MATH 20510 NOTES

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CONTENTS

LEBESGUE THEORY

Lecture 1

Lebesgue integration is a generalization of Riemann of the Riemann-Stieltjes integral. Idea is that we have some set and we can cover a set in $\mathbb{R}^n$ with $n-cells$ and then shrink the size of the $n-cells$ to decrease the size of the over-approximation.

Definition 0.1. *A family $\mathcal{A}$ of sets is called a* **ring** *if for every $A, B \in \mathcal{A}$*

(1) $A \cup B \in \mathcal{A}$
(2) $A \backslash B \in A$

A ring is called a σ ***-ring*** *if, for every $\{A_n\}_{n=1}^{\infty}$, then $\bigcup_1^{\infty} A_n \in \mathcal{A}$.*

Remark 0.2. *If $\mathcal{A}$ is a σ-ring and $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$.*

Definition 0.3. -

(1) ϕ is a set function defined on a ring $\mathcal{A}$ if $\phi(A) \in [-\infty, \infty] \forall A \in \mathcal{A}$.
(2) ϕ is ***additive*** *if $A \cap B = \emptyset, \phi(A \cup B) = \phi(A) + \phi(B)$.*
(3) ϕ is ***countably additive*** *if $\{A_n\}_1^{\infty}$, is pairwise disjoint (if $A_i \cap A_j = \emptyset$ if $i \neq j$), then $\phi(\bigcup_{n=1}^{\infty} An) = \sum_{n=1}^{\infty} \phi(A_n)$ (assuming we don't have $\phi(A) = \pm\infty$ for any $A \in \mathcal{A}$.*

Remark 0.4. *If ϕ is additive, then*

(1) $\phi(\emptyset) = 0$ (We know $\phi(\emptyset)$ is well-defined because $\phi(A \backslash A) = \phi(\emptyset)$)
(2) $\phi(A_1 \cup ... \cup A_n) = \sum_{k=1}^{n} \phi(A_k)$ if $A_1, ..., A_n$ are pairwise disjoint.
(3) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2), \forall A_1, ..., A_2 \in \mathcal{A}$.
(4) If ϕ is non-negative and $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$, then $\phi(A_1) \leq \phi(A_2) \leq \phi(A_3) \leq ...$
(5) If $B \subseteq A$ and $|\phi(B)| < \infty$, then $\phi(A \backslash B) = \phi(A) - \phi(B)$.

The proofs for all of these are exercises.

Theorem 0.5. *ϕ is countably additive on a ring $\mathcal{A}$. Suppose that $\{A_n\} \subseteq \mathcal{A}$, $A_1 \subseteq A_2 \subseteq A_3 ...$ and $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then $\phi(A_n) \to \phi(A)$ as $n \to \infty$.*
Note: $\mathcal{A}$ is not a $\sigma-$ring because the above holds for the sequence.

Proof. Set $B_1 = A_1$ and define $B_n = A_n \backslash A_{n-1}$. Note that

(1) $\{B_n\}$ are pairwise disjoint

(2) $A_n = \bigcup_{i=1}^n B_n$
(3) $A = \bigcup_{i=1}^\infty B_i$

We have $\phi(A_n) = \phi(\bigcup_{i=1}^n B_i) = \sum_{i=1}^n \phi(B_i)$ and $\phi(A) = \sum_{i=1}^\infty \phi(B_i)$.
The conclusion clearly follows. □

Construction of the Lebesgue Measure

Definition 0.6. -

1. *An interval of R^n (empty set is an interval)is a set of points $x = (x_1, ..., x_n)$ such that $a_i \leq x_i \leq b_i$ or $a_i \leq x_i < b_i$ or $a_i < x_i \leq b_i$ or $a_i < x_i < b_i \quad \forall 1 \leq i \leq n$.*
2. *If $A \subseteq \mathbb{R}^n$ is a union of a finite number of intervals, we say that A is **elementary**.*
3. *The set of all elementary sets is denoted by $\mathcal{E}$.*
4. *If $I = \{(a, b)\}^n$ is an interval of $\mathbb{R}^n$, define **vol**$(I) = \Pi_{i=1}^n (b_i - a_i)$.*
5. *If $A_k = \bigcup_{i=1}^k I_i$ and $\{I_k\}$ are pairwise disjoint, we define **vol**$(A) = \sum_{i=1}^k vol(I_i)$*

Lecture 2

Remarks 0.7. *(Prove these)*

1. *$\mathcal{E}$ is a ring, but not a $\sigma-$ring.*
2. *$A \in \mathcal{E}$, then A can be written as a finite union of disjoint intervals.*
3. *If $A \in \mathcal{E}$, then vol(A) does not depend on the decomposition of intervals.*
4. *Vol is additive on $\mathcal{E}$ (vol$(A \sqcup B)$ = vol(A) + vol(B)).*

Definition 0.8. *A nonnegative additive set function ϕ on $\mathcal{E}$ is* **regular** *if the following holds: For every $A \in \mathcal{E}$, $\quad \forall \epsilon > 0 \quad \exists$ open $G \supseteq A$ and $\exists$ compact set $F \subseteq A$ s.t. $\phi(G) - \epsilon \leq \phi(A) \leq \phi(F) + \epsilon$ (we will eventually talk about sets with no open subsets).*

Remarks 0.9. -

1. *Since $F \subseteq A \subseteq G$ and ϕ non-negative, then $\phi(F) \leq \phi(A) \leq \phi(G)$.*
2. *vol is regular*

Definition 0.10. -

1. *A **countable open cover** of $E \subseteq \mathbb{R}^n$ is a countable collection of open elementary sets $\{A_n\}_{n=1}^\infty$ s.t. $E \subseteq \bigcup_{n=1}^\infty A_n$.*
2. *Define the **Lebesgue outer measure** of any set $E \subseteq \mathbb{R}^n$ by $m^\star(E) = \inf \sum_{n=1}^\infty vol(A_n)$ where we take the* inf *over all countable open covers of E.*

Remarks 0.11. -

1. *$m^\star$ is well defined for every $E \subseteq \mathbb{R}^n$*
2. *$m^\star \geq 0$*
3. *If $E_1 \subseteq E_2$, then $m^\star(E_1) \leq m^\star(E_2)$.*

Theorem 0.12. *If A is an elementary set, then $m^\star(A) = vol(A)$.*

Proof. Let $A \in \mathcal{E}$, $\epsilon > 0$. Since vol is regular, $\exists$ open elementary set $G \supseteq A$ s.t. vol(G) $\leq$ vol(A)+ϵ. Since $G \supseteq A$ and $G \in \mathcal{E}$, then $m^\star(A) \leq$ vol$(A) + \epsilon$. By the definition of the outer measure, $\exists$ a collection of open elementary sets $\{A_n\}$ s.t. $A \subseteq \bigcup_{n=1}^\infty A_n$ and $\sum_{n=1}^\infty$ vol$(A) \leq m^\star(A) + \epsilon$. Call this result 1.
Recall that since vol is regular, $\exists$ compact elementary set F s.t. $F \subseteq A$ and vol$(A) \leq$ vol$(F) + \epsilon$. Since F is compact, $F \subseteq A_1 \cup ... \cup A_n$ for some sufficiently large N. We have vol$(A) \leq$ vol$(F) + \epsilon \leq$ vol$(A_1 \cup ... \cup A_n) + \epsilon \leq \sum_{i=1}^N$ vol$(A_i) + \epsilon \leq \sum_{i=1}^\infty$ vol$(A_i) + \epsilon \leq m^\star(A) + \epsilon + \epsilon$. Call this result 2.

From results 1 and 2, we have $\text{vol}(A) - 2\epsilon \leq m^\star \leq \text{vol(A)} + \epsilon$. Since ϵ is arbitrary, we are done. □

Lecture 3

Theorem 0.13. *If $E = \bigcup_{n=1}^{\infty} E_n$, then $m^\star(E) \leq \sum_{n=1}^{\infty} m^\star(E_n)$*

Proof. If $m^\star(E_n) = \infty$, for any $n \in \mathbb{N}$ then the conclusion is immediate. Otherwise, fix $\epsilon > 0$. For every $n, \exists \{A_{n,k}\}_k^\infty$ an open cover of E_n s.t. $\sum_{k=1}^{\infty}(\text{vol}(A_{n,k})) < m^\star(E_n) + \frac{\epsilon}{2^n}$. Since $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{n,k}$ is an open cover of E. Therefore, $m^\star(E) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \text{vol}(A_{n,k}) < \sum_{n=1}^{\infty}(m^\star(E_n) + \frac{\epsilon}{2^n}) = \sum_{n=1}^{\infty} m^\star(E_n) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \sum_{n=1}^{\infty} m^\star(E_n) + \epsilon$.

Since ϵ is arbitrary, we are done. □

We want not only countably sub-additivity, but countably additive. This is why we will use the Lebesgue measure.

Definition 0.14. *For any $A, B \subseteq \mathbb{R}^n$, define*

(1) $A \Delta B = (A \backslash B) \cup (B \backslash A)$ (symmetric difference)
(2) $d(A, B) = m^\star(A \Delta B)$
(3) Write $A_n \to A$ if $\lim_{n\to\infty} d(A_n, A) = 0$.

Definition 0.15. -

(1) If there is a seq $\{A_n\}$ of elementary sets s.t. $A_n \to A$, we say that A is ***finitely m-measurable*** *and write $A \in \mathsf{m}_F(m)$.*
(2) If A is the union of a countable collection of finitely m-measurable sets, we say that A is ***m-measurable*** *and write $A \in \mathsf{m}(m)$.*

Theorem 0.16. *$\mathsf{m}_F(m)$ is a $\sigma-$ring and $m^\star$ is countable additive on this class.*

Definition 0.17. *We let $m(A) := m^\star(A)$ for sets $A \in \mathsf{m}(m)$.*

Definition 0.18. *m is called the* ***Lebesgue measure****. This is only defined on Lebesgue Measurable Sets.*

Examples 0.19. -

(1) If $A \in \mathcal{E}$, then $A \in \mathsf{m}(m)$ because $\{A\} \to A$ (approaches itself).
(2) If $A \subseteq \mathbb{R}^n$ s.t. $m^\star(A) = 0$, then $A \in \mathsf{m}(m)$. $\emptyset \to A$ because $d(\emptyset, A) = m^\star(A) = 0$, so $A \in \mathsf{m}_F(m)$.
(3) For any $x \in \mathbb{R}^n$, $\{x\} \in \mathsf{m}(m)$, because $m^\star(\{x\}) = 0$.
(4) For any countable collection of points $x_1, x_2, ...$, then $\{x_1, x_2, ...\} \in \mathsf{m}(m).m(\{x_1, x_2, ...\}) = 0$ (countable additivity). This also means that $m(\{\mathbb{Q}\}) = 0$
(5) If $A \in \mathsf{m}(m)$, then $A^c \in \mathsf{m}(m)$.

Lecture 4

Definition 0.20. *$f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, \infty\}$. We say that f if* ***measurable*** *(Lebesgue measurable) if $\{x \in \mathbb{R}^n | f(x) > a\} \in \mathsf{m}(m)$ for every $a \in \mathbb{R}$, i.e., $f^{-1}((a, \infty]) \in \mathsf{m} \quad \forall a \in \mathbb{R}$ (this is pullback notation).*

Example 0.21. *f is continuous $\implies$ f is measurable because f^{-1} sends open sets to open sets.*

Theorem 0.22. *The following are equivalent:*

(1) $\{x | f(x) > a\}$ is measurable for every $a \in \mathbb{R}$

(2) $\{x|f(x) \geq a\}$ *is measurable for every* $a \in \mathbb{R}$

(3) $\{x|f(x) < a\}$ *is measurable for every* $a \in \mathbb{R}$

(4) $\{x|f(x) \leq a\}$ *is measurable for every* $a \in \mathbb{R}$

Proof. This will just be from $1 \to 2$. We can write $\{x|f(x) \geq a\} = \bigcap_{n=1}^{\infty}\{x|f(x) > a - \frac{1}{n}\}$ and we know that the right hand side is measurable from 1. Once idea that was used is that since measurable functions form a $\sigma-$ring, its closed under countable intersection. □

Theorem 0.23. *If* f *is Lebesgue measurable, then* $|f|$ *is measurable.*

Proof. Suffices to show that $\{x| \ |f(x)| < a\} \in \mathfrak{m} \quad \forall a \in \mathbb{R}$. This conclusion follows from $\{x| \ f(x) < a\} \cap \{x|f(x) > -a\}$ if $a > 0$. If $a < 0$, then $\{x| \ |f(x)| < a\} = \emptyset \implies$ measurable. □

Theorem 0.24. *Suppose* $\{f_n\}$ *is a sequence of measurable functions. Then* $g = \sup_n f_n$ *and* $h = \limsup_{n\to\infty} f_n$ *are measurable. The same is true for* inf *and* lim inf.

Proof. $\{x|g(x) > a\} = \bigcup_{n=1}^{\infty}\{x|f_n > a\}$, so g is measurable. This is true because all the sets in $\{x|f_n > a\}$ are measurable, and we are taking the countable union of them.

Let $g_m = \sup_{n\geq m} f_n$. This means that g is measurable (same reason as above). Let $h = \inf_m g_m$, so h is the infimum of measurable functions, which implies that h is measurable. □

Corollary 0.25. *(1) If* f, g *are measurable, then so are max(*f, g*) and min(*f, g*).*

(2) Define $f^+ =$ *max(*$f, 0$*) and* $f^- =$ *min(*$f, 0$*).* f^+ *and* f^- *are measurable.*

(3) If $\{f_n\}$ *is a sequence of measurable functions converging pointwise to* f*, then* f *is measurable (closed under pointwise convergence).*

Theorem 0.26. $f, g : \mathbb{R}^n \to \mathbb{R}$*, which are measurable.* $F : \mathbb{R}^2 \to \mathbb{R}$ *is continuous and* $h(x) = F(f(x), g(x))$*, then* h *is measurable.*

Remark 0.27. *This means that* $f + g$ *and* fg *are measurable.*

Note: Look at the proof for this in Rudin.

Lebesgue Integration

Definition 0.28. *Let* $f : \mathbb{R}^n \to \mathbb{R}$*. If the range of* f *is finite, we say that* f *is* ***simple***.

Example 0.29. *Let* $E \subseteq \mathbb{R}^n$*, define the* ***characteristic function*** *of* E *by*

$$\chi_E(x) = \begin{cases} 1 & \textit{if } x \in E, \\ 0 & \textit{if } x \notin E. \end{cases}$$

Suppose f is a simple function, so range(f) $= \{c_1, ..., c_n\}$. Let $E_i = \{x|f(x) = c_i\} = f^{-1}(\{c_i\})$. Then $f(x) = \sum_{i=1}^{m} \chi_{E_i}(x)c_i$.

If f is measurable, then E_i are measurable. Since E_i are measurable, then χ_E measurable. In fact, f measurable $\iff$ E_i's are measurable.

An example of a non-measurable function is a characteristic function of a non-measurable set.

Lecture 5

Theorem 0.30. *Let $f = \mathbb{R}^n \to \mathbb{R}$. There exists a sequence $\{f_n\}$ of simple functions s.t $f_n \to f$ pointwise. Moreover*

(1) If f is measurable, then $\{f_n\}$ can be chosen to be measurable.

(2) If $f \geq 0$, then $\{f_n\}$ can be chosen to be monotonically increasing.

Proof. Assume that $f \geq 0$.
Define $E_{n,i} = \{x \in \mathbb{R}^n | \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\}$ for $n = 1, 2, ..$ and $i = 1, 2, ..., n2^n$.
Define $F_n = \{x \in \mathbb{R}^n | f(x) \geq n\}$
Define $f_n = \sum_{i=1}^{n2^n} [\frac{i-1}{2^n} \chi_{E_{n,i}}] + n\chi_{F_n}$
We note that $E_{n,1}, E_{n,2}, ..., E_{n,n2^n}, F_n$ are disjoint that the union is $\mathbb{R}^n$. We also know that

$$f_n(x) = \begin{cases} \frac{i-1}{2^n} & \text{if } x \in E_{n,i} \\ n & \text{if } x \in F_n. \end{cases}$$

Note that $f_n \leq f$. Moreover, if $x \in E_{n,i}$, then $f(x) - f_n(x) \leq \frac{1}{2^n}$. Let $x \in \mathbb{R}^n$. Let $\epsilon > 0$ and $N \in \mathbb{N}$ s.t. $N > f(x)$ and $\frac{1}{2^N} < \epsilon$. Let $n \geq N$. By our choice of N, we have $x \in E_{n,i}$ for some i. It follows that $f(x) - f_n(x) \leq \frac{1}{2^n} \leq \frac{1}{2^N} < \epsilon$. Hence, $f_n \to f$ pointwise. This proves (2).

If f is measurable, then $E_{n,i} = \{x \in \mathbb{R}^n | f(x) \geq \frac{i-1}{2^n}\} \cap \{x \in \mathbb{R}^n | f(x) < \frac{1}{2^n}\}$ is measurable. This proves (1). ☐

Define $g(x) = \sum_{i=1}^{k} c_i \chi_{E_i}(x)$ where $c_i > 0$ is measurable. Let $E \in \mathsf{m}$.

Define $I_E(g) = \sum_{i=1}^{k} c_i \mathsf{m}(E_i \cap E)$ is the integral of g (g is a simple, non-negative measurable function on $\mathbb{R}^n$).

If f is measurable and non-negative, then $\int_E f\ dm = \sup I_E(g)$ =Lebesgue integral of f on E, where sup is taken over all non-negative simple measurable functions g such that $g \leq f$.

Remarks 0.31. *(1) $\int_E f\ dm$ can be ∞*

(2) $\int_E f\ dm = I_E(f)$ if f is measurable, simple, and non-negative.

Definition 0.32. *Let f be measurable, consider the two integrals $\int_E f^+$ and $\int_E f^-$. If at least one is finite, then define $\int_E f\ dm = \int_E f^+\ dm - \int_E f^-\ dm$.*

Example 0.33. *$f = 10\chi_{\mathbb{Q}}$ is simple, measurable, and non-negative. Then $\int_{\mathbb{R}} f\ dm =$* $10 \cdot \mathsf{m}(\mathbb{Q} \cap \mathbb{R}) = 10\mathsf{m}(\mathbb{Q}) = 0$

Example 0.34. *$f(x) = x$. Consider $\int_{[0,1]} f\ dm$. Note that f is not simple which implies that we use the f_n from the previous theorem, so $I_{[0,1]}(f_n) = \sum_{i=1}^{2^n} \frac{i-1}{2^n} \mathsf{m}(E_{n,i} \cap [0,1]) =$* $\sum_{i=1}^{2^n} \frac{i-1}{2^n} \cdot \frac{1}{2^n} = \frac{1}{2^{2n}} \cdot \sum_{i=1}^{2^n} (i-1) = \frac{1}{2^{2n}} (\frac{2^n(2^n+1)}{2} - 2^n) = \frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{2^n} = \frac{1}{2} - \frac{1}{2^{n+1}}$.

Lecture 6

Recall that if f is measurable and $E \in \mathsf{m}$ and at least one of $\int_E f^+ \ dm$ or $\int_E f^- \ dm$ is finite, then $\int_E f \ dm = \int_E f^+ \ dm - \int_E f^- \ dm$. If both are finite, then we say that f is **integrable** on E and $f \in \mathscr{L}$.

Remarks 0.35. *If f is measurable and $E \in \mathsf{m}$ (prove all of these for hw), then*

(1) *If $a \leq f \leq b$ for every $x \in E$ and $\mathsf{m}(E) < \infty$, then $a\mathsf{m}(E) \leq \int_E f \ dm \leq b\mathsf{m}(E)$.*

(2) *If f is bounded and $\mathsf{m}(E) < \infty$, then $f \in \mathscr{L}$(Corollary of 1).*

(3) *If f and g are integrable on E and $f(x) \leq g(x)\, \forall x \in E$, then $\int_E f \ dm \leq \int_E g \ dm$.*

(4) *If $f \in \mathscr{L}$ on E, then $cf \in \mathscr{L}$ on E, $\forall c \in \mathbb{R}$. Moreover, $\int_E cf \ dm = c\int_E f \ dm$.*

(5) *If $\mathsf{m}(E) = 0$, then $\int_E f \ dm = 0$.*

(6) *If $f \in \mathscr{L}$ on E, $A \in \mathsf{m}$, $A \subseteq E$, then $f \in \mathscr{L}$ on A.*

(7) *If f is Riemann integrable on $[a, b]$, then $f \in \mathscr{L}$ on $[a, b]$ and values agree (Lebesgue integral is more general than Riemann integrable).*

Proof of 1.

Proof. Assume $a \geq 0$. Recall that $\int_E f \ dm = \sup I_E(g)$, where sup is taken over all non-negative, simple, and measurable, g s.t. $0 \leq g \leq f$. Let $g(x) = a$, $\forall a \in \mathbb{R}^n$. Then $I_E(g) = a\mathsf{m}(E)$ and because $g \leq f$, then $\int_E f \ dm \geq I_E(g) = a\mathsf{m}(E)$.

Let g be any measurable simple function such that $0 \leq g \leq f$. Then, $g = \sum_{i=1}^k c_i \chi_{E_i}$, for some set $\{c_1, ..., c_n\}$ of real numbers and measurable disjoint sets E_i. $g(x) \leq f(x)$, $\forall x \in E$, hence $g(x) \leq b \ \forall x \in E$. Therefore, $c_i \leq b$, $\forall i$. So $I_E(g) = \sum_{i=1}^k c_i\mathsf{m}(E_i \cap E) \leq \sum_{i=1}^k b\mathsf{m}(E_i \cap E) = b\sum_{i=1}^k \mathsf{m}(E_i \cap E) = n\mathsf{m}(E)$. This means $\int_E f \ dm \leq b\mathsf{m}(E)$. □

(This is only for non-negative functions. For general case, split f into f^+ and f^-).

Theorem 0.36. -

(1) *Suppose f is measurable and non-negative. For any measurable set $A \in \mathsf{m}$, define $\phi(A) = \int_A f \ dm$. Then ϕ is countably additive on m.*

(2) *Same is true for $f \in \mathscr{L}$ on $\mathbb{R}^n$*

Note: 2 follows from 1 when considering f^+ and f^-.

Proof. Suppose $\{A_n\}$ is a sequence of measurable sets and $A_i \cap A_j = \emptyset$, $\forall i \neq j$. Let $A = \bigcup_{n=1}^\infty A_n$, we need to prove $\phi(A) = \sum_{n=1}^\infty \phi(A_n)$.

Step 1: Characteristic Functions

Suppose $f = \chi_E$ for some $E \in \mathsf{m}$. Then $\phi(A) = \int_A f \ dm = \mathsf{m}(E \cap A) = \mathsf{m}(E \cap (\bigcup_{n=1}^\infty A_n)) = \mathsf{m}(\bigcup_{n=1}^\infty (E \cap A_n))$[$E$ and A_n are pair-wise disjoint] $= \sum_{n=1}^\infty \mathsf{m}(E \cap A_n)$.

Note that $\phi(A_n) = \int_{A_n} f \ dm = \mathsf{m}(E \cap A_n)$, thus $\phi(A) = \sum_{n=1}^\infty \phi(A_n)$.

Step 2: Simple Functions

Suppose that f is measurable, non-negative, and simple where $f = \sum_{i=1}^{k} c_i \chi_{E_i}$, then $\phi(A) = \int_A f\ dm = \sum_{i=1}^{k} c_i \mathrm{m}(E_i \cap A) = \sum_{i=1}^{k} c_i \int_A \chi_{E_i}\ dm \implies \sum_{i=1}^{k} c_i (\sum_{n=1}^{\infty} \int_{A_n} \chi_{E_i}\ dm) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} (\int c_i \chi_{E_i}\ dm) = \sum_{n=1}^{\infty} \int_{A_n}\ dm = \sum_{n=1}^{\infty} \phi(A_n)$

Step 3: General

Suppose f is measurable and non-negative. Let g be a measurable, simple function such that $0 \le g \le f$. Then $\int_A g\ dm = \sum_{n=1}^{\infty} \int_{A_n} g\ dm \le \sum_{n=1}^{\infty} \int_{A_n} f\ dm = \sum_{n=1}^{\infty} \phi(A_n)$. Hence, $\phi(A) = \int_A f\ dm \le \sum_{n=1}^{\infty} \phi(A_n)$. If $\phi(A_n) = \infty$ for any n, the conclusion follows. So we assume $\phi(A_n) < \infty\ \forall n$.

Let $\epsilon > 0$. Choose a simple, measurable g such that $0 \le g \le f$ such that $\int_{A_1} g\ dm \ge \int_{A_1} f\ dm - \epsilon, ..., \int_{A_n} f\ dm - \epsilon$. Why is this possible? Because f is the sup of the functions. Therefore, $\phi(A_1 \cup ... \cup A_n) \ge \phi(A_1) + ... + \phi(A_n)$.

Lecture 7

Before, we showed that $\phi(A_1 \cup ... \cup A_n) \ge \phi(A_1) + ... + \phi(A_n)$ is true for FINITE union. We need this to be true for countable union. Since $A \supseteq (A_1 \cup ... \cup A_n),\ \forall n,\ \phi(A) \ge \sum_{n=1}^{k} \phi(A_n) \implies \phi(A) \ge \sum_{n=1}^{\infty} \phi(A_n)$.

(I believe the other direction is true by sub-additivity). □

Corollary 0.37. *If $A,\ B \in \mathrm{m}$ and $\mathrm{m}(A \backslash B) = 0$ and $B \subseteq A$, then $\int_A f\ dm = \int_B f\ dm,\ \forall f \in \mathscr{L}$.*

Theorem 0.38. *If $f \in \mathscr{L}$ on E, then $|f| \in \mathscr{L}$ on E and $|\int_E f\ dm| \le \int_A |f|\ dm$. Note: If a function is integrable, then it must be measurable.*

Proof. Let $A = \{x \in E | f(x) \ge 0\}$, $B = \{x \in E | f(x) < 0\}$. Note: $E = A \cup B$ and $A,\ B \in \mathrm{m}$. Then $\int_E |f|\ dm = \int_A |f|\ dm + \int_B |f|\ dm = \int_E f^+\ dm + \int_E f^-\ dm < \infty \implies |f| \in \mathscr{L}$. Since $f \le |f|$ and $-f \le |f|$, $\int_E f\ dm \le \int_E |f|\ dm$, $-\int_E f\ dm = \int_E -f\ dm \le \int_E |f|\ dm \implies |\int_E f\ dm| \le \int_E |f|\ dm$. □

Because Lebesgue integration is more general than Riemann integration, we can do a lot more things with it. Especially taking limits and other convergence properties.

Theorem 0.39. *(Lebesgue Monotone Convergence Theorem)*
Let $E \in \mathrm{m}$ be a measurable set, $\{f_n\}$ be a sequence of measurable functions with the following property: $0 \le f_1(x) \le f_2(x) \le ... \forall x \in E$.

Define $f(x) = \lim_{n\to\infty} f_n(x),\ \forall x \in E$ (this is well-defined). Then $\int_E f_n\ dm \to \int_E f\ dm$ as $n \to \infty$. In other words, the limit of the integral is the integral of the limit, or $\lim_{n\to\infty} \int_E f_n\ dm = \int_E \lim_{n\to\infty} f_n\ dm$.

Proof. First, if f_n is measurable, then f is measurable (Theorem 0.24). Because $\{f_n\}$ is a monotone sequence of non-negative measurable functions, $\{\int_E f_n\ dm\}$ is also a monotone sequence of extended real numbers, so $\exists \alpha \in \mathbb{R} \cup \{\infty\}$ such that $\lim_{n\to\infty} \int f_n = \alpha$. Since $f_n \le f$ for every n, it follows that $\alpha \le \int_E f\ dm$.

We just need to show the other direction.

Let $0 < c < 1$ and g be a simple, measurable function such that $0 \leq g \leq f$. For every $n \geq 1$, define $E_n = \{x \in E | f_n(x) \geq cg(x)\}$, which is clearly measurable. Since $\{f_n\}$ is increasing, $E_1 \subseteq E_2 \subseteq E_3 \subseteq ...$ Since $f_n \to f$ point-wise, $E = \bigcup_{n=1}^{\infty} E_n$.

For every n, $cg \leq f_n$ on E_n by definition, so $c\int_{E_n} g\ dm = \int_{E_n} cg\ dm \leq \int_{E_n} f_n\ dm$. As $n \to \infty$, $\int_{E_n} g\ dm \to \int_E g\ dm$ (Check this!!!). Therefore, $\alpha \geq \int_E g\ dm$ and because $c < 1$ was arbitrary, we know $\alpha \geq \int_E g\ dm$. By definition of integration, choose g close to f, then $\alpha \geq \int_E f\ dm$. □

Lecture 8

We will start with an application of the monotone convergence theorem.

Theorem 0.40. *Let $f = f_1 + f_2$, where $f_1, f_2 \in \mathscr{L}$(this means the integral exists, is well defined, and finite) on $E \in \mathsf{m}$. Then, $f \in \mathscr{L}$ on E and $\int_E f\ dm = \int_E f_1\ dm + \int_E f_2\ dm$.*

Proof. If f_1, f_2 are simple, measurable functions, then the conclusion is immediate (check this).

Assume $f_1, f_2 \geq 0$, Choose a monotonically increasing sequence of non-negative simple functions $\{g_n\}, \{h_n\}$ converging to f_1 and f_2 respectively. Let $s_n = g_n + h_n$. Then, $\int_E s_n\ dm = \int_E g_n\ dm + \int_E h\ dm$. Note: $\{s_n\}$ is a monotonic, increasing, non-negative, simply, and measurable.

Hence, by the monotone convergence theorem (MCT), $\int_E f\ dm = \lim_{n\to\infty}(\int_E s_n\ dm) = \lim_{n\to\infty}(\int_E g_n\ dm + \int_E h_n\ dm) = \int_E f_1\ dm + \int_E f_2\ dm$. Assume $f_1 \geq 0$ and $f_2 < 0$. Define $A = \{x \in E | f(x) \geq 0\}$ and $B = \{x \in E | f(x) < 0\}$. Since $f, f_1, -f_1 \geq 0$ on A and $f_1 = f + (-f_2)$, we have $\int_A f_1\ dm = \int_A f\ dm + \int_A -f_2\ dm = \int_A f\ dm - \int_A f_2\ dm \implies \int_A f\ dm = \int_A f_1\ dm + \int_A f_2\ dm$. Since $-f_1, f, -f_2 \geq 0$ on B, it follows that $\int_B -f_2\ dm = \int_B -f\ dm + \int_B f_1\ dm \implies \int_B f\ dm = \int_B f_1\ dm + \int_B f_2\ dm$.

Notice that $E = A \cup B$, where A and B are disjoint. Hence, $\int_E f\ dm = \int_A f_1\ dm + \int_B f_1\ dm + \int_A f_2\ dm + \int_B f_2\ dm = \int_E f_1\ dm + \int_E f_2\ dm$. Note: We are not allowed to split f into f^+ and f^-.

We want this to be true for all possible combinations of functions or

$E_1 = \{x \in E | f_1(x), f_2(x) \geq 0\}$, we did this.

$E_2 = \{x \in E | f_1(x) \geq 0,\ f_2(x) < 0\}$, we did this.

$E_3 = \{x \in E | f_1(x) < 0,\ f_2(x) \geq 0\}$, follows from E_2 by swapping f_1 and f_2.

$E_4 = \{x \in E | f_1(x), f_2(x) < 0\}$, E_1, but multiply by -1.

□

Lemma 0.41. *Fatoü's Lemma: Let $E \in \mathsf{m}$, $\{f_n\}$ sequence of non-negative and measurable functions. Let $f = \liminf_{n\to\infty} f_n$, then $\int_E f\ dm \leq \liminf_{n\to\infty} \int_E f_n\ dm$. (Recall $\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$)*

Before we move on to the proof, let make sure this statement makes sense. For the integral of f to exists, then f must be measurable, which follows from Theorem 0.24.

Proof. For ever $n \in \mathbb{N}$, define $g_n = \inf_{m\geq n} f_n$, where the g_ns are clearly measurable. Notice 3 things about the g_ns.

(1) $0 \leq g_1 \leq g_2 \leq ...$,

(2) $g_n \leq f_n$, $\forall n$,

(3) $\lim_{n\to\infty} g_n = f(x)$, $\forall x \in E$.

By the MCT, $\lim_{n\to\infty} \int g_n \ dm = \int f \ dm$. By 2, we know that $\int_E g_n \ dm \leq \int_E f_n \ dm$. Thus $\int_E f \ dm \leq \liminf_{n\to\infty} \int_E f_n \ dm$ □

One issue with Fatoü's Lemma are the restrictions on sequence. We can replace a lot of the conditions in Fatoü's lemma with the existence of a dominant function to f.

Theorem 0.42. *Dominated Convergence Theorem: Suppose $E \in \mathfrak{m}$ and $\{f_n\}$ be a sequence which each f_n is measurable on E such that $f_n \to f$ point-wise on E. Suppose $\exists g \in \mathscr{L}$ on E such that $|f_n(x)| \leq g(x)\ \forall x \in E$. Then $\int_E f \ dm = \lim_{n\to\infty} \int_E f_n \ dm$.*

Before we start the proof, lets make some notes. $\int_E f_n \ dm$ is valid because $|f_n|$ bounded bounded by $g(x)$. The same is true for f $(f_n, f \in \mathscr{L})$.

Proof. Since $f_n+g \geq 0\ \forall n$, we apply Fatoü's Lemma, so $\int_E (f_n+g) \ dm \leq \liminf_{n\to\infty} \int_E (f_n+g) \ dm \implies \int_E f \ dm + \int_E g \ dm \leq \liminf_{n\to\infty}(\int_E f_n \ dm + \int_E g \ dm) = \liminf_{n\to\infty}(\int_E f_n \ dm) + \int_E g \ dm \implies \int_E f \ dm \leq \liminf_{n\to\infty}(\int_E f_n \ dm)$. Since $g - f_n \geq 0$, we apply Fatoü's Lemma to obtain $\int_E (g-f) \ dm \leq \liminf_{n\to\infty}(\int_E (g-f_n) \ dm) \implies -\int_E f \ dm \leq \liminf_{n\to\infty}(-\int_E f_n \ dm) \implies \int_E f \ dm \geq \limsup_{n\to\infty}(\int_E f_n \ dm) \implies \int_E f \ dm = \lim_{n\to\infty}(\int_E f_n \ dm)$. □

Lecture 9

In this lecture, we will construct Vitali sets, which are non-measurable sets. For this, we need to assume the axiom of choice.

For every $a \in [-1,1]$, define $\tilde{a} = \{c \in [-1,1] | a - c \in \mathbb{Q}\}$. The way to visualize this is that we pick any point in the interval $[-1,1]$, and then we shift these points to rationals. Its unclear where this interval is, but we know its in $[-2,2]$ and that it has length 2. First, we will check if these form an equivalence relation.

Claim 1. *$\tilde{a} \cap \tilde{b} \neq \emptyset$, then $\tilde{a} = \tilde{b}$.*

Proof. Suppose $c \in \tilde{a} \cap \tilde{b}$, then $b - a \in \mathbb{Q}$ (because $(b-c)+(a-b)$). Let $d \in \tilde{a}$, then we want to show that $d \in \tilde{b}$, which means that $a - d \in \mathbb{Q}$. Since $a - d = (a-b)+(b-d) \implies (b-d) \in \mathbb{Q} \implies d \in \tilde{b}$, then we are done. □

Note that $[-1,1] = \bigcup_{a\in[-1,1]} \tilde{a}$. Let V be a set that contains exactly 1 element from each distinct $\tilde{a}$ (by the axiom of choice).

Let $r_1, r_2, ...$ be an enumeration of $\mathbb{Q} \cap [-2,2]$.

Claim 2. $[-1,1] \subseteq \bigcup_{k=1}^{\infty} V + r_k$.

Proof. $a \in [-1,1]$, then $\exists$(unique)$v \in V$ such that $v \in \tilde{a}$. Hence, $a - v \in \mathbb{Q} \cap [-2,2]$ and so $a - v = r_k$ for some k. In other words, $a = v + r_k \in V + r_k$. If we do this for every a, then the conclusion follows. □

By Claim 2, $2 = \mathsf{m}^\star([-1,1]) \leq \mathsf{m}^\star(\bigcup_{k=1}^{\infty} V+r_k) \leq \sum_{k=1}^{\infty} \mathsf{m}^\star(V+r_k) = \sum_{k=1}^{\infty} \mathsf{m}^\star(V)$(This last step is true because, as we proved in hw, measure is preserved under translation). This means $m^\star(V) > 0$.

Claim 3. *$V + r_1, V + r_2, \ldots$ are disjoint.*

Proof. If $d \in V + r_k \cap V + r_m$, then $d = v + r_k = v' + r_m$. This means that $v - v' \in \mathbb{Q}$ because $v - v' = r_m - r_k$, where $r_m, r_k \in \mathbb{Q}$. It also means $\tilde{v} = \tilde{v}'$ from Claim 1, which is a contradiction. □

Now, for every $n \in \mathbb{N}$ $\bigcup_{k=1}^{n} V + r_k \subseteq [-3, 3]$, so $\mathsf{m}^{\star}(\bigcup_{k=1}^{\infty} V + r_k) \leq 6$ for every $n \in \mathbb{N}$. We have choice over n, so we can choose n such that $n\mathsf{m}^{\star}(V) > 6$, then

$$\sum_{k=1}^{n} \mathsf{m}^{\star}(V + r_k) = n\mathsf{m}^{\star}(V) > 6 \geq \mathsf{m}^{\star}(\bigcup_{k=1}^{n} V + r_k).$$

Notice that this violates sub-additivity, implying that $V + r_1, V + r_2, \ldots$ cannot be measurable, which is a contradiction.

Fat Cantor sets.

Where instead of removing chunks that have length $\frac{1}{3}$, remove chunks of length 2ϵ, so like $[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ can be removed.

$\mathscr{L}^2$ functions.

Definition 0.43. *Let $E \in \mathsf{m}$, f be measurable. We write $f \in \mathscr{L}^2$ on E if $\int_E |f|^2 \, dm < \infty$.*

(Recall that if $f \in \mathscr{L} \implies \int_E |f| \, dm < \infty$)

Examples 0.44. *$f \in \mathscr{L}$, but $f \notin \mathscr{L}^2$. $E = (0, 1]$, $f(x) = x^{\frac{-1}{2}}$ which is integrable ($f \in \mathscr{L}$), but that $f \notin \mathscr{L}^2$ because $\int_0^1 |\frac{1}{x}| \, dm$ doesn't exist.*

$E = [1, \infty)$, $f(x) = \frac{1}{x}$, then $f \in \mathscr{L}^2$, but $f \notin \mathscr{L}$.

Theorem 0.45. *Let $E \in \mathsf{m}$, $\mathsf{m}(E) < \infty$, then $f \in \mathscr{L}^2$ on $E \implies f \in \mathscr{L}$.*

Fourier Series

Lecture 10

Intuition and History: Fourier wanted to prove that all functions can be represented as infinite sines and cosines. We want to represent functions as sines and cosines. Some questions we need to ask are "When can we do this?" and "What does it allow us to do?".

Let $f : \mathbb{R} \to \mathbb{C}$, where $f = f_{RE} + i f_{IM}$ and $f_{RE}, f_{IM} : \mathbb{R} \to \mathbb{R}$.

In this chapter, we will use Riemann Integrals, where $f \in R \implies f$ is Riemann integrable.

$f \in R$ if f_{RE} and $f_{IM} \in R$ and $\int_{-\infty}^{\infty} f \, dx = \int_{-\infty}^{\infty} f_{RE} \, dx + \int_{-\infty}^{\infty} f_{IM} \, dx$.

Definition 0.46. *A trigonometric polynomial is a function*

$$f(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx) = \sum_{-N}^{N} c_n e^{inx},$$

where $a_0, \ldots, a_N, b_1, \ldots, b_N, c_{-N}, \ldots, c_N \in \mathbb{C}$.

Note: We will be using Eulers formula a lot, which states $e^{ix} = \cos(x) + i\sin(x)$.
Note: The last equality makes sense when $a_i = c_i + c_{-i},\ b_i = (c_1 + c_{-1})i$.
We will be discussing:

(1) functions defined on finite intervals $[a, b]$, $(|a, b| < \infty)$ of length 2π,

(2) 2π periodic functions.

Definition 0.47. *Let $f \in R$ on $[a, a + 2\pi]$, $n \in \mathbb{N}$.*

*(1) The **nth Fourier coefficient of f** is*

$$\hat{f}(n) = \frac{1}{2\pi}\int_a^{a+2\pi} f(x)e^{-inx}dx.$$

*(2) The **Fourier series of f** is given by*

$$f \sim \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

(we use $\sim$ to not make any claims about convergence)

*(3) The **Nth partial sum of the Fourier series of f** is*

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}.$$

The question is does $S_N(f) \to f$?

Recall: If $n \in \mathbb{Z} - \{0\}$, then $\frac{d}{dx}\frac{e^{inx}}{in} = e^{inx}$, therefore

$$\frac{1}{2\pi}\int_a^{a+2\pi} e^{inx}dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Note that this works because e^{inx} is periodic in $\mathbb{C}$.

Example 0.48. *Suppose $f(x) = \sum_{n=-N}^{N} c_n e^{inx}$. Let $|m| \leq N$, then*

$$\begin{aligned} \hat{f}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{n=-N}^{N} c_n e^{inx})e^{-imx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{n=-N}^{N} c_n e^{ix(n-m)})dx \\ &= \frac{1}{2\pi}\sum_{n=-N}^{N} c_n \int_{-\pi}^{\pi} e^{ix(n-m)}dx \\ &= \frac{1}{2\pi}(c_m 2\pi) = c_m. \end{aligned}$$

Note that for the last equality, $m = n \implies \int = 1,\ m \neq n \implies \int = 0$.

If we let $|m| > N$, then $\hat{f}(m) = 0$. Hence, $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(x)e^{inx} = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = S_n(x)$.

This begs the question, in what sense does $S_N(x) \to f$ as $n \to \infty$?

There is a large class of functions which converge uniformly.

Example 0.49. *$f(x) = x$ on $[-\pi, \pi]$. $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x dx$.*

For the case when $n \neq 0$,

$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{inx}dx = \frac{1}{2\pi}[\frac{xe^{-inx}}{in}]_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} e^{-inx}dx = \frac{1}{2\pi}[\frac{xe^{-inx}}{in}]_{-\pi}^{\pi}$. *Using Integration by Parts.*

Fourier series of f is

$$\sum_{n\neq 0} \frac{(-1)^{n+1}}{in}e^{inx} = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n}.$$

Overview:

(1) Let $f \in R$ on $[a, a + 2\pi]$, does $S_N(f) \to f$? No, this is way to strong. To see why this is wrong, take a function where this is true, then change a point, so its still integrable, but convergence fails.

(2) What about if f is continuous and periodic? No, by Pubois-Raymond counter example.
(3) What if $f \in C^1$? Yes, this is true.
(4) We can ask the same questions for other types of convergence, like mean-square convergence.

The big theorem for this unit is that if $f \in \mathscr{L}^2$, then f has mean-square convergence.

Lecture 11

Our motivating question is if f is 2π periodic, when can we prove that $S_N(f) \to f$ pointwise? Uniformly?

Theorem 0.50. *Suppose f is integrable ($f \in R$) on $[0, 2\pi]$, f 2π periodic, and $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$ at which f is continuous.*

Corollary 0.51. *If f is continuous, 2π periodic, and $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$, then $f = 0$.*

Corollary 0.52. *If f, g are continuous, 2π periodic, and $\hat{f}(n) = \hat{g}(n)$ $\forall n \in \mathbb{Z}$, then $f = g$ ($f - g = 0$).*

Corollary 0.53. *Suppose f is continuous, 2π periodic, and that the Fourier series of f converges absolutely, i.e.*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)e^{inx}| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)||e^{inx}| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty,$$

then $\lim_{N\to\infty} S_N(f)(x) = f(x)$ uniformly.

Proof. Since $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$, the partial sums $S_N(f)$ converge uniformly. Define $g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} = \lim_{N\to\infty}\sum_{-N}^{N} \hat{f}(n)e^{inx}$ ($\hat{f}(n)$ is constant and e^{inx} is continuous). If we have a sequence of continuous functions converging uniformly, then the function they converge to is continuous, thus g is continuous. Moreover, $\forall n \in \mathbb{Z}$, $\hat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi}(\sum_{m=-\infty}^{\infty} \hat{f}(m)e^{imx})e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi}(\sum_{m=-\infty}^{\infty} \hat{f}(m)e^{ix(m-n)})dx$.
We can swap the integral and the sum if we have uniform limits, thus $= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(m)e^{ix(m-n)}dx =$

$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \hat{f}(m) \int_0^{2\pi} e^{ix(m-n)} dx = \hat{f}(n)$. We have shown that $\hat{f}(n) = \hat{g}(n)$, which implies that $f = g$. □

Lemma 0.54. *Suppose f is C^2 and 2π periodic, then $\exists c > 0$ such that for all sufficiently large $|n|$, then*

$$|\hat{f}(n)| \leq \frac{c}{|n|^2}.$$

The intuition for this is the more smoothness of a function, then the faster the decay of the fourier coefficients.

Note that the result means that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq \sum_{n=-\infty}^{\infty} \frac{c}{|n|^2} < \infty.$$

Proof. (Integration by Parts)
By using integration by parts twice we obtain

$$\begin{aligned}
2\pi \hat{f}(n) = \int_0^{2\pi} f(x) e^{-inx} dx &= -f(x) \frac{e^{-inx}}{in}\Big|_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} f'(x) e^{-inx} dx \\
&= \frac{1}{in} \int_0^{2\pi} f'(x) e^{-inx} dx \\
&= \frac{1}{in} [-f'(x) \frac{e^{-inx}}{in}]_0^{2\pi} + \frac{1}{(in)^2} \int_0^{2\pi} f''(x) e^{-inx} dx \\
&= \frac{-1}{n^2} \int_0^{2\pi} f''(x) e^{-inx} dx.
\end{aligned}$$

Call $2\pi \hat{f}(n) = \frac{-1}{n^2} \int_0^{2\pi} f''(x) e^{-inx} dx$ result 1.

We know that

$$\frac{-1}{n^2} \int_0^{2\pi} f''(x) e^{-inx} dx \leq \frac{1}{|n|^2} \int_0^{2\pi} |f''(x)||e^{-inx}| dx = \frac{1}{|n|^2} \int_0^{2\pi} |f''(x)| dx.$$

Because this (what is this???) is continuous and periodic, it follows that it is bounded.

Let

$$\frac{1}{|n|^2} \int_0^{2\pi} |f''(x)| dx \leq \frac{1}{|n|^2} 2\pi c,$$

where $c = \max_{x \in [0, 2\pi]} |f''(x)|$. It follows that $|\hat{f}(n)| \leq \frac{c}{|n|^2}$ $\forall n \neq 0$. □

If $f \in C^3$, then $|\hat{f}(n)| \leq \frac{c}{|n|^3}$. In fact, this generalizes. An important result from this is that if f is C^1, then $\hat{f'}(n) = in\hat{f}(n)$.

Lecture 12

Theorem 0.55. *Let f be a complex-valued, 2π periodic, Riemann integrable function. Then $\lim_{N \to \infty} \int_0^{2\pi} |f(n) - S_N f(x)|^2 dx = 0$.*

Note that this type of convergence is called mean-square or $\mathscr{L}^2$ convergence.

Definition 0.56. *A **vector space** is a set of vectors V equipped with operations $+$ and $\cdot$ such that $\forall x, y, z \in V$ and $\forall \lambda_1, \lambda_2 \in \mathbb{C}$.*

(1) $x + y \in V$

(2) $x + y = y + x$

(3) $x + (y + z) = (x + y) + z$

(4) $\lambda_1 x \in V$

(5) $\lambda_1(x + y) = \lambda_1 x + \lambda_1 y$

(6) $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$

(7) $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x.$

In addition, there exists 0 and 1 such that

(1) $x + 0 = 0 + x = x$

(2) $\exists(-x) \in V$ *such that* $x + (-x) = 0$

(3) $1 \cdot x = x \cdot 1 = x$

As great as this is, one thing we need to do is define the notion of orthogonality.

Definition 0.57. *An **inner product** on a vector space V associates to every pair of vectors $x, y \in V$ a number $(x, y) \in \mathbb{C}$ satisfying the following properties,*

(1) $(x, y) = (\bar{y, x})$

(2) $(\alpha x + \beta y, z) = \alpha(x, z) = \beta(y, z)$

(3) $(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$

(4) $(x, x) \geq 0$

Definition 0.58. *Given an inner product, we can define a **norm** by*

$$||x|| = \sqrt{(x, x)}$$

Definition 0.59. *An inner product is **strictly greater positive definite** if $||x|| = 0 \implies x = 0$*

Definition 0.60. *We say x and y are **orthogonal** if $(x, y) = 0$ and we write $x \perp y$.*

Example 0.61. $\mathbb{C}$: *define the **standard inner product** by (dot product in $\mathbb{R}^n$)*

$$(z, w) = z\bar{w}.$$

Example 0.62. *Let R be a set of complex values, 2π periodic, Riemann integrable functions. This is a vector space over $\mathbb{C}$ with*

$$(f + g)(x) = f(x) + g(x) \text{ and } (\lambda f)(x) = \lambda f(x)$$

and inner product

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\bar{g}(x)dx.$$

This induces a norm

$$||f|| = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx}$$

Vector space V over $\mathbb{C}$ with inner product has three properties.

(1) Pythagorean Theorem: If $x \perp y$, then $||x+y||^2 = ||x||^2 + ||y||^2$.

(2) Cauchy-Shwarz inequality: For any $x, y \in V$, then $|(x,y)| \leq ||x|| \cdot ||y||$.

(3) Triangle inequality: For any $x, y \in V$, $||x+y|| \leq ||x|| + ||y||$.

Let $e_n(x) = e^{inx}$. the family $\{e_n\}_{n\in\mathbb{Z}}$ is **orthonormal**, that is

$$(e_n, e_m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

(this uses the inner product defined in the example above)

This is like the standard basis vectors. If $n \neq m$, then $e_n \perp e_m$.
Note: $\forall f \in R$, $\forall n \in \mathbb{Z}$,

$$(f, e_n) = \frac{1}{2\pi}\int_0^{2\pi} f \cdot \bar{e}_n(x) dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx} dx = \hat{f}(n).$$

Observation: $S_N(f) = \sum_{n=-N}^{N} \hat{f}(n)e_n$. Since $\{e_n\}$ are orthonormal, $(f - \sum_{n=-N}^{N} \hat{f}(n)e_n) \perp e_m$, $\forall m \leq N$. Note that

$$f - \sum_{n=-N}^{N} \hat{f}(n)e_n = f - \sum_{n=-N}^{N} (f, e_n)e_n.$$

This means that

$$\begin{aligned}(f - \sum_{n=-N}^{N} (f, e_n)e_n, e_m) &= (f, e_m) - (\sum_{n=-N}^{N} (f, e_n)e_n, e_m) \\ &= (f, e_m) - \sum_{n=-N}^{N} (f, e_n)(e_n, e_m) \\ &= (f, e_m) - (f, e_m) \\ &= 0.\end{aligned}$$

Lecture 13

Recall

(1) R is a set of $2\pi-$periodic, (Riemann) integrable functions

(2) $(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)} dx$

(3) $e_n = e^{inx}$ and $(f, e_n) = \hat{f}(n)$ since $\{e_n\}$ is pairwise orthogonal

(4) $(f - S_N(f)) \perp e_m$ $\forall |m| \leq N$

Corollary 0.63. *$\forall$ $\{e_n\}_{n=-N}^{N}$, we have $(f - S_N(F)) \perp \sum_{n=-N}^{N} c_n e_n$.*

Some consequences are $f = f - S_N(f) + S_N(f)$, so by the Pythagorean Theorem,

$$||f||^2 = ||f - S_N(f)||^2 + ||S_N(f)||^2.$$

Also, we have

$$\begin{aligned} ||S_N(f)||^2 &= \sum_{n=-N}^{N} ||\hat{f}(n)e_n||^2 \\ &= \sum_{n=-N}^{N} |\hat{f}(n)|^2 ||e_n|| \\ &= \sum_{n=-N}^{N} |\hat{f}(n)|^2. \end{aligned}$$

Hence, $||f||^2 = ||f - S_N(f)||^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2$.

Lemma 0.64. *(Best Approximation Lemma) Let $f \in R$. Then $||f - S_N(f)|| \leq ||f - \sum_{n=-N}^{N} c_n e_b||$, $\forall$ complex numbers $\{e_n\}_{n=-N}^{N}$*

Theorem 0.65. *(Mean-Squared Convergence Theorem) If $f \in R$, then* $\lim_{N\to\infty} \int_0^{2\pi} |f - S_N(f)|^2 dx = 0$

Proof. (1) Let $f \in R$ be continuous. By a version of Stone-Weierstass, $\forall \epsilon > 0$, there exists a trigonometric polynomial P s.t. $|f(x) - P(x)| < \epsilon \; \forall x \in [0, 2\pi] \implies$

$$||f - P|| = (\frac{1}{2\pi} \int_0^{2\pi} |f(x) - P(x)|^2 dx)^{\frac{1}{2}} < (\frac{1}{2\pi} \int_0^{2\pi} \epsilon^2)^{\frac{1}{2}} = \epsilon.$$

Let M be the degree of P, so $P = \sum_{n=-M}^{M} c_n e_n$. By the Best Approximation Lemma, $\forall N \geq M$, $||f - S_N(f)|| \leq ||f - P|| < \epsilon$. Hence, $\forall \epsilon > 0$, $\exists M$ s.t. $\forall N \geq M$, $||f - S_N(f)|| < \epsilon$.

(2) Let $f \in R$. Let $\epsilon > 0$, then $\exists$ continuous g s.t. $\sup_{x\in[0,2\pi]} |g(x)| \leq \sup_{x\in[0,2\pi]} |f(x)| = B$ and $\int_0^{2\pi} |f(x) - g(x)| dx < \epsilon^2 \implies$

$$\begin{aligned} ||f - g|| &= (\frac{1}{2\pi} \int_0^{2\pi} |f(x) - g(x)|^2 dx)^{\frac{1}{2}} \\ &= (\frac{1}{2\pi} \int_0^{2\pi} |f(x) - g(x)||f(x) - g(x)| dx)^{\frac{1}{2}} \\ &\leq (\frac{B}{\pi} \int_0^{2\pi} |f(x) - g(x)| dx)^{\frac{1}{2}} \\ &< (\frac{B}{\pi}\epsilon^2)^{\frac{1}{2}} \\ &= \frac{\sqrt{B}}{\sqrt{\pi}}\epsilon. \end{aligned}$$

Since g is continuous, $\exists$ a trigonometric polynomial P s.t. $||g - P|| < \epsilon$. Thus, by triangle inequality

$$||f - P|| \leq ||f - g|| + ||g - P|| < \sqrt{\frac{B}{\pi}}\epsilon + \epsilon = \epsilon(1 + \sqrt{\frac{B}{\pi}}).$$

By the Best Approximation Lemma, $\forall N \geq \deg(P)$,

$$||f - S_N(f)|| < \epsilon(1 + \sqrt{\frac{B}{\pi}}).$$

Therefore,

$$\lim_{N\to\infty} \int_0^{2\pi} |f - S_N(f)|^2 dx = 0.$$

□

Corollary 0.66. *(Parseval's Identity) Let $f \in R$, then* $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = ||f||^2$.

Proof. $\forall N \in \mathbb{N}$, $||f||^2 \geq \sum_{n=-N}^{N} |\hat{f}(n)| dx$ by (2). By the previous theorem, $\forall \epsilon > 0$, $\exists M > 0$ s.t. $\forall N \geq M$, then $||f - S_N(f)|| < \epsilon$. Hence, by (2),

$$\sum_{n=-N}^{N} |\hat{f}(n)|^2 \geq ||f^2|| - \epsilon.$$

□

Corollary 0.67. *(Riemann-Lebesgue) If $f \in R$, then* $\lim_{|n|\to\infty} |\hat{f}(n)| = 0$.