Notes for PHYS 234: High Energy Astrophysics

Bill Wolf

April 22, 2013

Contents

1 Introduction

Monday, April 1, 2013

1.1 What is High Energy Astrophysics?

There are many definitions, but one is that high energy astrophysics is "the study of the violent and extreme processes in the universe." Another traditional definition is the study of processes that produce gamma rays and X-rays (possibly ultraviolet radiation as well). Unfortunately, such high energy photons are very effectively blocked by the Earth's atmosphere, so observations are limited to being taken from the upper atmosphere or space. As a result, high energy astrophysics is a relatively young subfield of astrophysics ($\lesssim 100$ yrs).

For our purposes, we will take an expanded view of what high energy astrophysics is, including white dwarfs (WDs), supernovae (SNe), and other phenomena.

1.2 How is the High Energy Sky Different?

In this course, we will deal with energy in terms of electron volts (eV). A common photon energy of interest to us has an energy of 1 MeV, which is roughly equivalent to 1 million optical photons (which are thus obviously about 1 eV in energy). If we fix energy, high energy (HE) events must produce fewer photons since each is carrying more energy. In addition to smaller photon counts, HE events are rarer, so there are far fewer sources in the sky at any given moment. There are on the order of 10^5 bright X-Ray sources (detectable by Chandra), on the order of 10^3 bright gamma ray objects (detectable by Fermi), about 50 events that are bright around 100 MeV, and dozens of objects with energies of TeVs. Additionally, we observe catastrophic fast and transient events that are here and gone again.

1.3 High Energy Photons

Traditionally 0.1 keV $-$ 100 TeV constitute "high energy photons". Some basic units are presented in Table 1. Related to those, a few other relations of interest are presented in Table 2. As we'll see, the messengers of HE processes are all over the EM spectrum, particularly at radio wavebands. Additionally, neutrinos, particles, and gravity waves get in the mix, giving rise to the phrase "multi-messenger" astrophysics.

Symbol	Value	Units
h	6.626×10^{-27}	erg s
1 eV	1.602×10^{-12}	erg
1 Å	10^{-8}	cm
r_{proton}	$\sim 10^{-13}$	cm

Table 1: Essential units in HE Astrophysics

1.4 History

R ontgen discovered X-rays in 1895, earning him the Nobel prize in 1901. In 1914, Henry Moseley developed the use of spectroscopy for HE processes through Moseley's Law in 1914. A bit earlier, Victor Hess discovered cosmic rays (1912) through the use of balloon experiments. He found that there was more ionization farther from Earth, indicating a new as of yet undiscovered source of ionizing radiation, earning him the Nobel Prize in 1936. In 1962, Giacconi used sounding rockets to contribute significantly to the study of X-rays. In particular, he discovered Sco X-I. His efforts in X-Ray astronomy earned him the Nobel prize in 2002.

In 1977, the first X-ray satellite went into orbit, HEAO-I, which had a bandwidth of 0.2 keV - 10 MeV. It detected 842 discrete sources. Later HEAO-2 (later renamed Einstein) was launched with a bandwidth of 0.1 keV - 20 keV. This was the first X-ray telescope capable of focussing incoming radiation. In 1992, the Compton Gamma Ray Observatory (CRGO) was launched. It allowed gamma ray bursts to be localized. Previously, gamma ray bursts were of unknown origins. In 1999, Chandra was launched, which is also able to focus X-Rays. More recently, Fermi was launched in 2008, which is a gamma ray focusing telescope. Of particular interest is the LAT (Large Area Telescope) which allows detections of extremely high energy photons. To date, the newest satellite is NuStar, which was launched in 2012 and excels in spectroscopy of high energy sources.

1.5 X-Ray Spectra and Moseley's Law

The *continuum* in X-Ray sources is caused by Bremsstrahlung "braking radiation" (free-free). Essentially an electron scatters off a proton and emits an X-ray in the process. The *lines* in the spectra are **K-shell** emissions. This is similar to Hα emission, but on heavier elements. Essentially, an electron from a low shell is ejected, allowing a higher n electron to fall to the ground (K) shell, emitting in the X-ray.

Moseley's Law states that the shell energy scales like the atomic number to the second power. It is then analogous to the Bohr atom (K$\alpha \sim$ Hα). Transition energies are then given by

$$h\nu = 13.6\,\mathrm{eV}(Z-1)^2 \left[\frac{1}{n_f^2} - \frac{1}{n_i}^2\right] \tag{1.1}$$

$$= 13.6\,\mathrm{eV}(Z-1)^2 \left[\frac{1}{1} - \frac{1}{2^2}\right] \tag{1.2}$$

Equation (1.2) is the Kα line for a ion with atomic number Z with an $n = 1$ electron kicked out ahead of time. This law accurately reflects the energies of lines we observe in X-Ray spectra.

Symbol	Value	Units
1 erg	624	GeV
$m_e c^2$	511	keV
$m_p c^2$	931	MeV
1 keV / k	11.6×10^6	K
1 keV / c	242×10^{17} Hz	

Table 2: Additional energy scales

1.6 Detectors

There are three main ways to stop HE photons

1. Photoelectric absorption:
$$\gamma + \text{atom} \to \text{atom}^+ + e^-$$

2. Compton Scattering:
$$\gamma + e^- \to \gamma + e^-$$

   This is essentially the high energy limit of Thomson scattering which allows for energy transfer

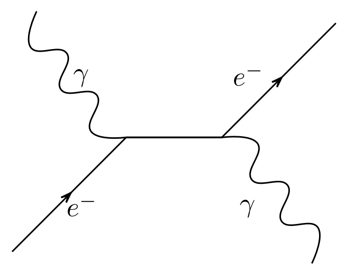

Figure 1: Schematic of Compton Scattering

3. Pair Production
$$\gamma + \text{nucleus} \to e^+ + e^- + \text{nucleus}$$

   The nucleus is required to appease conservation laws

All three of these have been used to make HE photon detectors since they can generate measurable electrons. To characterize the absorption of photons in a material through some characteristic wavelength ℓ, we'll assume there is some initial intensity I of HE photons. This intensity if attenuated according to

$$I = I_0 e^{-\left(\frac{\mu}{\rho}\right)\rho\ell} \tag{1.3}$$

where ρ is the density of the material and μ/ρ is the "mass attenuation" (similar to an opacity), given by

$$\frac{\mu}{\rho} = \underbrace{\frac{\sigma}{\rho}}_{\text{Compton}} + \underbrace{\frac{\tau}{\rho}}_{\text{photoelectric}} + \underbrace{\frac{\kappa}{\rho}}_{\text{pair production}} \tag{1.4}$$

Knowing how your detector material is sensitive to each of the HE photon stoppage sources is essential to designing an effective HE photon telescope. Now we'll discuss some different types of HE detectors.

Wednesday, April 3, 2013

Gas Proportional Counters A gas of Argon and (other stuff?) absorbs HE photons, which then give off free electrons which are collected and counted. This is essentially Geiger counter. Each interaction causes multiple ionizations, so we are able to detect individual photons.

Scintillation Counters This detector relies on a crystal with doping, providing so-called "actuator sites". Incoming photons create an electron-hole pair which gets re-radiated at the actuator site around 4200Å. This process is about 12% efficient. The produced light then interacts with a photomultiplier tube.

Solid State Solid State detectors function much like CCDs from optical astronomy, but using materials that are better suited to HE photons.

These detectors will count HE photons, but they do not provide directional information, like a camera with no telescope. So now we'll talk about different telescopes used for HE astrophysics. One big problem in such telescopes is that focussing is very difficult. In deed, HE photons will pass through mirrors unless glancing at an angle of three degrees or less. In 1952, Wolter came up with a way to effective focus X-rays using a barrel of parabolic and hyperbolic mirrors to gradually focus the X-rays through a series of many gentle reflections. As an example, NuStar will use 130 concentric mirrors to focus X-rays with energies as high as 79 keV. Chandra, XMM, HEAO-2/Einstein all work in this manner. Parabolic mirrors would focus an image onto a point, so hyperboloids are employed to focus images onto a plane, giving a field of view.

At higher energies, HE astronomers use **coded aperture masks**. These devices have a pattern of transparent and opaque partitions which will project different observed patterns at different angles. These differing patterns can be analyzed to reproduce an image. This technology is used by HETE-2, Swift, and Integral.

Additionally, we use Compton scattering and pair production devices to detect very high energy photons. Here, a gamma ray can interact with a Tungsten foil, causing pair production, which can be detected via a calorimeter. This is how the LAT instrument on Fermi works.

Energy	T_{thermal}	Process
$< 10\,\text{keV}$	$< 10^8$ K	K and L shell line emission, Bremsstrahlung, thermal blackbody
$10\,\text{keV} - 10\,\text{MeV}$	-	non-thermal processes: isomeric transitions from metastable isomers (gamma ray lines), e.g. $^{60}\text{Co} \rightarrow {}^{60}\text{Ni} + e^- + \bar{\nu}_e + 1.15\,\text{MeV}$ (present in supernova remnants)
$511\,\text{keV}$	10^9 K	$m_e c^2$ annihilation lines (seen in galactic center). E.g. $e^- + e^+ \rightarrow \gamma + \gamma$
140 MeV $-$ 10 GeV	10^{12} K	Pion decay (strong force)
$>$ 10 GeV	-	non-thermal processes: inverse Compton scattering, shock acceleration

Table 3: Different sources of high energy photons.

At the very highest energies, we must go back to the ground in order to use air showers. Such instruments include HESS, CANGAROO, and VRITAS. Here, a cosmic ray interacts with the air, moving faster than the local speed of light. This in tern, causes a cone of Cerenkov radiation. With an array of detectors, the air shower can be detected and the angle of the cone can be ascertained, giving the energy of the cosmic ray. Typically we get 100 optical photons per square meter for each 100 GeV photon.

Note that, beyond photons, HE processes produce other objects of interest. **Cosmic rays** are essentially high energy protons, electrons, nuclei, and other charged particles. Low energy neutrinos are produced through inverse beta decays ($p+e^- \rightarrow n+\nu_e$). High energy neutrinos come from… pion stuff ($\gamma + p \rightarrow n + \pi^+$, $\pi^+ \rightarrow \mu + \nu_\mu$). Finally, we can observe (ostensibly) gravity waves from rapidly changing gravitational fields.

1.7 Radioactivity

In 1898, Henri Becquerel discovered radioactivity by examining how uranium salts interacted with photographic plates. The actual term "radioactive" was coined by Marie and Pierre Curie. It was Rutherford in 1899 who first came up with the nomenclature of α, β, and γ rays, which are ordered in increasing penetration capability. Now we know that they are completely different things: helium nuclei, electrons, and… well… γ-rays.

1.8 Energetics

In stellar evolution, we learn that gravitational energy is not a primary source of energy except in early formation and late collapse phases. Even in the core collapse scenario, most of the gravitational energy is lost in the form of neutrinos. In general, then, gravitational binding energy release is unimportant to most stars most of the time. Other scales of energy release can be important though.

Chemical reactions give only a few eV per baryon. For instance 10 eV per baryon corresponds to 10^{13} erg/g. Chemical processes are not really important in astronomy as energy sources.

Nucleosynthesis, on the other hand, gives about 1 MeV per baryon, equivalent to 10^{18} erg/g. Note that the greatest gains in nucleosynthesis come from the simplest steps (like hydrogen to helium). Higher burning gives less and less energy, and eventually at Iron, there are no more energy gains to be had.

Accretion power is more efficient still since it scales as GM/R. For the sun, this corresponds to about 2×10^{15} erg/g. One proposed explanation for the energy source for the sun was gravitational contraction. If we take the gravitational energy of the sun and divide by its current luminosity, we get a lifetime on the order of 10^7 years– far too short to account for how long it has been shining on Earth. Quasi-stellar objects (QSOs), on the other hand, are powered by the accretion of matter (not nuclear powered!). We observe them at $L_{\text{QSO}} \sim 10^{47 \text{ erg/s}}$. If this was powered by nuclear reactions, there would need to be about 10^{12} $M_\odot$ of matter burned each Gyr to provide the power! Mass estimates for these objects are only as high as 10^{11}, so they are likely powered by accretion.

2 Accretion of Astrophysical Plasmas

2.1 Spherical Flow of Ionized Material onto Compact Objects

Now we will consider the spherically symmetric accretion of matter onto a compact object. This essentially ignores angular momentum, which will be included at a later time when we discuss accretion disks.

There is a fundamental limit on the rate at which matter can be accreted, called the **Eddington Limit**.

Monday, April 8, 2013

This limiting accretion rate is achieved when outward radiation pressure is perfectly balanced with inward gravity. Consider a fully-ionized medium of pure hydrogen (protons and electrons) The gravitational force for a given particle is

$$F_{\mathrm{grav}} \approx -\frac{GM(m_{\mathrm{particle}})}{r^2} \sim -\frac{GMm_p}{r^2} \tag{2.1}$$

Now we'll define a number flux of photons at a particular frequency via

$$S_\nu = \frac{L_{\mathrm{Edd}}}{4\pi r^2 h\nu} \tag{2.2}$$

We'll assume that radiative transfer is dominated by Thomson scattering, which gives an effective cross section of

$$\sigma_{\mathrm{T}} = \sum_i \frac{2}{3}\left(\frac{e_i^2}{m_i c^2}\right)^2 \tag{2.3}$$

Since the proton is ~ 2000 times more massive than an electron, the electron dominates the cross section. With this in hand, we want to find the force due to radiation pressure,

$$F_{\mathrm{rad}} = \sigma_{\mathrm{T}} S_\nu p = \frac{\sigma_{\mathrm{T}} L}{4\pi r^2 c} \tag{2.4}$$

where p is the momentum of a photon with frequency ν, namely $p = h\nu/c$. We can now solve for the Eddington Luminosity:

$$L_{\mathrm{Edd}} = \frac{4\pi G m_p c M}{\sigma_{\mathrm{T}}} \tag{2.5}$$

where we've set F_{grav} equal to F_{rad}. Parameterizing to solar masses, this gives a luminosity of

$$L_{\mathrm{Edd}} = 1.3 \times 10^{38}\left(\frac{M}{M_\odot}\right) \mathrm{erg\ s^{-1}} = 6.5 \times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot \tag{2.6}$$

This it the fundamental luminosity of a star or accreting object. Luminosities higher than this would result in mass loss. If an object shining at the Eddington luminosity radiates as a blackbody, we can find a characteristic temperature (at a given radius and mass):

$$T_{\mathrm{eff}} = \left(\frac{cGMm_p}{R^2 \sigma_{\mathrm{T}} \sigma_{\mathrm{B}}}\right)^{1/4} \tag{2.7}$$

For a neutron star, this corresponds to $kT \approx 1.9$ keV and for a white dwarf, $kT \approx 53$ eV. This is essentially the shock energy of infalling particles of accreted matter.

There is an effective accretion rate, $\dot{M}$ that corresponds to L_{Edd} if we assume that all of the luminosity is liberated at the edge of the star. The accretion luminosity is given by

$$L_{\mathrm{acc}} \approx \frac{GM\dot{M}}{R} \tag{2.8}$$

Equating this to the Eddington luminosity gives us the limiting accretion rate:

$$\dot{M}_{\mathrm{Edd}} = \frac{4\pi m_p c R}{\sigma_{\mathrm{T}}} \tag{2.9}$$

For a neutron star, this corresponds to $\dot{M} \sim 10^{-8}\, M_\odot/\mathrm{yr}$ and for a white dwarf, $\dot{M} \sim 10^{-5}\, M_\odot/\mathrm{yr}$. The accretion rate indicated by the capture rate of plasma by a central object must depend on properties at large distances as well as the mass of the central object. All other dynamics should be dictated by the background material as well as the pull of the large central mass.

We then define $\rho(\infty)$ to be the density at a sufficiently large distance away from the central object. Similarly, $c_s(\infty)$ and $T(\infty)$ are the asymptotic sound speed and temperature, respectively. We'll assume material at this sufficiently far enough distance starts at rest with respect to the bulk plasma. The capture rate can be modeled to be

$$\dot{M} = \pi r_{\mathrm{cap}}^2 v_{\mathrm{eff}} \rho(\infty) \tag{2.10}$$

where r_{cap} designates the radius to which the central mass is able to influence the background medium and v_{eff} is the effective velocity at the capture radius. If the particles are streaming by the mass, r_{cap} must be related to some typical velocity (for instance, the velocity of a passing wind). If at rest, though, r_{cap} is related to the sound speed, c_s. In fact, we know that c_s^2 is just the thermal energy per unit mass (up to a factor of two), so we may specify

$$c_s(\infty) \sim \sqrt{\frac{kT}{m_{\mathrm{H}}}} \sim 10\,\mathrm{km/s}\sqrt{\frac{T}{10^8\,\mathrm{K}}} \tag{2.11}$$

Setting the asymptotic sound speed equal to the escape velocity, $c_s \sim \sqrt{GM/r_{\mathrm{cap}}}$, we find the capture radius for static plasma to be

$$r_{\mathrm{cap}} = \frac{GM}{c_s^2(\infty)} \tag{2.12}$$

Now identifying v_{eff} and $c_s(\infty)$, we arrive at an accretion rate

$$\dot{M} \approx \frac{\pi G^2 M^2}{c_s^4(\infty)} c_s(\infty)\rho(\infty) = \frac{\pi G^2 M^2 \rho(\infty)}{c_s^3(\infty)} \tag{2.13}$$

To get some intuition, let's assume a number density of $n = 1\ \mathrm{cm}^{-3}$, which corresponds to $\rho = 1.6 \times 10^{-24}\ \mathrm{g/cm}^3$, and a sound speed of $c_s = 10^6\,\mathrm{cm/s}\sqrt{T/10^4\ \mathrm{K}}$. For a $1.4\,M_\odot$ neutron star, this give a capture rate of

$$\dot{M} = 1.8 \times 10^{11}\,\mathrm{g/s} \left(\frac{M}{1.4M_\odot}\right)^2 \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right) \left(\frac{T}{10^4\,\mathrm{K}}\right)^{-3/2} \sim 10^{-15}\, M_\odot/\mathrm{yr} \tag{2.14}$$

2.2 A More Detailed Calculation

Any treatment of accretion of plasma must satisfy the continuity equation (essentially mandating mass conservation), namely

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{2.15}$$

In a steady state, the time derivative must vanish, leaving us with simply

$$\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{2.16}$$

$\rho\mathbf{v}$ is essentially the mass flux. Its divergence is the net flux of mass per unit volume (a positive value would mean mass is accumulating at point and a negative value would mean that mass is being depleted from a point). We will assume spherically symmetric inflow ($\mathbf{v} = -v\,\hat{r}$). The continuity equation then tells us

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho v\right) = 0 \qquad \Rightarrow \qquad r^2 \rho v = \text{constant} \tag{2.17}$$

We can also write down an accretion rate given the mass flux:

$$\dot{M} = 4\pi r^2 \rho v \tag{2.18}$$

which allows us to eliminate the velocity via

$$v = \frac{\dot{M}}{4\pi r^2 \rho} \tag{2.19}$$

In addition to conserving mass, we must conserve momentum, introducing the Euler equation:

$$-\boldsymbol{\nabla} P + \mathbf{f} = \rho \frac{\partial \mathbf{v}}{dt} + (\rho \mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \tag{2.20}$$

where $\mathbf{f}$ is the external force density. Starting with the left side, we expand to get

$$-\boldsymbol{\nabla} P + \mathbf{f} = -\boldsymbol{\nabla} P - \frac{GM}{r^2}\rho\,\hat{r} \tag{2.21}$$

Now the right side of (2.20) is

$$\rho\frac{d\mathbf{v}}{dt} + \rho(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} P - \frac{GM}{r^2}\rho\,\hat{r} \tag{2.22}$$

Setting time derivatives to zero and simplifying, we get what we will refer to as our "Euler Equation":

$$\boxed{v\frac{dv}{dr} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0} \tag{2.23}$$

3 White Dwarfs

3.1 White Dwarf Structure

We'll begin by discussing some macroscopic equations governing pressure and density in white dwarfs. Recall that a white dwarf is just the degenerate core of an old star that is so dense that the pressure is due nearly entirely to electron degeneracy pressure. As a result of this, pressure is independent of temperature. Mass conservation tells us that

$$dm(r) = 4\pi r^2 \rho(r) \tag{3.1}$$

and hydrostatic balance mandates that

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{3.2}$$

We also want a relationship between the pressure P and the density ρ. We'll assume a polytropic equation of state

$$P = K\rho^\gamma = K\rho^{1+1/n} \tag{3.3}$$

where n is the polytropic index. The microscopic properties will give us the polytropic index or γ more precisely, but now we'll present a rough derivation as given by Landau.

3.1.1 Equations of State

Consider a uniform density of fermions (say, electrons). Define the number density as

$$n = \frac{3N}{4\pi R^3} \tag{3.4}$$

where R is the size of the star. The average spacing of the particles is then $\Delta x \approx n{-}1/3$. Heisenberg's uncertainty principle tells us

$$\Delta x \Delta p \sim \hbar \qquad \Rightarrow \qquad \Delta p \sim \frac{\hbar}{\Delta x} \approx \hbar n^{1/3} \tag{3.5}$$

Then the total energy, relativistically speaking, is

$$E^2 = p^2c^2 + m^2c^4 \tag{3.6}$$

If we assume $pc > mc^2$, we get an internal energy of (using our momentum relation from above)

$$E_i \approx pc = \hbar c \left(frac3N4\pi R^3\right)^{1/3} = \hbar c \left(\frac{3}{4\pi}\right)^{1/3} \frac{N^{1/3}}{R} \tag{3.7}$$

The gravitational energy per fermion of such a uniform density ball would be

$$E_g = -\frac{3GM}{5R} \tag{3.8}$$

where $M = Nm_{\text{particle}}$. The the total energy (the sum of the two) is

$$E_{\text{tot}} = E_i + E_G = \frac{!}{R}\left(\hbar c \left(\frac{3}{4\pi}\right)^{1/3} N^{1/3} - \frac{3}{5} GNm_{\text{particle}}\right) \tag{3.9}$$

We find equilibrium when there is no energy gradient, $\partial E/\partial r = 0$ when the right hand side vanishes:

$$\left(\frac{3}{4\pi}\right)^{1/3} N^{1/3}\hbar c = \frac{3}{5} G N m_{\mathrm{particle}} \tag{3.10}$$

$$N^{2/3} = \frac{\left(\frac{3}{4\pi}\right)^{1/3}\hbar c}{\frac{3}{5} G m_{\mathrm{particle}}} \tag{3.11}$$

$$N_{\mathrm{critical}} = \frac{\left(\frac{3}{4\pi}\right)^{1/2}(\hbar c)^{3/2}\left(\frac{5}{3}\right)^{3/2}}{G^{2/3} m_{\mathrm{particle}}^{3/2}} \tag{3.12}$$

$$= \left(\frac{3}{4\pi}\frac{5}{3}\right)^{1/2}\frac{5}{3}\left(\frac{\hbar c}{G m_{\mathrm{particle}}}\right)^{3/2} \tag{3.13}$$

$$= \frac{5\sqrt{5}}{6\sqrt{\pi}}\left(\frac{\hbar c}{G m_{\mathrm{particle}}}\right)^{3/2} \tag{3.14}$$

$$= 2.3 \times 10^{57} \text{ particles} \tag{3.15}$$

The corresponding mass is

$$M_{\max} = N_{\mathrm{crit}} * m_{\mathrm{particle}} = 1.05\left(\frac{\hbar c}{G}\right)^{3/2} m_{\mathrm{particle}}^{-2} \sim 1.9\, M_{\odot} \tag{3.16}$$

There have been a lot of approximations here, so now we'll do a more detailed calculation.

We'll still assume that the average spacing between particles is given by $\Delta x \sim n^{-1/3}$. For degeneracy pressure to be the dominant source of support, we must be in the regime where

$$\Delta p c \gg kT \tag{3.17}$$

We can effectively treat our white dwarfs as as zero temperature objects then, since kinetic pressure is negligible. Fermi-Dirac statistics give a number density (in phase space) of

$$\frac{dN}{d^3x\, d^3p} = \frac{g}{\hbar^3}\underbrace{\frac{1}{e^{(E-\mu)/kT}+1}}_{f(E)} \tag{3.18}$$

Here, g is a statistical weight that, for electrons is 2 due to the two spin states of the electron. E is the particle energy and μ is the chemical potential. When quantum effects are unimportant, $f(e) \approx \exp(\mu - E)/(kT)$, and the distribution function resembles that of a Maxwell-Boltzmann distribution. At lower temperatures, though, $f(E)$ resembles a step function:

$$f(E) = \begin{cases} 1 & : E \leq E_F \\ 0 & : E > E_F \end{cases} \tag{3.19}$$

where E_F is the Fermi Energy, which is the chemical potential at $T = 0$. There is a momentum associated with this energy, given by

$$E_F = \left(p_F^2 c^2 + m_e^2 c^4\right)^{1/2}. \tag{3.20}$$

Now we wish to find the number density, which requires we integrate the distribution function over all momenta:

$$\frac{N}{d^3x} = n_e = \int_0^\infty dn = \frac{2}{h^3}\int_0^{p_F} 4\pi p^2\,dp \tag{3.21}$$

This gives a number density of

$$n_e = \frac{8\pi}{3h^3}p_F^3 \tag{3.22}$$

Now solving for the Fermi momentum,

$$p_F = \left(\frac{3h^3}{8\pi}n_e\right)^{1/3} \tag{3.23}$$

Now we can relate the electron number density to the overall mass density via

$$n_e = \frac{Y_e\rho}{m_\mathrm{B}} \tag{3.24}$$

where Y_e is the number of electrons per nucleon (typically around a half for a C-O WD due to neutrons). Then the Fermi momentum is

$$p_F = \left(\frac{3h^3 Y_e\rho}{8\pi m_\mathrm{B}}\right)^{1/3} \propto \rho^{1/3} \tag{3.25}$$

Now we'll look at some limiting regimes. In the nonrelativistic case, we can assume that $p_F \ll m_e c$ and $E_F \approx m_e c^2$. For the relativistic case, $p_F \gg m_e c$ and $E_F \approx pc$. Back to our equation of state, $P = K\rho^\gamma$, we'll find that in the nonrelativistic limit, $\gamma = 5/3$, but in the relativistic regime, $\gamma = 4/3$. We'll now show where these number come from.

First, though, we need to revisit the general relation between pressure and energy density, which we'll denote as $\mathcal{E}$. The general relation is

$$P = (\gamma - 1)\mathcal{E} \tag{3.26}$$

This γ is the same as before, and is the adiabatic γ, the ratio of specific heats, $\gamma = c_P/c_V$. Now let's use this relation to investigate degeneracy in the nonrelativistic limit. The kinetic energy of a nonrelativistic electron is

$$E = \frac{1}{2}m_e v^2 = \frac{1}{2}m_e\left(\frac{p^2}{m_e^2}\right) = \frac{p^2}{2m_e} \tag{3.27}$$

Assuming a characteristic spacing distance of $\Delta x = a$, or $a = (m_p/\rho)^{1/3}$ this corresponds to

$$E \approx \frac{\hbar^2}{2m_e a^2} \tag{3.28}$$

The energy density then is

$$\mathcal{E} = E/a^3 \approx \frac{\hbar^2}{2m_e a^5} \tag{3.29}$$

Putting this in terms of the density, we get

$$\mathcal{E} = \frac{\hbar^2}{2m_e}\left(\frac{\rho}{m_{\rm B}}\right)^{5/3} \tag{3.30}$$

Here we've already shown that $\gamma = 5/3$, but we will use the pressure-energy density relation to get

$$P = \frac{\hbar^2}{3m_e}\left(\frac{\rho}{m_{\rm B}}\right)^{5/3} \tag{3.31}$$

A more careful derivation would give a pressure of

$$\boxed{P_{\rm non-rel} = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}Y_e^{5/3}\left(\frac{\rho}{m_e}\right)^{5/3}} \tag{3.32}$$

Fantastic. Now let's investigate the relativistic case, where $E \approx pc \approx hc/a$ (via Heisenberg Uncertainty Principle again). Then the energy density is

$$\mathcal{E} = \frac{E}{a^3} \approx \frac{hc}{a^4} = hc\left(\frac{\rho}{m_p}\right)^{4/3} \tag{3.33}$$

So now we see that $\gamma = 4/3$ in the relativistic case. Converting this to pressure in a more detailed calculation gives

$$\boxed{P_{\rm rel} = \frac{(3\pi^2)^{1/3}\hbar c Y_e^{4/3}}{4}\left(\frac{\rho}{m_{\rm B}}\right)^{4/3}} \tag{3.34}$$

3.1.2 Mass-Radius Relation for White Dwarfs

Linearizing the equation for hydrostatic equilibrium, (3.2), we find

$$\frac{P}{r} \sim \frac{m(r)\rho}{r^2} \sim r\rho^2 \tag{3.35}$$

Now mandating that $P \propto \rho^\gamma$, this gives us

$$\rho^\gamma \sim r^2\rho^2 \qquad \Rightarrow \qquad \rho \sim r^{2/(\gamma-2)} \tag{3.36}$$

Now assuming $M \sim \rho r^3$, we find

$$M \sim r^{\frac{3\gamma-4}{\gamma-2}} \tag{3.37}$$

So we see that as $\gamma \to 5/3$, we find

$$R \propto M^{-1/3} \tag{3.38}$$

and for the relativistic case ($\gamma = 4/3$), we find that mass is independent of radius!

3.2 The Chandrasekhar Mass

Monday, April 22, 2013

We'll only use the first two equations of stellar structure:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \qquad \frac{dM}{dr} = 4\pi r^2 \rho \tag{3.39}$$

Now we'll assume a polytropic equation of state, $P = K\rho^\gamma$ to combine these two equations and eliminate P in favor of ρ:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) + 4\pi G\rho r^2 = 0 \tag{3.40}$$

We now change variables to

$$w = \left(\frac{\rho(r)}{\rho_c}\right)^{1/n} \tag{3.41}$$

where $\gamma = 1 + 1/n$. Additionally, we'll express the radial coordinate as $r = az$ where

$$a = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right] \tag{3.42}$$

and z is a dimensionless distance. With that witchcraft behind us, eq. (3.40) becomes the **Lane-Emden Equation**:

$$\frac{1}{z^2}\left[\frac{d}{dz}\left(z^2\frac{dw}{dz}\right)\right] + w^n = 0. \tag{3.43}$$

This equation, in general, cannot be solved analytically, though we can sketch a general set of solutions. Solving the mass equation with this density solution, we get

$$M = 4\pi\rho_c \int_0^R w^n r^2\,dr = 1.437\,M_\odot\,(2Y_e)^2 \tag{3.44}$$

The radial coordinate is then

$$R = 3.347 \times 10^9 \left(\frac{\rho_c}{10^6\ \text{g/cm}^3}\right)^{-1/3} (2Y_e)^{3/2}\ \text{cm} \tag{3.45}$$

for the nonrelativistic case and

$$R = 1.122 \times 10^9 \left(\frac{\rho}{10^6\,\text{gm/cm}^3}\right)^{-1/6} (2Y_e)^{5/6}\ \text{cm} \tag{3.46}$$

for the relativistic case. We can express the nonrelativistic mass in terms of the central density or the radius via

$$M(\rho_c) = 0.4964 \left(\frac{\rho_c}{10^6\,\text{g/cm}^3}\right) 61/2\,(2Y_e)^{1/2}\ M_\odot \tag{3.47}$$

$$M(R) = 0.7011 \left(\frac{R}{10^9\,\text{cm}}\right)^{-3} (2Y_e)^5\ M_\odot \tag{3.48}$$

3.3 White Dwarf Cooling

Typically a WD isn't just a ball of degenerate matter. There is usually a very thin (by mass) layer of nondegenerate H/He that actually dominates the cooling process of the WD. This is much how the luminosity able to exit a star sets the rate at which fusion occurs at the center. Before we press on, here are some fun facts about WDs. 97% of stars will end up as WDs. That is, 97% of stars are less than 8 $M_\odot$. The peak in the WD mass function is around 0.6 $M_\odot$, so we would expect to see a lot of WDs near that mass.

So the H/He layer is an optically thick insulating layer that keeps the underlying WD hot. The interior is largely isothermal due to the efficiency of heat transport by electrons, and is supported by electron degeneracy. An understanding of the WD cooling process, we must first understand the luminosity of the WD (that is, the rate at which it is losing thermal energy since there is no nuclear energy to power the WD). Radiative diffusion tells us that

$$L = -4\pi r^2 \frac{c}{3\kappa\rho}\frac{d}{dr}\left(aT^4\right) \tag{3.49}$$

where a is the radiation constant (it's nasty, but built from fundamental constants). $(\kappa\rho)^{-1}$ is the mean free path of photons diffusing through medium, with κ being the opacity, with dimension of area per mass (essentially the cross section per unit mass). Simply carrying through the derivative, we get

$$L = -16\pi r^2 \frac{acT^3}{3\kappa\rho}\frac{dT}{dr} \tag{3.50}$$

Solving for the temperature gradient, we get

$$\frac{dT}{dr} = -\frac{3\kappa\rho}{4acT^3}\frac{L}{4\pi r^2} \tag{3.51}$$

If we assume a Kramers' Law dominates the opacity (where $\kappa \propto \rho T^{-3/2}$), we can get a more precise handle on this. For now we'll assume the following parameters

$$\kappa_0 = 4.31 \times 10^{24} Z(1+X)\,\mathrm{cm}^2\,\mathrm{g}^{-1} \tag{3.52}$$

$$\tag{3.53}$$

where $\kappa = \kappa_0 \rho T^{-3/2}$. Now, because we know where we're going, we do some bizarre math, where we divide the condition for hydrostatic equilibrium by the temperature gradient to give us dP/dT:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \qquad \Rightarrow \qquad \frac{dP}{dT} = \frac{16\pi acGMT^{-6.5}}{3\kappa_0 L\rho} \tag{3.54}$$

where we've approximated $M(r) \approx M$ in the outermost layers of the WD. We can eliminate ρ by using the relation between pressure, density, and temperature. We will assume an ideal gas equation of state to do so, meaning that $\rho = P\mu m_\mathrm{B}/(kT)$. This gives us

$$P\,dP = \frac{16\pi acGMk}{3\kappa_0 L\mu m_\mathrm{B}} T^{7.5}\,dT \tag{3.55}$$

Integrating this from the surface inwards, we get

$$\int_0^P p\,dp = \mathcal{G}\int_0^T \tau^{7.5}\,d\tau \tag{3.56}$$

where we're assuming a $P = T = 0$ boundary condition at the "surface", and $\mathcal{G}$ holds all those stupid constants. The resulting equation relating density to temperature is

$$\boxed{\rho = \left(\frac{64\pi acGM\mu m_{\mathrm{B}}}{51\kappa_0 Lk}\right)T^{13/4}} \tag{3.57}$$

Gross, right? Now at some point, the density is high enough so that the matter becomes degenerate and these approximations fail. At this point, $T_* \sim T_c$ due to the isothermal nature of degenerate material. Additionally,

$$\frac{\rho_* kT_*}{\mu_e m_{\mathrm{B}}} = 1.0\times 10^{13}\left(\frac{\rho_*}{\mu_e}\right)^{5/3} \tag{3.58}$$

where $\mu_e = 1/Y_e$. This is essentially stating that the pressure due to an ideal gas (the LHS) is equation to that from degenerate electrons (the RHS). We can solve for this critical density at which degeneracy takes over to find

$$\rho_* \approx 48\,\mathrm{g\,cm^{-3}}\left(\frac{\mu_e}{2}\right)\left(\frac{T_*}{10^6\ \mathrm{K}}\right)^{3/2} \tag{3.59}$$

Now solving for the luminosity as a function of the core temperature, we find

$$L = 1.43\times 10^{-7}\,L_\odot\,\frac{\mu}{\mu_e}\frac{1}{Z(1+X)}\left(\frac{M}{M_\odot}\right)\left(\frac{T_c}{10^6\,\mathrm{K}}\right) \tag{3.60}$$

Or more succinctly, $L = CMT_c^{7/2}$ where $CM_\odot = 2\times 10^6$ erg/s. Now we might ask what the characteristic cooling time is. What is the timescale on which the temperature will change significantly? If we have the luminosity, we should also figure out what the thermal energy of the WD is, since that is what is initially being radiated away, though crystallization energy losses eventually become important. Assuming ideal gas internal energy, the thermal energy is approximately

$$E_{\mathrm{thermal}} = \frac{3}{2}NkT_c = \frac{3}{2}\left[\frac{M}{23m_{\mathrm{B}}}\right]kT_c \tag{3.61}$$

assuming pure carbon-12 in the core. A typical energy at birth would be about $E_{\mathrm{birth}} \sim 10^{58}$ ergfor a 0.4 $M_\odot$ WD at $T_c \sim 10^8$ K. Assuming that the luminosity is the time rate of change of the thermal energy, or

$$-\frac{dE}{dt} = L \qquad \Rightarrow \qquad \frac{3}{2}\frac{km}{Am_{\mathrm{B}}}\frac{dT_c}{dt} = CMT_c^{7/2} \tag{3.62}$$

(where A is the average mass number of core material.) Integrating, we find

$$\frac{3}{5}\frac{k}{Am_{\mathrm{B}}}\left(T^{-5/2} - T_0^{5/2}\right) = C(t-t_0) \tag{3.63}$$

For some constant C. If we assume that at early times, $T_0 \gg T$ and defining $t_0 = 0$, we get

$$\frac{3kT^{-5/2}}{5Am_{\mathrm{B}}} \approx Ct \tag{3.64}$$

Solving for this time, we get

$$t = \frac{3kMT}{5Am_{\mathrm{B}}L} \tag{3.65}$$

Parameterizing this result, we have

$$t \approx 10^8 \text{ yrs } A^{-1} \left(\frac{M/M_\odot}{L/L_\odot} \right)^{5/7} \tag{3.66}$$