MATH 20510 NOTES

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Contents

Lebesgue Theory

Lecture 1

Lebesgue integration is a generalization of Riemann of the Riemann-Stieltjes integral. Idea is that we have some set and we can cover a set in $\mathbb{R}^n$ with $n - cells$ and then shrink the size of the $n - cells$ to decrease the size of the over-approximation.

Definition 0.1. *A family $\mathcal{A}$ of sets is called a* **ring** *if for every $A, B \in \mathcal{A}$*

1. $A \cup B \in \mathcal{A}$
2. $A \backslash B \in A$

A ring is called a ***σ -ring*** *if, for every $\{A_n\}_{n=1}^{\infty}$, then $\bigcup_1^{\infty} A_n \in \mathcal{A}$.*

Remark 0.2. *If $\mathcal{A}$ is a σ-ring and $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$.*

Definition 0.3. -

1. *ϕ is a set function defined on a ring $\mathcal{A}$ if $\phi(A) \in [-\infty, \infty] \forall A \in \mathcal{A}$.*
2. *ϕ is* ***additive*** *if $A \cap B = \emptyset, \phi(A \cup B) = \phi(A) + \phi(B)$.*
3. *ϕ is* ***countably additive*** *if $\{A_n\}_1^{\infty}$, is pairwise disjoint (if $A_i \cap A_j = \emptyset$ if $i \neq j$), then $\phi(\bigcup_{n=1}^{\infty} An) = \sum_{n=1}^{\infty} \phi(A_n)$ (assuming we don't have $\phi(A) = \pm\infty$ for any $A \in \mathcal{A}$.*

Remark 0.4. *If ϕ is additive, then*

1. *$\phi(\emptyset) = 0$ (We know $\phi(\emptyset)$ is well-defined because $\phi(A \backslash A) = \phi(\emptyset)$)*
2. *$\phi(A_1 \cup ... \cup A_n) = \sum_{k=1}^{n} \phi(A_k)$ if $A_1, ..., A_n$ are pairwise disjoint.*
3. *$\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2), \forall A_1, ..., A_2 \in \mathcal{A}$.*
4. *If ϕ is non-negative and $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$, then $\phi(A_1) \leq \phi(A_2) \leq \phi(A_3) \leq ...$*
5. *If $B \subseteq A$ and $|\phi(B)| < \infty$, then $\phi(A \backslash B) = \phi(A) - \phi(B)$.*

The proofs for all of these are exercises.

Theorem 0.5. *ϕ is countably additive on a ring $\mathcal{A}$. Suppose that $\{A_n\} \subseteq \mathcal{A}$, $A_1 \subseteq A_2 \subseteq A_3 ...$ and $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then $\phi(A_n) \to \phi(A)$ as $n \to \infty$.*
Note: $\mathcal{A}$ is not a $\sigma-$ring because the above holds for the sequence.

Proof. Set $B_1 = A_1$ and define $B_n = A_n \backslash A_{n-1}$. Note that

1. $\{B_n\}$ are pairwise disjoint
2. $A_n = \bigcup_{i=1}^{n} B_n$
3. $A = \bigcup_{i=1}^{\infty} B_i$

We have $\phi(A_n) = \phi(\bigcup_{i=1}^n B_i) = \sum_{i=1}^n \phi(B_i)$ and $\phi(A) = \sum_{i=1}^\infty \phi(B_i)$.
The conclusion clearly follows. □

Construction of the Lebesgue Measure

Definition 0.6. -

1. *An interval of R^n (empty set is an interval)is a set of points $x = (x_1, ..., x_n)$ such that $a_i \leq x_i \leq b_i$ or $a_i \leq x_i < b_i$ or $a_i < x_i \leq b_i$ or $a_i < x_i < b_i \quad \forall 1 \leq i \leq n$.*
2. *If $A \subseteq \mathbb{R}^n$ is a union of a finite number of intervals, we say that A is **elementary**.*
3. *The set of all elementary sets is denoted by $\mathcal{E}$.*
4. *If $I = \{(a, b)\}^n$ is an interval of $\mathbb{R}^n$, define **vol**$(I) = \Pi_{i=1}^n (b_i - a_i)$.*
5. *If $A_k = \bigcup_{i=1}^k I_i$ and $\{I_k\}$ are pairwise disjoint, we define **vol**$(A) = \sum_{i=1}^k vol(I_i)$*

Lecture 2

Remarks 0.7. *(Prove these)*

1. *$\mathcal{E}$ is a ring, but not a $\sigma-$ring.*
2. *$A \in \mathcal{E}$, then A can be written as a finite union of disjoint intervals.*
3. *If $A \in \mathcal{E}$, then vol(A) does not depend on the decomposition of intervals.*
4. *Vol is additive on $\mathcal{E}$ (vol($A \sqcup B$) = vol(A) + vol(B)).*

Definition 0.8. *A nonnegative additive set function ϕ on $\mathcal{E}$ is* **regular** *if the following holds: For every $A \in \mathcal{E}$, $\quad \forall \epsilon > 0 \quad \exists$ open $G \supseteq A$ and $\exists$ compact set $F \subseteq A$ s.t. $\phi(G) - \epsilon \leq \phi(A) \leq \phi(F) + \epsilon$ (we will eventually talk about sets with no open subsets).*

Remarks 0.9. -

1. *Since $F \subseteq A \subseteq G$ and ϕ non-negative, then $\phi(F) \leq \phi(A) \leq \phi(G)$.*
2. *vol is regular*

Definition 0.10. -

1. *A **countable open cover** of $E \subseteq \mathbb{R}^n$ is a countable collection of open elementary sets $\{A_n\}_{n=1}^\infty$ s.t. $E \subseteq \bigcup_{n=1}^\infty A_n$.*
2. *Define the **Lebesgue outer measure** of any set $E \subseteq \mathbb{R}^n$ by $m^\star(E) = \inf \sum_{n=1}^\infty vol(A_n)$ where we take the* inf *over all countable open covers of E.*

Remarks 0.11. -

1. *$m^\star$ is well defined for every $E \subseteq \mathbb{R}^n$*
2. *$m^\star \geq 0$*
3. *If $E_1 \subseteq E_2$, then $m^\star(E_1) \leq m^\star(E_2)$.*

Theorem 0.12. *If A is an elementary set, then $m^\star(A) = vol(A)$.*

Proof. Let $A \in \mathcal{E}$, $\epsilon > 0$. Since vol is regular, $\exists$ open elementary set $G \supseteq A$ s.t. vol(G) $\leq$ vol(A)+ϵ. Since $G \supseteq A$ and $G \in \mathcal{E}$, then $m^\star(A) \leq$ vol$(A) + \epsilon$. By the definition of the outer measure, $\exists$ a collection of open elementary sets $\{A_n\}$ s.t. $A \subseteq \bigcup_{n=1}^\infty A_n$ and $\sum_{n=1}^\infty$ vol$(A) \leq m^\star(A) + \epsilon$. Call this result 1.
Recall that since vol is regular, $\exists$ compact elementary set F s.t. $F \subseteq A$ and vol$(A) \leq$ vol$(F) + \epsilon$. Since F is compact, $F \subseteq A_1 \cup ... \cup A_n$ for some sufficiently large N. We have vol$(A) \leq$ vol$(F) + \epsilon \leq$ vol$(A_1 \cup ... \cup A_n) + \epsilon \leq \sum_{i=1}^N$ vol$(A_i) + \epsilon \leq \sum_{i=1}^\infty$ vol$(A_i) + \epsilon \leq m^\star(A) + \epsilon + \epsilon$. Call this result 2.
From results 1 and 2, we have vol$(A) - 2\epsilon \leq m^\star \leq$ vol(A) $+ \epsilon$. Since ϵ is arbitrary, we are done. □