3

Relativistic Kinematics

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3.1 Introduction

The understanding of the dynamics of particles and their interactions rests on a foundation of relativistic kinematics. Kinematic calculations allow us to determine the energies, masses, and momenta of elementary particles entering into any kind of interaction with each other or with atoms and nuclei given certain measured or known quantities. With these kinematic techniques we do not deal with the nature of the interaction itself, its strength, range, spin dependence, etc., but rather with a description of the energies and momenta of the initial and final state particles. No experiment in nuclear or particle physics could be interpreted without the techniques of relativistic kinematics. In this chapter we shall investigate these techniques and apply them to a variety of collision and decay processes. We will demonstrate how information on the nature of particles and interactions can be obtained through a study of the kinematics of the reactions. Relativistic kinematics is based on the special theory of relativity and on the conservation of energy and momentum. Its basis and elementary applications can be found in many texts.[1–4] Here we will be concerned with its application to particle reactions and decays. We will calculate energies, momenta, decay angles, decay lengths, and cross sections. Treatments relevant to particle physics are found in brief summaries in books on particle and nuclear physics[5–8] and in several monographs.[9–11] The Review of Particle Properties[12] includes a short summary of kinematics and contains useful information on properties of particles and particle interactions in materials.

3.2 Space-Time Transformations

3.2.1 The Invariant Interval

It is shown in elementary treatments of special relativity that the space-time interval $\Delta s^2 = c\Delta t^2 - \Delta x^2$ is invariant under Lorentz transformations. If $\Delta s^2 < 0$, we call the interval *spacelike*; and if $\Delta s^2 > 0$, we call the interval *timelike*. If $\Delta s^2 = 0$, then the two space-time points can be linked by a light ray and the interval is said to be *lightlike;* the two points lie on the *light cone.*

3.2.2 The Space-Time Four Vector

We may define a four element space-time quantity which transforms as a four-vector under Lorentz transformations. This four-vector is $X = (ct, x_1, x_2, x_3)$ where x_i are the three space components of the position of an event and t is the time of an event. In general we will write a four-vector $Q = (q_0, q_1, q_2, q_3)$ with the multiplication rule

$$PQ = p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3. \tag{3.1}$$

In particular, the square of the space-time four-vector is $X^2 = c^2t^2 - x_1^2 - x_2^2 - x_3^2$. The product of two four-vectors is invariant under Lorentz transformations, just as the scalar product of ordinary three-vectors is invariant under rotations in three dimensional space.

3.2.3 The Lorentz Transformation

The Lorentz transformation between two reference frames may be written in matrix notation as $X = \mathcal{L} X^*$, where X^* refers to the four-vector in a particular reference frame, usually the "center-of-mass" frame (CM system). $\mathcal{L}$ is explicitly

$$\mathcal{L} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

$$\beta = v/c \tag{3.3}$$

$$\gamma = 1/\sqrt{1 - \beta^2}, \tag{3.4}$$

where v is the relative velocity of the two frames, here assumed to be along the x_1 axis. The inverse transformation is

$$\mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.5}$$

and it is easy to see that $\mathcal{L}\mathcal{L}^{-1} = \mathcal{I}$, the identity matrix.

For relative motion along the x-axis the Lorentz transformation between two space-time four-vectors can be written as (starred variables refer to the CM frame and unstarred variables refer to the Lab frame)

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct^* \\ x^* \\ y^* \\ z^* \end{pmatrix} \tag{3.6}$$

or in the form of linear equations as

$$ct = \gamma(ct^* + \beta x^*) \tag{3.7}$$

$$x = \gamma(x^* + \beta ct^*) \tag{3.8}$$

$$y = y^* \tag{3.9}$$

$$z = z^*. \tag{3.10}$$

The components of position perpendicular to the direction of motion of the reference frames are invariant. Recall the example of a train entering a tunnel. The vertical clearance between the train and the tunnel will not change at relativistic speeds.

3.2.4 Example: Lifetime and Decay Length

This example is important in its own right and will make clear which frames we are using and the sign of the relative velocity.

A short-lived particle decays in a time τ in its own rest frame (CM system). This is the time we would measure if it were at rest in our reference frame, the Lab system. Suppose, however, it is moving relative to our Lab frame with a speed v in the positive x direction. *Where* does the decay take place in the Lab and *when* does it takes place as measured by our Lab clocks?

Call t the time of decay in Lab and call x the position of the decay in the Lab. In this case $\beta = v/c$ and the space coordinates of the particle in its rest frame are $x^* = y^* = z^* = 0$. The Lorentz transformation to the Lab system is

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\tau \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.11}$$

or written as linear equations:

$$t = \gamma\tau \tag{3.12}$$

$$x = \beta\gamma c\tau. \tag{3.13}$$

This gives us the observed lifetime and decay length for a moving particle. The lifetime as observed in the lab has been extended by a factor, γ. Of course in the Lab system, the point of decay, $x = \beta ct = vt$ as we should expect.

3.3 Energy-Momentum Transformations

An energy-momentum four-vector, P, has the ordinary three-momentum vector as the "space" components and the total energy as the fourth component, $P = (E, \vec{p}c)$. The justification for this comes from a detailed study of collisions between particles and the space-time transformations.[1,2,4] The four-vector constructed this way has the same transformation properties as the space-time four-vector.

The square of a four-momentum vector is $P^2 = E^2 - |\vec{p}|^2c^2$, where $\vec{p}$ is the three-momentum and E is the total energy. In the rest frame of a particle

$E = mc^2$ and $\vec{p} = 0$ so $P = (mc^2, 0)$ and $P^2 = E^2 = m^2c^4$. Since the square of a four-vector is invariant under Lorentz transformations we see that in any frame $m^2c^4 = E^2 - |\vec{p}|^2c^2$ so that

$$E^2 = |\vec{p}|^2c^2 + m^2c^4. \tag{3.14}$$

From now on we will drop (set equal to 1) the factors of c that appear in the above equations. The factors of c can always be restored at the end of a calculation by correcting the dimensions. This is equivalent to choosing a set of units in which $c = 1$. For example, we often speak of the "mass" of an electron being about 0.5 MeV; what we really mean is that mc^2 is about 0.5 MeV or m is 0.5 MeV/c^2. Of course, "MeV" has dimensions of energy not mass. With this choice of dimensions:

$$\beta = v \tag{3.15}$$
$$P = (E, \vec{p}) \tag{3.16}$$
$$P^2 = E^2 - |\vec{p}|^2 \tag{3.17}$$
$$E^2 = |\vec{p}|^2 + m^2 \tag{3.18}$$
$$P^2 = m^2. \tag{3.19}$$

For massless particles such as photons and neutrinos $E = |\vec{p}|$ and $P^2 = 0$. The product of two four-vectors is $P_1P_2 = E_1E_2 - \vec{p}_1 \cdot \vec{p}_2$. The product of two four-vectors and the square of a four-vector are Lorentz invariant and consequently can be evaluated in any reference frame.

3.3.1 Lorentz Transformation

The Lorentz transformation for the momentum four-vector is

$$\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E^* \\ p_x^* \\ p_y^* \\ p_z^* \end{pmatrix}. \tag{3.20}$$

Written as linear equations:

$$E = \gamma(E^* + \beta p_x^*) \tag{3.21}$$
$$p_x = \gamma(p_x^* + \beta E^*) \tag{3.22}$$
$$p_y = p_y^* \tag{3.23}$$
$$p_z = p_z^*, \tag{3.24}$$

where the starred quantities refer to the CM system and p_x is the component of momentum along the direction of motion of the CM system. The perpendicular components of momentum are the same in both reference frames. In the above equations β refers to the velocity of the CM system as seen from the Lab; γ is $1/\sqrt{1-\beta^2}$.

Important relationships for a particle of mass m moving with speed $\beta = v$ can be derived from the Lorentz transformation between the rest frame of the particle and the Lab frame. p is the component of momentum parallel to the velocity, v:

$$\begin{pmatrix} E \\ p \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix}. \tag{3.25}$$

Written as linear equations this becomes

$$E = \gamma m \tag{3.26}$$

$$p = \beta\gamma m = \beta E. \tag{3.27}$$

3.4 Collisions and Decays

3.4.1 Introduction

Center of Mass Energy

Consider a system of two particles, a and b, with masses, m_a and m_b. These could be two particles which will collide or the two decay products of an unstable particle. Their four-momenta are $P_a = (E_a, \vec{p}_a)$ and $P_b = (E_b, \vec{p}_b)$. Let's define

$$s \equiv (P_a + P_b)^2. \tag{3.28}$$

Since s is the square of a four-momentum it is invariant under Lorentz transformations. First evaluate s in the CM system of the two particles. In the CM system $\vec{p}_a^{\,*} + \vec{p}_b^{\,*} = 0$ and $|\vec{p}_a^{\,*}| = |\vec{p}_b^{\,*}|$. Then

$$s = (P_a + P_b)^2 = (E_a^* + E_b^*)^2 - (\vec{p}_a^{\,*} + \vec{p}_b^{\,*})^2 = (E_a^* + E_b^*)^2, \text{ and} \tag{3.29}$$

$\sqrt{s}$ is the total energy in the CM system.

In a reaction the magnitude of $\sqrt{s}$ determines whether new particles or excited states can be created.

Energy and Momentum in the CM system

Suppose we have a short-lived particle of mass M, which decays into two secondaries, a and b. In the rest frame of the parent particle we know the CM energy; it is just $\sqrt{s} = M$. Now, if we know the CM energy, $\sqrt{s}$, for a system of two particles we can calculate their momenta and energies. In the CM system define $p^* \equiv |\vec{p}_a^{\,*}| = |\vec{p}_b^{\,*}|$ and from conservation of energy, $M = E_a^* + E_b^* = \sqrt{s}$.

If two outgoing particles, a and b, are observed in some reaction, we can define a mass, M, which plays the role of the mass of the parent particle. This may be the mass of the actual decaying particle or of a fictitious particle.

Then $M = \sqrt{|p^*|^2 + m_a^2} + \sqrt{|p^*|^2 + m_b^2}$, rearrange and square this and solve for $E_b^* = \sqrt{|p^*|^2 + m_b^2}$. We find

$$E_b^* = \frac{M^2 + m_b^2 - m_a^2}{2M}$$

$$E_a^* = \frac{M^2 + m_a^2 - m_b^2}{2M}$$

$$p^* = \frac{\sqrt{(M^2 - (m_a - m_b)^2)(M^2 - (m_a + m_b)^2)}}{2M}. \tag{3.30}$$

Notice that when a massive particle decays into two light or massless particles, then p^* is simply related to the mass of the decaying particle. We will take advantage of this observation to determine the mass of a heavy decaying particle (such as the intermediate vector boson, $W^{\pm}$) from the angular distribution of the decay products.

CM Energy Calculated in the Lab system

We can also evaluate s in the Lab system: $s = (P_a + P_b)^2 = (E_a + E_b)^2 - (\vec{p}_a + \vec{p}_b)^2$.

As an important special case we will consider a beam particle colliding with a stationary target particle. Assume that particle b is at rest, so $\vec{p}_b = 0$, $E_b = m_b$. In this case we call a the beam particle and b the target particle; so $P_a = (E_a, \vec{p}_a), P_b = (m_b, 0)$ and $P_a^2 = m_a^2, P_b^2 = m_b^2$. Then $s = (P_a + P_b)^2 = P_a^2 + P_b^2 + 2P_aP_b = m_a^2 + m_b^2 + 2E_am_b$. Alternatively $P_a + P_b = (E_a + m_b, \vec{p}_a)$. In terms of the Lab kinetic energy $T_a = E_a - m_a$ we have

$$s = (m_a + m_b)^2 + 2T_am_b. \tag{3.31}$$

This is the CM energy written in terms of masses and Lab kinetic energy.

How fast is the CM system moving as seen from the Lab? Write the four-vector for the combined system of two particles in the CM systems and in the Lab system and write the Lorentz transformation between them.

$$\begin{pmatrix} E_a + m_b \\ p_a \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \sqrt{s} \\ 0 \end{pmatrix}. \tag{3.32}$$

Equivalently, $E_a + m_b = \gamma\sqrt{s}$ and $p_a = \beta\gamma\sqrt{s}$. So we have then

$$\beta = \frac{p_a}{E_a + m_b} \tag{3.33}$$

$$\gamma = \frac{E_a + m_b}{\sqrt{s}}. \tag{3.34}$$

In general $\beta = p/E$ and $\gamma = E/M$, where E is the total energy seen in the Lab and $M = \sqrt{s}$ is the mass of the (possibly fictitious) CM particle.

Invariant Mass

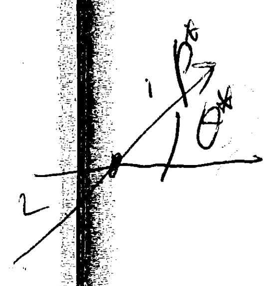
A particle, M, decays into several secondaries, $m_1, m_2, m_3, \ldots$. Square $P = P_1 + P_2 + P_3 + \ldots$ to obtain

$$M^2 = (\Sigma P_i)^2 = (\Sigma E_i)^2 - (\Sigma \vec{p}_i)^2. \tag{3.35}$$

So if we know the energies and momenta of all the decay products we can find the mass of the particle which produced them.

This is a very powerful tool for analyzing particle reactions, especially when short-lived or neutral particles decay without being directly observed. An example of the application of this is in the decay of the very short-lived neutral ρ meson, which decays via the strong interaction into two charged pions. Even if there are many pions in the final state of a reaction we can calculate the invariant mass for all pairs of pions. A distribution of this quantity which shows a peak at the mass of the ρ above a continuous background confirms the presence of a ρ meson even though it can not be observed directly.

3.4.2 Two Secondary Particles



An unstable particle of mass, M, decays into two secondary particles of masses m_1 and m_2. In the rest frame of the decaying particle $\vec{p}_1^{\,*} + \vec{p}_2^{\,*} = 0$ and $M = \sqrt{s} = E_1^* + E_2^*$. The energies and momenta of the secondary particles are given by Eq. 3.30.

Suppose the decaying particle is moving in the x direction in the Lab system with speed $\beta = v$. If a secondary particle, m_1, is produced with decay angle θ^* in the CM frame, we can calculate its momentum ($\vec{p}_1$), energy (E_1), and angle (θ), in the Lab system. The angles are measured from the direction of relative motion of the frames.

If two particles collide to produce two final state particles, we can consider this collision as producing an intermediate CM particle which then decays into the final state products. Thus collisions and decays can be treated in similar ways. Conversely a decay can be viewed as a "collision" with no target particle.

Decay Angles

We transform the energy and momentum of the secondary particle, m_1, from the CM system to the Lab system. Let $p_\perp$ represent the component of momentum perpendicular to the motion of M and $p_\parallel$ the component along the direction of motion.

$$E_1 = \gamma(E_1^* + \beta p_\parallel^*) \tag{3.36}$$

$$p_\parallel = \gamma(p_\parallel^* + \beta E^*) \tag{3.37}$$

$$p_\perp = p_\perp^*. \tag{3.38}$$

In terms of the decay angle, $\theta*$, we have

$$E_1 = \gamma(E_1^* + \beta p^* \cos\theta^*) \tag{3.39}$$

$$p\cos\theta = \gamma(p^* \cos\theta^* + \beta E^*) \tag{3.40}$$

$$p\sin\theta = p^* \sin\theta^*. \tag{3.41}$$

Divide the last two equations and define $\beta^* = p^*/E^*$, the speed of the secondary particle in the CM system:

$$\tan\theta = \frac{p^*\sin\theta^*}{\gamma(p^*\cos\theta^* + \beta E^*)} = \frac{\sin\theta^*}{\gamma(\cos\theta^* + \beta/\beta^*)} \tag{3.42}$$

$$\cot\theta = \gamma(\cot\theta^* + (\beta/\beta^*)/\sin\theta^*). \tag{3.43}$$

The maximum Lab angle occurs when $d\theta/d\theta^* = 0$.

For the special case where the secondary particle happens to be massless (a photon or neutrino), then $\beta^* = 1$ and $\cot\theta = \gamma(\cot\theta^* + \beta/\sin\theta^*)$. Notice that when $\theta^* = \pi/2$, $\cot\theta = \beta\gamma$. In this case θ can take on any value between 0 and π and there is no maximum Lab angle. An example we will look at in some detail is the decay of the neutral mesons (the π^0 is a familiar example) into two gamma rays.

Beam and Target

Suppose particle a is the beam particle with momentum $\vec{p}_a$ and the target particle with mass m_b is at rest in the Lab system. In the Lab system the total four-momentum is $P = P_a + P_b = (E_a + m_b, \vec{p}_a)$. In the center-of-mass system, $P^* = (\sqrt{s}, 0)$, where $\sqrt{s}$ is the CM energy and the total three-momentum is zero by definition. A Lorentz transformation between these frames gives $E_a + m_b = \gamma\sqrt{s}$ and $P_a = \beta\gamma\sqrt{s}$, where β is the velocity of the CM system as seen from the Lab. Then it follows that

$$\gamma = (E_a + m_b)/\sqrt{s} \quad \text{and} \quad \beta = p_a/(E_a + m_b),$$

where p_a is the magnitude of the three-momentum of particle a in the Lab.

In general the Lorentz transformation is

$$E = \gamma(E^* + \beta p_\parallel^*) \quad \text{and} \quad p_\parallel = \gamma(p_\parallel^* + \beta E^*),$$

where the starred quantities refer to the CM system and $p_\parallel$ is the component of momentum along the direction of motion of the CM system. β and γ refer to the velocity of the CM system as seen from the Lab.

Distributions of Energy and Angle

Suppose a particle of mass M decays into two secondaries. In the rest frame of the decaying particle (CM system) the energy and momenta of the secondaries are unique and depend only on the masses of the particles. These are given in Eq. 3.30. If the decaying particle is moving, then the energies of the secondaries will have some distribution in the Lab system, since they are emitted over a range of angles in the CM system. The energy and momentum of a secondary in the Lab depends on the CM decay angle relative to the direction of motion of the parent particle. We will assume that the decay in the rest frame of the parent particle is isotropic, *i.e.*, that all directions are equally likely. Isotropic means that the angular distribution is

$$dw = \frac{d\Omega^*}{4\pi} \quad \text{where} \quad d\Omega^* = d(\cos\theta^*)d\phi \tag{3.44}$$

$$S^*(\cos\theta^*, \phi)d(\cos\theta^*)d\phi = \text{const } d(\cos\theta^*)d\phi. \quad (3.45)$$

where S^* is the angular distribution in the CM system. We usually integrate over ϕ, the azimuthal angle, giving a factor of 2π, to get $S^*d(\cos\theta^*) = \frac{1}{2}d(\cos\theta^*)$.

The energy distribution in the Lab, $S(E)$, is such that $S(E)dE = S^*(\cos\theta^*)d(\cos\theta^*)$. This means that the number of events within an energy interval in the Lab is equal to the number of events within some appropriate $\cos\theta^*$ interval in the CM system. Since $S^* = \frac{1}{2}$ is a constant, we have $S(E) = \frac{1}{2}d(\cos\theta^*)/dE$. From the Lorentz transformation—$E = \gamma E^*(1 + \beta\beta^*\cos\theta^*)$—we see that there is a linear relationship between E and $\cos\theta^*$, so that $S(E)$ is also a constant within some range in E. Setting $\cos\theta^*$ to either 1 or -1 we have

$$E_{max} = \gamma E^*(1 + \beta\beta^*) \quad (3.46)$$

$$E_{min} = \gamma E^*(1 - \beta\beta^*). \quad (3.47)$$

The energy distribution, $S(E)$, is flat within these limits.

Similarly the Lab angular distribution is given by $S(\cos\theta) = d(\cos\theta^*)/d(\cos\theta)$. This derivative may be calculated from the Lorentz transformation equations (Eq. 3.41).

3.4.3 π^0 Decay

Introduction

The ability to detect neutral pions is still a detector design consideration for current and future experiments in high energy physics. Neutral pions are produced in strong interactions and are a component of hadronic showers. They are also produced through the decays of neutral and charged kaons, the eta meson, the rho meson, the tau lepton, etc. About 30 percent of the decays of the short-lived neutral kaon, K_s^0, are into two neutral pions, so their detection is of great importance in the study of CP violation. It has been known since 1957 that weak decays are not symmetric under either the parity (P) transformation or the charge conjugation (C) transformation.[5] In the decay of neutral kaons even the product of these two transformations (CP) fails to be conserved. Attempts to understand the underlying cause of this CP symmetry violation, so far seen only in kaon decays, constitute an important area of current research. There is hope that answers may emerge from more precise experiments on the kaon system soon to be carried out or from studies of neutral B mesons which include a b-quark. To pursue this latter approach a "B-factory" is under construction at Stanford.

The neutral pion decays electromagnetically into two photons ($\pi^0 \longrightarrow \gamma\gamma$) with a mean life of about 10^{-16} seconds. This very characteristic decay mode (about 99 percent of all π^0 decays) is the key to its recognition.

Detectors must be able to distinguish between the π^0 and other particles decaying into two photons and also must distinguish a true two-photon decay pattern from random photon hits. In typical high energy experiments this discrimination is often rather difficult as a consequence of the high multiplicity of tracks and the small angular cone in which decay products are usually found. The need for this kind of pattern recognition places requirements on the energy resolution and spatial resolution of the detector. We will investigate the decay pattern of the π^0 both analytically and using the computer programs and will try to establish criteria for a detector which is efficient for identifying the π^0 through its decay.

Angular Distribution of Gamma Rays

Since the final state particles are massless the transformations are slightly simpler. We need

$$\frac{d(\cos\theta^*)}{d(\cos\theta)} = \frac{\sin\theta^*}{\sin\theta}\frac{d\theta^*}{d\cot\theta^*}\frac{d\cot\theta^*}{d\theta}. \tag{3.48}$$

From the Lorentz transformation for a final state gamma ray,

$$E = p = \frac{1}{2}E_{\pi^0}(1 + \beta\cos\theta^*) \tag{3.49}$$

$$p\cos\theta = \frac{1}{2}E_{\pi^0}(\cos\theta^* + \beta). \tag{3.50}$$

Then $\cos\theta^* = (\cos\theta - \beta)/(1 - \beta\cos\theta)$. Trigonometry gives

$$\cot\theta^* = \gamma(\cot\theta - \beta/\sin\theta). \tag{3.51}$$

Then, taking the derivative: $d\cot\theta^*/d\theta = -\gamma(1 - \beta\cos\theta)/\sin^2\theta$. Finally,

$$S(\theta) = \frac{\sin\theta}{2\gamma^2(1 - \beta\cos\theta)^2} \tag{3.52}$$

$$\frac{dw}{d\Omega} = \frac{1}{4\pi\gamma^2(1 - \beta\cos\theta)^2}. \tag{3.53}$$

Decay Opening Angle

The angle between the two gamma rays from the decay of a neutral particle such as the π^0 is a particularly important parameter, as it contains information on the decaying particle and consequently becomes a design parameter for particle detectors. Define $M = m_{\pi^0}$ and let β and γ refer to the velocity of the neutral pion in the Lab. The gamma ray Lab momenta are $\vec{p}_1$ and $\vec{p}_2$, and the angle between the gamma rays (opening angle) is θ (Fig. 3.1).

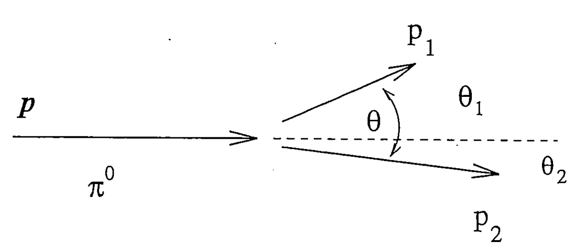


Figure 3.1: π^0 Decay.

$$M^2 = (P_1 + P_2)^2 = P_1^2 + P_2^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 \tag{3.54}$$

$$M^2 = 2E_1E_2(1 - \cos\theta) \tag{3.55}$$

$$M = 2\sqrt{E_1E_2}\sin\frac{\theta}{2}. \tag{3.56}$$

Use $E^* = |\vec{p}^*| = M/2$ for the gamma rays and write the Lorentz transformation:

$$E_1 = \frac{1}{2}\gamma M(1 + \beta\cos\theta_1^*) \tag{3.57}$$

$$E_2 = \frac{1}{2}\gamma M(1 - \beta\cos\theta_1^*). \tag{3.58}$$

Combining these with Eq. 3.56, we find

$$\sqrt{E_1E_2} = \frac{1}{2}\gamma M\sqrt{1 - \beta^2\cos^2\theta_1^*} = \frac{M}{2\sin\frac{\theta}{2}}. \tag{3.59}$$

Solve this for $\cos\theta_1^*$ to obtain

$$\cos\theta_1^* = \pm\frac{1}{\beta}\sqrt{1 - \frac{1}{\gamma^2\sin^2\frac{\theta}{2}}}. \tag{3.60}$$

The distribution of opening angles, θ, is

$$\frac{dn}{d\theta} = \frac{dn}{d\cos\theta_1^*}\frac{d\cos\theta_1^*}{d\theta}. \tag{3.61}$$

Since the distribution in the CM frame is isotropic—$dn/d(\cos\theta_1^*)$ is a constant—we just need to evaluate the derivative $d(\cos\theta_1^*)/d\theta$, which gives

$$\frac{dn}{d\theta} = \frac{1}{2\beta\gamma}\frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}\sqrt{\gamma^2\sin^2\frac{\theta}{2} - 1}}. \tag{3.62}$$

The minimum value of θ is when $\gamma\sin\frac{\theta}{2} = 1$ so that $\sin\frac{1}{2}\theta_{min} = M/E_{\pi^0}$. In the extreme relativistic limit $\theta_{min} = 2M/E_{\pi^0} = 2/\gamma$. The distribution of opening angles has an interesting shape, which can be studied using the kinematic simulation.

3.5 Compton Scattering

The Compton effect is one of the three principal modes of interaction of photons with matter and plays a key role in the experimental detection of photons. The other interactions are photo-electric effect, important at low photon energies, and pair production, which becomes dominant somewhat above the threshold for producing an electron-positron pair. The Compton effect dominates at energies well above the

K shell atomic binding energies. It is always important near 0.5 to 1 MeV but for low Z materials (beryllium, for example) it dominates even at tens of kilovolts.

In a simple model of the Compton scattering reaction,

$$\gamma + e \longrightarrow \gamma' + e' \tag{3.63}$$

we assume that the initial electron is unbound, but at rest in the Lab system. We will calculate the energies of the outgoing electron and gamma ray, (Figure 3.2). In particular, the minimum and maximum values of these energies are interesting from the point of view of understanding the response of a detector to incident photons.

Call the angle between the incoming and outgoing gamma rays θ; so $\vec{p}_\gamma \cdot \vec{p}'_\gamma = E_\gamma E'_\gamma \cos\theta$. In this case $P_\gamma^2 = 0$, $P_e^2 = m_e^2$, $\vec{p}_e = 0$, $E_\gamma = |\vec{p}_\gamma|$, and $P_e = (m_e, 0)$. Use four-momentum conservation:

$$P_\gamma + P_e = P'_\gamma + P'_e. \tag{3.64}$$

Rearrange Eq. 3.64 and square it so that the outgoing electron variables drop out.

$$P_\gamma - P'_\gamma + P_e = P'_e \tag{3.65}$$

$$(P_\gamma - P'_\gamma)^2 + 2P_e(P_\gamma - P'_\gamma) + m_e^2 = m_e^2 \tag{3.66}$$

$$-E_\gamma E'_\gamma + E_\gamma E'_\gamma \cos\theta + m_e(E_\gamma - E'_\gamma) = 0. \tag{3.67}$$

Finally, we have for the outgoing gamma ray energy:

$$E'_\gamma = \frac{m_e E_\gamma}{m_e + E_\gamma(1 - \cos\theta)}. \tag{3.68}$$

This is maximum when $\cos\theta = 1$ so that $E'_\gamma = E_\gamma$ and is minimum when $\cos\theta = -1$, so

$$E'_\gamma = \frac{m_e E_\gamma}{m_e + 2E_\gamma} \approx \frac{1}{2} m_e. \tag{3.69}$$

The kinetic energy of the outgoing electron is

$$T'_e = E_\gamma - E'_\gamma \tag{3.70}$$

$$T'_e = \frac{E_\gamma^2(1 - \cos\theta)}{m_e + E_\gamma(1 - \cos\theta)} \tag{3.71}$$

$$= \frac{2E_\gamma^2}{m_e + 2E_\gamma} \quad \text{for} \quad \cos\theta = -1. \tag{3.72}$$

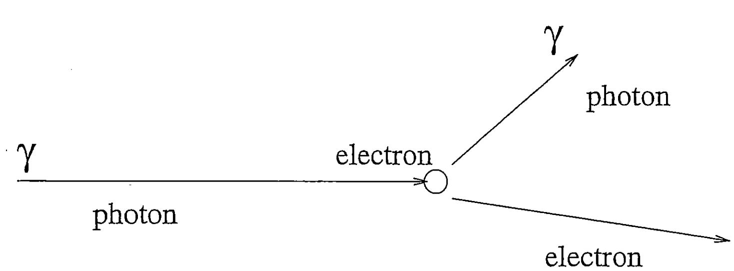


Figure 3.2: Compton scattering.

Thus the maximum kinetic energy of the final state electron is

$$T_e' \approx E_\gamma(1 - m_e/2E_\gamma) = E_\gamma - \frac{1}{2}m_e. \tag{3.73}$$

The Compton edge, or maximum value of the outgoing electron energy, is approximately 0.25 MeV below the incident gamma ray energy. This shows up as a sharp feature in the response of a detector to low energy gamma rays, since the detector response depends on measuring ionization in a material due to the secondary electron.[13,14,15]

The minimum gamma ray energy occurs in the backward direction ($\cos\theta = -1$) and one gets approximately the same values of E_γ' for a range of angles near 180°. This has the effect of producing a "backscatter" peak in the energy spectrum at approximately $E_\gamma' = \frac{1}{2}m_e$ for large values of E_γ.

3.6 Elastic, Inelastic, Quasi-Elastic Scattering

3.6.1 Introduction

In elastic scattering the outgoing particles are of the same type and have the same mass and quantum numbers as the incoming particles Figure 3.3. Thus, in a collision with a nucleus there is no internal excitation of the nucleus and in collisions between elementary particles there is no radiation of photons or secondary particle production. Four-momentum conservation gives $P_1 + P_2 = P_3 + P_4$, where the masses are $m_1, m_2, m_3 = m_1, m_4 = m_2$. Define $m = m_1 = m_3$.

3.6.2 Momentum Transfer

We have already defined the invariant $s = (P_1 + P_2)^2$, the square of the CM energy (Eq. 3.29). We now define a second invariant, t, the square of the four-momentum transfer,

$$t = (P_1 - P_3)^2. \tag{3.74}$$

This is a generalization of the non-relativistic three-momentum transfer, $\vec{Q} = \vec{p}_1 - \vec{p}_3$.

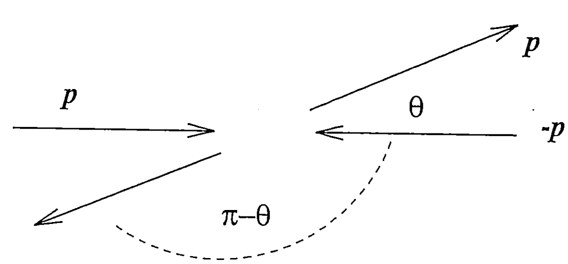


Figure 3.3: Elastic scattering.

In the CM system, since the initial and final particles have the same mass,

$$k \equiv |p_1^*| = |p_3^*| \tag{3.75}$$

$$p \equiv |p_2^*| = |p_4^*| \tag{3.76}$$

$$E^* \equiv E_1^* = E_2^* \tag{3.77}$$

$$(E^*)^2 = k^2 + m^2. \tag{3.78}$$

Now we evaluate the momentum transfer, t.

$$t = (P_1 - P_3)^2 = P_1^2 + P_3^2 - 2P_1P_3 = 2m^2 - 2(E^*)^2 + 2k^2 \cos\theta^* \tag{3.79}$$

$$t = -2k^2(1 - \cos\theta^*) = -4k^2 \sin^2\frac{\theta^*}{2}. \tag{3.80}$$

An alternate way to write t is

$$t = (P_1 - P_3)^2 = (E_1^* - E_3^*)^2 - (\vec{p}_1^{\,*} - \vec{p}_3^{\,*})^2 = -(\vec{p}_1^{\,*} - \vec{p}_3^{\,*})^2 \tag{3.81}$$

$$t = -2k^2(1 - \cos\theta^*). \tag{3.82}$$

So we see that $\sqrt{-t}$ is the three-momentum transfer:

$$\sqrt{-t} = 2k \sin\frac{\theta^*}{2}. \tag{3.83}$$

In the Lab system with a target particle, M, at rest we can write

$$t = (P_2 - P_4)^2 = (E_2 - E_4)^2 - (\vec{p}_2 - \vec{p}_4)^2, \tag{3.84}$$

but $E_2 = M$, the target particle mass, and $\vec{p}_2 = 0$. Then t can be written in terms of the kinetic energy, T_4, of the outgoing target particle:

$$t = (M - E_4)^2 - |\vec{p}_4|^2 = 2M^2 - 2ME_4 = -2MT_4, \tag{3.85}$$

also

$$t = m_1^2 + m_3^2 - 2(E_3E_1 - \vec{p}_3 \cdot \vec{p}_1). \tag{3.86}$$

3.6.3 Inelastic Scattering

In high energy lepton-nucleon scattering, $l + N \longrightarrow l + X$, the outgoing hadronic state may be an excited nucleon or a "jet" which subsequently breaks up into a shower of hadronic particles, pions, protons, etc. (Fig. 3.4).

Thus, the outgoing hadronic state has some effective mass, usually much larger than a nucleon mass. Such experiments have played an important role in understanding the structure of nucleons and nuclei. Electrons, muons, and neutrinos have all been used as the probe leptons for such investigations. If the final state hadronic system is a nucleon resonance, for example, the N^* or $\Delta(1238)$ resonance, then the process is called **quasi-elastic**.

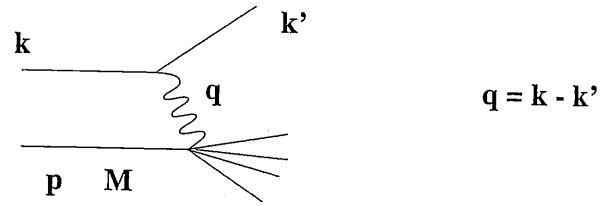


Figure 3.4: Inelastic scattering.

Let K_1 and K_2 be the four-momenta of the initial and final state leptons and P_N that of the initial nucleon, of mass M. Four-momentum conservation is

$$K_1 + P_N = K_2 + P_X; \tag{3.87}$$

rearrange and square to obtain W^2, the effective mass-squared of X:

$$W^2 = (K_1 - K_2 + P_N)^2. \tag{3.88}$$

This can be simplified if the lepton masses can be neglected compared with initial lepton energy. Then $W^2 = (K_1 - K_2)^2 + M^2 + 2P_N(K_1 - K_2)$:

$$W^2 = t + M^2 + 2M\nu; \tag{3.89}$$

where $\nu = E_1 - E_2$ is the change in the lepton's energy or the energy transferred from the initial lepton to the hadronic system. If the energy and momentum transfer to the hadronic system is very large ($Q^2 = -t >> M^2$ and $\nu >> M$) then the process is called **deep-inelastic** scattering.[16]

3.7 The Intermediate Vector Boson, $W^{\pm}$

When studying the decays of very short-lived particles, particularly massive particles, important information may be obtained from the distribution of transverse momentum of a decay product. The intermediate vector boson, $W^{\pm}$, has a mass of more than 80 GeV, is extremely short-lived, and decays into a charged lepton and a neutrino or a quark and anti-quark. We will calculate the transverse momentum of the charged lepton and see how it can be used to provide information on the W. The intermediate vector bosons, both $W^{\pm}$ and Z^0, can be produced (and were first discovered), in $\bar{p}p$ collisions. The Z^0 has been studied in great detail in e^+e^- collisions at CERN(LEP) and at SLAC(SLC) and $W^{\pm}$ pairs are being produced that way at CERN(LEP2).

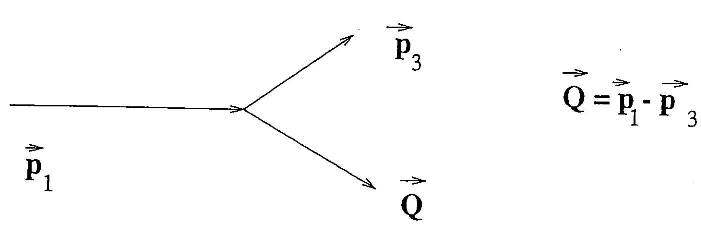


Figure 3.5: Three-momentum transfer.

3.7.1 Transverse Momentum Distribution

If the angular distribution in the rest frame of a decaying particle is isotropic, then $dw = d\Omega^*/4\pi$, where $d\Omega^* = d(\cos\theta^*)\,d\phi$ is the element of solid angle in the CM system. Then, integrating over ϕ,

$$dw = \frac{1}{2}d(\cos\theta^*). \tag{3.90}$$

As transverse momenta are invariant, we have $p_\perp = p_1^* \sin\theta_1^*$, from which

$$\cos\theta_1^* = \pm\sqrt{1 - \frac{p_\perp}{p_1^*}} \tag{3.91}$$

$$\left|\frac{d(cos\theta^*)}{dp_\perp}\right| = \frac{p_\perp}{p_1^*\sqrt{(p_1^*)^2 - p_\perp^2}}. \tag{3.92}$$

Now, to find the distribution in $p_\perp$ use

$$\frac{dw}{dp_\perp} = \frac{dw}{d(\cos\theta^*)}\frac{d(\cos\theta^*)}{dp_\perp} \tag{3.93}$$

$$\frac{dw}{dp_\perp} = \frac{p_\perp}{p_1^*\sqrt{(p_1^*)^2 - p_\perp^2}}. \tag{3.94}$$

The factor of 2 has canceled as a result of the double-valued square-root function; the distribution is folded over on itself. The range of $p_\perp$ is from zero to p_1^* and at the upper limit the density distribution becomes singular. The Jacobian of the transformation is singular, but the density is normalized to 1. There is a concentration of events in the Lab frame near the maximum allowed value of $p_\perp$. For massive decaying particles p_1^*, and therefore $p_\perp$ has a very simple form and the $p_\perp$ distribution gives us information on the mass of the decaying particle.

3.8 Neutrinos

3.8.1 High Energy Neutrino Beams

High energy neutrino beams[17] are produced at accelerators from the decays of charged pions and kaons. The π and K mesons are produced when high energy protons strike a target and they subsequently decay primarily via the reactions $\pi \longrightarrow \mu + \nu_\mu$ and $K \longrightarrow \mu + \nu_\mu$. The decay modes yielding electrons instead of muons are much less probable. Which decay modes would you expect to be the most important sources of electron-type neutrinos, ν_e? At medium energies (about 100 GeV) there are about 10% kaons in the meson beam.

In order to form an intense neutrino beam the mesons must be allowed a long decay path, their lifetime in the Lab frame being extended as a result of relativistic time dilation.

An additional important design criterion for a neutrino beam is that it not be contaminated with the muons which accompany the neutrinos from the decay reaction. This is accomplished by interposing a very thick muon absorber between the decay region and the neutrino detector. As muons do not have strong interactions and lose energy via ionization of the material (electromagnetic interaction), this shield must be quite thick. This puts the detector at a considerable distance from the original source of neutrinos and the angular divergence of the neutrino beam becomes an important consideration in designing the transverse dimensions of the detector. The detector, itself, must be as large as the budget permits since the neutrino cross section is extremely small.

A beam of either positive or negative mesons may be chosen according to whether one wants neutrinos or antineutrinos. Which beam polarity gives which? Which neutrino interaction channels are possible in each case?

The neutrino interaction with hadronic matter can be viewed as a fundamental weak neutrino interaction with a quark, mainly the u and d "valence" quarks of the nucleons. The outgoing quark then materializes into an hadronic shower. The outgoing lepton may be either a muon (charged-current interaction) or a neutrino (neutral-current interaction). Of course, the outgoing neutrino is never observed, since only an extremely small fraction of neutrinos interact in a typical detector. Can you estimate the fraction of neutrinos that interact per 1000 kg of material? The fundamental interactions are $\nu_\mu + d \longrightarrow \mu^- + u$ and $\overline{\nu}_\mu + u \longrightarrow \mu^+ + d$. Write the reactions for neutrons and protons.

Neutrino detectors are usually hadronic calorimeters[18,19] with tracking capabilities to observe outgoing muons. A typical "narrow band" neutrino beam and detector arrangement has a primary meson production target, a spectrometer region to momentum-select the mesons, followed by a decay region of length L_d, followed by a muon shield of length, L_s. Beyond the shield sits the neutrino detector of length, L, and diameter, D.

3.8.2 Neutrino Interactions

When the beam particle is a very high energy neutrino we can usually neglect the mass of the outgoing muon (that is $E_\mu = p_\mu$). Then $E_\mu = \gamma E^*_\mu(1 + \cos\theta^*)$. If the target mass is also small we have $\gamma = E_\nu/\sqrt{s}$ and $E^*_\mu = \frac{1}{2}\sqrt{s}$. So $E^*_\mu = \frac{1}{2}E_\nu$ and $E_\mu = \frac{1}{2}E_\nu(1 + \cos\theta^*)$

The inelasticity is $y = 1 - E_\mu/E_\nu$. Then $1 - y = \frac{1}{2}(1 + \cos\theta^*)$. So measurement of laboratory energies is sufficient to determine the CM scattering angle and the cross section. In the case of deep inelastic scattering the square of the momentum transfer is

$$Q^2 = 2E_\nu E_\mu(1 - \cos\theta_\mu), \tag{3.95}$$

where the quantities are evaluated in the Lab system. The energy transferred from the lepton system to the hadronic system is $\nu = E_\nu - E_\mu$.

3.8.3 Astrophysical and Solar Neutrinos

The most intense part of the neutrino spectrum from astrophysical or solar sources is between 0.5 and 50 MeV, although solar neutrinos fall into the lower part of this range.[20,21,22] A dominant component of the solar neutrino spectrum from the reaction $p + p \longrightarrow d + e^+ + \nu_e$ is below 0.42 MeV. Neutrinos from other reactions in the solar cycle extend up to about 19 MeV with the important ^{8}B reaction neutrinos having an endpoint at about 15 MeV.

Most of the earlier detectors of solar neutrinos depended on neutrino-induced nuclear reactions and chemical separation of the nuclear species. The very well known chlorine experiment by Davis and collaborators which has been running for nearly 30 years depends on the neutrino conversion of ^{37}Cl to ^{37}Ar which then decays back to ^{37}Cl. More recent experiments (SAGE and GALLEX) operate on the same principle but with ^{71}Ga as the neutrino target. These chemical extraction experiments have virtually no sensitivity to neutrino energy, direction, or time of arrival except insofar as there is an energy threshold for the reaction, and the chemical extraction process is carried out periodically.

Other recent detectors and those under construction or being proposed detect neutrinos in "real time" by observing neutrino-electron scattering, $\nu + e- \longrightarrow \nu + e-$. This two-body reaction can take place for any neutrino flavor. For neutrino energies larger than about 6 MeV the recoil electron direction is strongly correlated with the incident neutrino direction. One can also take advantage of the two-body kinematic constraints to establish the neutrino energy spectrum. The outgoing electron can be detected on an event-by-event basis by taking advantage of Čerenkov radiation or ionization. The angular resolution for the low energy secondary electron may be as large as 27°, as in the Kamiokande series of water Čerenkov detectors in Japan, but still permits a valuable constraint on the direction of the incident neutrino and on its energy. Furthermore, the neutrino time of arrival is known and is especially interesting for neutrinos of astrophysical origin from the galactic center or from supernovae.

The SNO detector in Sudbury, Ontario, Canada, utilizing heavy water as the neutrino target has the unique capability to observe neutral-current events through the process $\nu + d \longrightarrow \nu + p + n$ in addition to charged-current events $\nu_e + d \longrightarrow p + p + e^-$. The neutral-current reaction is accessible for all three neutrino flavors (e, μ, τ). It is planned to detect the neutron emerging from the neutral-current neutrino reaction using ^{3}He neutron detectors.

3.9 Exercises

3.1 **Cosmic Ray Pions and Muons**

A charged pion is produced in the upper atmosphere with an energy of 5 GeV. The mean lifetime of the pion is 2.6×10^{-8} seconds and the mean lifetime of the muon is 2.2 μs.

a. What is the mean decay length of the pion?
b. Suppose a pion decays into a muon at an altitude of 10^5 feet. What must be the γ and momentum of the muon if it can reach the surface of the Earth within one mean lifetime?

3.2 **Classical CM Energy**

A simple classical analogue for the role of center-of-mass energy is the following: two blocks are moving toward each other; one of them has a spring attached to the side which compresses during the collision. Work this out in detail for the special case of equal-mass blocks with one block at rest initially.

a. What is the maximum compression of the spring? This is very easy to work out if one first calculates the energy in the CM system. This problem illustrates the value of separating the total energy into energy *in* the CM and energy *of* the CM motion.
b. Show that the behavior of the system after the collision is much easier to understand when viewed in the CM frame.

3.3 **Creation and Annihilation of Pairs**
The reactions $\gamma \longrightarrow e^+ + e^-$ and $e^+ + e^- \longrightarrow \gamma$ are allowed by conservation of charge, lepton number, etc. Show that they are not consistent with energy-momentum conservation. Pair production and annihilation are observed, however; how does that come about?

3.4 **Colliders vs. Fixed-Target Accelerators**
The magnitude of the center of mass energy, $\sqrt{s}$, determines what excitations or reactions can take place. Which has a higher CM energy: a 10 GeV proton colliding with a target electron or a 10 GeV electron colliding with a target proton? What is the advantage of a "collider" in which 45 GeV electrons and positrons collide as compared with a beam of 90 GeV positrons hitting a stationary electron? In each case what is the CM energy available to create new particles?

3.5 **Reaction Thresholds**
In this problem you can use the program RELKIN to investigate reaction thresholds. You can check the results with pencil-and-paper calculations.

a. Find the threshold energy for producing K^0 mesons plus Λ^0 in pion-proton collisions. Is there a lower threshold for producing K^0 mesons using other reactions?
b. Early synchro-cyclotrons were built to be above the threshold for production of pions. Although beryllium targets were often used (proton-Be collisions) you may assume that the target is a proton: what proton beam energy is required?
c. The mass of the tau lepton[23] was recently (1993) measured with very high precision at the Beijing Electron Synchrotron by researchers studying the tau production threshold using electron-positron collisions. They obtained $m_\tau = 1776.9 \pm 0.2 \pm 0.2\,\mathrm{MeV/c^2}$, where the first uncertainty is statistical and the second is systematic. What electron and positron energies are required? Assume that the electrons and positrons have the same energy. If you knew the tau mass only approximately, what strategy would you choose for running the experiment? That is, what electron and positron energies would you choose?

3.6 **Asymmetric Colliders**
Although most particle colliders run with equal beam energies, there are some interesting exceptions. An asymmetric B meson "factory" is being built at the Stanford Linear Accelerator Center (SLAC). In this machine

the positron and electron beam energies will be 3.1 GeV and 9 GeV, respectively.

a. What is the center-of-mass (CM) energy?
b. At LEP where the electron and positron beams each have energies of about 45 GeV, tau lepton pairs have a unique energy (except for small effects due to electromagnetic radiation). What is the mean separation in space between the decay points of the two tau leptons?
c. At the SLAC B factory suppose that one of the tau leptons is produced in the forward direction of the high energy beam. What will be the energies of the two tau leptons and what will be the mean separation between their decay points? The program RELKIN may be adapted to problems involving asymmetric colliders by considering the decay of a fictitious particle whose mass is equal to the CM energy.

3.7 **Calculating Invariant Mass**
In this problem you can use the program RELKIN to provide values of kinematic variables that you need to calculate the invariant mass of a pair of decay products. The program will also plot the invariant mass of decay products.

a. Use the computer program to observe the decay of a particle (for example, the K^0). Record the information you will need to compute the invariant mass and calculate it.
b. Observe the decay of the ρ meson into two pions and use the program to plot the invariant mass. Explain the distribution you observe.

3.8 **Maximum Angle**
Based on the expression for $\tan\theta$ in two-body decay decide under what conditions there is a maximum angle in the Lab. Use the program to study pion decay ($\pi \longrightarrow \mu\nu$) and establish the conditions for a maximum muon Lab angle. Does the neutrino have a maximum Lab angle? Explain.

3.9 **Classical Energy Distribution**
In this problem you can explore scattering in the non-relativistic limit. This approximation is often adequate, for example, for neutron scattering in the MeV energy range.

a. Use the program to study low energy ("low" means in the non-relativistic regime) neutron-proton or neutron-nucleus scattering. Since the nucleon mass is of order 1000 MeV, energies in the tens of MeV are non-relativistic. What are β and γ? Observe the energy distribution. Based on observations you make with the program, explain why water is a better neutron moderator than lead.
b. Work out the relationship between CM angle and Lab angle for classical non-relativistic scattering of two masses. Assume isotropic scattering in the CM system and deduce the Lab energy distribution. What are the minimum and maximum energies? Partial answer: $\tan\theta = \sin\theta^*/(\cos\theta^* + m_1/m_2)$. $E_{min} = E_1(m_1 - m_2)^2/(m_1 + m_2)^2$.
c. Suppose a neutron makes two successive collisions with a proton. What is the resulting energy distribution? Remember that the secondary energy distribution after one collision has a very simple form.

Write a short Pascal program to calculate the outcome of two collisions. Generalize to multiple collisions. This is very important in nuclear reactors and was worked out and discussed by Fermi in his lecture notes on nuclear physics.[24,25]

3.10 **Proton Decay**

Some reactions permitted in the program menu have not been observed. What conservation laws are operative? In this exercise you can set a crude limit on the proton decay rate into $e^+\pi^0$ based on the number of protons in your body. You can estimate the number of protons in your body from your mass and suppose that these protons decayed at some finite rate. Suppose also that the total energy deposition rate due to e^+ from this decay is no larger than that due to cosmic radiation. The average energy deposition due to cosmic radiation in the US[12] is approximately 1 mGy (1 milligray). A "gray" (Gy) corresponds to 1 joule per kilogram, or 6.24×10^{12} MeV/kg. What, then, is the limit on the proton lifetime? Of course, only a crude estimate is sufficient.

3.11 **Opening Angle in π^0 Decay**

What is the minimum allowed θ for a given γ? What is $\cos\theta_1^*$ for the minimum value of θ? What are the gamma energies at this minimum? Use the program to observe π^0 decays. Under what conditions is the opening angle a minimum?

3.12 **The Decay Seen in the Lab Frame**

A π^0 has a momentum of 5 GeV/c in the Lab system.

a. Use the program RELKIN to observe typical decay angles in the lab and typical photon energies. Note the range of energies and angles.
b. From the **Plot Variables** menu set up the program to plot the gamma ray opening angle; observe the characteristic shape and then use the plot to help you reconstruct (calculate) the π^0 mass. In Chapter 4 this exercise is extended to include properties of the gamma ray detector.
c. If you measure the gamma ray energies and angles, do you have enough information to determine the decay point of the π^0?

3.13 **Compton Edge**

Compton scattering is an important process when photons interact in matter. In laboratories, photons in the MeV range are readily available from radioactive sources and are often used to calibrate and evaluate particle detectors.

a. Describe the Compton effect for photon energies available in the laboratory, for example, 0.66 MeV ^{137}Cs, and also for very low energy photons, for example, 50 keV. Where is the Compton edge and what is the energy range of secondary photons? Is the Compton effect likely to be important for such photons in copper? in beryllium? You should look up graphs of photon interaction probabilities as a function of the photon energy and the atomic number of the material.
b. Use the program RELKIN to determine the energy of the Compton edge for ^{24}Na gamma rays (2.75 MeV).

3.14 **Backscatter**
Use the program to find the range of Lab angles near the backward direction that would contribute to a backscatter peak. For ^{60}Co gamma rays (there are two closely spaced lines whose average energy is approximately 1.25 MeV) what is the energy of the backward peak?

3.15 **Three-Momentum Transfer**
Derive $Q^2 = 2k^2(1 - \cos\theta^*)$ where k is the incoming and also outgoing momentum of a particle in the CM system and θ^* is the scattering angle. $\vec{Q} = \vec{p}_1 - \vec{p}_2$. Work in the CM system (Fig. 3.5).

3.16 **$W^{\pm}$ Decay**
For the charged intermediate vector boson W, assuming an isotropic decay in the CM system, use the program to estimate the fraction of decay events with $p_\perp > \frac{1}{2}p_1^*$. What is the maximum value of $p_\perp$ in this case? This technique is commonly used to identify heavy mesons containing b quarks and distinguish them from particles containing only lighter quark species (u, d, s, c).

3.17 **Charged Pion Decay** Use the program to investigate the $p_\perp$ distribution for $\pi \longrightarrow \mu\nu$ and compare with the decay of a heavier particle such as the W boson.

3.18 **Neutrinos from Meson Decay** Assume that the primary π and K mesons have a momentum of 10 GeV/c in the Lab system. In practice there is always some spread in momentum but we can ignore that for now. Use the program RELKIN to observe typical decay angles in the lab and typical muon and neutrino energies. Note the range of energies and angles. You will have to set an appropriate scale on the plots to record the very small angles. Use the **Control/Plot Variables** menu for this purpose. Derive an analytic expression for the maximum neutrino energy.

3.19 **Narrow Band Neutrino Beams**
"Narrow band" neutrino beams are produced from nearly mono-energetic pion and kaon beams in which the pions and kaons are allowed to decay.

a. What are the decay processes which give rise to neutrinos? Do they provide muon-type or electron-type neutrinos?
b. Do the neutrinos in the beam come mainly from pion decay or kaon decay? What is the fraction from each of the mesons? Consider the lifetimes of the mesons and their decay modes.
c. Are there any electron-type neutrinos in the beam? Why or why not? Knowledge about the electron-neutrino contamination is crucial if the beams are used to study muon neutrino oscillations.
d. Use the program RELKIN to observe the decay angles and energies of the neutrinos.
e. Use the program to help you estimate the size and momentum spread of such neutrino beams.

3.20 **Tagged Beams**
"Tagged" neutrino beams are being designed. What is involved in being able to tag the energy of each neutrino? Use the programs to observe

the energy and angle of the muon used as a tag. In chapter 4 this series of exercises is extended to include properties of the neutrino beam and detector.

3.21 Solar Neutrinos

In a proposed solar neutrino experiment (Hellaz) the low energy neutrinos scatter on electrons in high pressure cold helium gas. The gas is inside a *time projection chamber*, so that the direction and range of the outgoing charged track can be measured. Assume that the direction of the sun is known.

a. Derive an expression for the neutrino energy as a function of the secondary electron angle and kinetic energy.

b. For neutrinos in the few MeV range verify your result with the computer program.

c. Work out the maximum neutrino energy in the dominant solar neutrino production process $p + p \longrightarrow d + e^+ + \nu_e$.

d. For neutrinos above about 6 MeV the secondary laboratory angle tends to be small (of order $\sqrt{m_e/E_\nu}$). Verify this with the computer program. How does this help in identifying neutrino events?

3.22 Astrophysical Neutrinos; Quasi-Elastic Nucleon Scattering

Assume that the reaction is $\overline{\nu} + p \rightarrow e^+ + n$ and that you have detailed information on the final positron. How well can you determine the neutrino energy and direction? What advantages and disadvantages does this reaction have as compared with elastic electron scattering?

3.23 Neutrino Mass

The muon neutrino mass can be determined from pion decay at rest.

a. Consider the kinematics of pion decay and derive an expression for the neutrino mass in terms of other masses and measurable quantities.

b. Three-body decays (kaon decay and tau lepton decay) have been proposed as effective ways to set limits on the neutrino mass. Comment on the advantages and disadvantages of three-body decay modes. How would you use a three-body mode as in kaon decay to determine the neutrino mass? Compare this approach with what was done in the case of tritium beta-decay to study the electron neutrino mass.[26]

c. Good measurements have also been done using pion decay in flight. Try to find references (the "Review of Particle Properties"[12] may be helpful) for these experiments and comment on the advantages and disadvantages. Use the program to simulate conditions that have been used.

d. Modify the particle data table (file PART.DAT) to give the neutrino a mass and observe changes in the kinematics. You will need to give the neutrino a substantial mass in order to be able to observe the effects.

3.10 Programs

3.10.1 Relativistic Kinematics—RELKIN

Features

This program calculates center-of-mass frame (CM) and laboratory frame (Lab) kinematic quantities for two-body scattering and two-body decay reactions of elementary particles. The initial and final state may be chosen from among options listed in menus. It is easy to include additional options. CM and Lab quantities are calculated and diagrams showing momentum vectors for final state particles in both frames are displayed along with numerical values of the various momenta, energies, angles, etc.

A Monte Carlo simulation is provided with $\cos\theta^*$ and ϕ^* in the CM system randomized for an isotropic CM distribution of final state particles. The Monte Carlo simulation generates a preselected number of events and displays the vector diagrams for each event while also plotting kinematic quantities. A scatter plot of one kinematic quantity against another and a histogram of one of these quantities are updated as the events are generated. The variables to be plotted can be chosen with a menu. The variables include momentum, energy, θ, $\cos\theta$, transverse momentum, kinetic energy, angle difference, invariant mass. The user may select these variables for either of the final state particles in either reference frame. The variable that is histogrammed is the one displayed on the x-axis of the scatter plot.

Particle masses, lifetimes, and other data are read from an input file, PART.DAT. The initial and final state particles for a number of reactions and decays are read from the file REACT.DAT. The format of the reaction list file is as follows:

1. A comment line.
2. Initial state particles.
3. Number of different final states (maximum is 4).
4. A line for each set of final state particles.
5. A blank line.
6. Repeat sequence starting with initial state particles.

Standardized names are used to designate the particles, for example, neutrino, pizero, kzero, etc. See the file PART.DAT for the names of particles known by the program.

Control of the Monte Carlo run and resetting of plots are implemented with hot keys. Pull-down menus provide choices of initial and final state particles as well as plot variables and scales.

The Six-Window Display

- Vectors representing the momenta of the final state particles in the CM and Lab frames are drawn (Fig. 3.6).
- Numerical values of the kinematic quantities for CM and Lab frame are displayed below the vector diagrams.

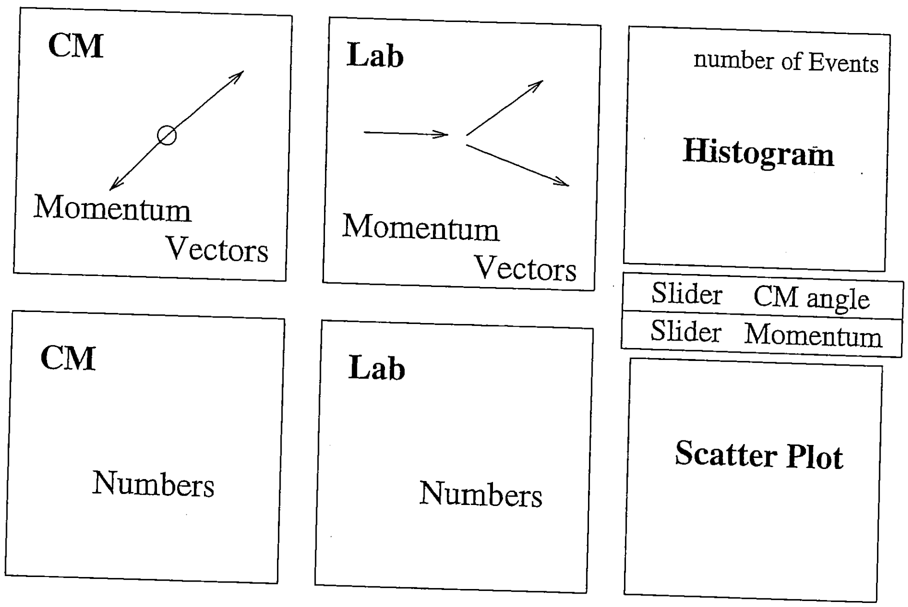


Figure 3.6: The six windows.

- A scatter plot and a histogram are displayed for variables selected from a predefined set.

User-Selected Quantities

- The initial state particle(s) and final state particles can be chosen from a menu.
- The momentum of the beam or decaying particle and the CM angle of an outgoing particle may be controlled with sliders or specified in an input screen.
- Display of vectors may be rescaled to an automatically chosen size.
- Single events may be processed or the Monte Carlo run may be enabled.
- The speed of Monte Carlo and the number of events to be generated are specified in an input screen.
- Variables and reference frames for the plots as well as plot scales may be specified on input screen.
- Writing an output file of variables may be enabled.
- Plots and the event counter may be cleared.
- Particle masses and lifetimes may be displayed on a help screen.

Implementation

- Unit RELKIN. The main program.
- Unit RELSCAT. Initialization, menus, input screens, key handling.
- Unit PSCAT12. Kinematic calculations, plot momentum vectors, list numerical values.
- Unit PLOT2. Histogram and scatterplot.
- Unit RELUTIL. Utilities for plotting and listing numbers. Functions.

3.10.2 Running the Program RELKIN

Menus

An arrow, ⟶, following the menu item means that a pull-down menu will be opened.

The top level menu items are

File	Initial	Final	Control	MonteCarlo	Help

- **File**: ⟶ "About," output file, exit.
  - — **About Program**
  - — **About CUPS**
  - — **Enable Output File**
  - — **Close, Disable File**
  - — **Exit Program**
- **Initial**: ⟶ Select the collision or decay initial state. Choices are listed.
- **Final**: ⟶ Final state; choices are dynamically loaded by program and listed.
- **Control**: ⟶ Control of scales and parameters.
  - — **Redraw**: Generate and draw event.
  - — **Rescale**: Rescale vectors automatically.
  - — **Choose Input**: Initial conditions: CM angle, beam particle momentum.
  - — **Choose Scale**: ⟶Change vector scale manually.
  - — **Clear Plots**: Clear plots.
  - — **Plot Variables**: ⟶Choose plot variables and scales.
  - — **Particle data**: Display particle data file.
- **MonteCarlo**: ⟶ Control of Monte Carlo parameters.
  - — **Fast MC**: Run Monte Carlo at high speed.
  - — **Slow MC**: Run Monte Carlo at low speed.
  - — **Parameters**: ⟶ Change Monte Carlo event limit, pause time between events.

- **Help** : ⟶ Help files.
  - — **Reactions**
  - — **Display**
  - — **Plots**
  - — **Input**

Hot Keys

F1–Help	Display help screens.
F5–Draw Evt	Generate and display a new event.
F6–Run MC	Start Monte Carlo generation of events.
F7–Stop MC	Stop Monte Carlo generation of events.
F8–Clr/Rescl	Clear plots, rescale vectors.
F10–Menu	Return to menu.

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