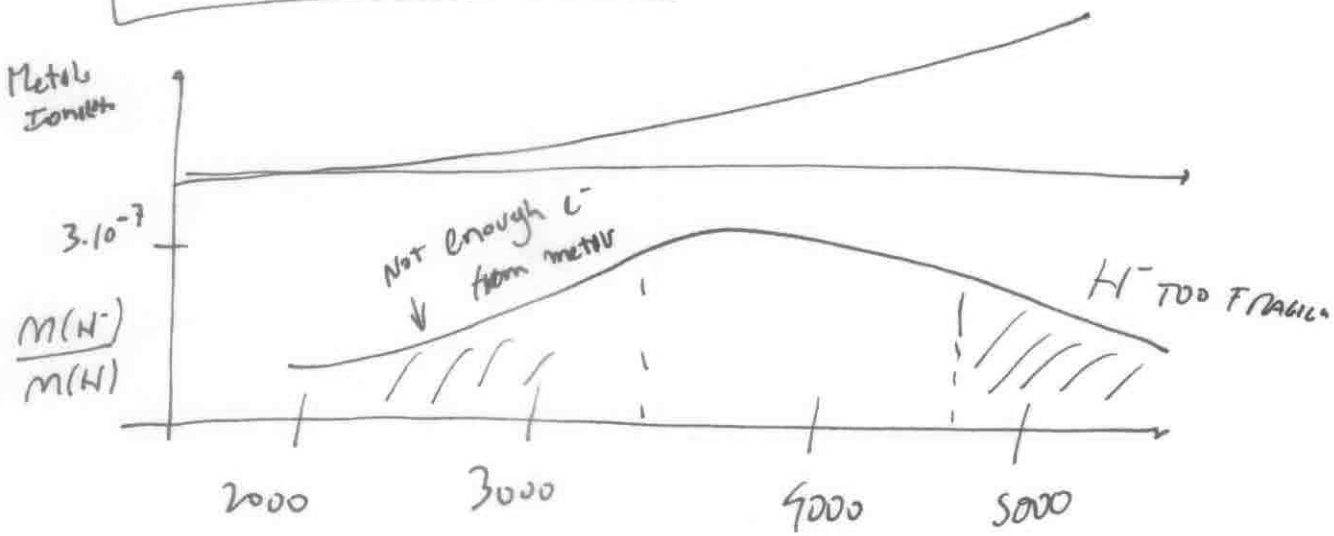
[H^- dominant source of opacity in the sun]

$$\boxed{\mu_e + \mu_H \rightleftharpoons \mu_{H^-}}$$

$$\frac{n(H)}{n(H^-)} = \frac{n_a}{n_e} \exp\left(-\frac{0.75\,eV}{kT}\right)$$

$$\frac{n(H^-)}{n(H)} = \frac{n_e}{n_a}\left[\frac{8700}{T}\right] \qquad n_e \simeq 10^{13}\,cm^{-3}$$

$$\Rightarrow \boxed{\frac{n(H^-)}{n(H)} \approx 10^{-8}\, e^{8700/T}}$$



@ $T = 3000K \approx 10^{-7}$ of H is in H^- (peak)

Answers to some of the questions came up during the class

- We defined $\boxed{n_Q \equiv \left(\frac{2\pi m K T}{h^2}\right)^{3/2}}$, what is its physical interpretation?

$$\left(\frac{m K T}{h^2}\right)^{3/2} = \frac{P}{h^3} = \frac{1}{\lambda_{DB}^3} \quad \Rightarrow \quad n_Q \simeq \frac{1}{\lambda_{DB}^3}$$

n_Q is the density for which electron clouds of atoms start overlapping. For $n \sim n_Q$ we can no more apply the SAHA equation, as pressure ionisation has an important effect (see e.g. solar interior calculation).

To use SAHA we want first to check that $n \ll n_Q$, equivalent to $\lambda \gg \lambda_{DB}$ (and equivalent to $\exp\left(\frac{E_p - \mu}{KT}\right) \gg 1$)

- NLTE (and HeI, HeII lines)

Breakdown of LTE assumption occur, for example, if the rate of photon absorptions becomes $\gg$ rate of electron collisions. In this case one has to explicitely calculate rate equations, instead of simply using BOLTZMANN/SAHA eqs. It is then clear that LTE tend to be valid for low T & high ρ while NLTE departures occur for high T & low ρ (the case for massive stars)

For example it seems NLTE departures in the characterisation of He lines can be important for $T_{eff} \gtrsim 30000$ K even in subdwarf B stars (NAPIWOTKI, ApJ 322, 1987)